Solving the homogenous Bethe-Salpeter equation with a quantum annealer

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Introduction to the physics problem

Our aim is to apply a variational method, tailored to a Quantum Annealer, for solving the **homogeneous Bethe-Salpeter equation (hBSE)** in Minkowski momentum-space

[E.E Salpeter and H.A. Bethe, Phys. Rev. 84, 1232 (195)]

$$\phi_b(k,p) = G_0^{(12)}(k,p)\int \frac{d^4k'}{(2\pi)^4} i\mathcal{K}(k,k',p)\phi_b(k',p), \text{ with } p = p_1 + p_2 \text{ and } k = \frac{p_1 - p_2}{2}$$

In **relativistic quantum-field-theory**, this equation has the same role in the **bound state** description that the **Schrödinger equation** has in non **relativistic quantum mechanics**

We consider the hBSE describing a **bound system** composed by **two massive scalars** interacting through the **exchange of a massive boson**.

The hBSE in this case can be written as a **Non Symmetric Generalized Eigenvalue Problem (GEVP)** adopting a standard discretization method*

*T.Frederico, G.Salmè, and M.Viviani, Phys. Rev. D 89, 016010 (2014)

hBSE as a Generalized Eigenvalue Problem (GEVP)

[T.Frederico, G.Salmè and M.Viviani, Phys. Rev. D 89, 016010 (2014)]

$$A\ \mathbf{v}_i = \lambda_i B\ \mathbf{v}_i$$

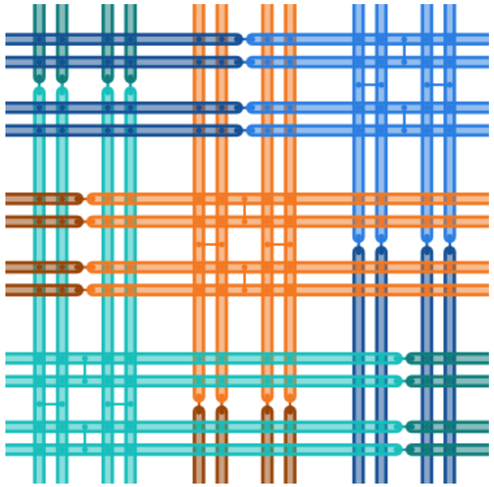
- The eigenvalues are $\lambda_i = \dfrac{1}{\alpha_i}$, where $\alpha_i = \dfrac{g_i^2}{32\pi^2}$,with g_i the **coupling constant** of the interaction.
- We are interested **only** in the in the eigenpair corresponding to the **largest, real and positive eigenvalue** λ_n, since $\lambda_n = \dfrac{1}{\alpha_n}$
- α_n is the **minimal coupling constant** that allows the existence of a **bound system** with a given mass $M = 2m - B$

We are using $(n \times n)$ **matrices** obtained by the **discretization of the hBSE** with binding energy $\dfrac{B}{m} = 1.0$ and with an exchanged boson of mass $\dfrac{\mu}{m} = 0.15$

Advantage 4.1

Thanks to the **D-Wave-Cineca** agreement, as part of an international project approved by Q@TN(INFN-UNITN-FBK-CNR), we used the QA **Advantage 4.1** provided by **D-Wave Systems**

The transverse-Ising Hamiltonian that represents the problem to minimize is **encoded** into the target topology (**PEGASUS**) through an **embedding procedure**, with a heuristic algorithm provided by the D-Wave software



To properly translate the problem on the topology of the hardware, the embedding needs to represent the logical qubits of the original problem with a **larger number** of physical qubits

With the **Simulated Annealing (SA)**, the original problem can be directly solved without the an embedding procedure

Formulate a QUBO problem

Goal: Write the **GEVP** as a problem **suitable** by the QA

First step: write the non symmetric **GEVP** as a **variational problem***

$$f(A, B, \mathbf{v}, \tilde{\lambda}) = \mathbf{v}^T[A - \tilde{\lambda}B]^T[A - \tilde{\lambda}B]\mathbf{v} \geq 0$$

The **objective function (OF)** is minimized by all the eigenvectors $\mathbf{v}$:

$$f(A, B, \mathbf{v}, \tilde{\lambda}) = 0 \Rightarrow \lambda^R(\mathbf{v}) = \mathbf{v}^T \frac{AB^T + B^T A}{2}\mathbf{v}$$

$$\lambda(\mathbf{v}) = \lambda^R(\mathbf{v}) \pm i\lambda^I(\mathbf{v}) \Rightarrow \text{We want } \lambda^I(\mathbf{v}) = 0$$

