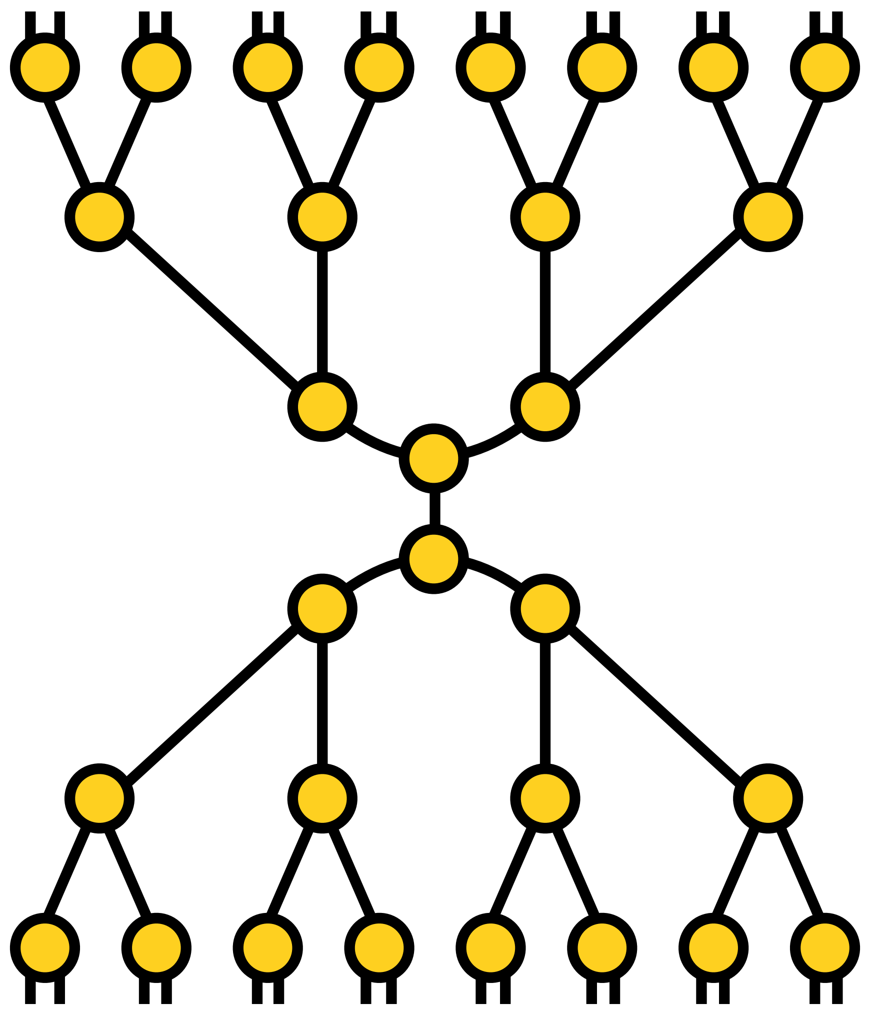
Entanglement in finite-temperature Rydberg arrays

with tensor networks

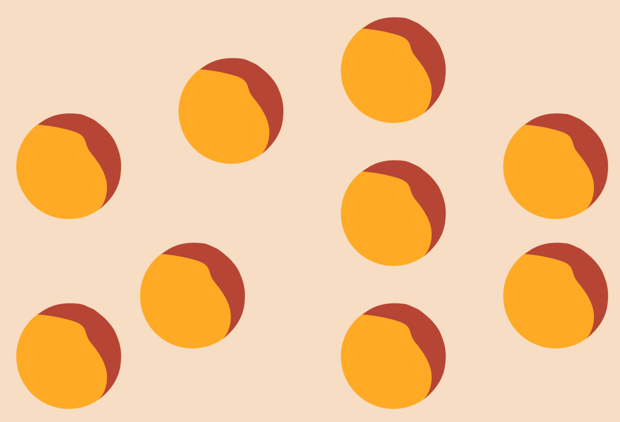


Nora Reinić @ Quantum Computing @ INFN

University of Padova

Simulating quantum many-body systems

The quantum many-body problem



System:

- N particles on a lattice
- Local Hilbert space dimension d

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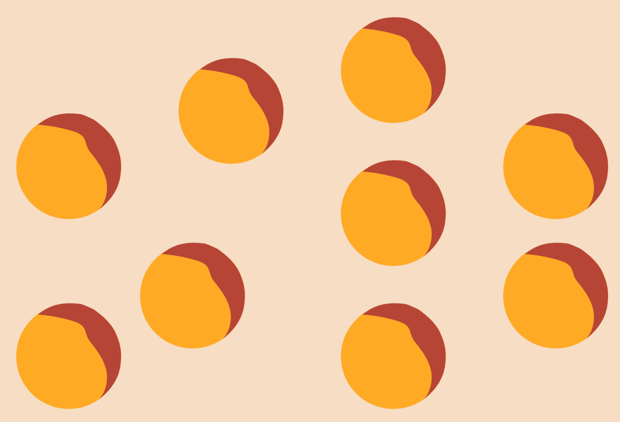

System:

- N particles on a lattice
- Local Hilbert space dimension d

Dimension of the total Hilbert space is d^N

Too big to be simulated exactly for large number of particles!

Tensor network methods

Number of degrees of freedom of quantum many-body state depends on the amount of entanglement in the system

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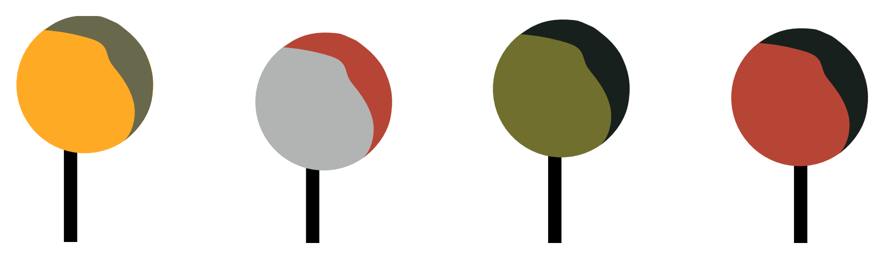
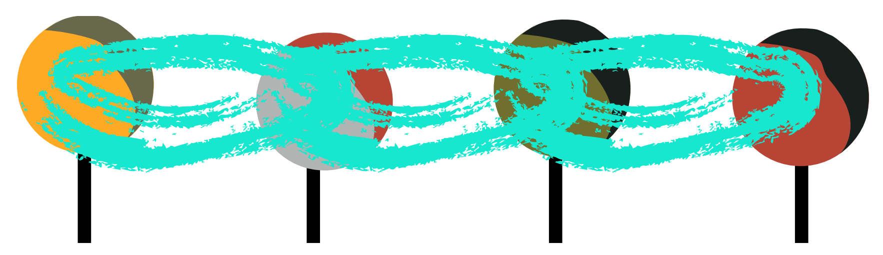
No entanglement



Easy to simulate

A lot of entanglement



Exponential number of degrees of freedom

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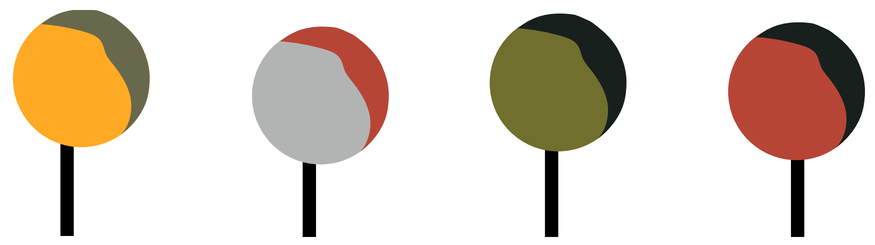

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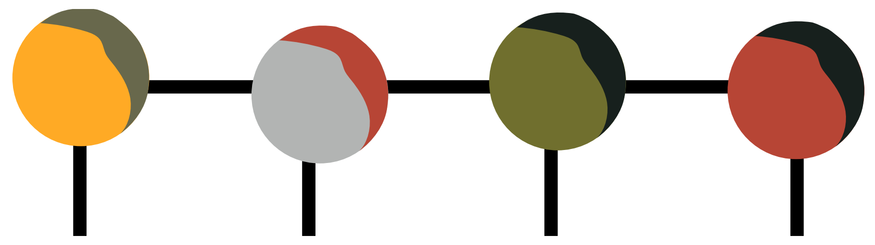
Mean field approximation

This scenario is usually not the case

Many realistic physical systems:

the amount of entanglement is sufficiently low

(area law of entanglement)



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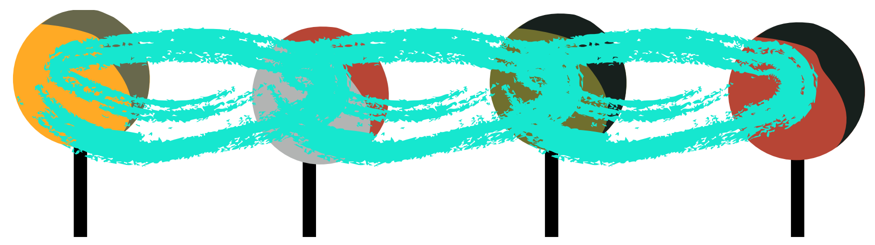

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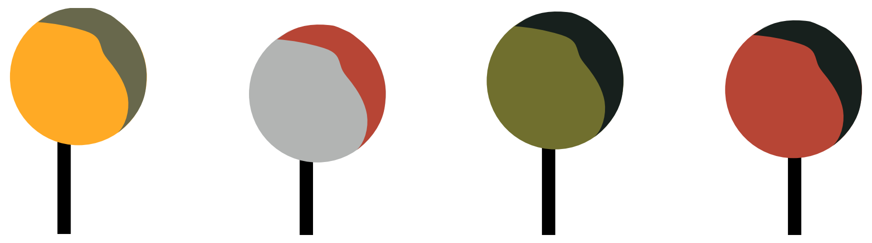

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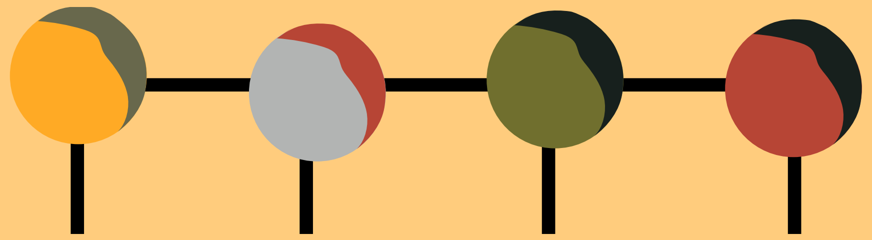
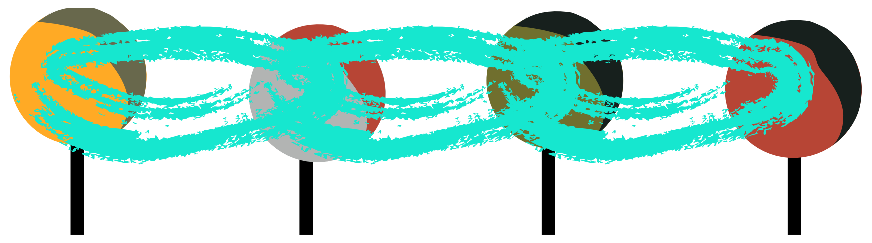
TENSOR NETWORK METHODS



Keep only certain amount of entanglement in the system and discard the rest

A lot of entanglement



Exponential number of degrees of freedom

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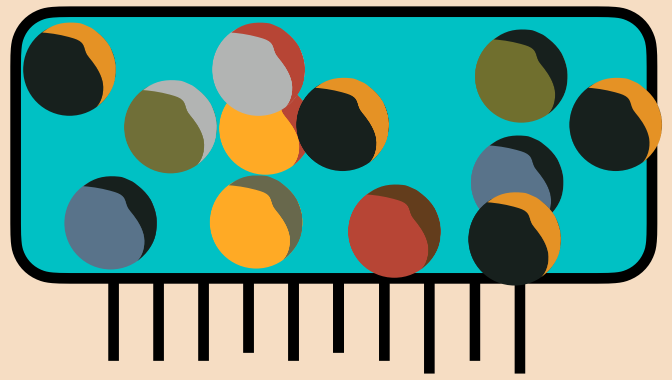
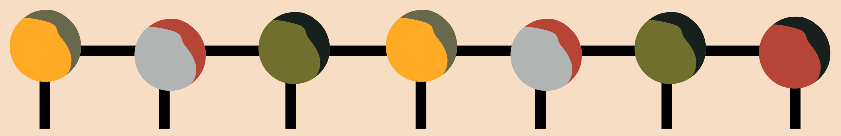
Tensor network ansatz idea

One vector of exponentially scaling dimension → Ansatz: multiple interconnected tensors of “small” dimensions

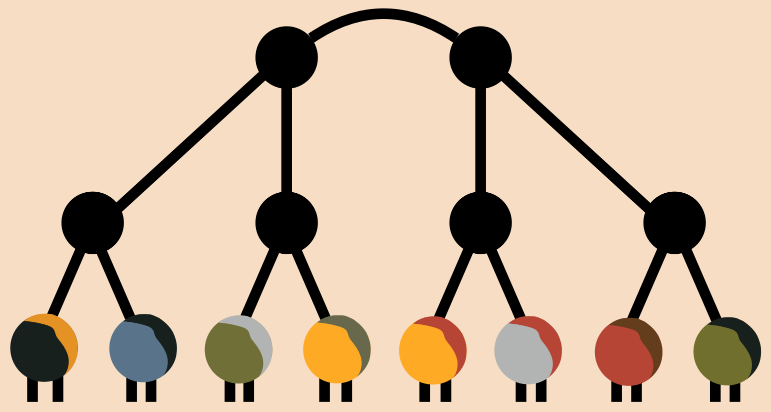
$|\psi > =$



d^N

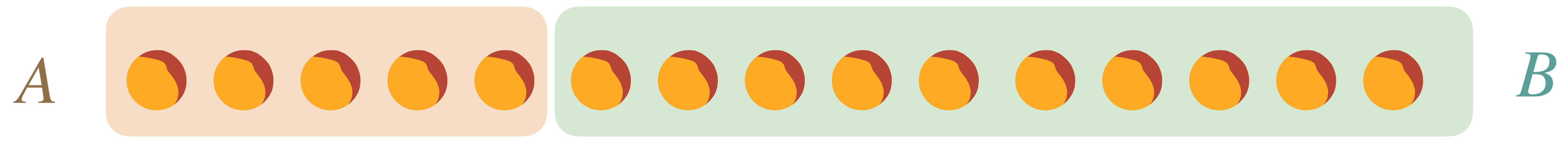


Matrix Product State



Tree Tensor Network

Bipartite entanglement measures



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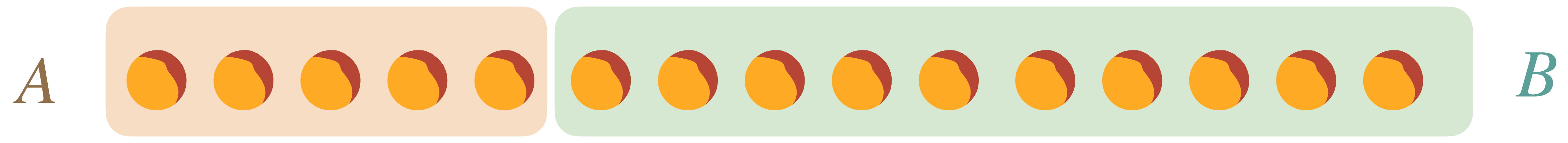

