FAULT-TOLERANT SIMULATION OF LATTICE GAUGE THEORIES WITH GAUGE COVARIANT CODES

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Nuclear
STructure and
REaction

GOALS AND OBJECTIVES

Quantum error correction

Quantum computers can undergo errors.

We can define symmetries and conserved quantities.

If something is violated, we know an error occurred.

arXiv:2405.19293

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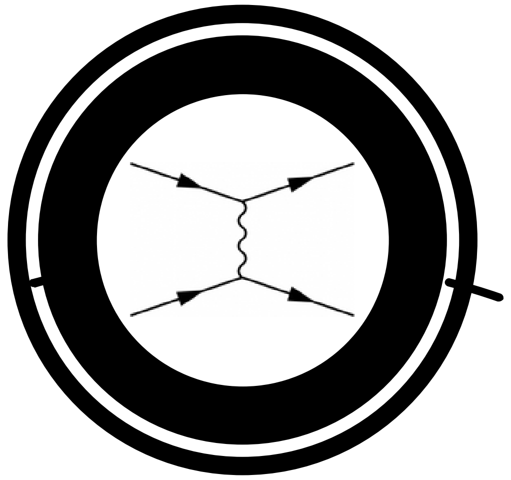
Gauge Theories

Gauge Theories are physical theories with a gauge symmetry, which is a local symmetry.



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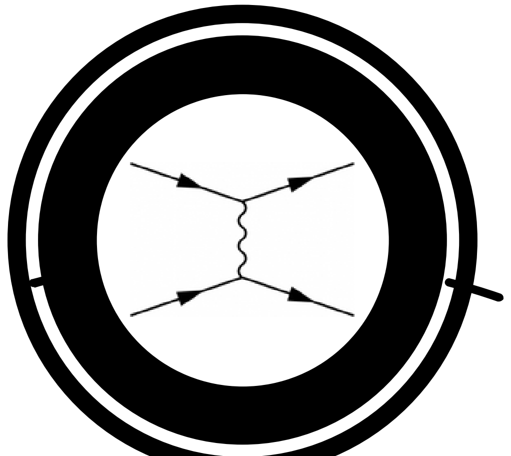

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Linking two fields

We can use the Gauge symmetry as a symmetry to do error correction.

If it is violated, an error occurred.

QUESTIONS: what type and how many errors can we correct with the gauge symmetry?

QUANTUM ERROR CORRECTION

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Classical computes:

bit: $0, 1$

Possible errors:

bit-flip: $$0 \to 1$$ $$1 \to 0$$

We can correct errors adding redundancy:

$$0_L = 000$$

$$010 \to 000$$

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Quantum computes:

qbit: $|\psi\rangle = a|0\rangle + b|1\rangle$

Possible errors:

bit-flip: $|0\rangle \rightarrow |1\rangle$
$|1\rangle \rightarrow |0\rangle$

phase-flip: $|0\rangle \rightarrow |0\rangle$
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$$|0\rangle_{LL} \to |0_L 0_L 0_L\rangle$$

$$|0\rangle_L \to |+++\rangle$$

We cannot measure the state to know if an error happened

EXAMPLE: BIT-FLIP

Which operators are we alloed to measure without making the wavefunction collapse?

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Remember:

$$Z|0\rangle = |0\rangle$$

$$Z|1\rangle = -|1\rangle$$

We can measure the parity between 2 qubits:

$$ZZ|00\rangle = |00\rangle$$

$$ZZ|01\rangle = -|01\rangle$$

$$ZZ|10\rangle = -|10\rangle$$

$$ZZ|11\rangle = |11\rangle$$

Without destroying superpositions:

$$ZZ(|00\rangle + |11\rangle) = (|00\rangle + |11\rangle)$$

$$ZZ(|01\rangle + |10\rangle) = -(|01\rangle + |10\rangle)$$

EXAMPLE: BIT-FLIP

Which operators are we alloed to measure without making the wavefunction collapse?