Second step: decompose the B matrix to transform the GEVP into a **non symmetric eigenvalue problem** with the standard **LDL** decomposition

$$CL^T\mathbf{v}_i = \lambda_i L^T\mathbf{v}_i\,, \quad \mathbf{w}_i = L^T\mathbf{v}_i$$

$$f(C, \mathbf{w}, \tilde{\lambda}) = \mathbf{w}^T[C - \tilde{\lambda}I]^T[C - \tilde{\lambda}I]\mathbf{w} = \mathbf{w}^T S(\tilde{\lambda})\mathbf{w} \geq 0$$

We have to exploit the **non singularity** of B

*S. Alliney, F. Laudiero and M. Savoia, Applied mathematical modelling 16, 148 (1992)

Formulate a QUBO problem

The **QA** is designed to deal with transverse-Ising model and **QUBO** problems in an hypercube $C_{n,b} = [0,1]^{n*b}$. After a **single annealing cycle** the QA returns:

$$\mathbf{x} = \arg\min_{\mathbf{x} \in C_{n,b}} \mathbf{x}^T Q \mathbf{x}$$

We need to approximate our **quadratic form** into a **QUBO** one, rewriting the matrices elements in a **binary basis**

$$\mathbf{w} \simeq P^T \mathbf{x} \Rightarrow w_\alpha = -q_{b,\alpha} + \sum_{i=1}^{b-1} \frac{q_{i,\alpha}}{2^i} \quad P^T = diag(\mathbf{p}^T, \ldots, \mathbf{p}^T) \quad \mathbf{p} = (-1, \frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2^{b-1}})^T$$

$$\mathbf{w} = (w_1, w_2, \ldots, w_n)^T \in [-1,1[^n \Rightarrow \mathbf{x} = (q_{1,1}, \ldots, q_{b,1}, \ldots, q_{1,n}, \ldots, q_{b,n})^T \in C_{n,b}$$

$$f(C, \mathbf{w}, \tilde{\lambda}) = \mathbf{w}^T S(\tilde{\lambda}) \mathbf{w} \simeq \mathbf{x}^T P S(\tilde{\lambda}) P^T \mathbf{x} = \mathbf{x}^T Q \mathbf{x}$$

Gershgorin Theorem

$$f(C, \mathbf{w}, \tilde{\lambda}) = \mathbf{w}^T[C - \tilde{\lambda}I]^T[C - \tilde{\lambda}I]\mathbf{w} = \mathbf{w}^T S(\tilde{\lambda})\mathbf{w} \geq 0$$

We **must set** $\tilde{\lambda} \simeq \lambda_n$ if we want to obtain $\mathbf{w}_n$ from a QA cycle

We can guide the "guess phase" by using the **Gershgorin circle theorem***:

$$\mathscr{C}_i : |y - c_{ii}| = \leq \sum_{j \neq i} |c_{ij}| = R_G(c_{ii})$$

c_{ij} elements of C

If a circle $\mathscr{C}_j$ is **disconnected** from the others then it contains **one ad only one real eigenvalue**

If $\mathscr{C}_N$ is **disconnected** from the **others**, we can set $\tilde{\lambda} = c_{11} > c_{22} > c_{33} \ldots$ and we select **only the solutions** $\in \mathscr{C}_N$

* **R. S. Varga, "Gershgorin and his circles", Vol. 36 (Springer Science & Business Media, 2010)**

Algorithm I: Guess Phase

$$f(C, \mathbf{w}, \tilde{\lambda}) = \mathbf{w}^T S(\tilde{\lambda}) \mathbf{w} \simeq \mathbf{x}^T P S(\tilde{\lambda}) P^T \mathbf{x} = \mathbf{x}^T Q \mathbf{x}$$

After $\alpha = 1, \ldots, N_A^{GP}$ annealing cycles, the QA or the SA returns N_A^{GP} **binary vectors** $\{\mathbf{x}_\alpha\}$

We analyze the set $\{\mathbf{w}_\alpha, f(C, \mathbf{w}, \tilde{\lambda} = c_{11}), \lambda^R(\mathbf{w}_\alpha), \lambda^I(\mathbf{w}_\alpha)\}$