Pure states

Statevector description

$|\psi>$

Mixed states

Density matrix description

$$\rho = \sum_j p_j |\psi_j> < \psi_j|$$

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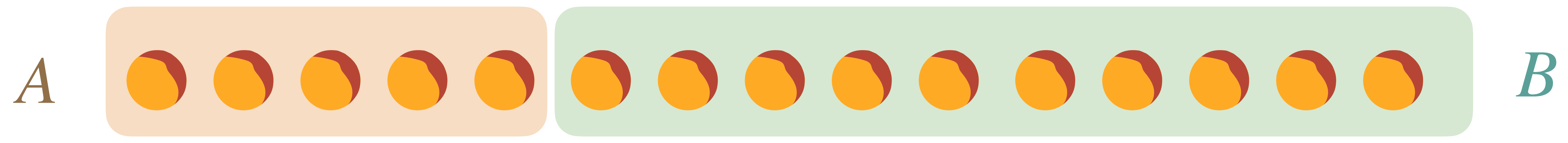

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Statevector description

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Well-defined via reduced density matrix entropies

Entanglement entropy

$$\mathcal{S} = Tr(\rho_A \ln \rho_A) = \sum_i \lambda_i \ln \lambda_i$$

Straightforward to measure!

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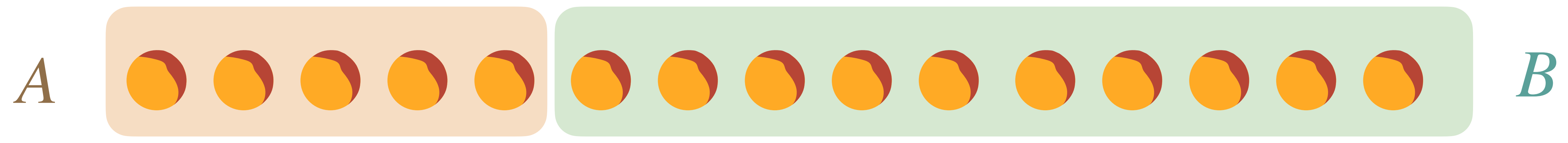

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No unique way to measure mixed-state entanglement!

Entanglement negativity: $E_N = \dfrac{||\hat{\rho}^{\mathsf{T}_{N/2}}||_1 - 1}{2}$

Entanglement of formation: $E_F \inf_{\{\psi_j,p_j\}} \left\{ \sum_j p_j \mathcal{S}_E(|\psi_j>) \ : \ \hat{\rho} = \sum_j p_j |\psi_j> < \psi_j| \right\}$

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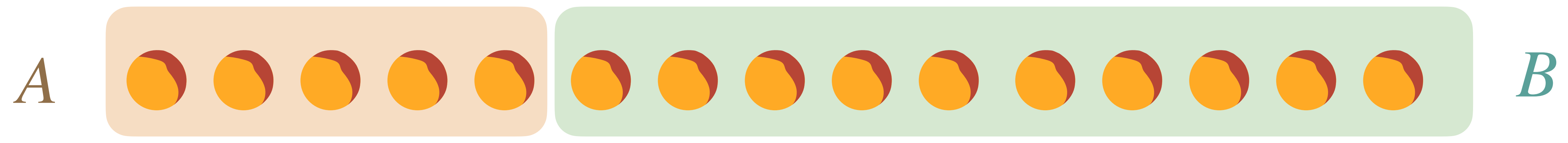

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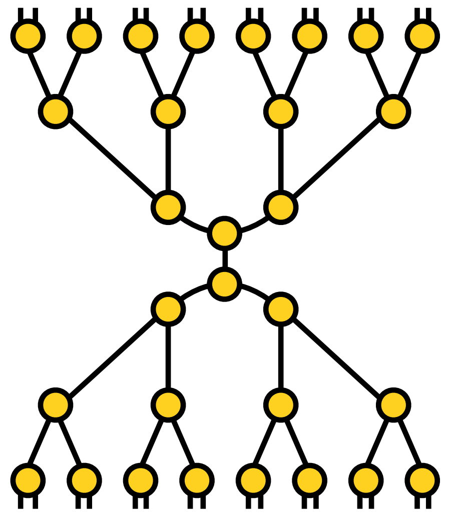
How to efficiently measure mixed-state entanglement of a quantum many-body system?

Simulating systems at finite temperature

Talk plan:

Part 1.

Tensor network toolbox for finite-temperature systems



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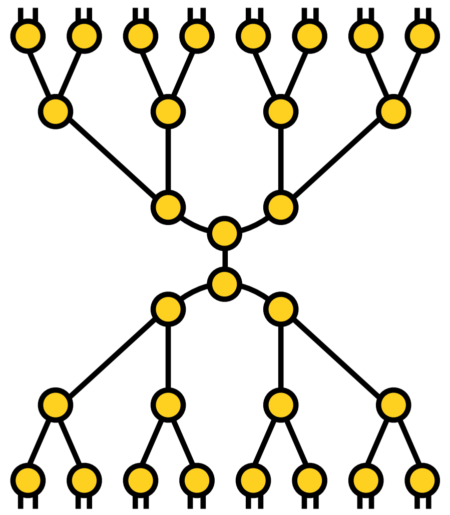
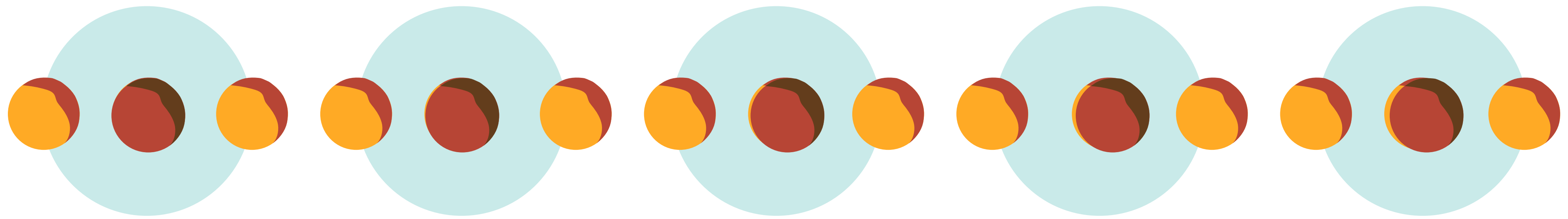


Part 2.

Finite-temperature Rydberg array analysis



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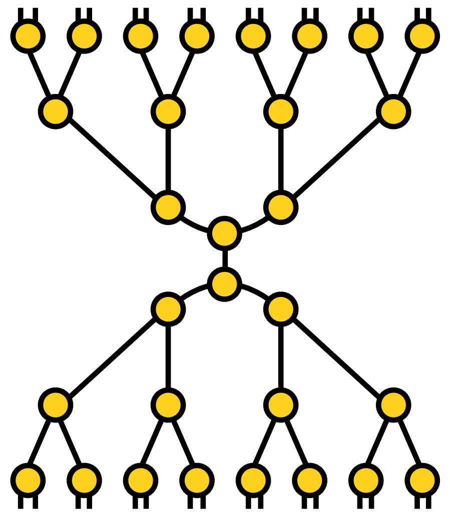
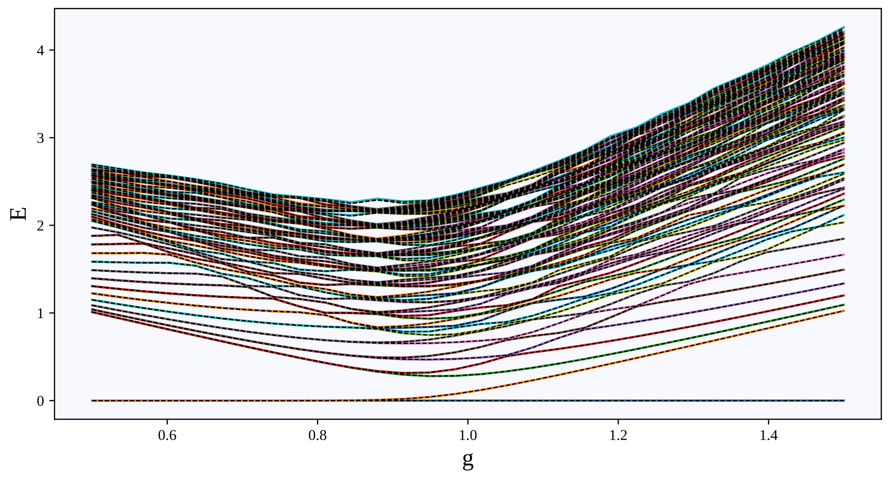


Part 3.

New project!

Extracting the **energy spectrum** of a quantum many-body system



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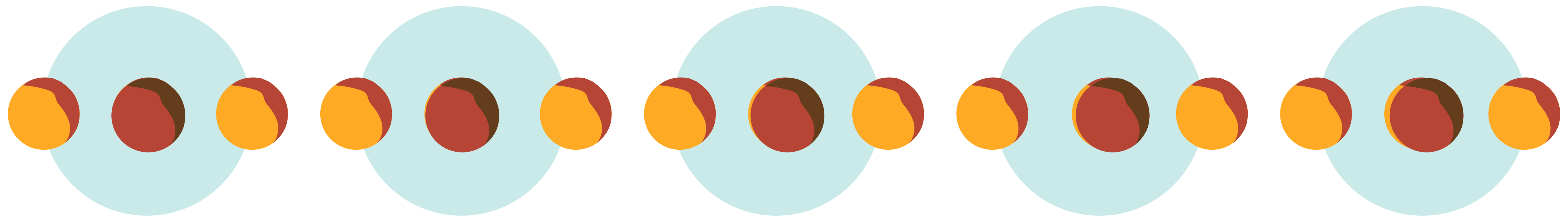

Part 1: **The algorithm**

Part 1: Finite-temperature density matrix

Setup:	**Quantum many-body system**	**+**	**Thermal equilibrium**

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How do we know in which state will the physical system be?

Statistical physics: we can assign a classical probability to finding a system in each of the Hamiltonian eigenstates

Part 1: Finite-temperature density matrix

Setup: **Quantum many-body system** **+** **Thermal equilibrium**

How do we know in which state will the physical system be?

Statistical physics: we can assign a classical probability to finding a system in each of the Hamiltonian eigenstates

System is described with a **density matrix**

$$\rho = \sum_{j=1}^{d^N} p_j |\psi_j> < \psi_j|$$

$$p_j = \frac{e^{-\beta E_j}}{Z}$$

$|\psi_i>$ = eigenstates of Hamiltonian

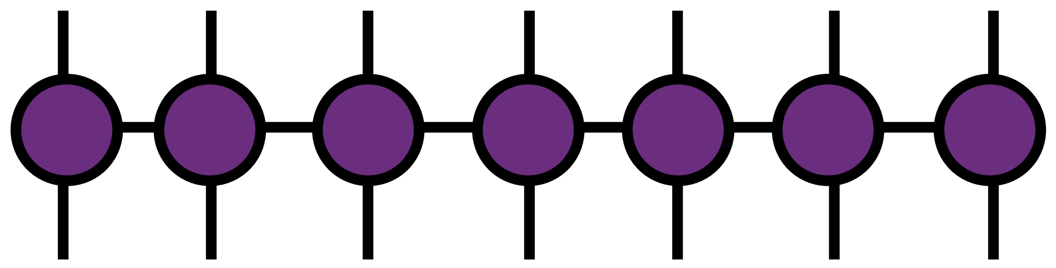
E_i = eigenenergies of Hamiltonian

$$\beta = \frac{1}{k_B T} \qquad Z = \sum_{i=1}^{d^N} e^{-\beta E_i}$$

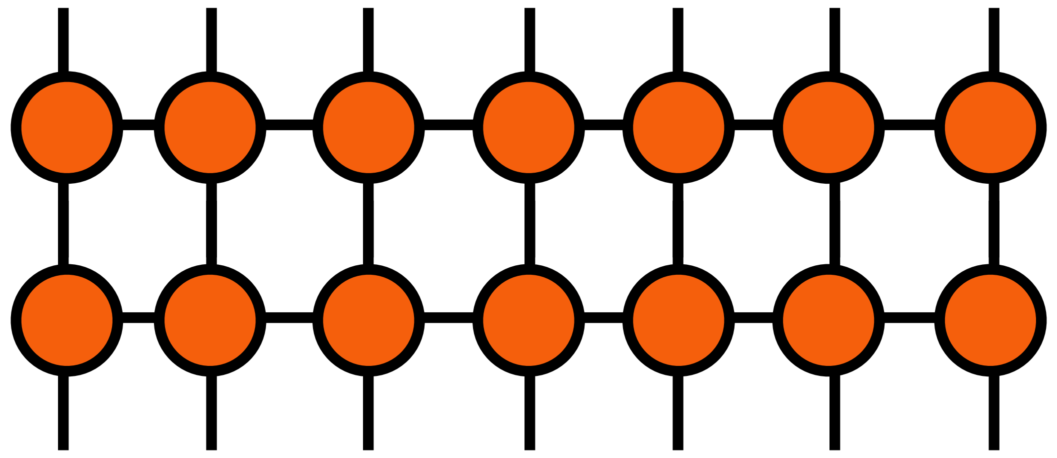
Common density matrix ansatzes

$$\rho = \sum_j p_j |\psi_j > < \psi_j|$$

Matrix product operator
(MPO)