The logical states:

$$|0\rangle_L \to |000\rangle$$
$$|1\rangle_L \to |111\rangle$$

Stabilizers:

$$S_1 = Z_1 Z_2$$
$$S_2 = Z_2 Z_3$$

Remember:

$$Z|0\rangle = |0\rangle$$
$$Z|1\rangle = -|1\rangle$$

S_1	S_2	error
-	+	1
-	-	2
+	-	3

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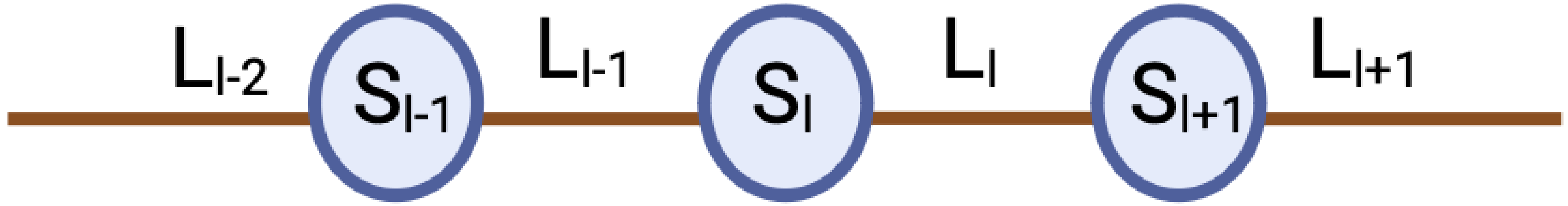
LATTICE GAUGE THEORIES

Consider an abelian, truncated, lattice gauge theory



On links there is the Gauge field with 2 possible values:

zero $|0\rangle$

one $|1\rangle$

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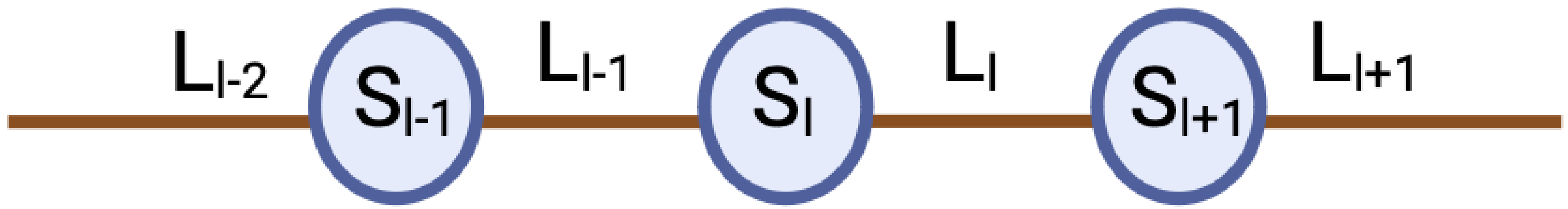

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Assuming no particles in the system, the incoming and outgoing fields have to be the same:

$$Z_{L_{l-1}} Z_{L_l} |\psi\rangle = |\psi\rangle$$

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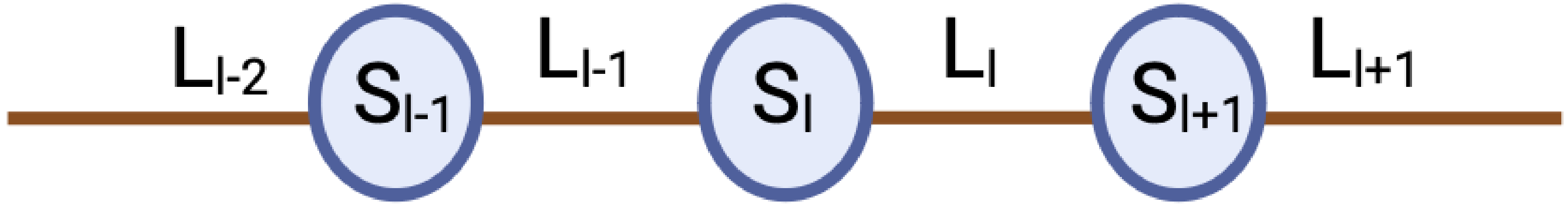