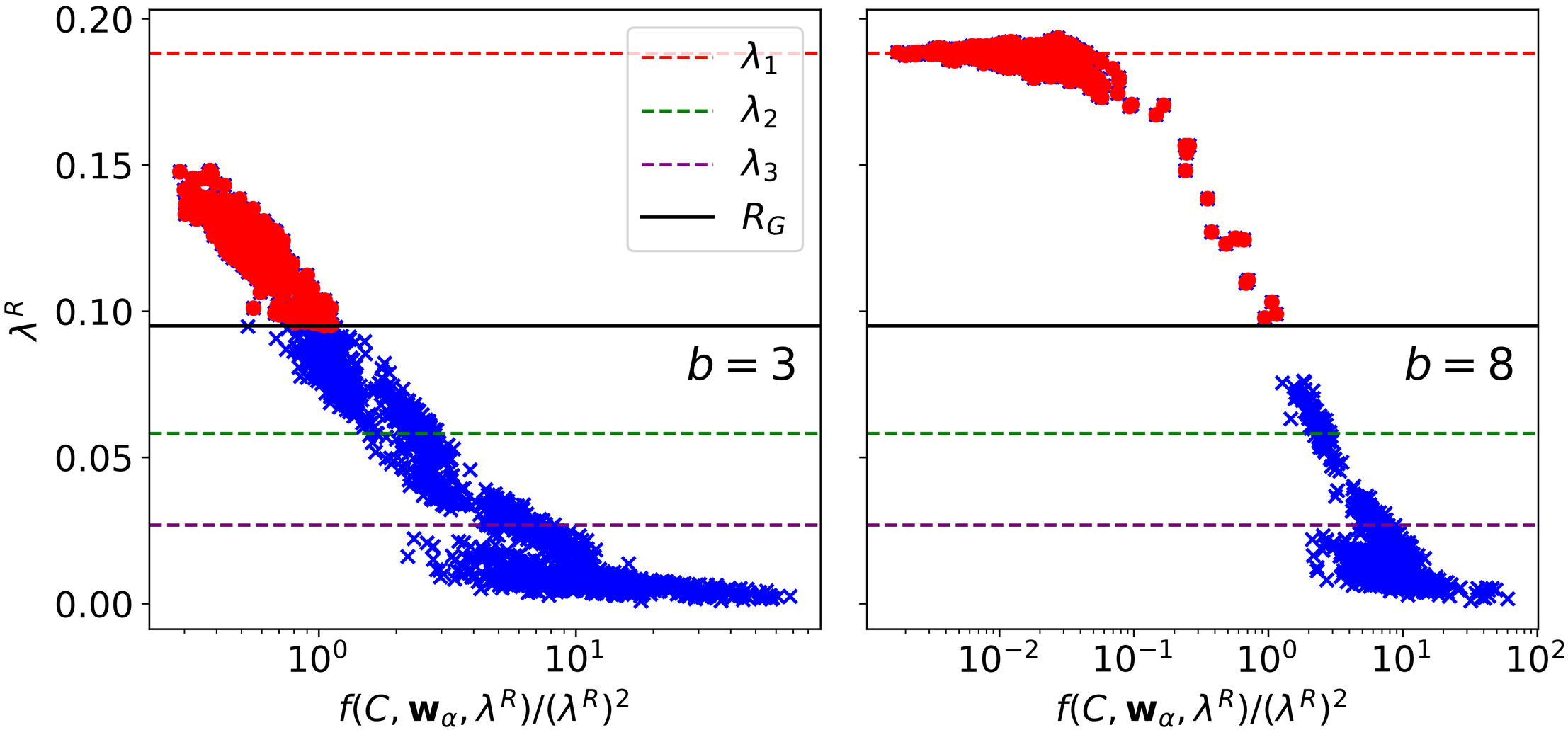
We eliminate the **solutions outside the disc** $\mathscr{C}_n$

Among the surviving solutions, we take the one that satisfies:

$$f_{best}^{GP} = \min_{\mathbf{w}_\alpha} f(A, \mathbf{w}_\alpha, \tilde{\lambda} = \lambda^R(\mathbf{w}_\alpha))$$

At the end of the **Guess Phase** the best eigenpair $(\mathbf{w}_{\alpha_{GP}}, \lambda^R(\mathbf{w}_{\alpha_{GP}}))$ is passed to the **Gradient-Descent phase** in order to improve the precision on that solution

F.F., A.Gnech, T.Frederico, F.Pederiva, M.Rinaldi, A.Roggero, G.Salmè, S.Scopetta and M.Viviani, *Phys.Rev.D* 110 (2024) 5, 056012



- Single run on the SA ($N_A^{GP} = 2000$) for a 32x32 matrix

Algorithm II: gradient-descent

At the **end of the GP** of the algorithm we find a solution $\mathbf{w}^{(z=0)} = \mathbf{w}_{\alpha_{GP}}$