Locally purified tensor network
(LPTN)



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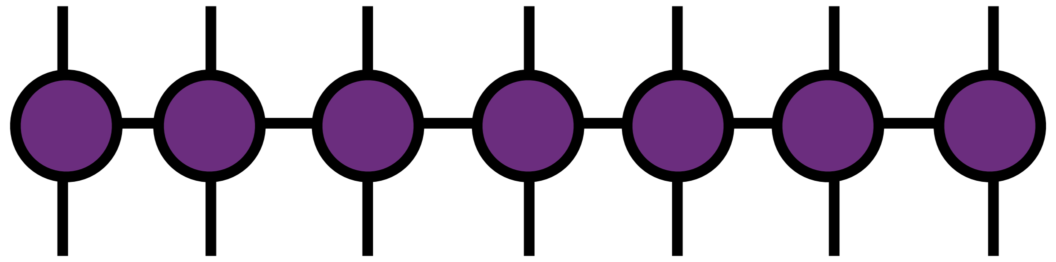
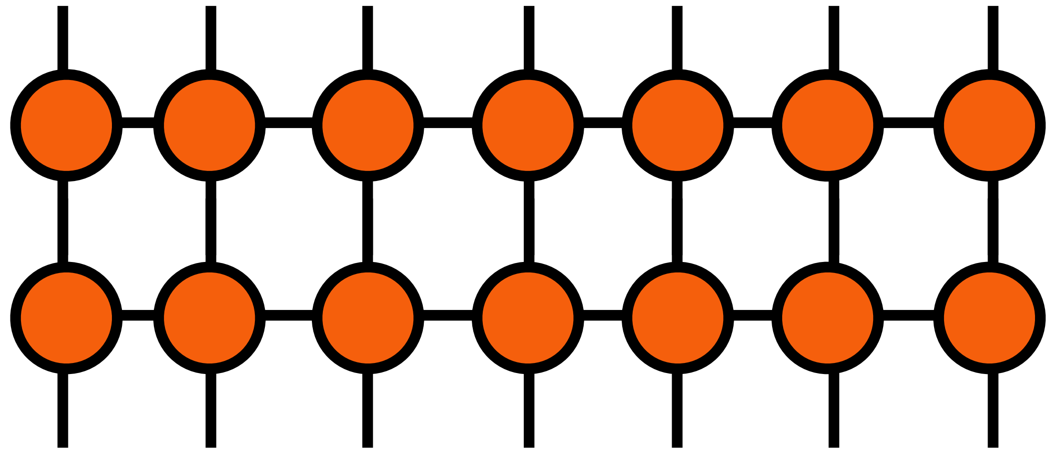
	Matrix product operator (MPO)	Locally purified tensor network (LPTN)
		
Preserves **positivity**	✗	✓
No restriction on **mixedness**	✓	✓
Access to **local observables**	✓	✓

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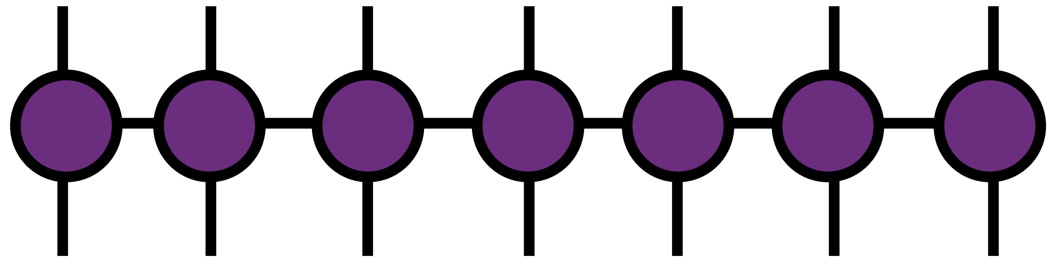
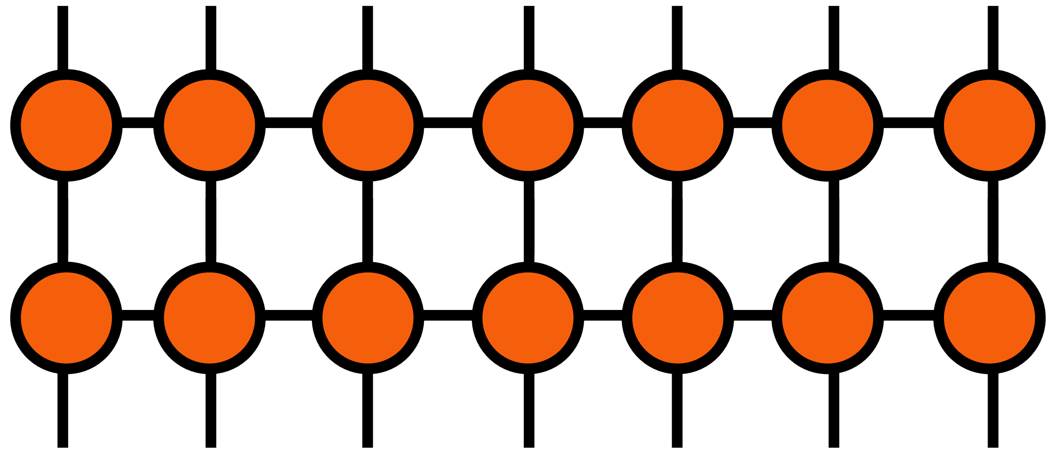


Locally purified tensor network (LPTN)

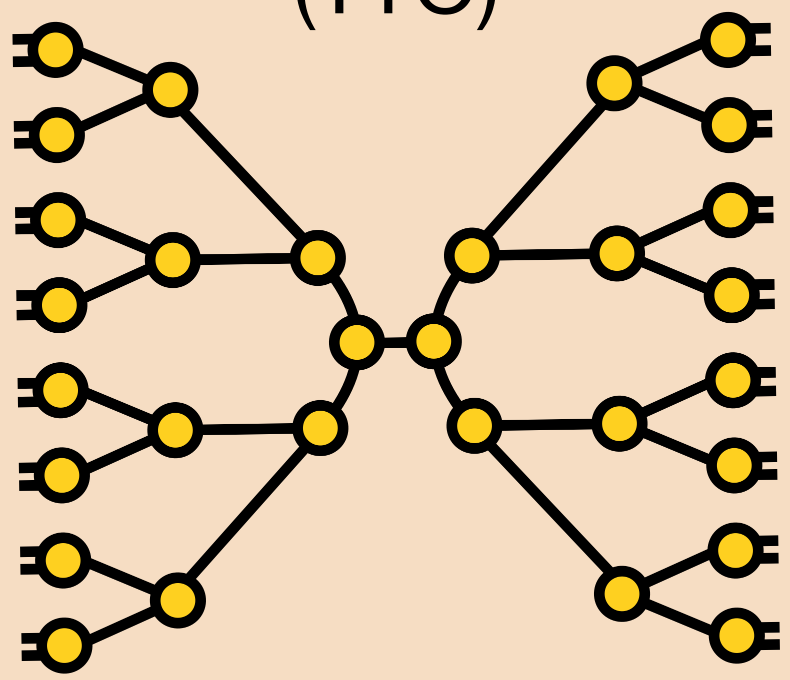


	MPO	LPTN
Preserves **positivity**	✗	✓
No restriction on **mixedness**	✓	✓
Access to **local observables**	✓	✓
Access to **entanglement**	✗	✗
Access to **eigenvalues** of density matrix	✗	✗

Common density matrix ansatzes

Our proposal:

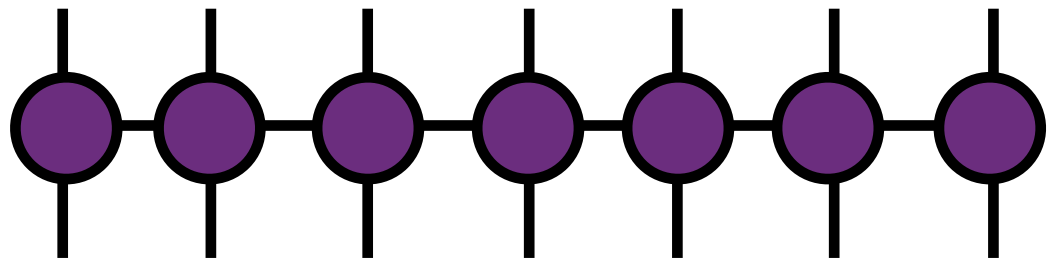
Tree tensor operator (TTO)



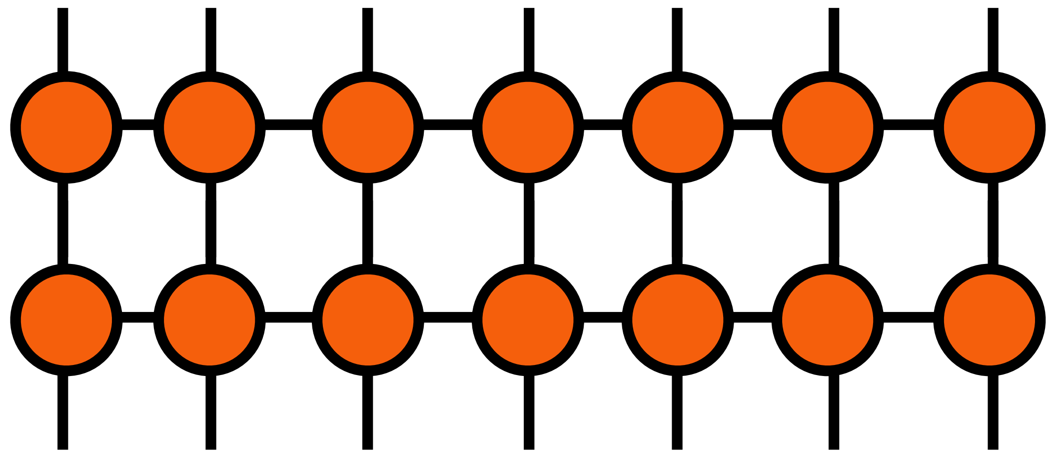
See: L. Arceci, P. Silvi, and S. Montangero Phys. Rev. Lett. 128, 040501 (2022)

$$\rho = \sum_j p_j |\psi_j> < \psi_j|$$

Matrix product operator (MPO)



Locally purified tensor network (LPTN)



	Matrix product operator (MPO)	Locally purified tensor network (LPTN)	Tree tensor operator (TTO)
Preserves **positivity**	✗	✓	✓
No restriction on **mixedness**	✓	✓	✗
Access to **local observables**	✓	✓	✓
Access to **entanglement**	✗	✗	✓
Access to **eigenvalues** of density matrix	✗	✗	✓

$$\rho = \sum_j p_j |\psi_j > < \psi_j|$$

Tree tensor operator

L. Arceci, P. Silvi, and S. Montangero
Phys. Rev. Lett. 128, 040501 (2022)

$$\rho = XX^\dagger$$

Columns of X are given with $\sqrt{p_j} |\psi_j >$

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(Assuming isometrization towards the orange tensor)

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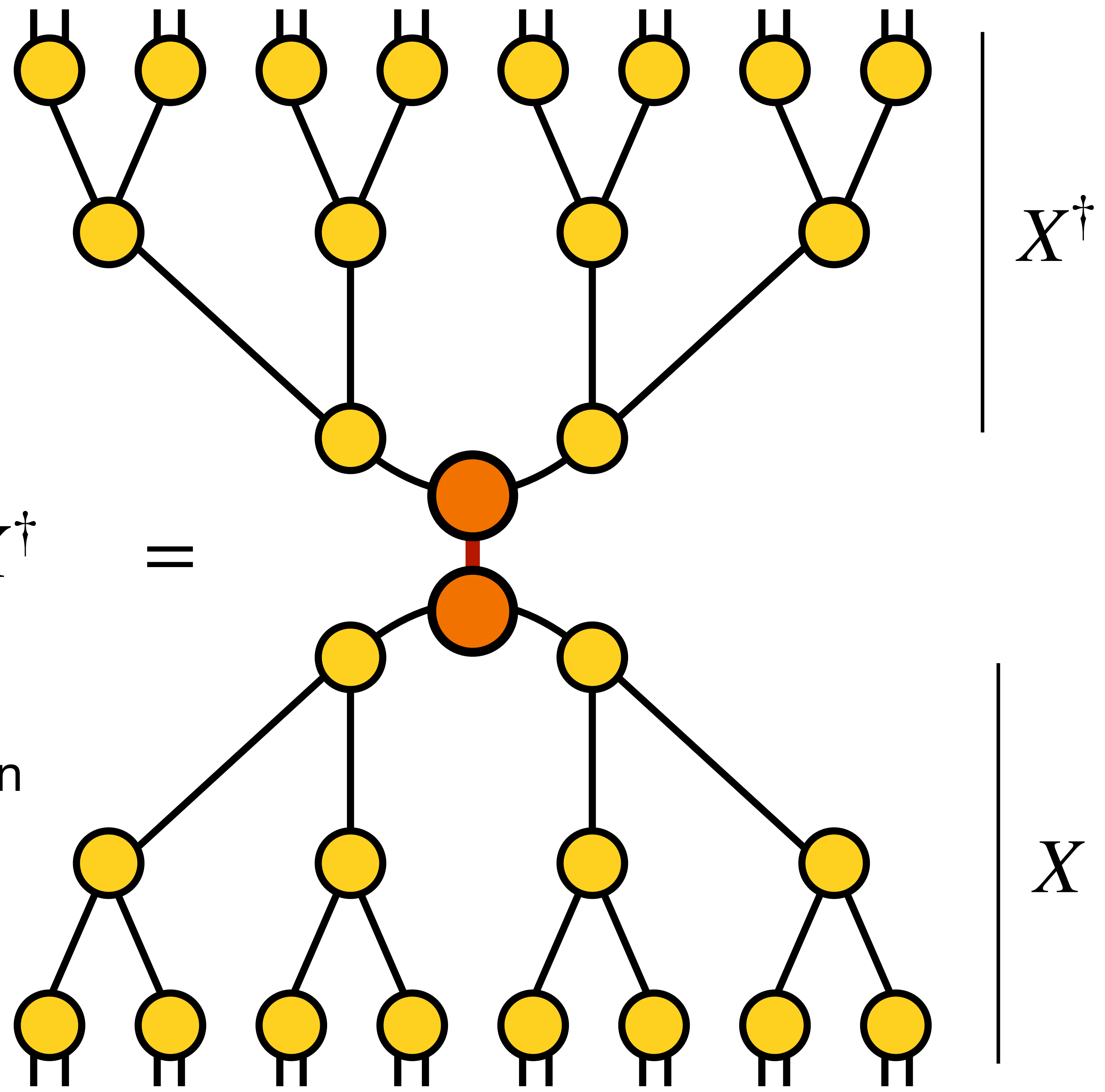

