Dynamical attractors in 1D and 3D in Relativistic Boltzmann Transport

Vincenzo Nugara

based on: V. Nugara, S. Plumari, L. Oliva, and V. Greco, *Eur.Phys.J.C 84 (2024) 8, 861*;
V. Nugara, S. Plumari, V. Greco (*in preparation*)

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Laboratori Nazionali del Sud

Meeting SIM 2024
Catania, September 10th

Outline

- Attractors in uRHICs
- Relativistic Boltzmann Transport Approach
- 1D systems
- 3D systems. Moments
- 3D systems. Anisotropic flows
- Summary and outlook

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Attractors

What is an attractor?

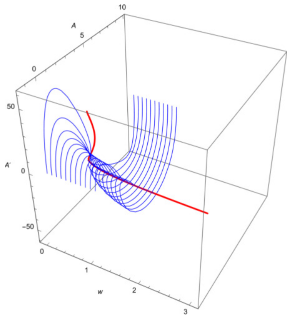
Subset of the phase space to which all trajectories converge after a certain time.

Why do we look for attractors?

- Uncertainties in initial conditions affect final observables? Memory of initial conditions?
- Appearance of attractors and hydrodynamisation. The issue of small systems, as produced in *pp* or *pA*

Where do we look for attractors?

- Full distribution function $f(x, p)$
- Moments of $f(x, p)$, probing regions of the phase-space



Jankowski, Spalinski, *Hydrodynamic attractors in ultrarelativistic nuclear collisions*, 2023

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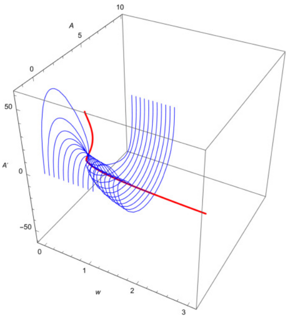

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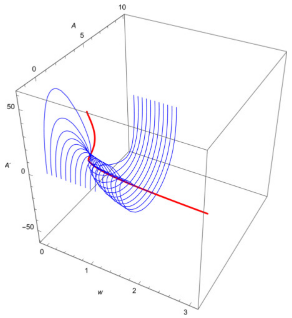

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Normalized moments

Moments $M^{nm}(x)$ of the distribution function $f(p)$

$$M^{nm}(x) = \int \frac{d^3\vec{p}}{(2\pi)^3 p^0} (p \cdot u)^n (p \cdot z)^{2m} f(x,p)$$

They carry information about the $f(x,p)$ (M. Strickland *JHEP* 12, 128, (2010)) .

All moments $\Longleftrightarrow$ whole $f(x,p)$

Attractors spotted in the normalized moments

$$\overline{M}^{nm}(x) = \frac{M^{nm}[f(x,p)]}{M^{nm}[f_{eq}(T_{eff}, \Gamma_{eff}, u^\mu)]}$$

$f_{eq} = \Gamma_{eff} \exp(-(p \cdot u)/T_{eff})$). Matching conditions imply: $M^{10} = n$, $M^{20} = \varepsilon$, $M^{01} = P_L$.
System equilibrates at large $\tau \implies \lim_{\tau \to \infty} \overline{M}^{nm}[f] = 1$.

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Boltzmann Equation

Solve the Relativistic Boltzmann Equation with the full collision integral:

$$p^{\mu}\partial_{\mu}f(x,p) = C\left[f(x,p)\right]_{\mathrm{p}}\,, \tag{1}$$

Only binary elastic $2 \leftrightarrow 2$ collisions:

$$C[f]_{\mathrm{p}} = \int \frac{d^3p_2}{2E_{\mathrm{p}_2}(2\pi)^3}\int \frac{d^3p_{1'}}{2E_{\mathrm{p}_{1'}}(2\pi)^3}\int \frac{d^3p_{2'}}{2E_{\mathrm{p}_{2'}}(2\pi)^3}\left(f_{1'}f_{2'} - f_1 f_2\right) \times |\mathcal{M}|^2\,\delta^{(4)}\left(p_1 + p_2 - p_{1'} - p_{2'}\right) \tag{2}$$

$\mathcal{M}$: transition amplitude. $|\mathcal{M}|^2 = 16\pi\, s\, s^2\, d\sigma/dt \overset{\text{isotr.}\sigma}{=} 16\pi\, s\, \sigma$.

How to solve the Boltzmann Equation with the **full collision integral** $C[f]$?

Numerical solution with test particle method: simulation of propagating particles which collide with locally fixed cross-section σ_{22}.

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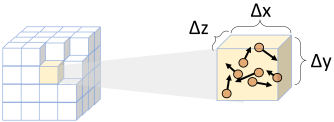
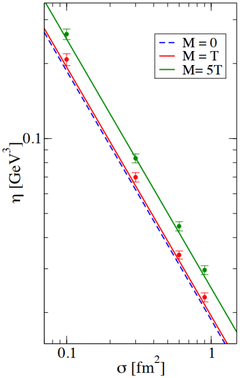
Relativistic Boltzmann Transport (RBT) Code

- C language: high performance (up to $3 \cdot 10^8 N_{\text{particles}}$)
- Stochastic Method to implement collisions (Xu, Greiner, *PRC 71* (2005), Ferini, Colonna, Di Toro, Greco, *PLB 670* (2009))
- Space discretisation: particles in the same cell can collide elastically with probability $P_{22} \propto \sigma_{22}$
- $2 \leftrightarrow 2$ collisions $\Rightarrow$ Particle conservation: Fugacity $\Gamma \neq 1$
- Fix η/s by computing σ_{22} locally via the Chapman-Enskog formula (Plumari, Puglisi, Scardina, Greco, PRC 86 (2012)):

$$\eta = f(m/T)\frac{T}{\sigma_{22}} \overset{m=0}{\simeq} 1.2\frac{T}{\sigma_{22}}$$





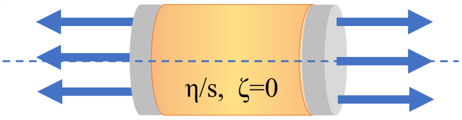
Code setup for 1D boost-invariant systems

- Conformal system $(m = 0)$
- One-dimension
  Homogeneous distribution and periodic boundary conditions in the transverse plane.
- Boost-invariance. No dependence on η_s!
  $dN/d\eta_s = \text{const.}$ in $[-\eta_{s\max}, \eta_{s\max}]$, $\eta_{s\max}$ large enough to avoid propagation of information from boundaries. (We will relax this hypothesis later)

Romatschke-Strickland Distribution Function

$$f_0(\mathrm{p}; \gamma_0, \Lambda_0, \xi_0) = \gamma_0 \exp\left(-\frac{1}{\Lambda_0}\sqrt{p_\perp^2 + p_w^2(1+\xi_0)}\right),$$

where $p_\perp^2 = p_x^2 + p_y^2$ and $p_w = (p \cdot z)$.
ξ_0 fixes initial P_L/P_T, γ_0 and Λ_0 fix initial ε and n



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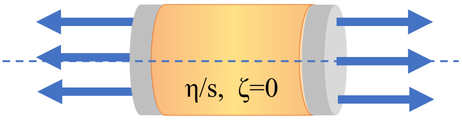

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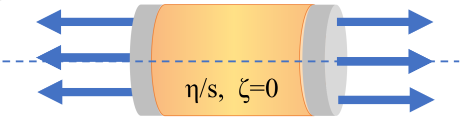

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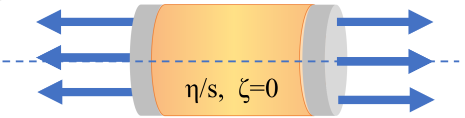

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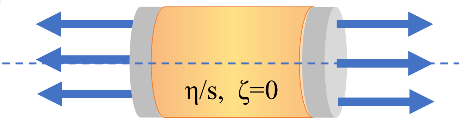

Forward Attractor

- 0+1D $\implies \overline{M}^{nm}(x) = \overline{M}^{nm}(\tau)$
- Change initial anisotropy ξ_0 (and thus P_L/P_T).
- Fix τ_0, T_0, η/s.