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Assuming no particles in the system, the incoming and outgoing fields have to be the same:

$$Z_{L_{l-1}} Z_{L_l} |\psi\rangle = |\psi\rangle$$

It is the repetition code, with a number of copies equal to the number of links

$$G_l = Z_{L_{l-1}} Z_{L_l}$$

So, we can correct every bit-flip error

FUTURE WORK

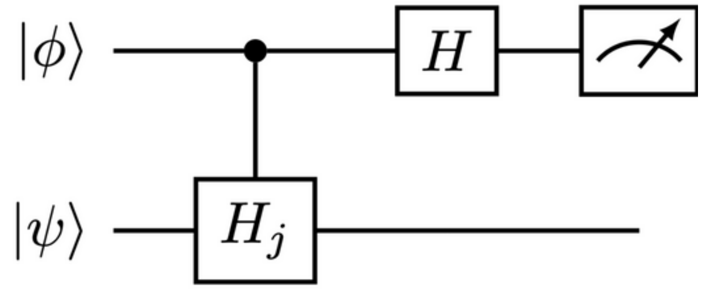
arXiv:2405.19293

- ✓ fermions
- ✓ more spatial dimensions
- ✓? higher bond dimension
- ? non-abelian theories

✓ Fault-tolerant quantum simulation (Trotter)

$$e^{iHt} = e^{it\sum_j c_j H_j} \approx \prod_j e^{itc_j H_j}$$

$$e^{itc_j H_j} =$$



CONCLUSIONS

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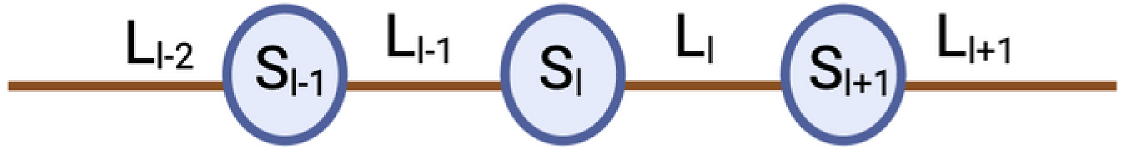

We want to simulate a system with a gauge symmetry

$$G_l = Z_{L_{l-1}} Z_{L_l}$$

We can use the gauge symmetry to detect and correct every X error

In this way we can save memory, and easily perform quantum simulations

- ✓ fermions
- ✓ more spatial dimensions
- ✓? higher bond dimension
- ? non-abelian theories

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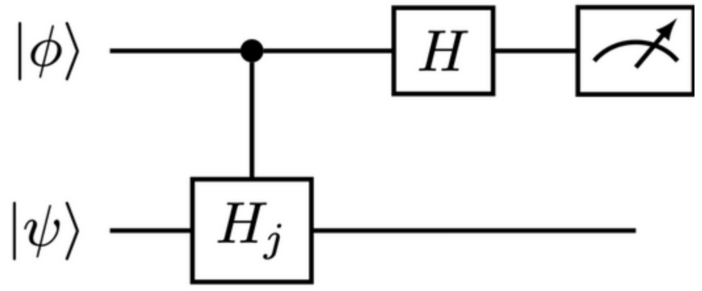
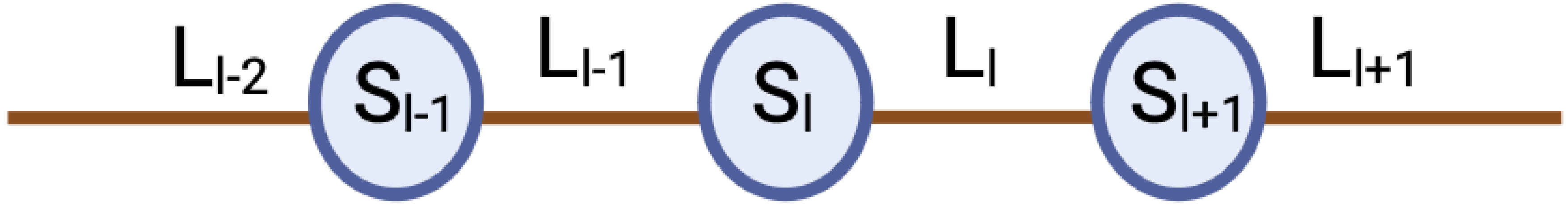