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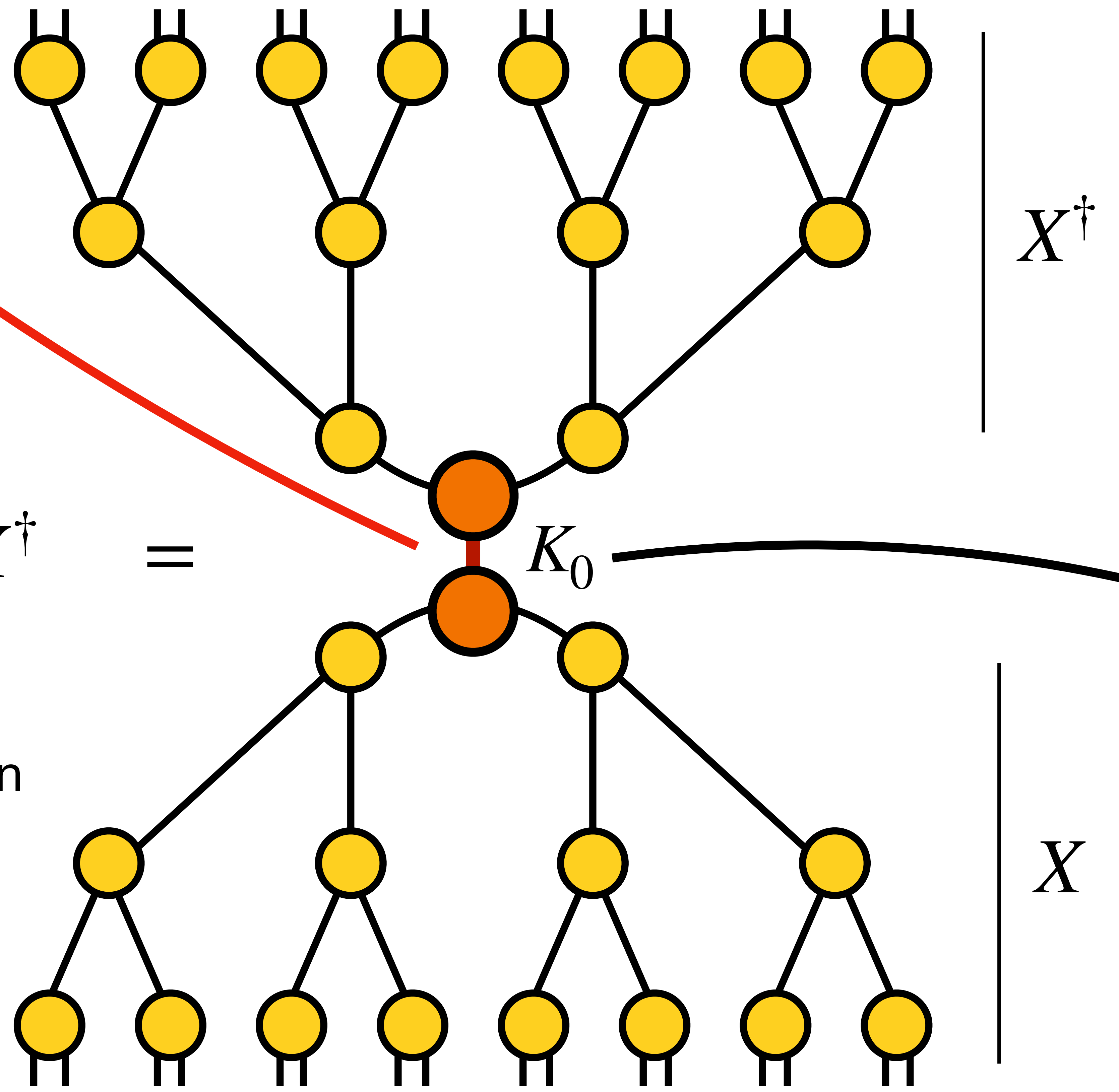

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K_0 - number of eigenstates contributing to the density matrix

$\rho = \sum_j p_j |\psi_j> < \psi_j|$

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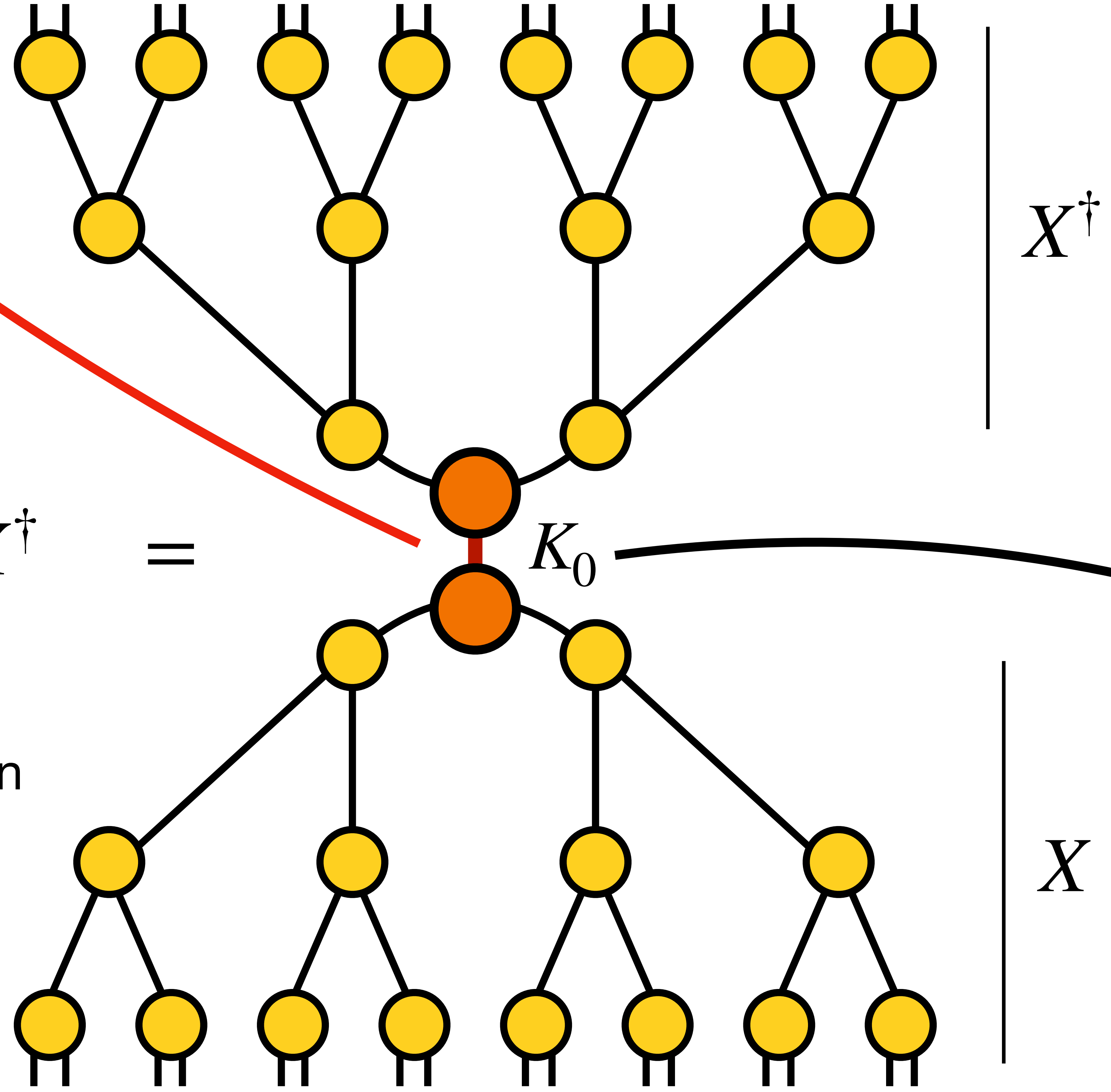
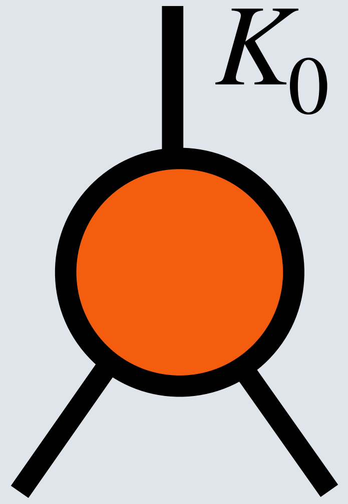


Columns of X are given with $\sqrt{p_j}|\psi_j>$

Root tensor = compressed density matrix

All information about **bipartite entanglement** is compressed into a single tensor



Singular values of this tensor are $\sqrt{p_j}$

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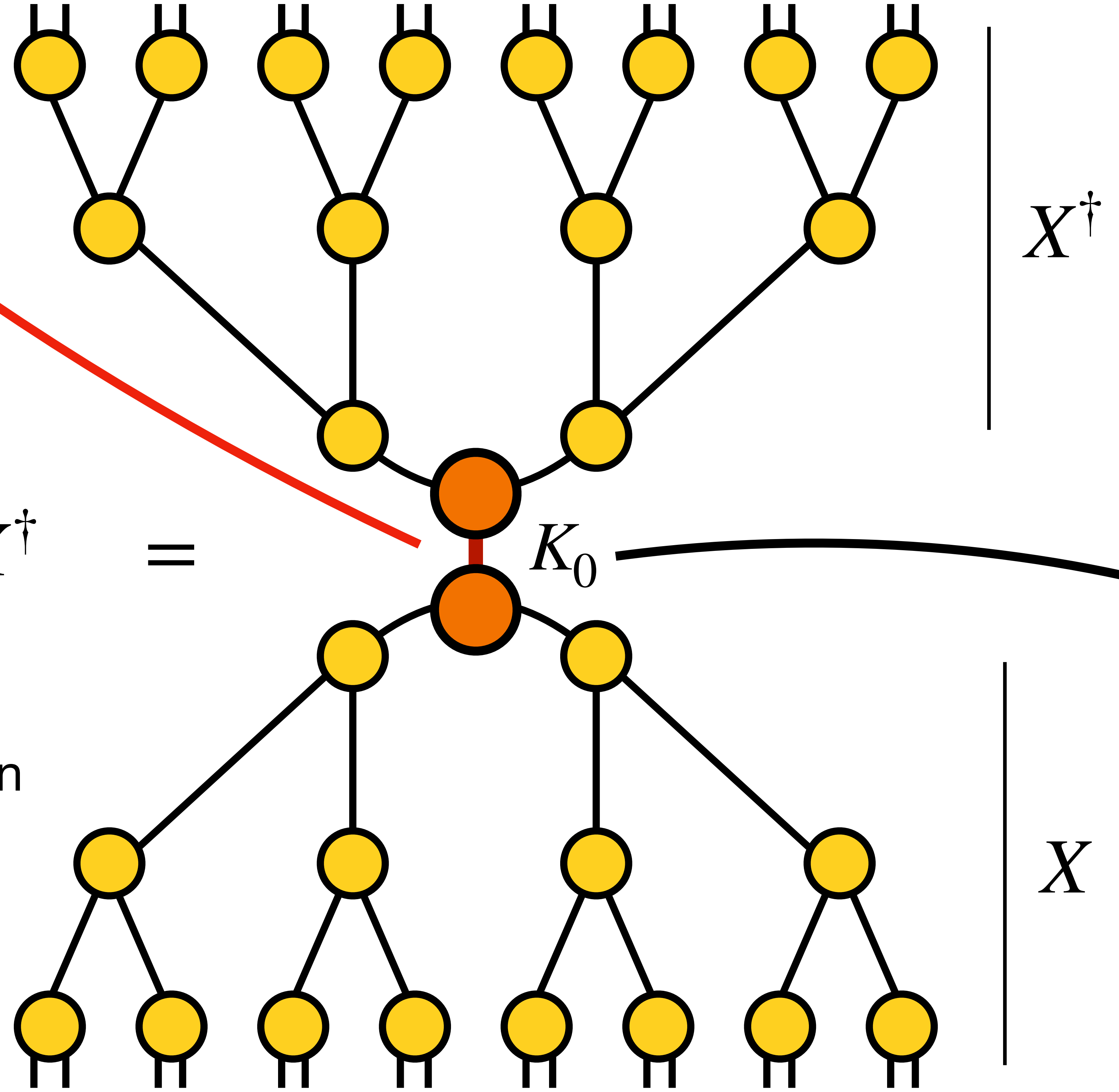

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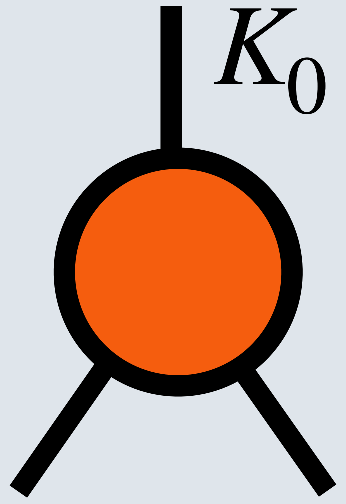

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Question: how to efficiently compute the thermal density matrix in a TTO ansatz?

How to obtain a TTO thermal density matrix?

N. Reinić, D. Jaschke, D. Wanisch, P. Silvi, S. Montangero
Phys. Rev. Research 6, 033322 (2024)

2-step process:

How to obtain a TTO thermal density matrix?