Gradient-descent (GD): iterative algorithm that founds a new solution $\mathbf{w}^{(z)}$

$$\mathbf{w}^{(z)} = \mathbf{w}^{(z-1)} + \frac{1}{2^z}\delta(z)$$

The **OF** can be expanded around $\mathbf{w}^{(z-1)}$:

$$f(C, \mathbf{w}^{(z)}, \tilde{\lambda}) = f(C, \mathbf{w}^{(z-1)}, \tilde{\lambda}) + \mathbf{w}^{(z-1)T} S(\tilde{\lambda})\delta(z) + \delta(z)^T \frac{S(\tilde{\lambda})}{2}\delta(z)$$

New **OF**: $\hat{f}(C, \delta(z), \tilde{\lambda}) = \delta(z)^T \mathcal{Q}(z, \tilde{\lambda})\delta(z)$

$$\mathcal{Q}(z, \tilde{\lambda})_{ij} = \frac{1}{2}S(\tilde{\lambda})_{ij} + \delta_{ij}[\mathbf{w}^{(z-1)T} S(\tilde{\lambda})]_i$$

Algorithm II: gradient-descent

$\lambda_{best} = \lambda(\mathbf{w}^{(z-1)})$ and $\mathbf{w}_{best} = \mathbf{w}^{(z-1)}$

At **each zoom step** z $(z = 1,\ldots, z_{max})$ an **inner loop is opened** $i^z = 1,2,\ldots$

The QA or the SA returns en **ensemble of** $\alpha_i = 1,\ldots, N_A^{GD}$ **qubits states** $\{\mathbf{x}_{\alpha_i}\}$

Among the ensemble we select the one with the minimal energy:

$$\hat{f}^{DP}_{best;i}(z) = \min_{\delta_{\alpha_i}(z)} \hat{f}(C, \mathbf{w}^z_{\alpha_i}, \tilde{\lambda} = \lambda_{best})$$

If $\lambda^R(\mathbf{w}^z_{best;i}) \geq \lambda_{best}$ we pass the **next zoom step** $z + 1$. $i^z_{max} = i^z$

i^z_{max}: iterations needed to find the **best solution for each** z

Total annealing time: $T \propto N_A^{GD} \sum_{z=1}^{z_{max}} i^z_{max}$

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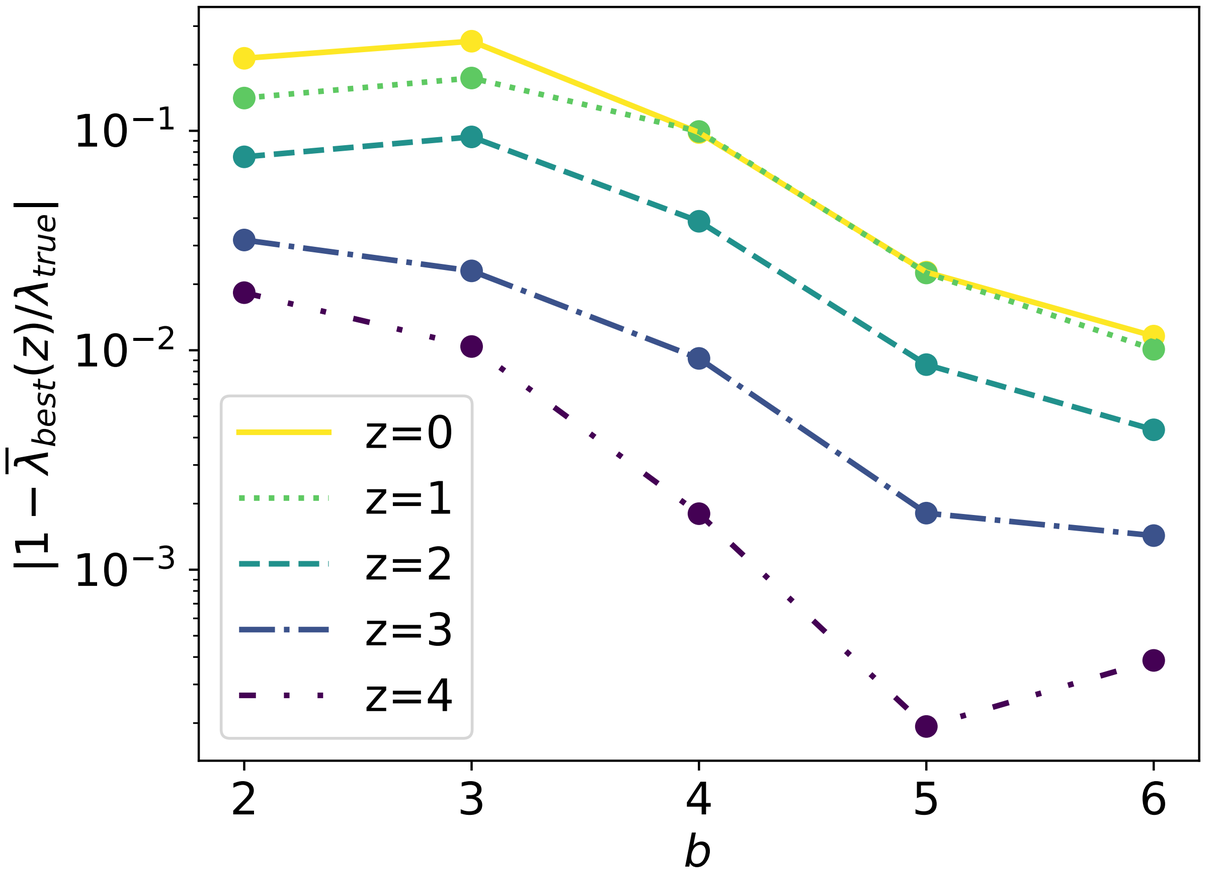