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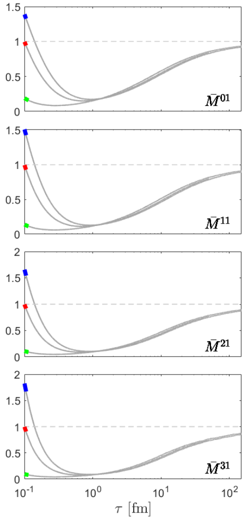
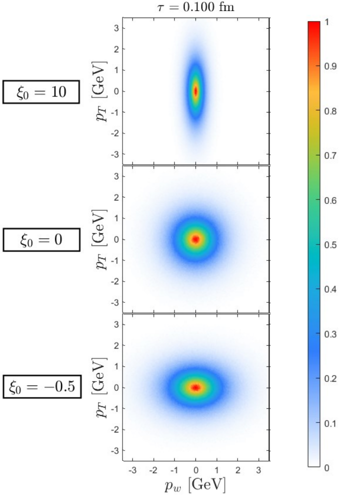
Distribution function evolution: Forward attractor vs τ, $\eta/s = 10/4\pi$.

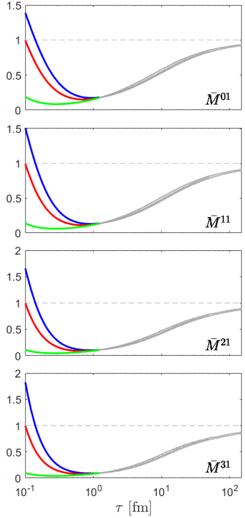
- At $\tau = \tau_0$, three different distributions in momentum space: oblate ($\xi_0 = 10$), spherical ($\xi_0 = 0$) and prolate ($\xi_0 = -0.5$).

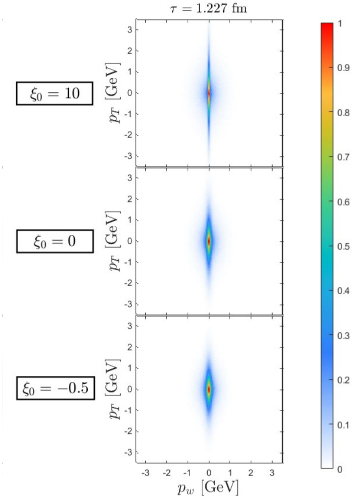




Distribution function evolution: Forward attractor vs τ, $\eta/s = 10/4\pi$.

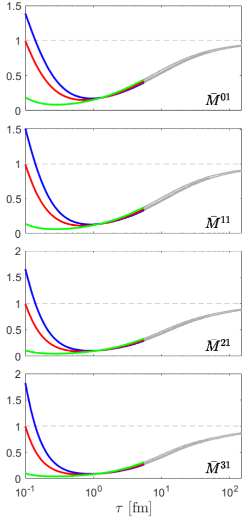
- Already at $\tau \sim 1$ fm, strong initial longitudinal expansion brings the system away from equilibrium
- Distribution functions have similar (but not identical) shape.

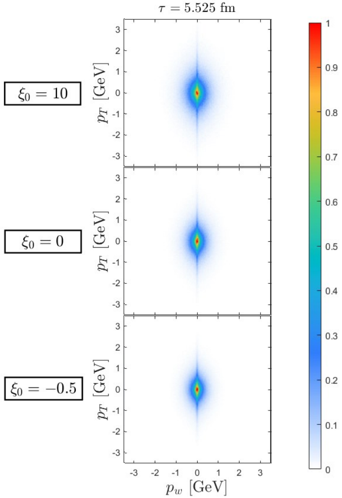




Distribution function evolution: Forward attractor vs τ, $\eta/s = 10/4\pi$.

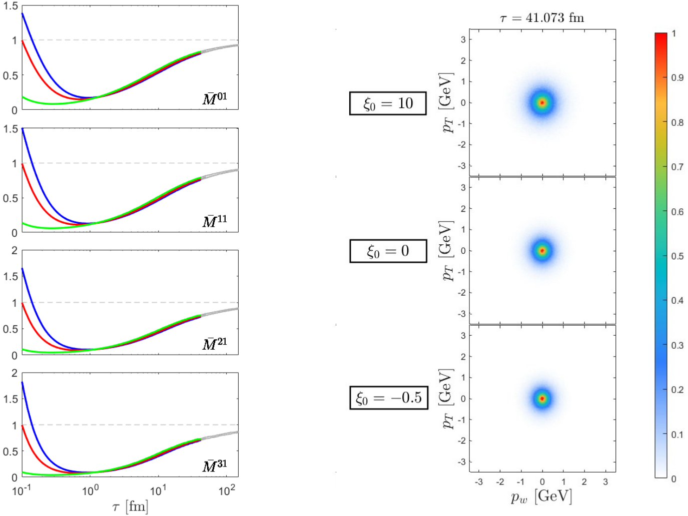
- At $\tau \sim 5$ fm, clear universal behaviour also for the distribution functions.
- Two components: strongly peaked p_w distribution and a more isotropic one (Strickland, *JHEP* 12, 128)





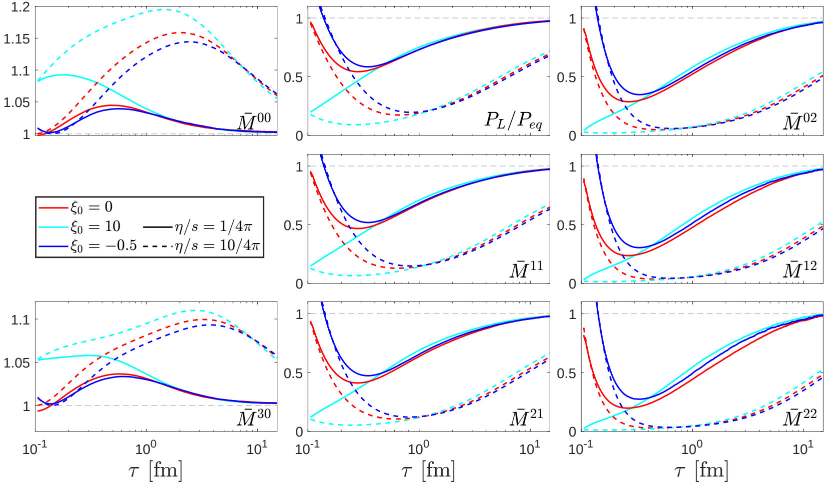
Distribution function evolution: Forward attractor vs τ, $\eta/s = 10/4\pi$.