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2-step process:

Temperature ∞

$\hat{\rho} = \frac{1}{d^N} \cdot \mathbb{1}$

$e^{-d\beta\hat{H}/2}\; e^{-d\beta\hat{H}/2} \dots e^{-d\beta\hat{H}/2}$

Temperature T

$\hat{\rho} = e^{-\beta\hat{H}}$

Cool down via imaginary time evolution

Mathematically equivalent to **time evolution** with the substitution:

$$\frac{it}{\hbar} \rightarrow \frac{\beta}{2} = \frac{1}{2T}$$

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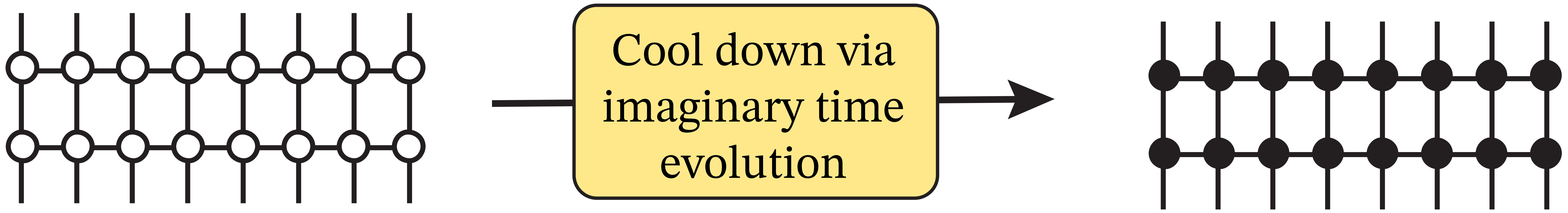
$\rho =$



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Locally Purified Tensor Network

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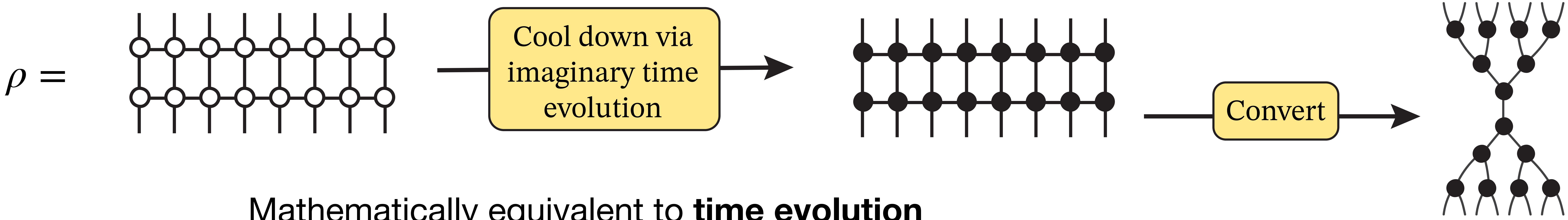

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Conversion using an iterative procedure of tensor decompositions and manipulations

Locally Purified Tensor Network

Tree Tensor Operator

Part 2: **Application to finite temperature Rydberg arrays**

Part 2: Finite-temperature Rydberg arrays

$$n_z = \frac{1}{2}(\sigma_z + 1)$$

Ω - Rabi frequency

Δ - Detuning

$$\hat{H} = \frac{\Omega}{2}\sum_{i}^{N}\sigma_x^i - \Delta\sum_{i}^{N}n_z^i + \sum_{i\neq j}\frac{C_6}{R_{ij}^6}n_z^i n_z^j$$

Laser terms

Rydberg blockade mechanism

1D chain with open boundaries

N. Reinić, D. Jaschke, D. Wanisch, P. Silvi, S. Montangero, Phys. Rev. Research 6, 033322 (2024)

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$$\hat{H} = \frac{\Omega}{2}\sum_i^N \sigma_x^i - \Delta \sum_i^N n_z^i + \sum_{i\neq j} \frac{C_6}{R_{ij}^6} n_z^i n_z^j$$

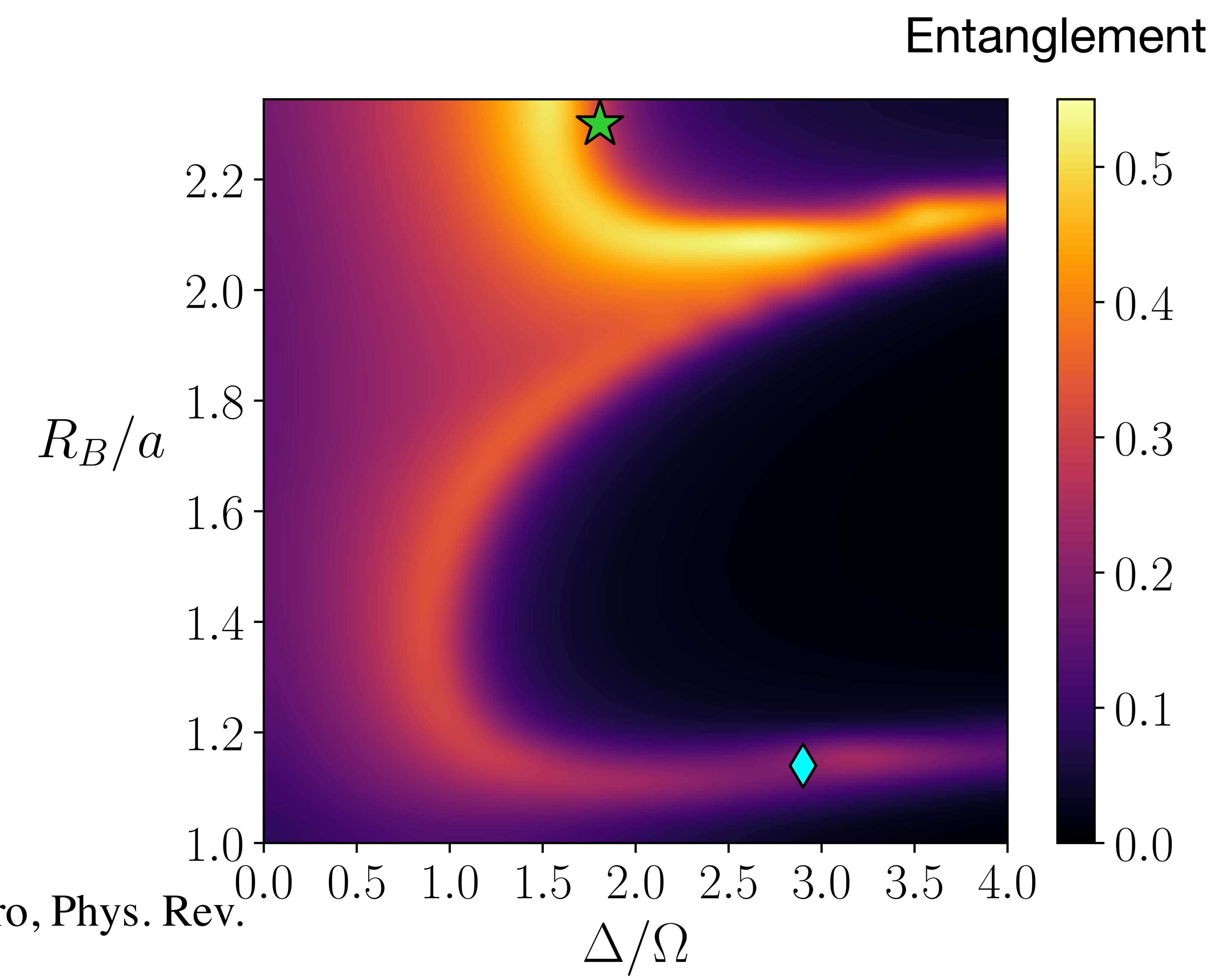
Laser terms

Rydberg blockade mechanism

Blockade radius:

$$R_B = \sqrt[6]{\frac{C_6}{\Omega}}$$

1D chain with open boundaries



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Part 2: Finite-temperature Rydberg arrays

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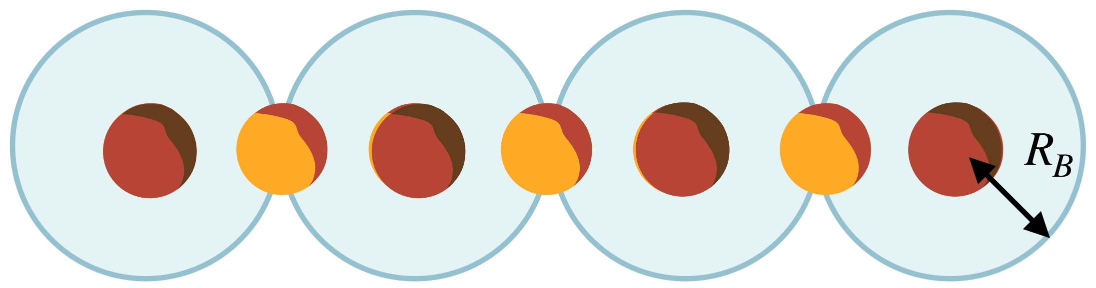
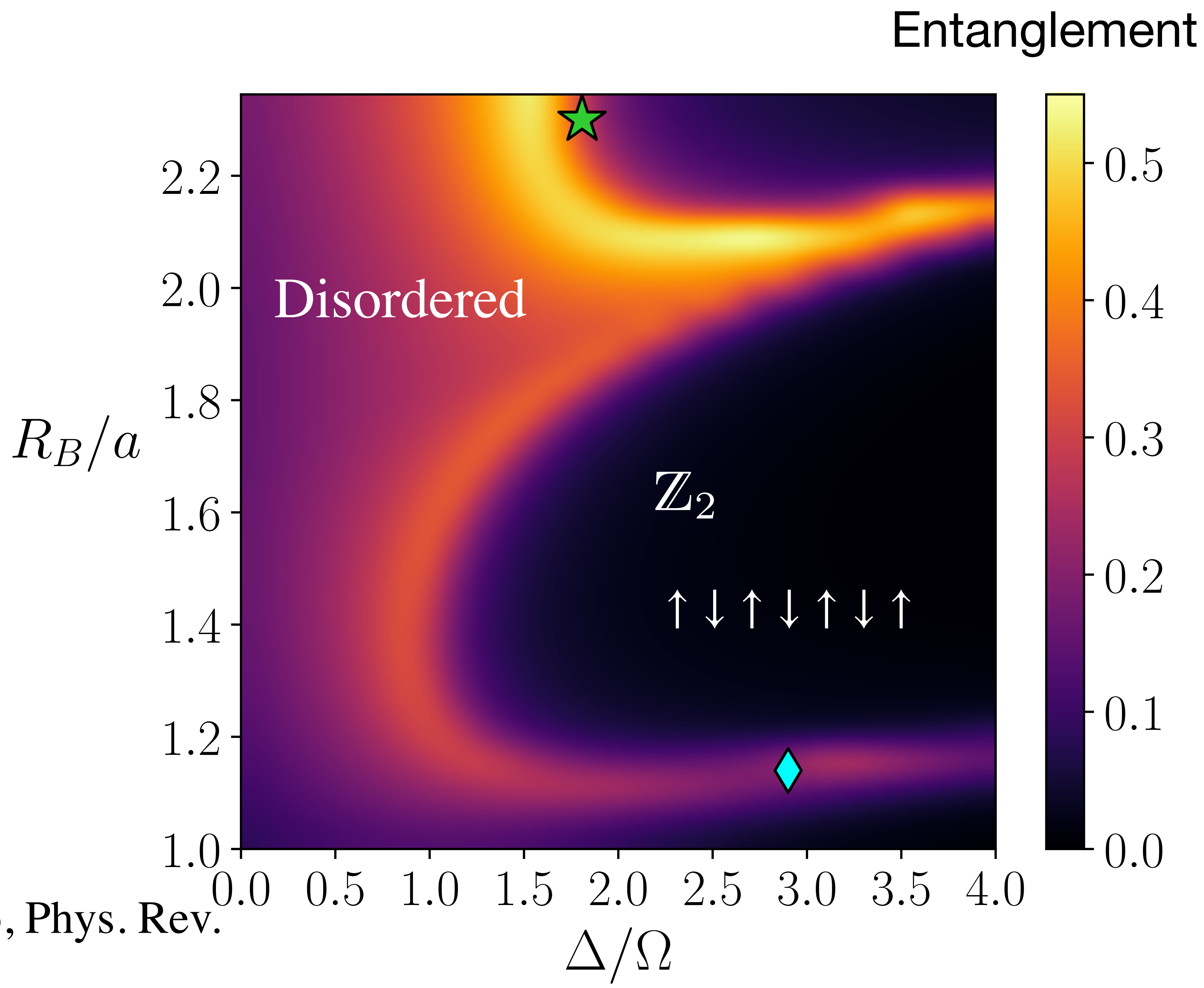
Period-2





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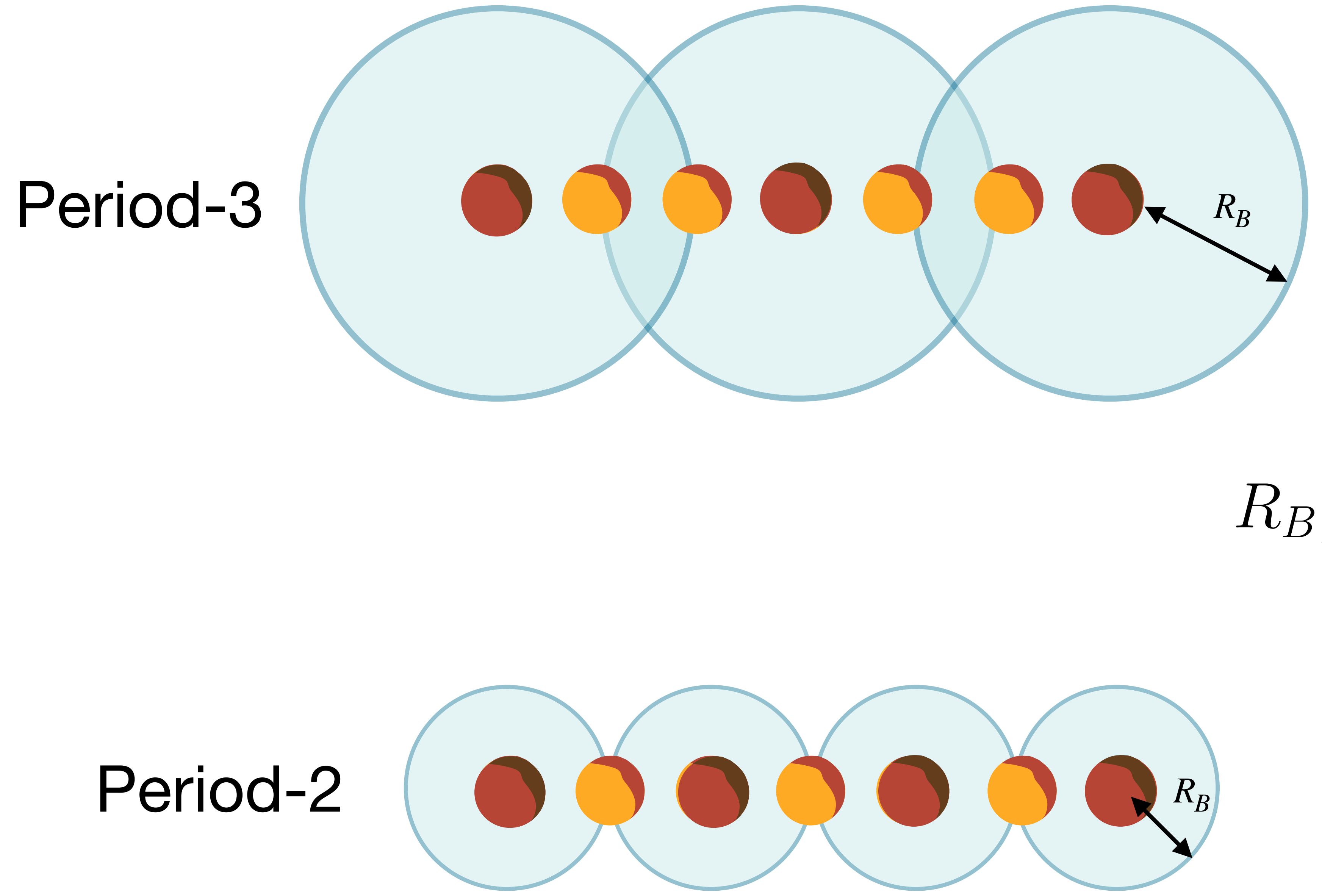