- **500 independent samples** on the SA ($N_A^{GP} = 200, N_A^{GD} = 20$) for a 32x32 matrix
- **Trade-off between b and z_{max}**, already found in the symmetric case *

*** B. Krakoff, S. M. Mniszewski and C. F. A. Negre, arXiv:2104.11 (2021)**

F.F., A.Gnech, T.Frederico, F.Pederiva, M.Rinaldi, A.Roggero, G.Salmè, S.Scopetta and M.Viviani, *Phys.Rev.D* 110 (2024) 5, 056012

n_M	b	N_{run}	λ_{true}	$\bar{\lambda}_{best}$	$\bar{\lambda}^I_{best}/\bar{\lambda}_{best}$	$\overline{\lvert\lvert \mathbf{v}_{true} - \mathbf{v}_{best} \rvert\rvert}$
4	3	80	0.188026	$0.188012^{+1\cdot10^{-5}}_{-7\cdot10^{-6}}$	$0.00024^{+2\cdot10^{-5}}_{-5\cdot10^{-6}}$	$0.00024^{+3\cdot10^{-5}}_{-2\cdot10^{-5}}$
8	3	80	0.188204	$0.18820^{+2\cdot10^{-5}}_{-2\cdot10^{-5}}$	$0.0003^{+1\cdot10^{-4}}_{-1\cdot10^{-4}}$	$0.0003^{+1\cdot10^{-4}}_{-1\cdot10^{-4}}$
12	3	80	0.188203	$0.18821^{+2\cdot10^{-5}}_{-2\cdot10^{-5}}$	$0.0005^{+1\cdot10^{-4}}_{-1\cdot10^{-4}}$	$0.0006^{+2\cdot10^{-4}}_{-1\cdot10^{-4}}$
16	2	80	0.188203	$0.18820^{+4\cdot10^{-5}}_{-3\cdot10^{-5}}$	$0.0009^{+1\cdot10^{-4}}_{-1\cdot10^{-4}}$	$0.0011^{+2\cdot10^{-4}}_{-2\cdot10^{-4}}$
24	2	80	0.188225	$0.18822^{+5\cdot10^{-5}}_{-4\cdot10^{-5}}$	$0.0013^{+1\cdot10^{-4}}_{-3\cdot10^{-4}}$	$0.0015^{+3\cdot10^{-4}}_{-2\cdot10^{-4}}$
32	2	200	0.188225	$0.18823^{+4\cdot10^{-5}}_{-3\cdot10^{-5}}$	$0.0016^{+2\cdot10^{-4}}_{-2\cdot10^{-4}}$	$0.0018^{+4\cdot10^{-4}}_{-3\cdot10^{-4}}$

- $z_{max} = 9$
- **Euclidean distance** and magnitude of the **imaginary part** slightly increase with $n_M \times b$