- For large τ the system is almost completely thermalized and isotropized.



Forward Attractor vs τ

Different initial anisotropies $\xi_0 = -0.5,\ 0,\ 10,\ \infty$, for $\eta/s = 1/4\pi$ and $\eta/s = 10/4\pi$.



- $\eta/s = 1/4\pi$: attractor at $\tau \sim 0.5$ fm
- $\eta/s = 10/4\pi$: attractor at $\tau \sim 1.0$ fm
- Not 10 times larger!
- Less collisions to reach the attractor?
- **Different attractors for different η/s?**

Definition of the relaxation time

Only one relevant time-scale in our simulation.

Mean free time

$$\tau_{coll} = \frac{1}{2}\left(\frac{1}{N_{\text{part}}}\frac{\Delta N_{\text{coll}}}{\Delta t}\right)^{-1}$$

Notice: $\tau_{coll} \propto \lambda_{\text{mfp}}$.

$$\tau_{eq}^{RBT} = \frac{3}{2}\tau_{coll} = \tau_{tr} = \tau_{eq}^{\text{RTA}} = \frac{5\eta/s}{T}$$

τ_{eq}^{RTA} used in RTA kinetic theory and hydro.

(Denicol *et al.PRD* 83, 074019)

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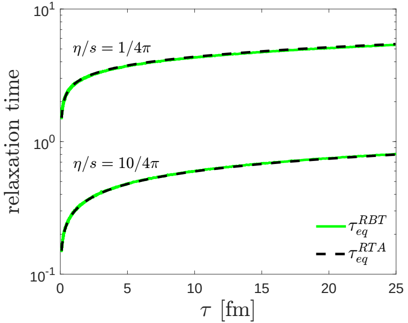
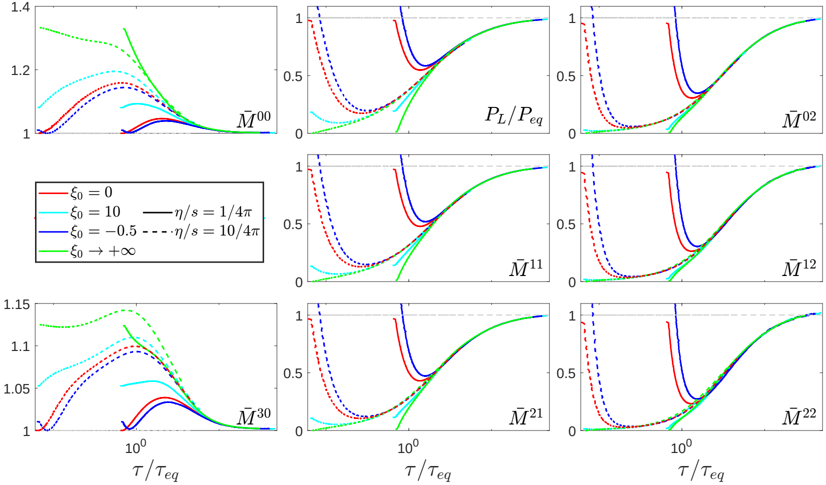


Comparison between τ_{eq}^{RTA} and τ_{eq}^{RBT}.

Pull-back attractor

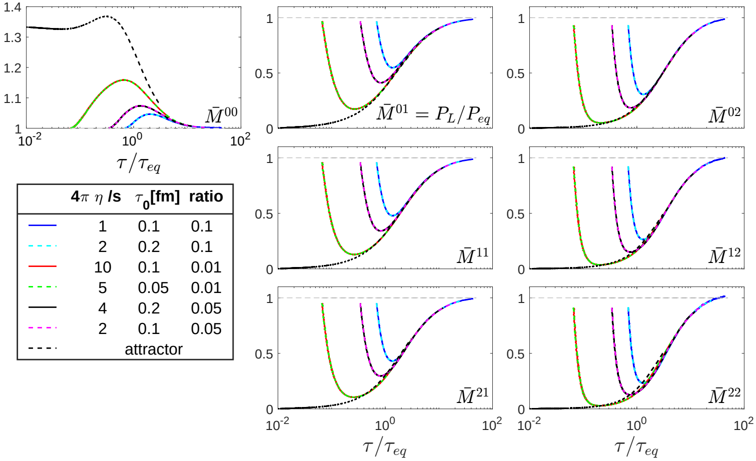
Different initial anisotropies $\xi_0 = -0.5,\ 0,\ 10, \infty$ for $\eta/s = 1/4\pi$ and $\eta/s = 10/4\pi$.



- **Unique attractor!**
- $\eta/s = 1/4\pi$: attractor at $\tau \sim 1.5\,\tau_{eq}$
- $\eta/s = 10/4\pi$: attractor at $\tau \sim 0.2\,\tau_{eq}$
- Less collisions per particle to reach the attractor?

Pull-back attractor

Change η/s and τ_0: three values for the ratio $\tau_0/(4\pi\eta/s)$: 0.1, 0.01, 0.05 fm.

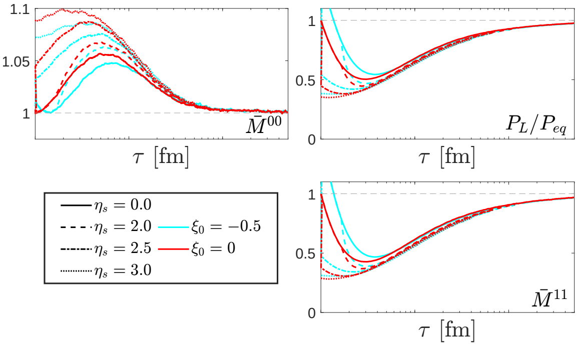


- Curves depend only on $\tau_0 T_0/(\eta/s) \propto (\tau/\tau_{eq})_0$ ratio (same universality found in hydrodynamics and RTA)
- Equilibration achieved at same τ/τ_{eq}
- Attractor reached at different τ/τ_{eq}
- Initial $\sim$ free streaming

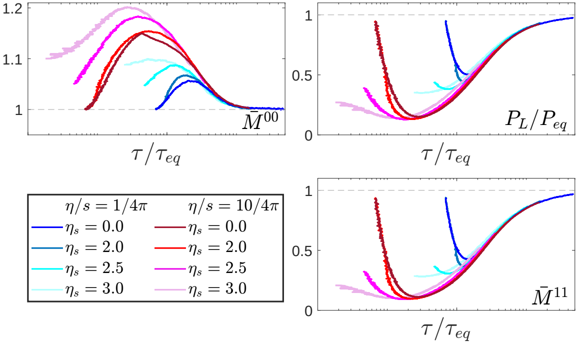
Breaking boost-invariance. Attractors at finite rapidity

Finite and non-homogenous initial distribution in η_s. 1+1D $\implies \overline{M}^{nm}(x) = \overline{M}^{nm}(\tau, \eta_s)$

Forward attractor. Fixed $\eta/s = 1/4\pi$.

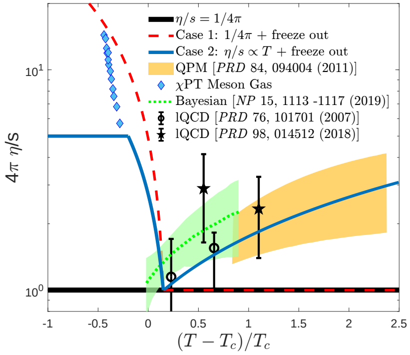


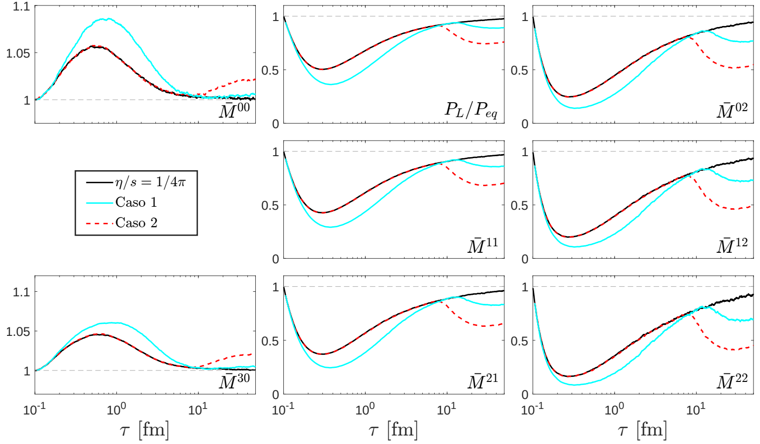
Pull-back attractor. Fixed $\xi_0 = 0$.



Universal behaviour even at $\eta_s = 3$, outside the initial distribution range!

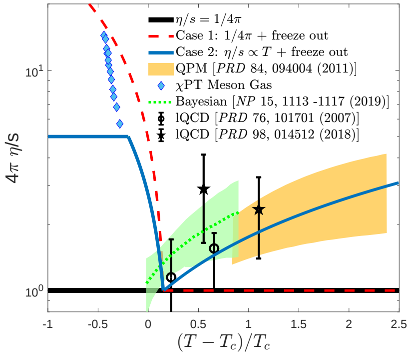
T-dependent η/s: Plot with respect to τ

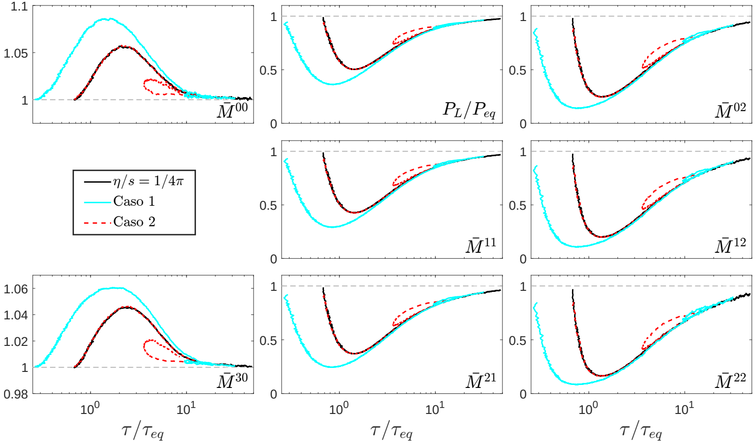




Universal behaviour lost at different τ (depend on local T)

T-dependent η/s: Plot with respect to τ/τ_{eq}





Universal behaviour restored after ‘loops’.

Code setup for 3D systems

- Conformal system ($m = 0$)
- Relax boundary conditions in the transverse plane $\Longrightarrow$ Transverse expansion

Romatschke-Strickland Distribution Function

$$f_0(x, p) = \gamma_0 \exp\left(-\frac{1}{\Lambda_0}\sqrt{p_x^2 + p_y^2 + p_w^2(1+\xi_0)}\right) e^{-x_\perp^2/R^2}\,\theta(2.5 - |\eta_s|)$$

- γ_0 and Λ_0 fix initial ε and n (Landau matching conditions).
- ξ_0 fixes initial longitudinal anisotropy (P_L/P_T)
- in the transverse plane with r.m.s
- Uniform distribution in η_s: $[-2.5, 2.5]$

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Transverse expansion

$0 < t < R$

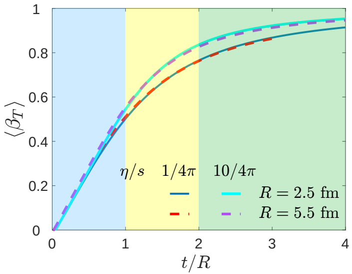
Longitudinal expansion ($\sim$ 1D)

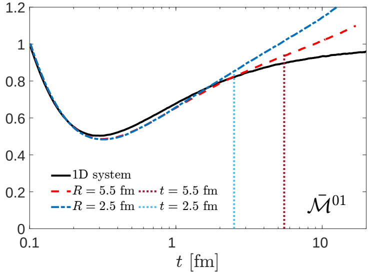
$t > R$

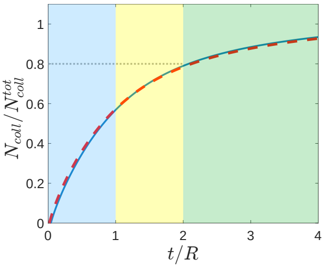
Onset of transverse expansion

$t > 2R$

Quasi free streaming ($\langle \beta_\perp \rangle > 0.8$)







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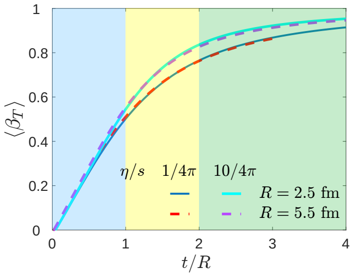

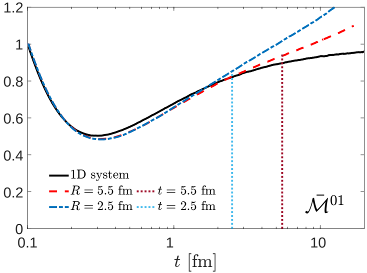

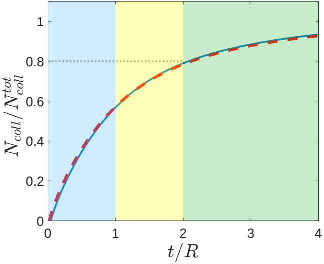

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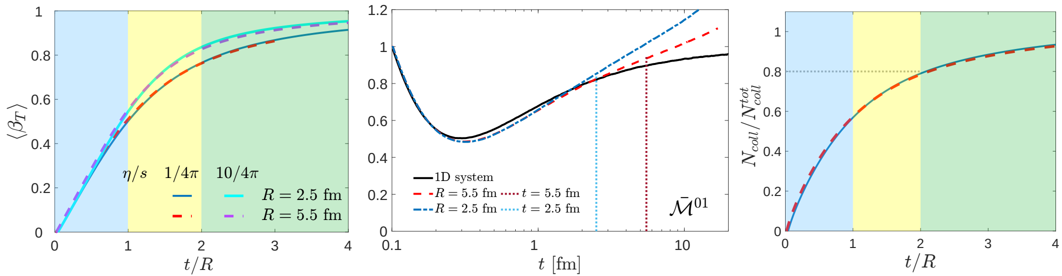

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New time scale: R. If we plot R/λ_{mfp}, simulations cluster in universality classes

In Relaxation and Isotropization Time Approximation, opacity $\hat{\gamma}$ emerges in solving the Boltzmann equation as the *only scaling parameter*. (Kurkela et al., PLB 783, 274 (2018); Ambrus et al. PRD 105, 014031 (2022))

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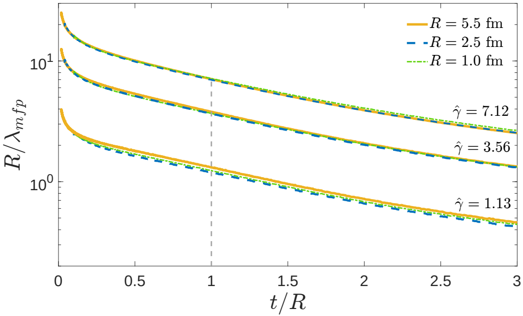

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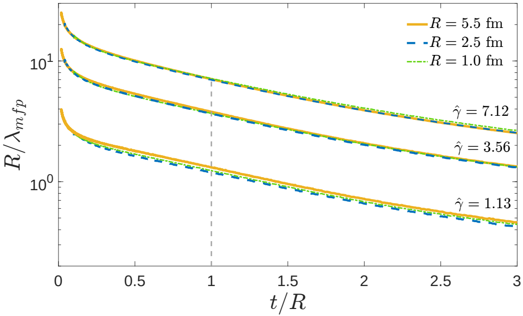

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Forward attractors

3+1D, but azimuthal symmetry and look at $\eta_s \sim 0 \implies \overline{M}^{nm} = \overline{M}^{nm}(t, x_\perp)$.
Fix $\eta/s = 1/4\pi$. Change ξ_0 (P_L/P_T) and R.

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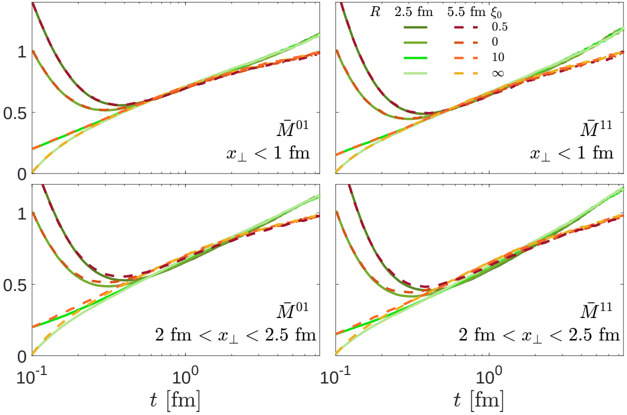

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Pull-back attractors

We do not have a unique time-scale any more.
How do we rescale time? Do we expect pull-back attractors at all?

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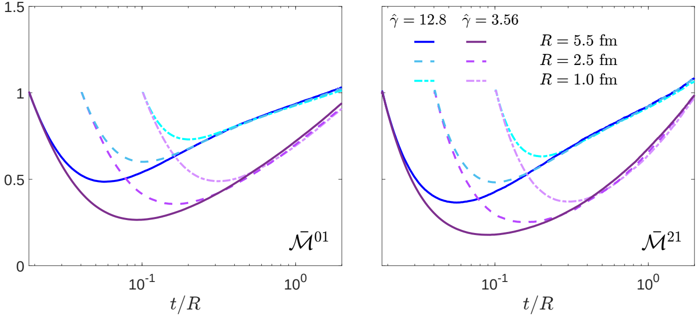

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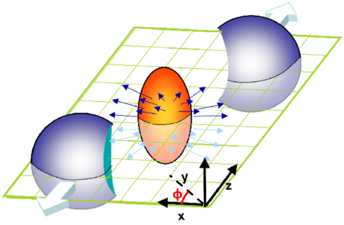
Eccentricities and anisotropic flows

Reproduce eccentricity in coordinate space by shifting (x, y):

$$z = x + iy \to z' = z - \alpha \bar{z}^{n-1}$$

$$\epsilon_n = \frac{\sqrt{\langle x_\perp^n \cos(n\phi)\rangle^2 + \langle x_\perp^n \sin(n\phi)\rangle^2}}{\langle x_\perp^n \rangle} \overset{\alpha \ll 1}{\simeq} n\alpha \frac{\langle x_\perp^{2(n-1)}\rangle}{\langle x_\perp^n\rangle}.$$

(S. Plumari, G. L. Guardo, V. Greco, J.-Y. Ollitrault, Nucl. Phys. A 941, 87 (2015))



Viscosity converts space anisotropies in momentum space. Expand distribution function as:

$$\frac{dN}{d\phi\, p_\perp\, dp_\perp} \propto 1 + 2\sum_{n=1} v_n(p_\perp)\cos[n(\phi_p - \Psi_n(p_\perp))].$$

Anisotropic flows $v_n = \langle \cos(n\phi)\rangle$

How efficiently does this conversion happen? How does it depend on η/s, R and $\hat{\gamma}$?

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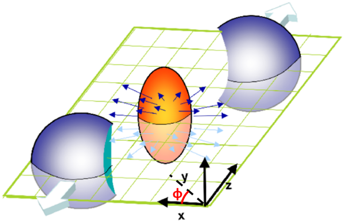

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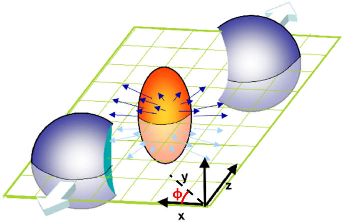

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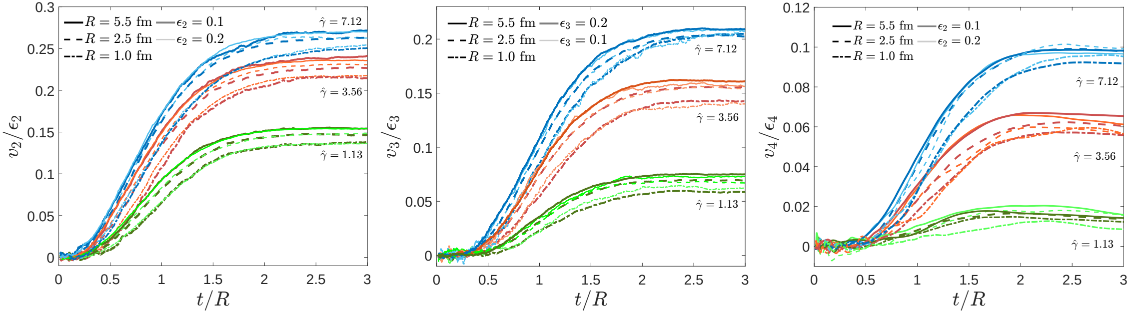
Response functions v_n/ϵ_n



- No dependence on ϵ_n
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- For fixed $\hat{\gamma}$, monotonic ordering in R
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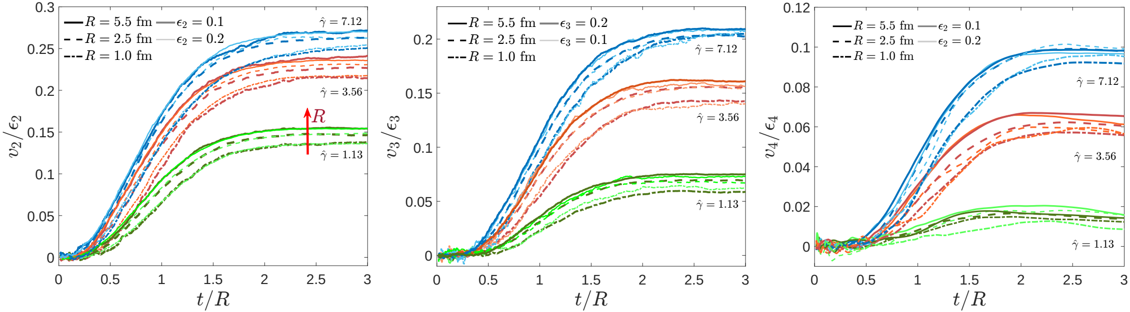

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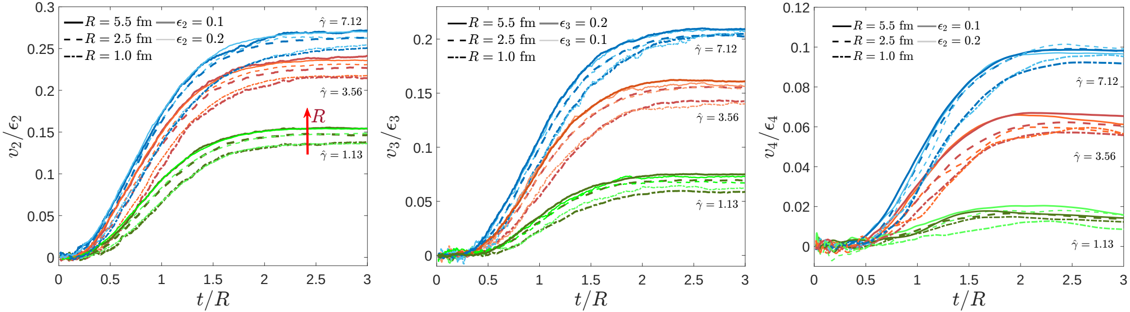

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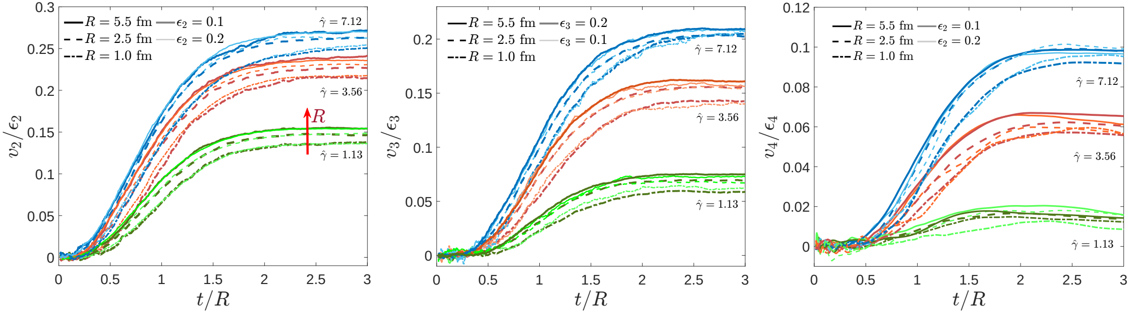

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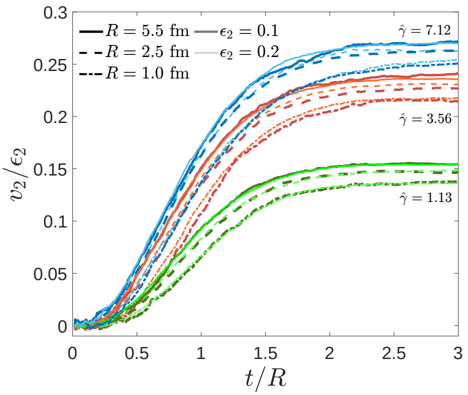
Opacity estimates



$\hat\gamma$	R [fm]	$4\pi\eta/s$	
1.13	1.0	3.18	*pp*
	2.5	6.33	
	5.5	11.4	
3.56	1.0	1.00	*pp*
	2.5	2.00	*pA*
	5.5	3.61	
7.12	1.0	0.503	
	2.5	1.00	*pA*
	5.5	1.81	*AA*

Summary

1D systems

- Attractors in conformal boost-invariant case in the distribution function and its moments.
- Non boost-invariant systems show universal behaviour, also at large η_s.
- If $\eta/s = \eta/s(T)$, only temporary breaking of the universal behaviour ('loops').

3D systems

- ✓ Forward and pull-back attractors ($\sim$ 1D)
- ✓ Difference w.t.r. 1D for $t > R$
- ✓ Opacity $\hat{\gamma}$ quite good universal parameter (especially for large $\hat{\gamma}$)

Outlook

- Non-conformal simulation in progress
  - Pre-hydrodynamic transport + transport without discontinuity in bulk viscosity

Thank you for your attention.

LRF and matching conditions

Define the Landau Local Rest Frame (LRF) via the fluid four-velocity:

$$T^{\mu\nu}u_\nu = \varepsilon u^\mu,$$
$$n = n^\mu u_\mu$$

ε and n are the energy and particles density in the LRF.
Fluid is not in equilibrium $\implies$ define locally effective T and Γ via Landau matching conditions:

$$T = \frac{\varepsilon}{3\,n}, \qquad \Gamma = \frac{n}{d\,T^3/\pi^2},$$

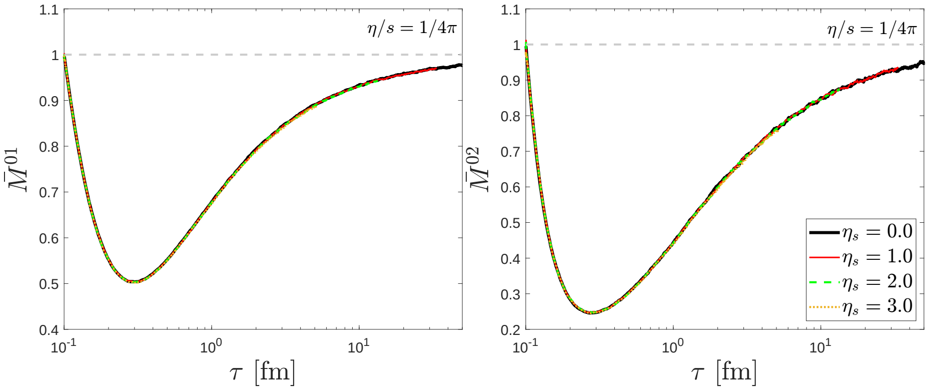
d is the # of dofs, fixed $d = 1$.

Code setup

- Cell: $\Delta x = \Delta y = 0.12$ fm, $\Delta\eta_s = 0.25$. Results taken in one-cell-thick slices in η_s.
- Test particles: from 10^7 up to $3 \cdot 10^8$.
- Time discretization: to avoid causality violation ($\sim 10^3$ time steps).
- Performance: 1 core-hour per 10^6 total particles in $2 \cdot 10^3$ time steps.

Testing boost-invariance

Compute normalized moments at different η_s's within an interval $\Delta\eta_s = 0.04$.



No dependence on η! We look for them at midrapidity: $\eta \in [-0.02, 0.02]$

Boltzmann RTA Equation for number-conserving systems

Boltzmann equation in Relaxation Time Approximation (RTA) (Strickland, Tantary, JHEP10(2019) 069)

$$p^\mu \partial_\mu f_p = -\frac{p \cdot u}{\tau_{eq}}(f_{eq} - f_p).$$

Exactly solvable, by fixing number and energy conservation.
Two coupled integral equations for $\Gamma_{eff} \equiv \Gamma$ and $T_{eff} \equiv T$:

$$\Gamma(\tau) T^4(\tau) = D(\tau,\tau_0)\Gamma_0 T_0^4 \frac{\mathcal{H}(\alpha_0\tau_0/\tau)}{\mathcal{H}(\alpha_0)} + \int_{\tau_0}^{\tau} \frac{d\tau'}{2\tau_{eq}(\tau')} D(\tau,\tau')\Gamma(\tau') T^4(\tau')\mathcal{H}\left(\frac{\tau'}{\tau}\right),$$

$$\Gamma(\tau) T^3(\tau) = \frac{1}{\tau}\left[D(\tau,\tau_0)\Gamma_0 T_0^3 \tau_0 + \int_{\tau_0}^{\tau} \frac{d\tau'}{\tau_{eq}(\tau')} D(\tau,\tau')\Gamma(\tau') T^3(\tau')\tau'\right].$$

Here $\alpha = (1+\xi)^{-1/2}$. System solvable by iteration.

vHydro equations

Second-order dissipative viscous hydrodynamics equations according to DNMR derivation, starting from kinetic theory (G. S. Denicol *et al.*, *PRL*105, 162501 (2010)) :

$$\partial_\tau \varepsilon = -\frac{1}{\tau}(\varepsilon + P - \pi),$$
$$\partial_\tau \pi = -\frac{\pi}{\tau_\pi} + \frac{4}{3}\frac{\eta}{\tau_\pi \tau} - \beta_\pi \frac{\pi}{\tau},$$

where $\tau_\pi = 5(\eta/s)/T$ and $\beta_\pi = 124/63$.
Solved with a Runge-Kutta-4 algorithm.

aHydro for number-conserving systems

Formulation of dissipative anisotropic hydrodynamics with number-conserving kernel (Almaalol, Alqahtani, Strickland, PRC 99, 2019).

System of three coupled ODEs:

$$\partial_\tau \log \gamma + 3\partial_\tau \log \Lambda - \frac{1}{2}\frac{\partial_\tau \xi}{1+\xi} + \frac{1}{\tau} = 0;$$

$$\partial_\tau \log \gamma + 4\partial_\tau \log \Lambda + \frac{\mathcal{R}'(\xi)}{\mathcal{R}(\xi)}\partial_\tau \xi = \frac{1}{\tau}\left[\frac{1}{\xi(1+\xi)\mathcal{R}(\xi)} - \frac{1}{\xi} - 1\right];$$

$$\partial_\tau \xi - \frac{2(1+\xi)}{\tau} + \frac{\xi(1+\xi)^2\mathcal{R}^2(\xi)}{\tau_{eq}} = 0.$$

Solved with a Runge-Kutta-4 algorithm.

Computation of moments in other models

- RTA:

$$M^{nm}(\tau) = \frac{(n+2m+1)!}{(2\pi)^2}\Big[D(\tau,\tau_0)\alpha_0^{n+2m-2}T_0^{n+2m+2}\Gamma_0\frac{\mathcal{H}^{nm}(\alpha\tau_0/\tau)}{[\mathcal{H}^{20}(\alpha_0)/2]^{n+2m-1}} + \\ + \int_{\tau_0}^{\tau}\frac{d\tau'}{\tau_{eq}(\tau')}D(\tau',\tau')\Gamma(\tau')T^{n+2m+2}(\tau')\mathcal{H}^{nm}\left(\frac{\tau'}{\tau}\right)\Big];$$

- DNMR:

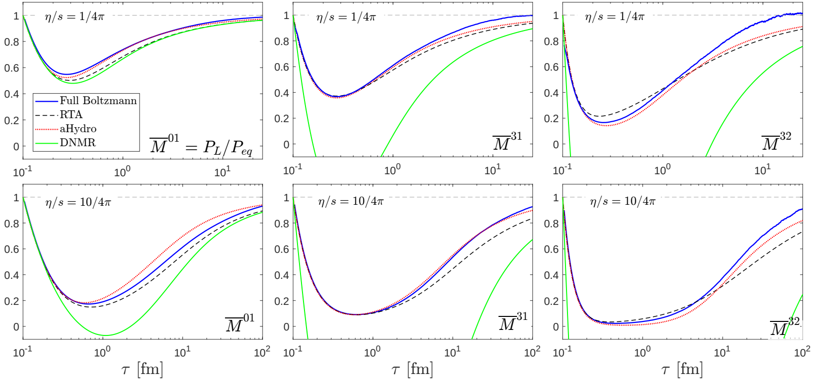
$$\overline{M}^{nm}_{\mathrm{DNMR}} = 1 - \frac{3m(n+2m+2)(n+2m+3)}{4(2m+3)}\frac{\pi}{\varepsilon};$$

- aHydro:

$$\overline{M}^{nm}_{\mathrm{aHydro}}(\tau) = (2m+1)(2\alpha)^{n+2m-2}\frac{\mathcal{H}^{nm}(\alpha)}{[\mathcal{H}^{20}(\alpha)]^{n+2m-1}};$$

Comparison with other models

Compute normalized moments with DNMR, anisotropic hydrodynamics (aHydro) and Relaxation Time Approximation (RTA) Boltzmann Equation.

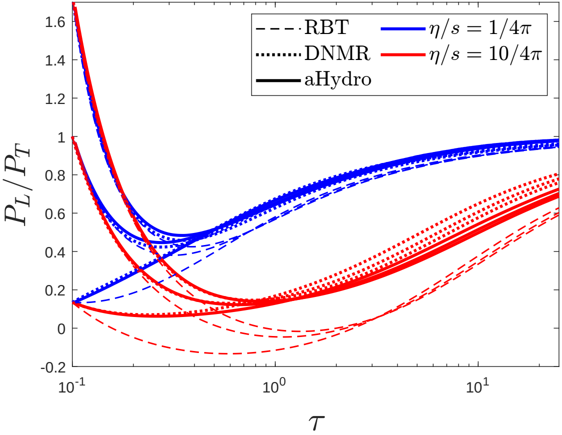


- Better agreement with RTA and aHydro for lower order moments
- Better agreement with DNMR for lower η/s

(V. Ambrus *et al.*, PRD 104.9 (2021))

Pressure anisotropy in different frameworks

For $\eta/s = 1/4\pi$ and $\eta/s = 10/4\pi$, compute P_L/P_T from three different initial anisotropies: $\xi_0 = -0.5,\ 0,\ 10$.



- RTA (not showed) really similar to aHydro
- aHydro attractor reached ∼ time than RBT
- vHydro attractor reached at later time, especially for larger η/s

Who is *the* attractor?

All curves scale to a universal behaviour. Which is the curve they converge to?

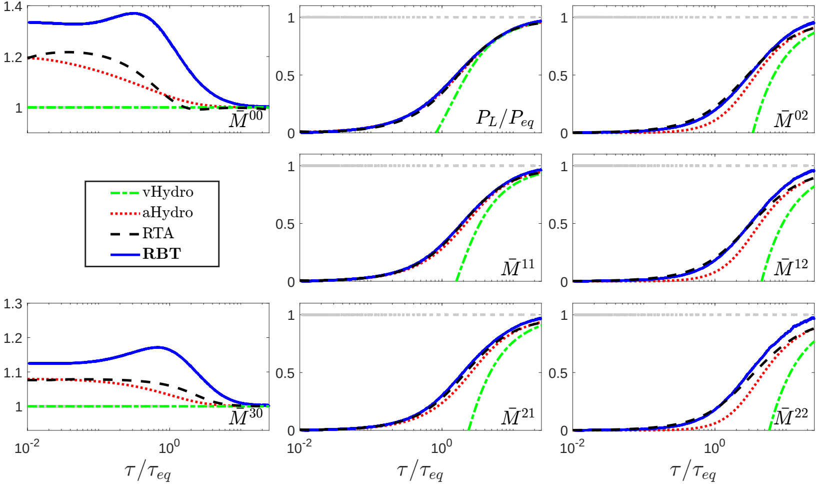
- Viscous (vHydro) and Anisotropic (aHydro) Hydrodynamics: analytical solution (M. Strickland *et al.PRD*, 97, 036020 (2018)) ;
- Relaxation Time Approximation (RTA) Boltzmann Equation (P. Romatschke *PRL* 120, 012301 (2018)) : $\tau_0 \ll 1$ and $\xi_0 \to \infty$ (in accordance with aHydro).

Infinitely oblate distribution $\xi_0 \to \infty$, initial scaled time $\tau_0 T_0/(\eta/s) \to 0$.

Is it the RBT attractor, too? It is.

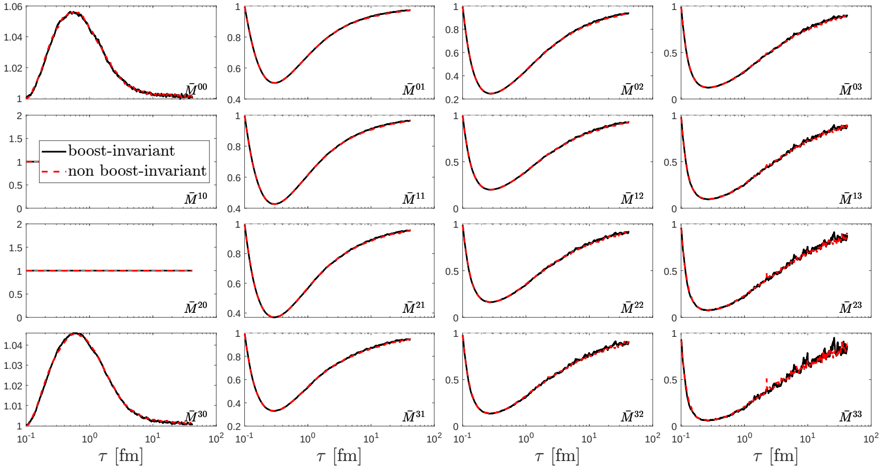
The system initially is dominated by strong longitudinal expansion.

Attractors in different models



- $\overline{M}^{nm}$, $m > 0$: very good agreement
- Higher order moments $\rightarrow$ stronger departure between models
- **RBT** thermalizes earlier
- No agreement for $\overline{M}^{n0}$

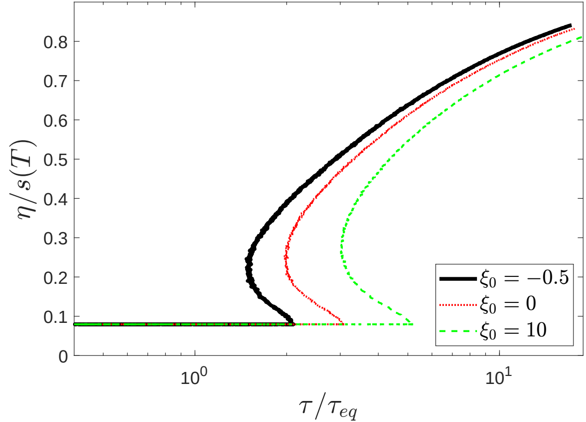
Midrapidity

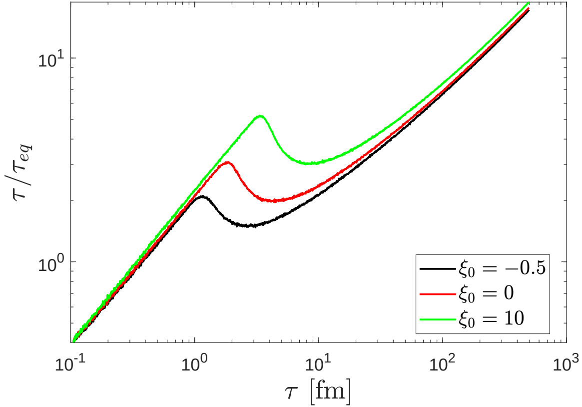


At midrapidity no difference w.r.t. the boost invariant case.

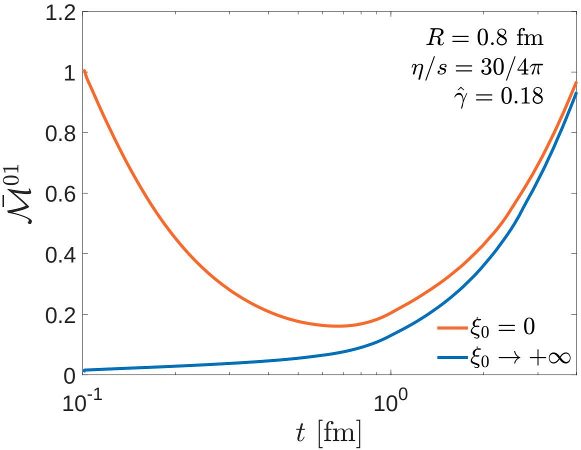
Non-monotonic τ/τ_{eq} for Case 1

Loops when τ/τ_{eq} is no more a monotonic function: $\tau_{eq} \propto \eta/s(T)/T$ grows faster than τ.





Loss of attractors for small $\hat{\gamma}$



Attractor do not reached even for $t = 4$ fm $\approx 5R$!