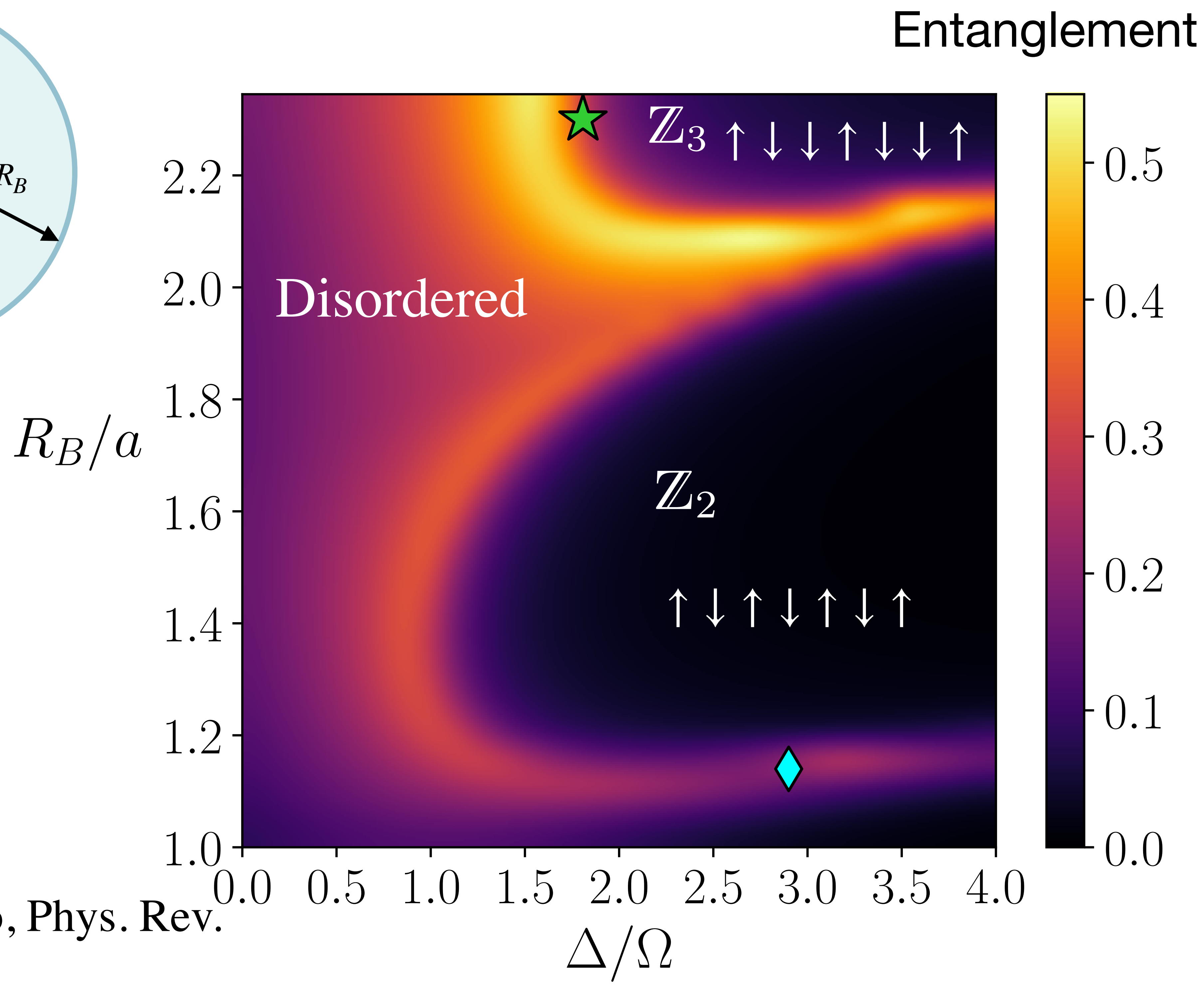

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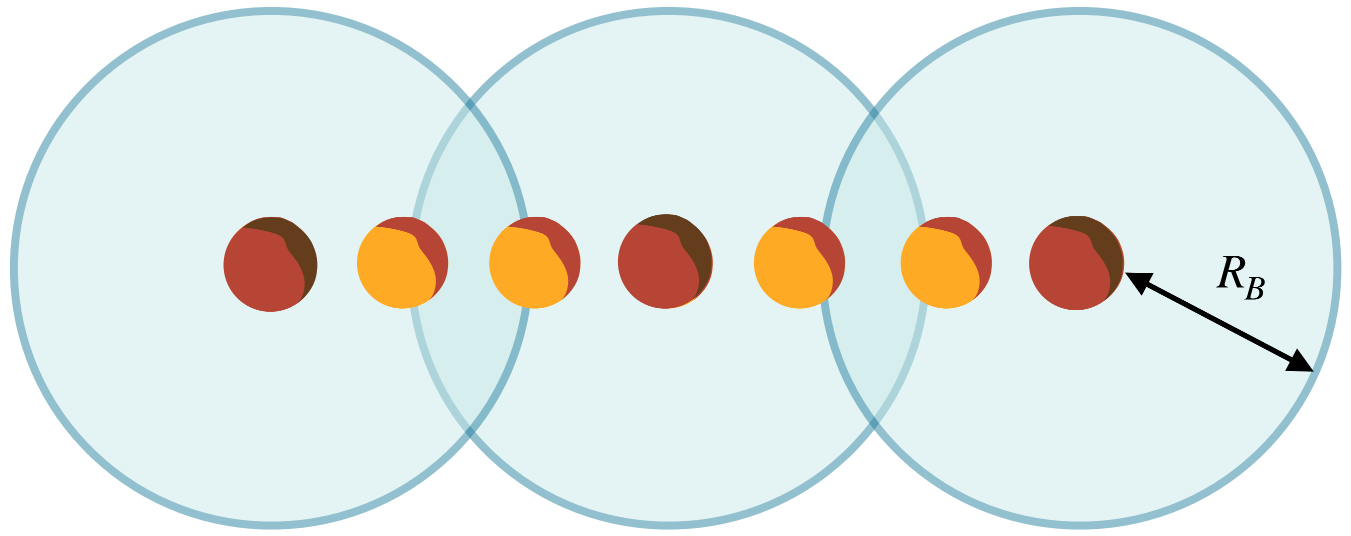
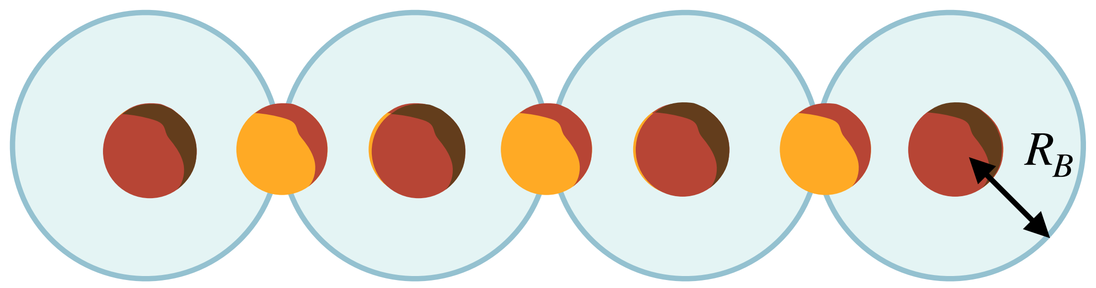
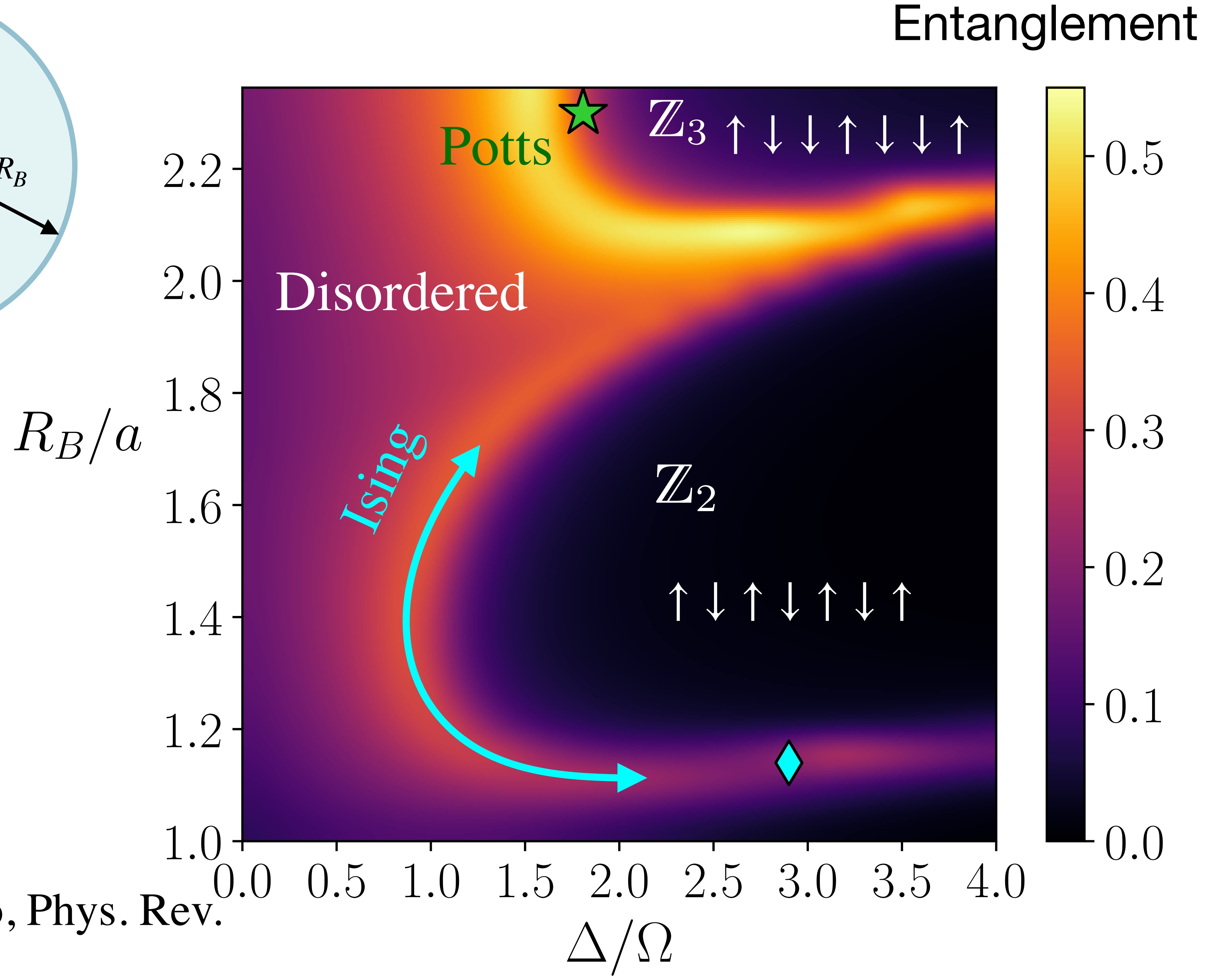
Period-3



Period-2





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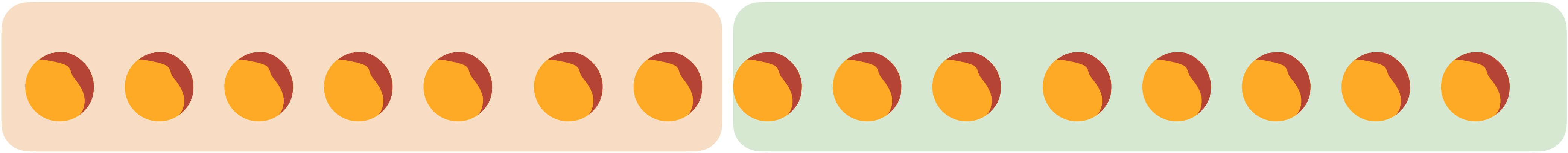
Finite-temperature entanglement in Rydberg chain



Entanglement between the system's halves

Zero temperature:

Pure-state entanglement:
entanglement entropy $\mathscr{S}$

At quantum critical points:

$$\mathscr{S} \propto \frac{c}{6} log(N)$$

Conformal field theory result
for 1D with open boundaries

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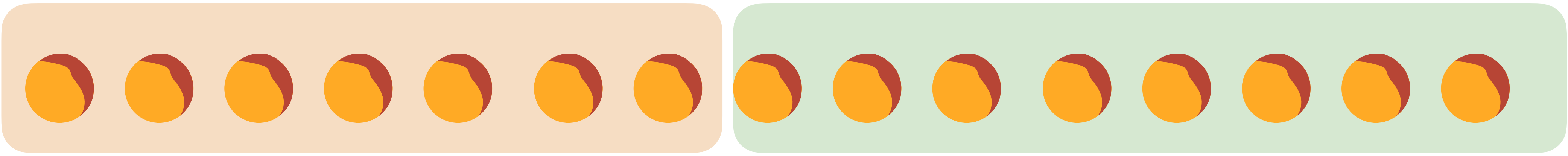