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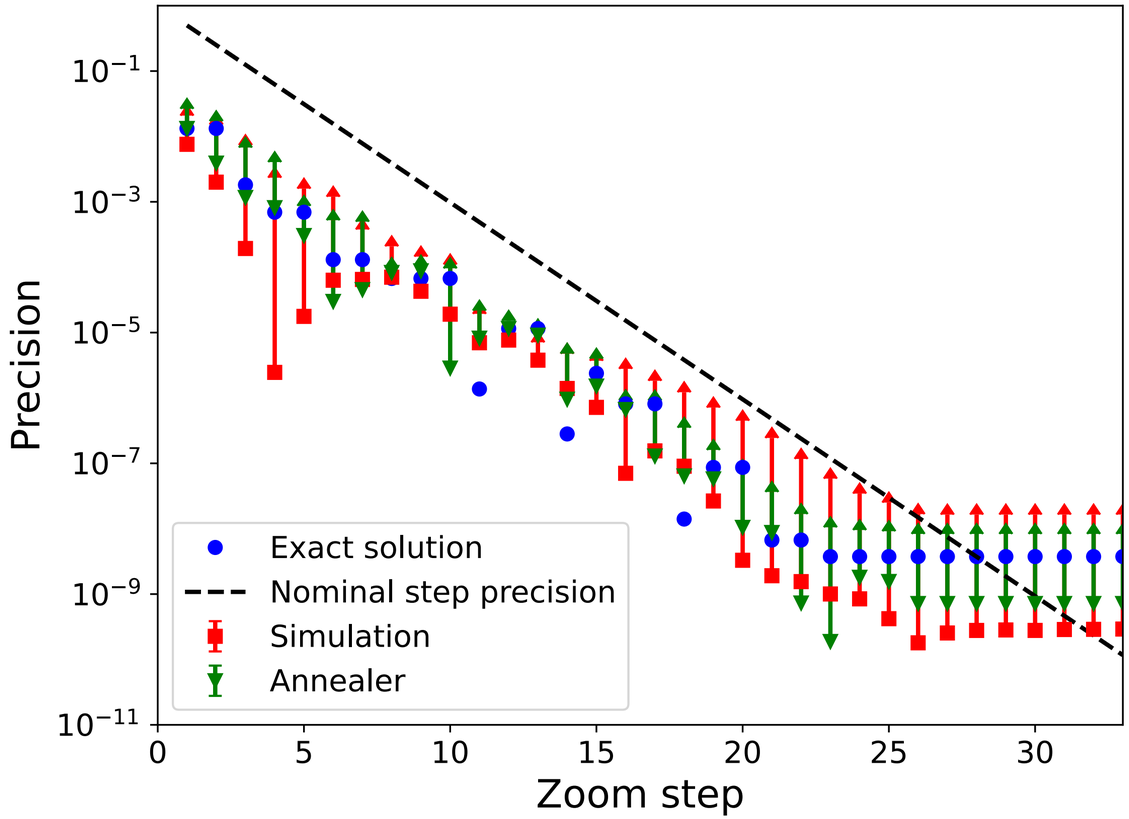

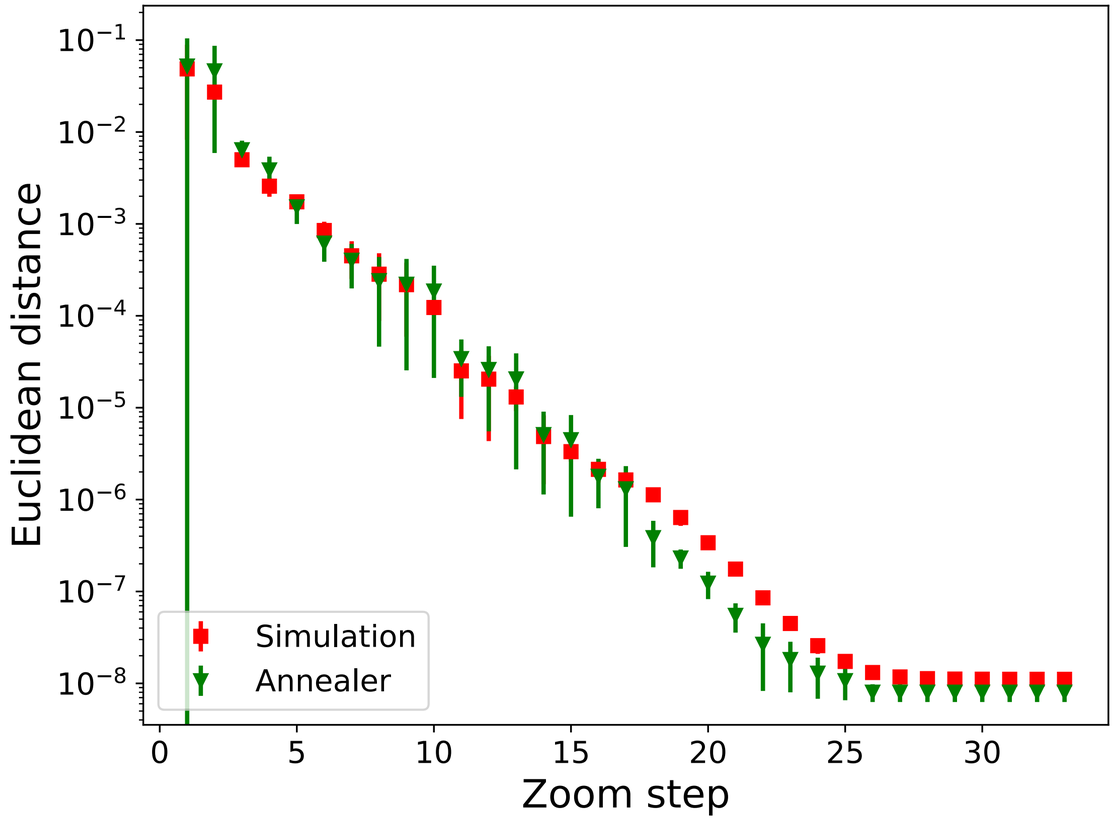

- 4×4 matrices with $b = 3$. $N_{run} = 1000$ on the SA and $N_{run} = 10$ on the QA
- **Actual** 0 of the **OF** $\simeq 10^{-16} \Rightarrow$ **Plateau** at 10^{-8}

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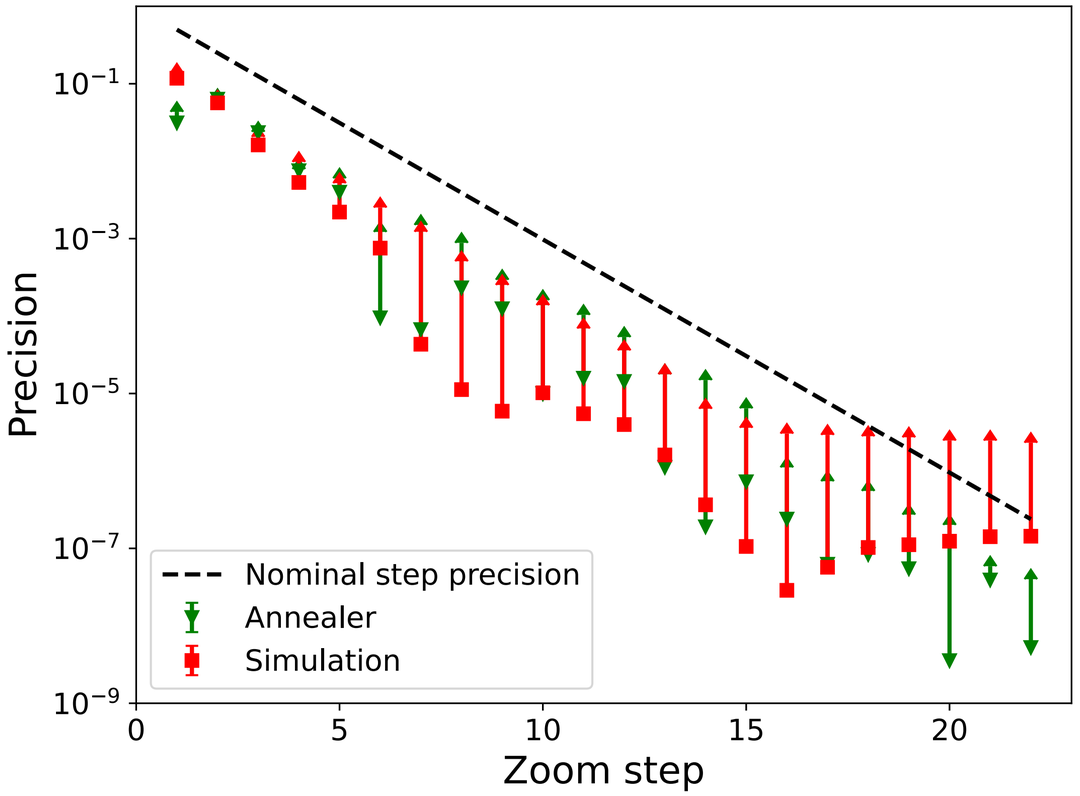