THANK YOU

FERMIONS



If the site is empty:

$$Z_{S_l}|\psi\rangle = |\psi\rangle$$

Incoming and outgoing fields have to be the same:

$$Z_{L_{l-1}}Z_{L_l}|\psi\rangle = |\psi\rangle$$

If the site is full:

$$Z_{S_l}|\psi\rangle = -|\psi\rangle$$

Incoming and outgoing fields have to be different:

$$Z_{L_{l-1}}Z_{L_l}|\psi\rangle = -|\psi\rangle$$

Sites can be:

empty $|0\rangle$

full $|1\rangle$

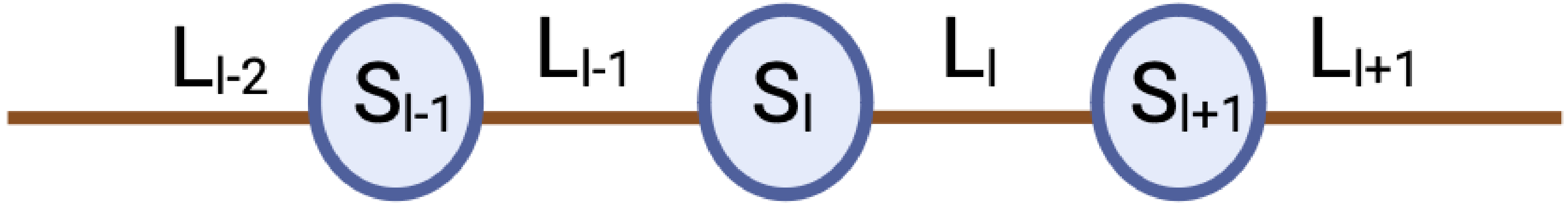
On links there is the Gauge field with 2 possible values:

zero $|0\rangle$

one $|1\rangle$

$$\boxed{G_l = Z_{L_{l-1}}Z_{S_l}Z_{L_l}}$$

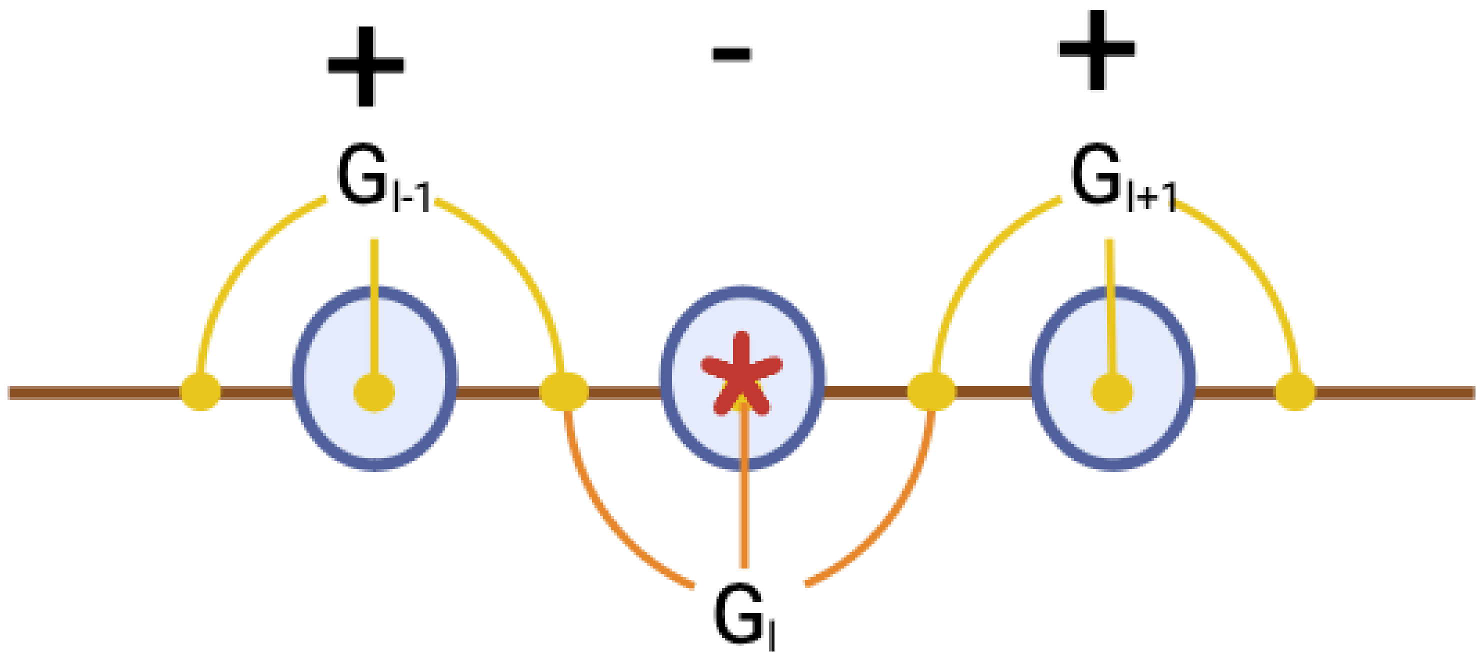
MEASUREMENT AND CORRECTION

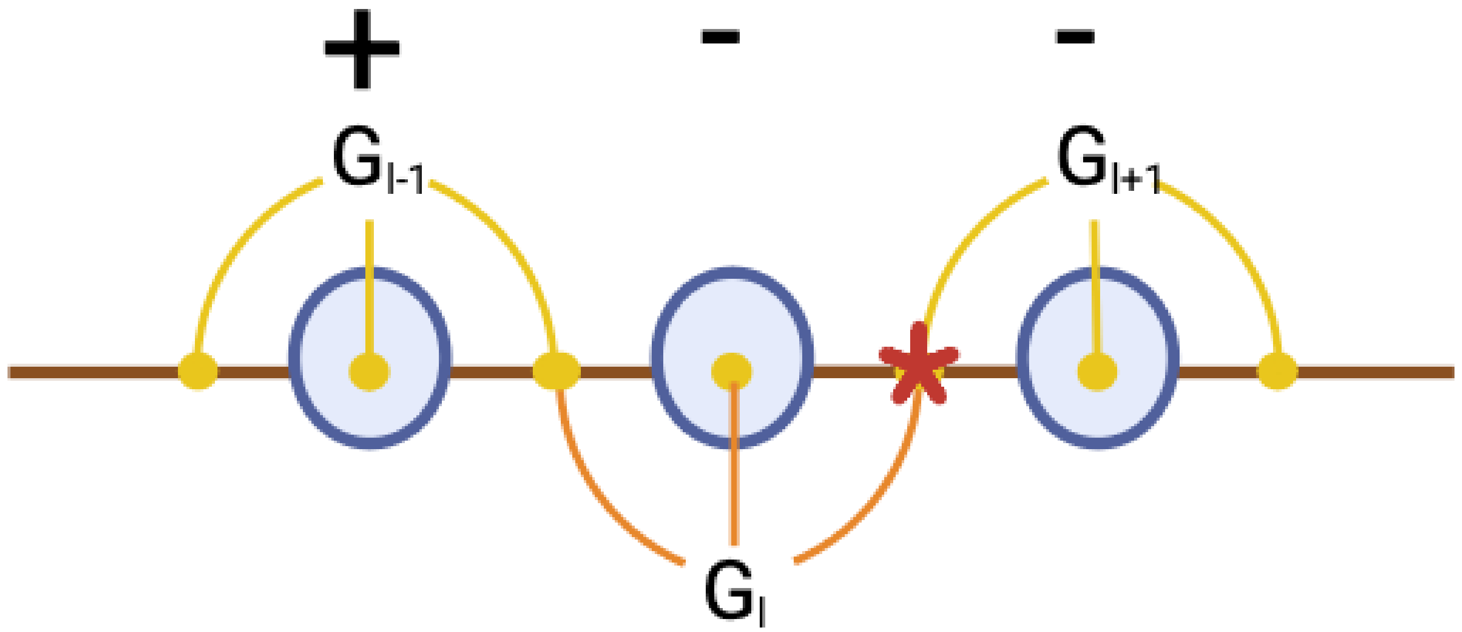


$$G_l = Z_{L_{l-1}} Z_{S_l} Z_{L_l}$$

$$G_l|\psi\rangle = |\psi\rangle$$

$$G_l(X_{S_l}|\psi\rangle) = -X_{S_l} G_l|\psi\rangle = -X_{S_l}|\psi\rangle$$





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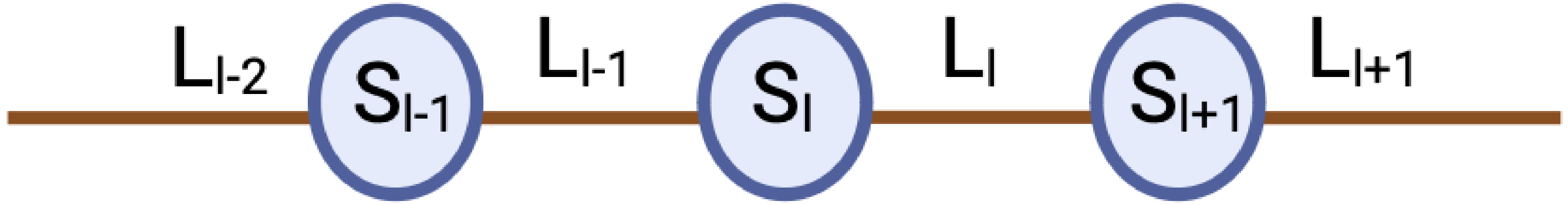


$$G_l = Z_{L_{l-1}} Z_{S_l} Z_{L_l}$$

$$G_l|\psi\rangle = |\psi\rangle$$

G_{l-1}	G_l	G_{l+1}	error location
$+$	$+$	$+$	none
$-$	$-$	$+$	L_{l-1}
$+$	$-$	$+$	S_l
$+$	$-$	$-$	L_l

We can correct every X error in the system, but we cannot detect Z errors

To correct Z errors we can use more layers of redundancy

TIME EVOLUTION

The hamiltonian, can be written as

$$H = \sum_j c_j H_j$$

The time evolution operator we want to apply is

$$e^{iHt} = e^{it \sum_j c_j H_j}$$

To simplify the implementation, we can break up the operator, approximating it:

$$e^{it \sum_j c_j H_j} \approx \prod_j e^{itc_j H_j}$$

This is the first-order Trotter formula, and the error is:

$$\left\| e^{itH} - \prod_j e^{itc_j H_j} \right\| \leq t^2 \sum_j \left\| \sum_k [H_j, H_k] \right\|$$

So we need a way to implement the single exponentials

But we can apply easily on the system only the logical operations

They correspond to Pauli matrices on the logical qubits

TIME EVOLUTION

How do we implement

$$e^{itc_jH_j}$$

Assuming the Hamiltonian is a sum of Pauli matrices:

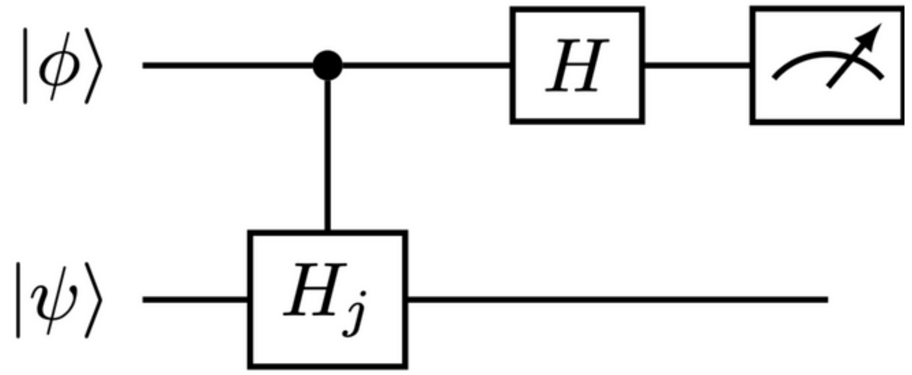
$$e^{itc_jH_j} = \cos(tc_j) + i\sin(tc_j)H_j$$

In this way we move the problem of applying the exponential, to the problem of preparing an ancilla qubit state

To do this, let us assume to have an ancilla qubit on which we can do arbitrary rotations, prepared in the following state:

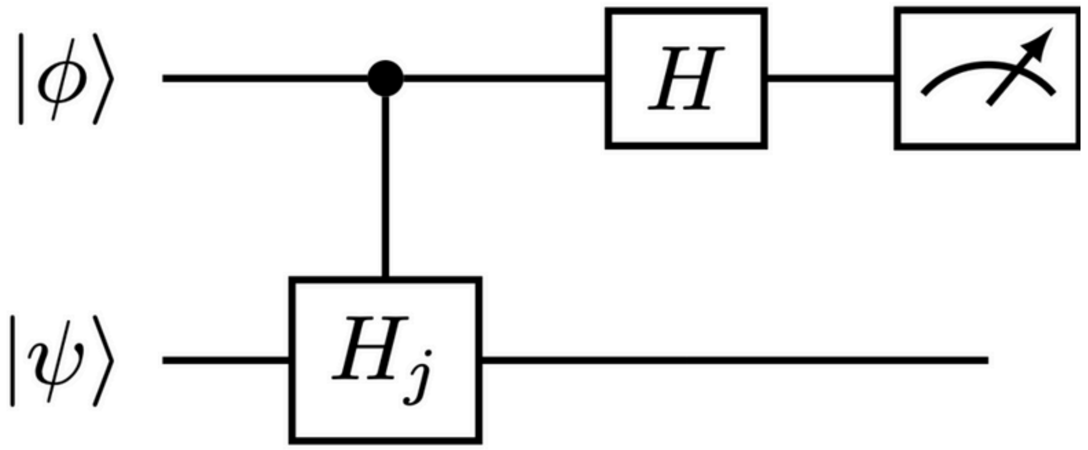
$$|\phi\rangle = \cos(tc_j)|0\rangle + i\sin(tc_j)|1\rangle$$

Then, the following circuit applies the right exponential:



IMPLEMENTATION OF TROTTER

Why the following circuit implements the right exponential?



$$|\phi\rangle = \cos(tc_j)|0\rangle + i\sin(tc_j)|1\rangle$$

$$|\phi\rangle|\psi\rangle = \cos(tc_j)|0\rangle|\psi\rangle + i\sin(tc_j)|1\rangle|\psi\rangle$$

$$CH_j \to \cos(tc_j)|0\rangle|\psi\rangle + i\sin(tc_j)|1\rangle H_j|\psi\rangle$$

$$\begin{aligned} H &\to \cos(tc_j)\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)|\psi\rangle + i\sin(tc_j)\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)H_j|\psi\rangle \\ &= \frac{1}{\sqrt{2}}(\cos(tc_j) + i\sin(tc_j)H_j)|0\rangle|\psi\rangle + \frac{1}{\sqrt{2}}(\cos(tc_j) - i\sin(tc_j)H_j)|1\rangle|\psi\rangle \\ &= \frac{1}{\sqrt{2}}|0\rangle e^{itc_jH_j}|\psi\rangle + \frac{1}{\sqrt{2}}|1\rangle e^{-itc_jH_j}|\psi\rangle \end{aligned}$$

The circuit applies with probability 1/2 the right exponential, with probability 1/2 its hermitian conjugate

We can apply always the right exponential with a cycle of oblivious amplitude amplification

HAMILTONIAN

The starting Hamiltonian:

$$H = m\sum_l (-1)^l \psi_l^\dagger \psi_l + \epsilon \sum_l (\psi_l^\dagger Q_l \psi_{l+1} + \psi_{l+1}^\dagger Q_l^\dagger \psi_l) + 2\lambda_E \sum_l P_l$$

Fermionic operators $\psi = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \frac{1}{2}(1+Z)X$

Field operators

$$Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = X$$

$$P = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = Z$$

In terms of Pauli matrices:

$$H = \frac{m}{2}\sum_l (-1)^l (1 - (-1)^l Z_{S_l}) + \frac{\epsilon}{2}\sum_l (1 + Z_{S_l} Z_{S_{l+1}}) X_{S_l} X_{L_l} X_{S_{l+1}} + 2\lambda_E \sum_l Z_{L_l}$$

Logical operations:

$$\bar{Z}_l = Z_{L_l}$$

$$\bar{X}_l = X_{S_l} X_{L_l} X_{S_{l+1}}$$

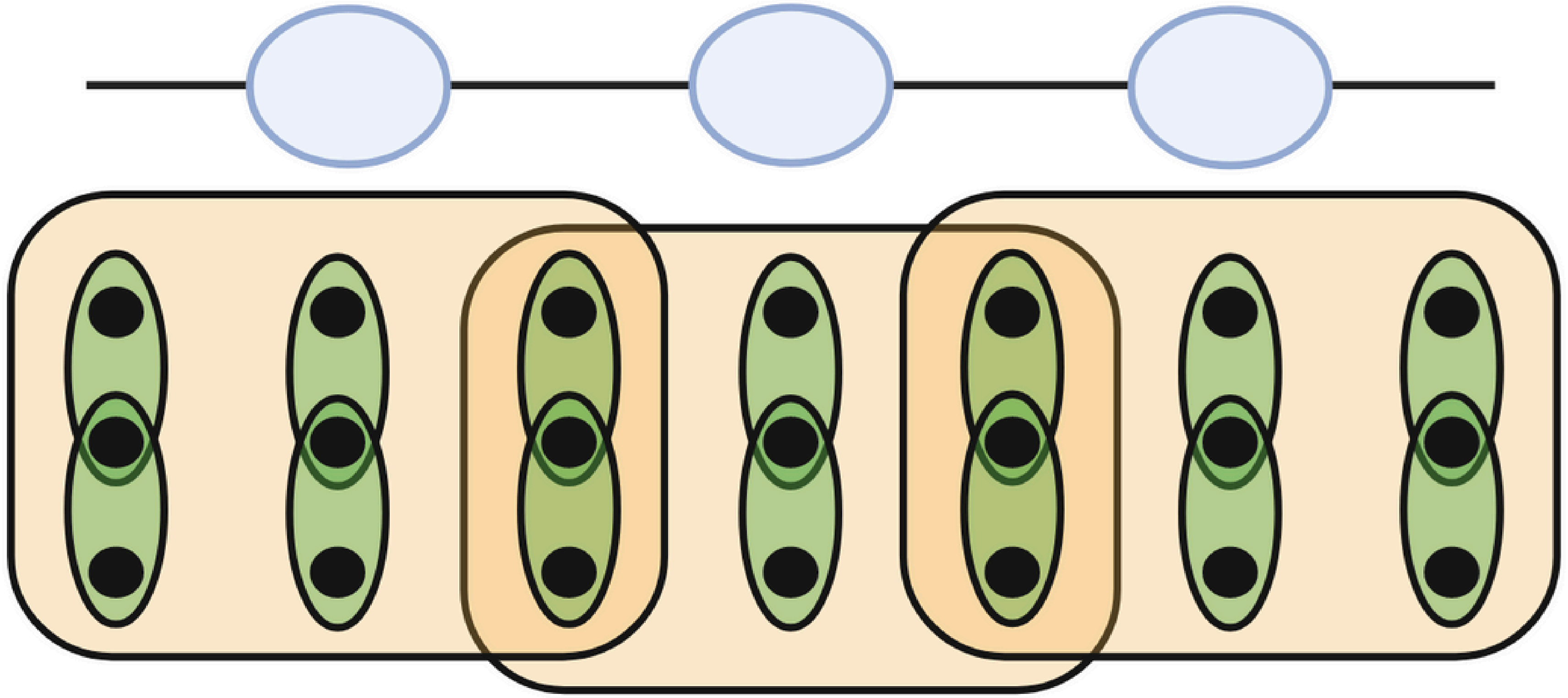
$$\bar{X}|0\rangle_L = |1\rangle_L$$
$$\bar{X}|1\rangle_L = |0\rangle_L$$

$$\bar{Z}|0\rangle_L = |0\rangle_L$$
$$\bar{Z}|1\rangle_L = -|1\rangle_L$$

In terms of logical operations:

$$H = \frac{m}{2}\sum_l (-1)^l (1 - \overline{Z}_{l-1}\overline{Z}_l) + \frac{\epsilon}{2}\sum_l \left(1 - \overline{Z}_{l-1}\overline{Z}_{l+1}\right)\overline{X}_l + 2\lambda_E \sum_l \overline{Z}_l$$

FULL ENCODING



3 qubits per site
3 qubits per link

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$
$$|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

codewords

$$|0\rangle_L \to |+++\rangle$$
$$|1\rangle_L \to |---\rangle$$

stabilizers

$$S_1 = X_1 X_2$$
$$S_1 = X_2 X_3$$

$$X|+\rangle = |+\rangle$$
$$X|-\rangle = -|-\rangle$$

STABILIZER CODES

Let "P" be the n-qubits Pauli group

Define "S" the stabilizer group as an abelian subgroup of P

A codword is a state such that, for every element of S

$$S_i|x\rangle = |x\rangle$$

The element of S are traceless, with eigenvalues +1 or -1

By adding an element to S, we half the Hilbert space of codewords

If we start with n physical qubits

we define n-k stabiliser operators

we will have a number of codewords equal to

$$2^n/2^{n-k} = 2^k$$

So we will have k logical qubits

Logical operators are elements of P that commute with S

They are 2k operators. Every operator commute with all other operators but one that has to anti commute