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$$E_F = \frac{c}{6} log(N) + g_F(TN^z)$$

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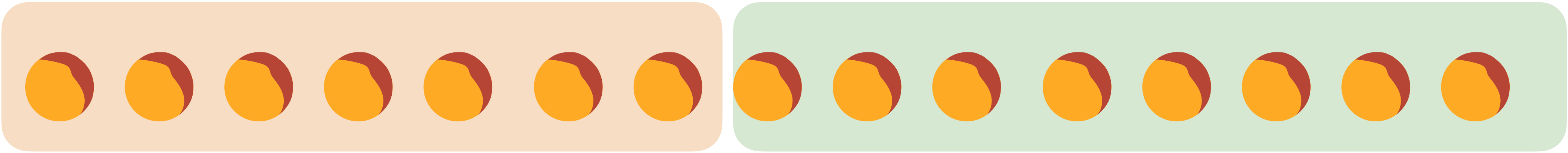

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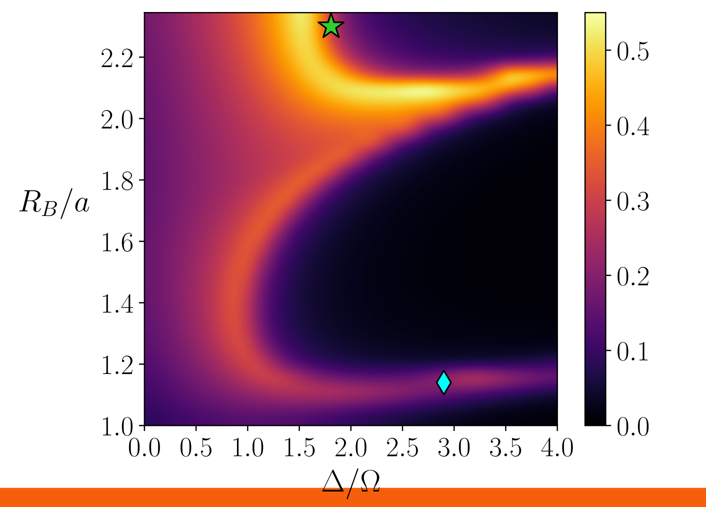
We measure:

- ***entanglement negativity***
- ***entanglement of formation***

At **Ising** and **Potts** quantum critical points



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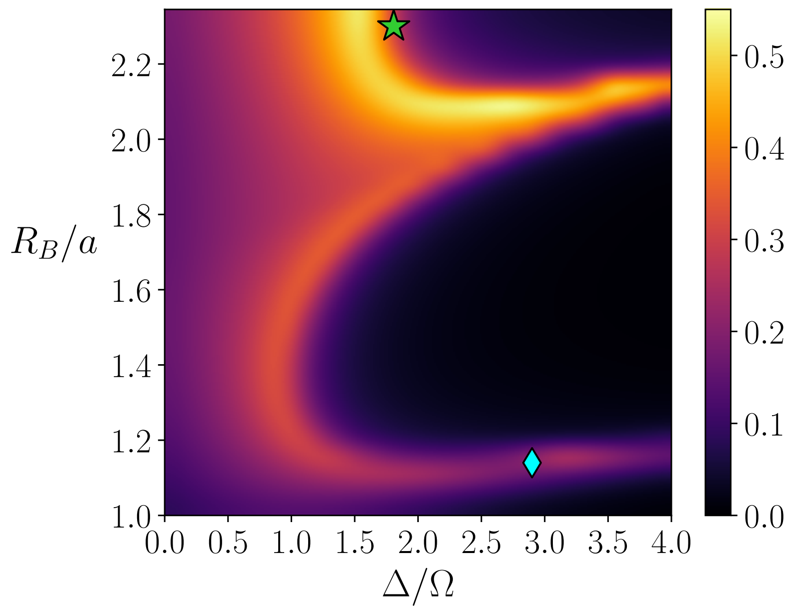

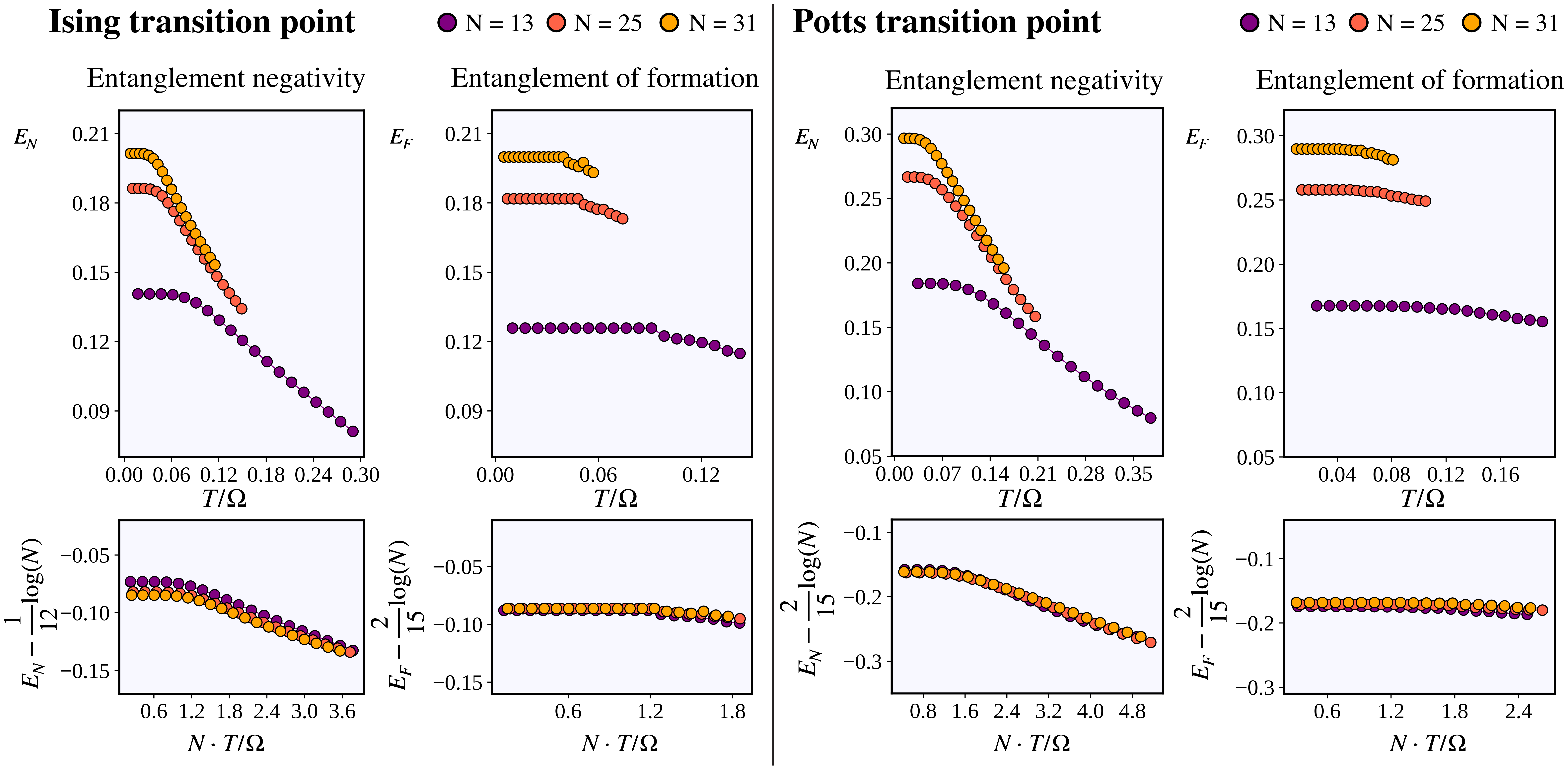

$$E_N = \frac{c}{6} log(N) + g_N(TN^z)$$

$$E_F = \frac{c}{6} log(N) + g_F(TN^z)$$

Part 1& 2 - **Conclusions**

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Phys. Rev. Research 6, 033322 (2024)



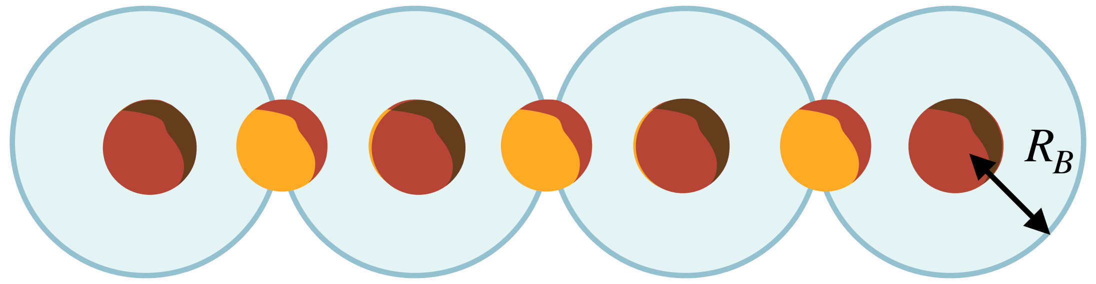

- **We developed the tree tensor operator numerical toolbox**

Mixed state entanglement, thermal probabilities

Applicable to a general quantum many-body model

- Applied the algorithm on **Rydberg atom systems**

- Confirmed the **finite-temperature entanglement scaling law** at quantum critical points

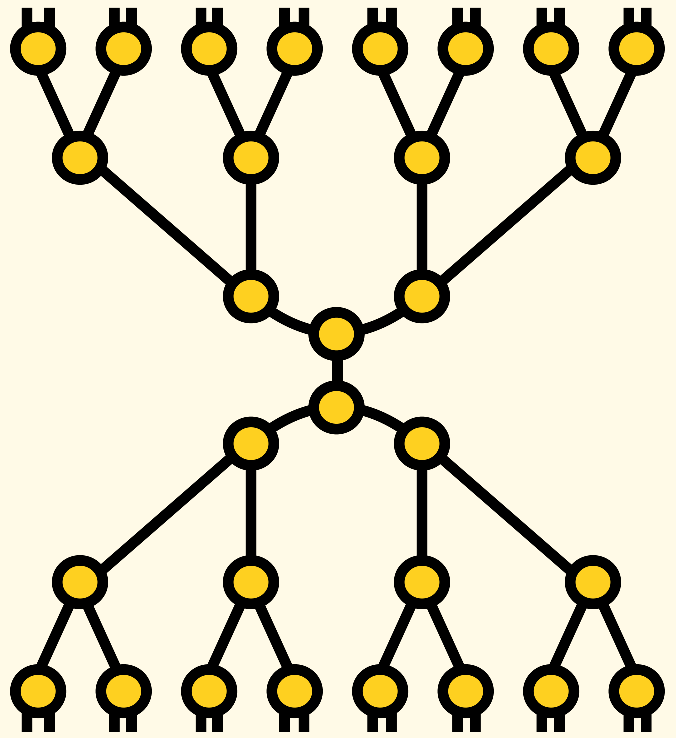


Part 3: **Energy spectrum extraction**

(Preliminary)

Excited states from thermal density matrix

Idea:



$$\rho = \sum_{j=1}^{K_0} p_j |\psi_j> < \psi_j|$$

With tree tensor operator algorithm we have:

thermal probabilities p_j

We know:

$$p_j = \frac{e^{-\beta E_j}}{Z} \qquad \beta = \frac{1}{T}$$

E_i = eigenenergies of Hamiltonian

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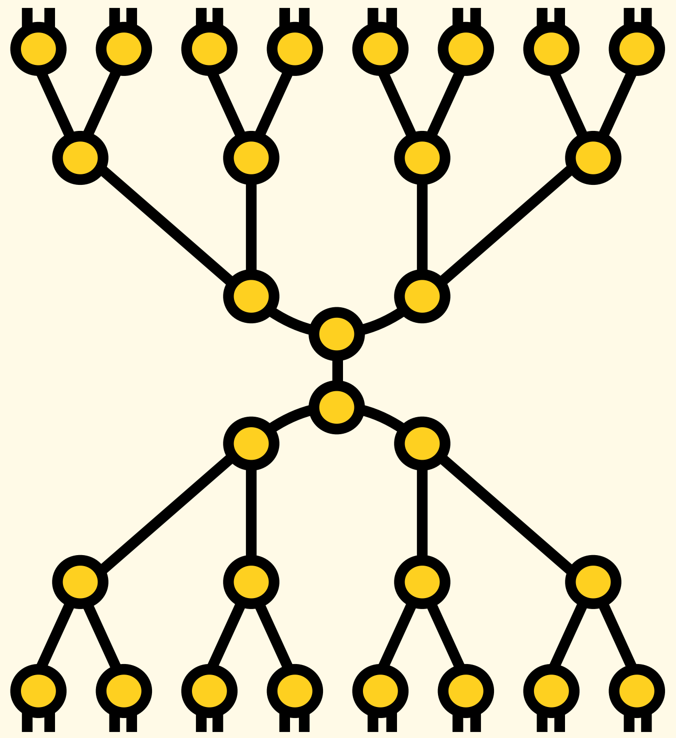

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$$\frac{p_{j+1}}{p_j} = e^{-\frac{1}{T}(E_{j+1} - E_j)}$$

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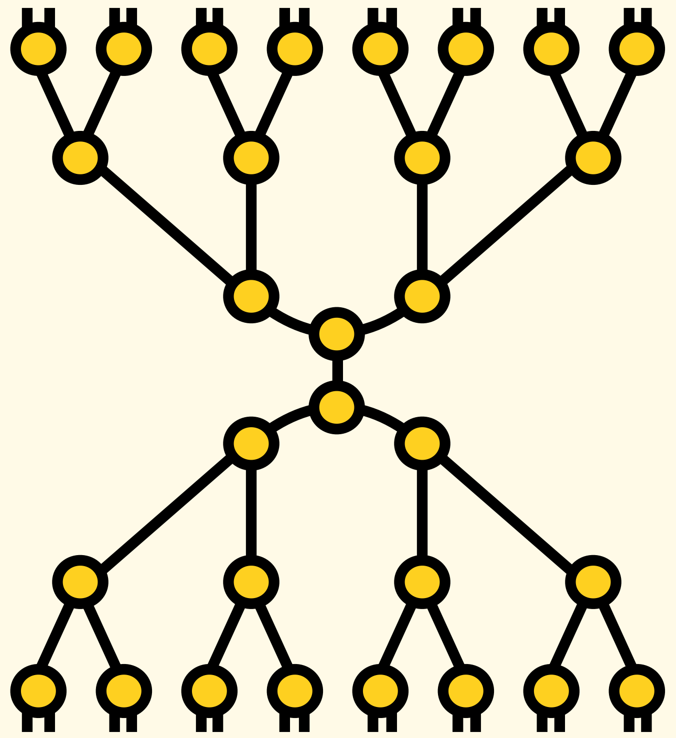