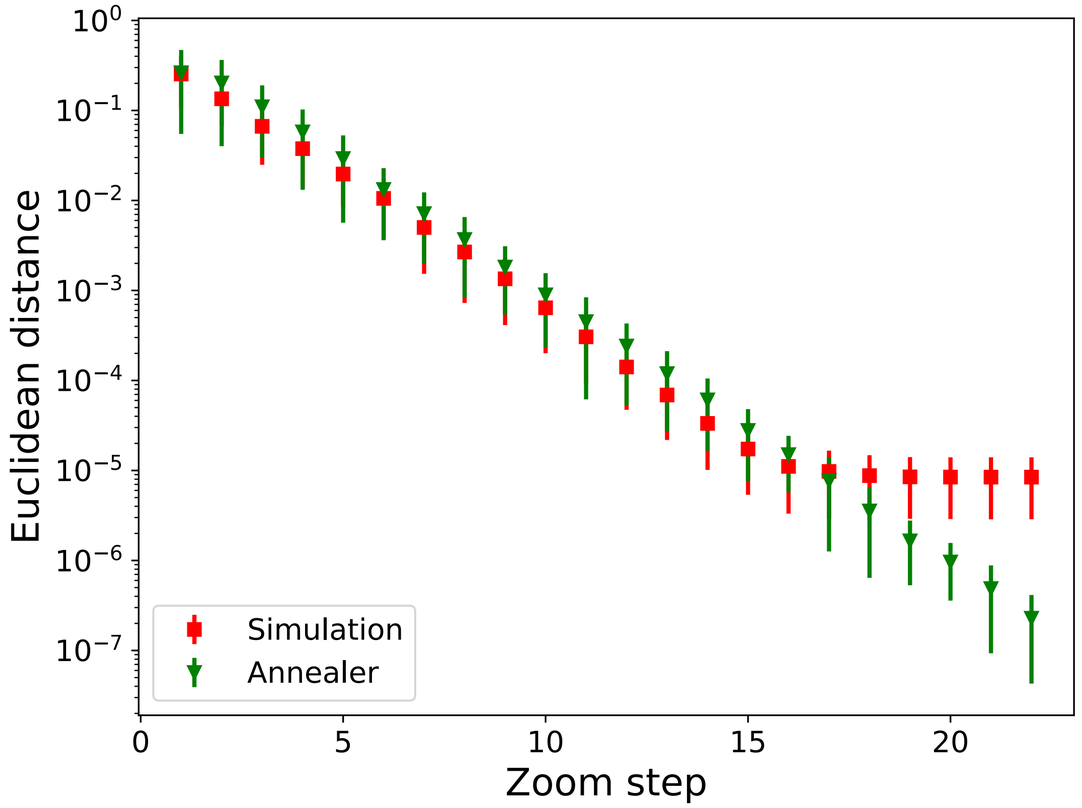

- 32×32 matrices with $b = 2.$ $N_{run} = 1000$ on the SA and $N_{run} = 10$ on the QA
- **Lower SA**'s performance in managing large matrices

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$n_M \times b$	N_{run}	$T[ms]$	N_{qubits}
$12(4 \times 3)$	80	$18.5^{+0.9}_{-0.7}$	$24.1^{+0.9}_{-1.1}$
$24(8 \times 3)$	80	$21.7^{+1.3}_{-0.9}$	$81.3^{+2.1}_{-2.3}$
$32(16 \times 2)$	80	$23.0^{+1.4}_{-1.0}$	$140.9^{+1.9}_{-1.9}$
$36(12 \times 3)$	80	$23.0^{+1.4}_{-1.3}$	$177.6^{+6.4}_{-6.6}$
$48(24 \times 2)$	80	$23.0^{+1.0}_{-1.0}$	$306.6^{+11.8}_{-15.6}$
$64(32 \times 2)$	200	$23.6^{+1.2}_{-1.2}$	$529.8^{+30.6}_{-33.8}$

- **Total annealing time** T and **total number of physical qubits** N_{qubits}, averaged on N_{run}
- T is **slightly increasing** when the matrix dimension $n_M \times b$ increase
- Quadratic growth of N_{qubits} with the total dimension $n_M \times b$

Conclusions and perspectives

- A hybrid algorithm, suitable for a quantum annealer, was implemented to evaluate the **largest real eigenvalue and corresponding eigenvector** of a **GEVP** for the **discretization of the hBSE**
- Numerical results obtained by running our **two-phase algorithm** both on **Advantage 4.1** and a **SA**
- The results obtained by the **SA** established a **practical set of input parameters** (b, N_A^{GP}, N_A^{GD})
- We successfully approached the target eigenpair by **running the code on the D-Wave QA**, obtaining very encouraging results, up to a matrix with dimension $n = 32$ and $b = 2$
- **The next challenge** is to **improve the algorithm** in order to address the GEVP without exploiting the **non singularity of the symmetric matrix** B