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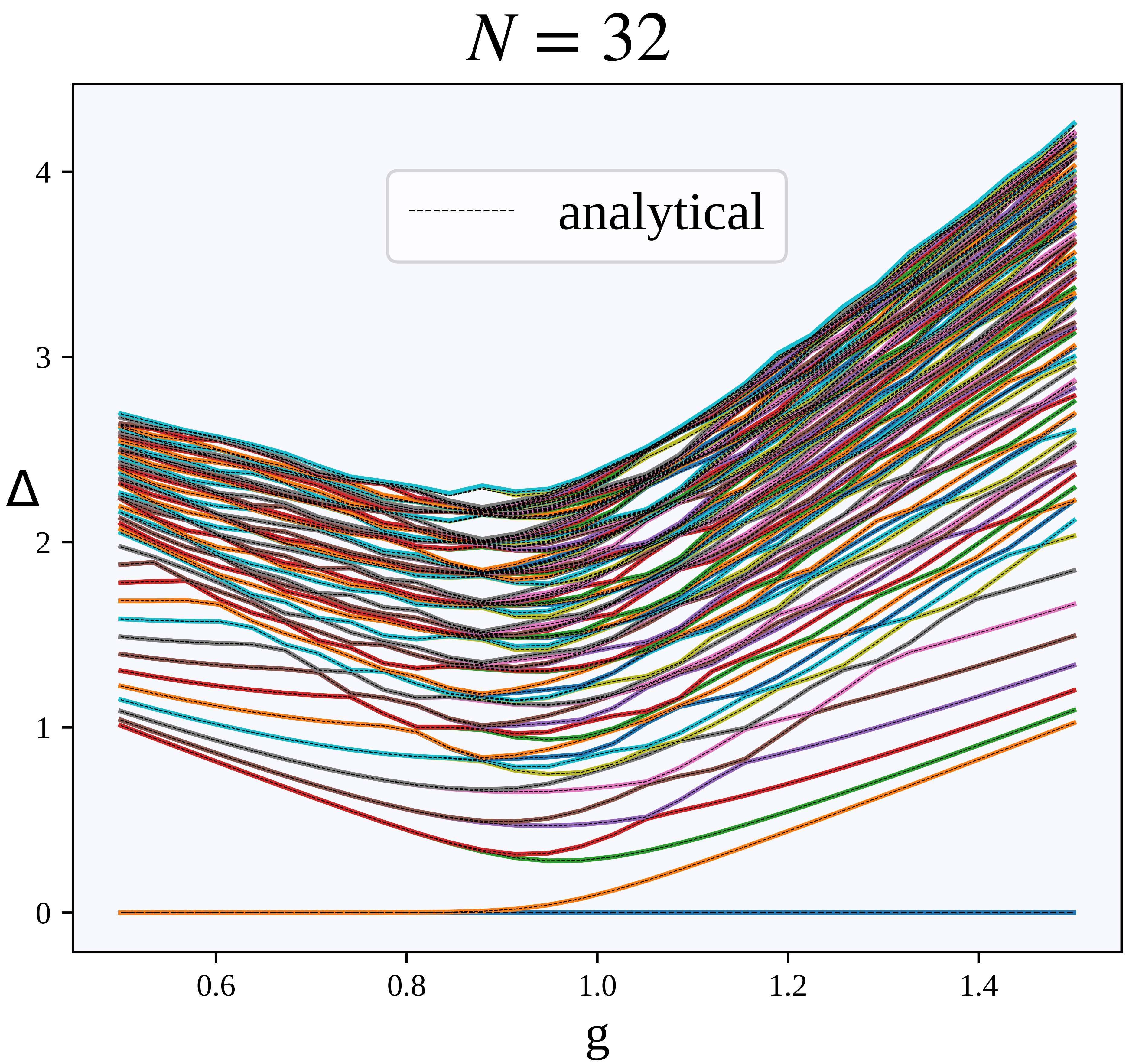
$$\frac{p_{j+1}}{p_j} = e^{-\frac{1}{T}(E_{j+1}-E_j)} \qquad E_{j+1} - E_j = \Delta_{j,j+1} = T \cdot ln\left(\frac{p_j}{p_{j+1}}\right)$$

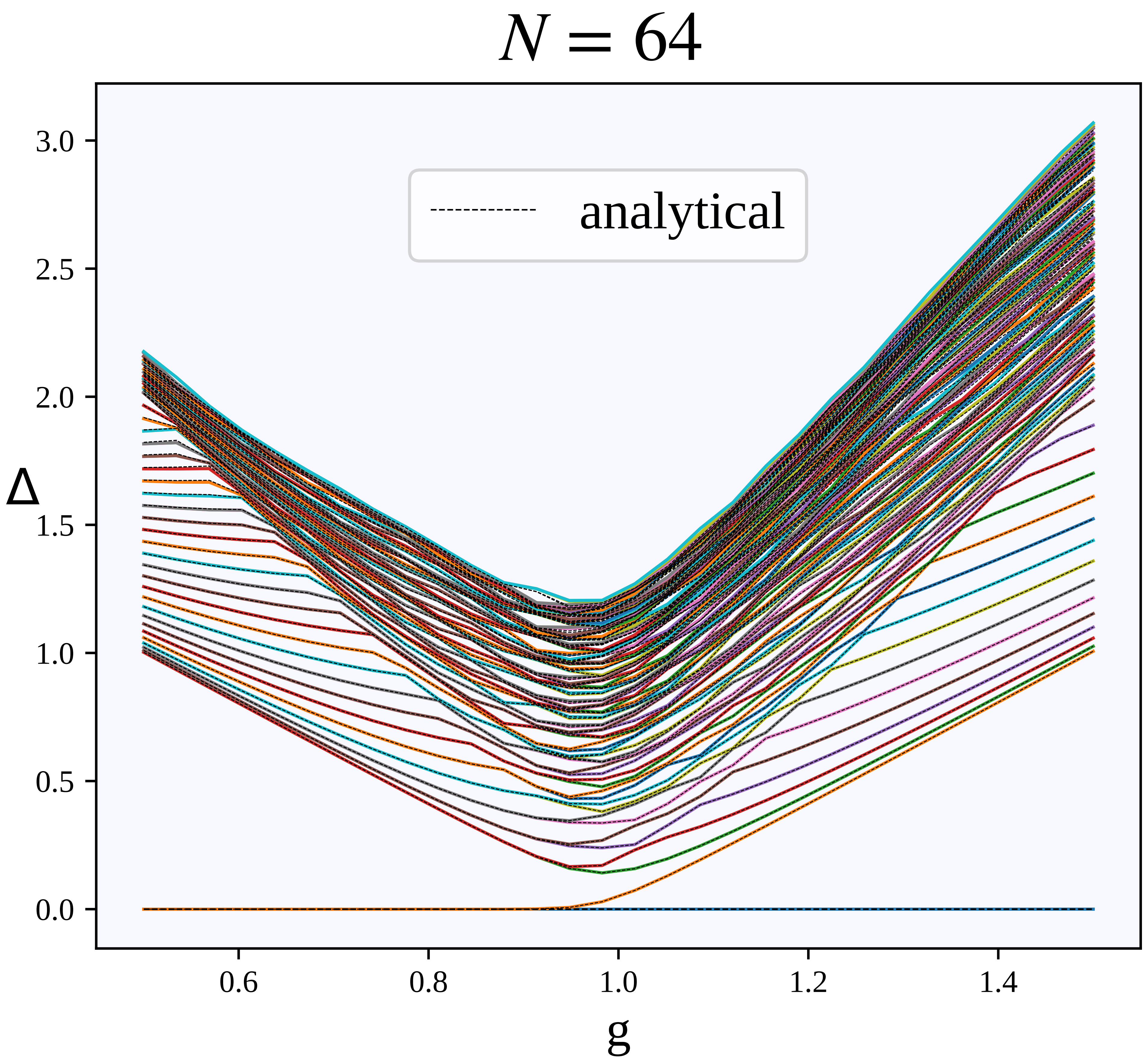
We have the direct access to the first (K_0-1) energy gaps!

Results

First 100 energy levels of quantum Ising 1D

$$H = -\sum_i \sigma_x^i \sigma_x^{i+1} + g \sum_i \sigma_z^i$$





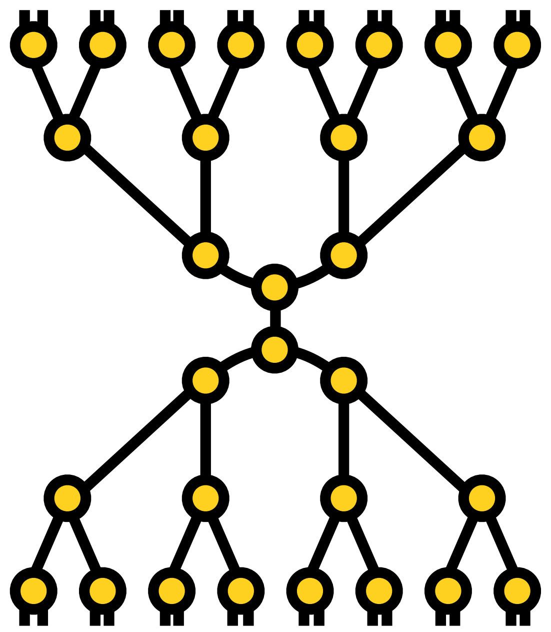
Thank you for the attention!

*The research leading to these results has received funding from European Union's Horizon 2020 research and innovation programme under the Marie-Sklodowska Curie grant agreement no 101034319 and from the European Union - NextGenerationEU.

$\rho = \sum_j p_j |\psi_j > < \psi_j|$

Restriction on mixedness

Tree tensor operator (TTO)



Preserves **positivity** ✓

No restriction on **mixedness** ✗

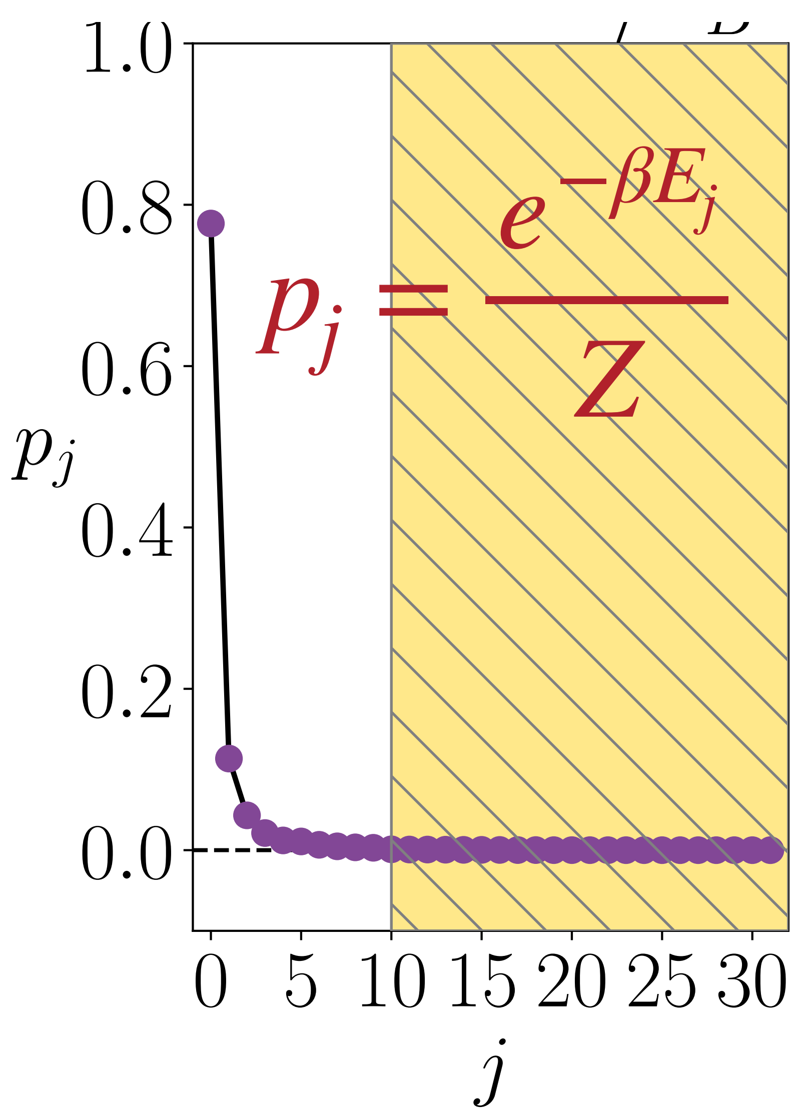
Access to local **observables** ✓

Access to **entanglement** ✓

Access to **eigenvalues** of density matrix ✓

Density matrix of thermal equilibrium states:

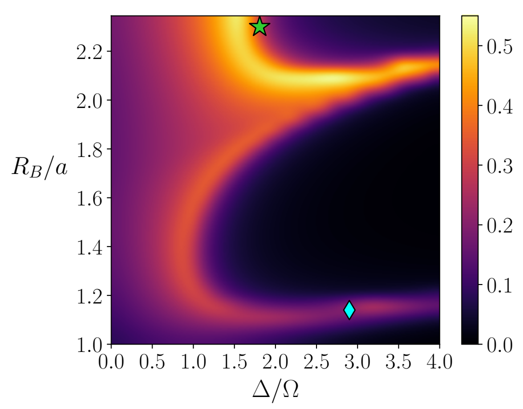
Thermal probabilities **decay exponentially** with the eigenenergies



At low temperatures: only the low-energy excited states will have non-negligible contribution

✓ **Our physical states of interest are usually not very mixed**

Ground state resilience to temperature



$T \approx 0$:
Pure ground state

$T \gg$:
Mixture of states

Up to which temperature can we expect the system to be in a ground state?

Temperature range: $[0.025,\ 0.8]\ \frac{\Omega}{k_B}$

$\Omega = 2.5$ MHz

Temperature range: $[3,\ 96]\ \mu K$

Usual experimental temperatures: $\sim 10\ \mu K$

Temperature at which the **purity** falls below $\gamma_{cut} = 0.9$: