# **Heavy-quark transport and hadronization, including multi-heavy-flavour: large & small collision systems**

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High Luminosity LHC and Hadron Colliders, LNF, Frascati, 1-4 Ottobre 2024



Finanziato

# **Outline \_**

- ² **Basic concepts & motivation for HQ physics in HotQCD matter**
- ² **Results from the first stage:** 
	- strong <u>non-perturbative</u> HQ dynamics [agreement to LQCD?!, close to AdS/CFT limit?]
	- non-universal hadronization  $\neq$  e<sup>+</sup>e<sup>-</sup> in AA, but seems even in pp@TeV
- $\diamond$  Why precise measurement at low  $p_T$ , extension to bottom & access to new **observables allow for a breakthrough**
- $\diamond$  The relevance of multi-charm production and scan from PbPb  $\rightarrow$  OO :

# **Basic Scales and specific of HQ**



### **Why Heavy?**

- $\triangleright$  *PARTICLE Physics*:  $m_{c,b}$  >>  $\Lambda_{\text{OCD}}$  pQCD initial production Ø *PLASMA Physics*:
	- **- mc,b >> TRHIC,LHC** no thermal production
	- $m_{c,b}$  >>  $gT_{RHIC, LHC}$  soft scatterings  $\rightarrow$  Brownian motion

#### **Specific Features:**

- Ø t**0≈ 1/2mQ <<** t**QGP** witness of all the QGP evolution
- $\triangleright \tau_{th} \approx \tau_{OGP}$   $\triangleright$   $\tau_{q,g}$  carry more information of their evolution

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 $\triangleright$  **τ**<sub>th</sub> ≈ **τ**<sub>QGP</sub> >> **τ**<sub>q,g</sub> carry more information of their evolution

- For HQ we know initial  $p_T$  distribution at variance with light quark & gluons
- **HQ not created at hadronization**  $m_{b,c}$  **>>** $\Lambda_{QCD}$ **,T :**

# **HQ link to Lattice QCD at finite T**

### v **Ab-initio Diffusion Transport Coefficient**

Spectral function  $\rho_F$  extracted **from euclidean color-electric correlator**  $D_F(\tau) \rightarrow$ 

Kubo formula diffusion in the  $p\rightarrow 0$  limit:

$$
\frac{D_p}{T^3} = \lim_{\omega \to 0} \frac{T \rho_E(\omega)}{\omega} \longrightarrow D_s = \frac{T^2}{D_p} = \frac{T}{M_Q} \tau_{th}
$$



 $D<sub>s</sub>$  determines diffusion (brownian limit) and by fluctuation-dissipation theorem: HQ momentum drag  $\gamma \rightarrow$  thermalization time

$$
p\big\rangle = p_0 e^{-\gamma t} \qquad \left\langle \Delta p^2 \right\rangle = 3D_p / \gamma (1 - e^{-2\gamma t}) \qquad D_s = \frac{T}{M\gamma} = \frac{T^2}{D_p} = \frac{T}{M} \tau_{th}
$$

#### Approximations/limitations:

- *Extraction of*  $\rho_F(\omega)$  from  $D_F(\tau)$  is not a well posed problema with *a finite limited # of points*
- *infinite HQ massvs. charm quark, continuum extrapolation…*
- *quenched Nf =0*à *to non quenched QCD (2023-24)*

HQ allow for developing a NRQCD EFT at finite T & many-body T-matrix from  $V(r,T)$  by LQCD

# **Standard Dynamics of Heavy Quarks in the QGP**



- $\div$  This is the main set up at least at  $p < 8-10$  GeV
- $\Diamond$  Brownian motion challenged for charm (M<sub>c</sub> ~ 3 T~ gT): Relativistic Boltzmann dynamics
- $\triangle$  At  $p_T$ > 10 GeV radiative E<sub>loss</sub>, q<sub>hat</sub>, jet physics [Cunqueiro Mendez, previous talk]

# Studying the HF in uRHIC



## **R<sub>AA</sub> and v<sub>2</sub> evolution & correlation**

No interaction means  $R_{AA}=1$  and  $v_2=0$ . more interaction decrease  $R_{AA}$  and increase  $v_2$ 



### $R_{AA}$  is "generated" faster than  $v_2$



The relation between RAA and time is **not trivial** and depend on the time (temperature) dependence of the interaction.

### **Diffusion Coefficient of Charm Quark:first stage**

uRHIC created matter is the **Hot QCD matter not in perturbative regime**!



X. Dong and VG, Prog.Part.Nucl.Phys. (2019)

v Largely non-perturbative Ds (close AdS/CFT) Non perturbative interaction even if  $M_{\rm O} >> \Lambda_{\rm OCD}$  and  $M_{\rm Q} >> m_q$ 

$$
\tau_{th} = \frac{M}{2\pi T^2} (2\pi T D_s) \approx 1.8 \frac{2\pi T D_s}{(T/T_c)^2} \text{ fm/c}
$$

**pQCD, Asymptotic free regime**

**Not a model fit to lQCD data! Phenomenology**  $R_{AA}$  &  $v_2 \approx$  Lattice QCD

**Infinite Strong Coupling (AdS/CFT)**



# **Diffusion of Charm Quark: first stage**  $\sum_{x \text{. Dong & VG, Prog. Part. Nucl. Phys. (2019)}}$



\*Main differences in comparing to LQCD-AdS/CFT:

- quenched QCD (Yang-Mills) +  $M_0 \rightarrow \infty$
- phenomenology at intermediate  $p_T LQCD(AdS/CFT)$  at  $p \rightarrow 0$

### \*Main sources of differences in models:

- impact of hadronization («unexpected» large baryon production)
- momentum depedence of matrix elements
- data not enough precise/observable not enough constraining

# **Diffusion of Charm Quark: first stage**  $\sum_{x \text{. Dong & VG, Prog. Part. Nucl. Phys. (2019)}}$



New LQCD 2023-24 at least a factor of 2 smaller

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**QGP diffuse Charm quarks like a "perfect fluid"**

# **Impact of HF in-medium Hadronization**



an effect that brings up toward experimental data, allows to disentagle the two

 $f_M(P_H = p_1 + p_2) \approx f_q(p_1) \otimes f_{\overline{q}}(p_2) \otimes \Phi_M(\Delta x, \Delta p)$ Phase-space coalescence: quark recombination Independent Fragmentation  $f_H(P_H = zp_T) = f_{q,g}(p_T) \otimes D_{q,g \to H}(z)$ ,  $z < 1$ 

 $\rightarrow$ Add momenta:  $P_T^H$  from low  $p_T$  quark  $\rightarrow$  Enhance elliptic flow v<sub>2</sub> by n<sub>q</sub> scaling  $n_q v_2 (n_q p_T)$  Hadronization play an important role in AA to determine  $R_{AA}$  and  $v_2$  of D meson

 $\rightarrow$  Determintation of transp. coeff. D<sub>s</sub>(T)

… but there has been a surprise both in AA but even in pp@TeV

## **In-medium modification of hadronization even in pp@TeV**



- Large Heavy Baryon to Meson production  $\sim$  a factor of 10 larger than in e+e- or PYTHIA
- Breaking of Universal Fragmentation Function already in pp in HF sector

# HF hadronization has stimulated several developments

- **PYHTIA** beyond Leading Color  $(LC) \rightarrow$  Color Reconnection  $(CR)$  in pp
- Coalescence+Fragmentation approach applied also to pp
- Local Color Recombination: **POWLANG** in AA and in pp
- Ø Inclusion of HF Coalescence+ Fragmentation in **EPOS** (pp &AA)



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# PYTHIA Color Reconnection/ Local Color neutralization



### $(\bar{\bar{q}})$  $(\bar{\bar{q}})$  $(q)$

### Altmann et al., arXiv 2405.19137 Leading Color (N<sub>c</sub> $\rightarrow \infty$ ): Prob. of Local Color neutralization $\rightarrow 0$

- $\Box$  When string color reconnection is switched-on in pp  $\rightarrow$  Very large baryon  $\Lambda_c$ ,  $\Sigma_c$  enhancement
	- $\rightarrow$  not so relevant for D, like coalescence+fragmentation
- Ø Not indipendent strings **Local reconnection** à  **string energy minimization**à **smaller invariant mass** close to D meson states

(a) Mesonic reconnection.

(b) Baryonic reconnection.

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Needed switch-off of **diquark** *l=1* junction suppression *(set for e<sup>+</sup>e<sup>-</sup>). Removing it*  $\rightarrow$  *Agreeement to data of*  $\Lambda_c \leftarrow \Sigma_c$ It goes in the direction of simply recombine according to SU(3)

#### **POWLANG Local Color Neutralization**   $\overline{a}$  and  $\overline{a}$  and  $\overline{a}$  and  $\overline{a}$  are  $\overline{a}$  and  $\overline{a}$  are  $\overline{a}$  are  $\overline{a}$  and  $\overline{a}$  are  $\overline{a}$  and  $\overline{a}$  are  $\overline{a}$  and  $\overline{a}$  and  $\overline{a}$  and  $\overline{a}$  and  $\overline{a}$  and  $\overline{a}$  and

A. Beraudo et al., EPJC82(2022) [AA] A. Beraudo et al., PRD109(2024) [pp]



Charm recombine *locally* with quarks & diquarks assumed thermally distributed  $+$  radial flow:

$$
n_l \cong g_s g_l \frac{T_H m_l^2}{2\pi^2} K_2 \left(\frac{m_l}{T_H}\right) \qquad l = q, \bar{q}, s, \bar{q}, (ud)_0, (sq)_0, (sq)_1, \dots
$$

**Dense medium** (pp &AA)  $\rightarrow$  **local** color statistical neutralization **Narrow invariant M distribution close to D meson masses** not large M string breaking with large y endpoints

 $\rightarrow$  Qualitatively similar to PYTHIA with local CR Coalescence or Resonance Recombination including strong impact on  $v_2(p_T)$  from  $c \rightarrow D$ ,  $\Lambda_c$  (all recomb.)

# Studying the HF in uRHIC after Run2



- $\triangleright$  Most models studies at p<sub>T</sub>>1.5-2 GeV and mainly not including impact of hadroning into  $\Lambda_c$
- $\triangleright$  To be done a new assesement of D<sub>s</sub>(T) with upgraded approach:
	- $\rightarrow$  compare to LQCD & AdS/CFT **need data**  $p_T \rightarrow 0$
	- $\rightarrow$  need precision data at low  $p_T$  not only for D, necessary  $\Lambda_c$ , important  $\Xi_c$   $\Omega_c$
	- $20$  $\rightarrow$  need not only R<sub>AA</sub> and v<sub>2</sub> but also more esclusive observables  $\rightarrow$  needed HL-LHC

# "See" Hadronization mechanism through elliptic flow

If the enhancement of the yield comes from quark coalescence it should be associated to  $\rightarrow$  Large v<sub>2</sub> of  $\Lambda_c \sim n_q v_{2q} (n_q p_T)$ , visible at intermediate p<sub>T</sub> Effect to be measured in AA; will it be seen also in pp?  $\qquad \qquad$  [for AA Run3-4]



- $\checkmark$  It should be also confirmed for  $\Xi_c$  [Run 5-6]
- $\triangleright$  Would PYHTIA-CR predict finite  $v_2$  of D,  $\Lambda_c$  in pp? by String shoving? Can it predict D,  $\Lambda_c$  systematics?

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Minissale, Plumari, VG, in preparation **Example 22** and the set of t

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Methods/tools of AA allow better insight into Hadronization in pp.

### **Able to «see» even the local Temperature. fluctuations of the QGP**

### **Transverse view**



Relativistic HIC in '90s, '00 till about 2005 Anisotropies only with **even** parity due to symmetry  $\rightarrow$  v<sub>2</sub> elliptic flow



#### **Transverse view of HIC,nowdays**



**All harmonics** appearing with different weights.

$$
v_n = \langle \cos(n\varphi) \rangle
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When including fluctuations, all moments appear:



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**Transverse view of HIC, nowdays** A powerful not yet exploited for HQ especially at low  $p_T$  lack statistics



# HL-LHC allows to access v<sub>n</sub> light-HQ correlation

Event-by-event coupling of the anisotrpy of the bulk (light) and the charm (heavy) one  $\rightarrow$  Much more precise determination of the strength interaction: drag  $\Gamma \sim 1/D_s$ 

![](_page_25_Figure_2.jpeg)

![](_page_26_Picture_0.jpeg)

A very solid and high precision comparison to LQCD, development of NRQCD-EFT, quantification of interaction only by  $D_s$  full Brownian motion) requires a full HQ, but  $M_c \sim gT$ ,  $\langle p \rangle$  at T $\sim$  300-500 MeV  $\rightarrow$  full Heavy is Bottom

#### **Relevance of direct Bottom measurements \_**

- Quite close to **M**→ ∞ & **Non Relativistic** limit
	- $\rightarrow$  more solid comparison to LQCD/NRQCD for D<sub>s</sub>(T)
- $\triangleright$  M<sub>O</sub>(T)  $\gt$  T, gT full **Brownian motion**, satisfy fluctuations dissipation theorem
	- $\rightarrow$  damps uncertainties in transport evolution (Langevin, Boltzmann, Kadanoff-Baym...)
- $\triangleright$  Impact of **hadronization** on dN/dp<sub>T</sub> & v<sub>n</sub>(p<sub>T</sub>) moderate and less different by fragmention

![](_page_27_Figure_6.jpeg)

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![](_page_28_Figure_6.jpeg)

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- $\triangleright$  Larger  $\tau_{th}^{b} \sim M/T$   $\tau_{th}^{c}$  more sensitive to dynamical evolution: carry more info

![](_page_29_Figure_7.jpeg)

# **Extension of QPM to bottom dynamics:**  $\mathbf{R}_{AA}$  $\vee$ **<sub>2</sub>,**  $\vee$ **<sub>3</sub>**

![](_page_30_Figure_1.jpeg)

![](_page_30_Figure_2.jpeg)

- $\triangleright$  No parameters changed wrt charm (only  $M_b$ ), but :
	- agreement within still large uncertainty
	- no direct B data (semileptonic decay)
	- $-$  lack  $v_3$
	- $v_n$ (hard)- $v_n$ (soft) correlation

 $\rightarrow$  Need for luminosity of Run  $5-6$ 

M.L. Sambataro et al., *PLB* 849(2024)

### **HQ probe of CGC/Glasma phase** 0+<t<0.3 fm/c

Color Glass Condensate (CGC) is the high-energy limit of QCD in the BFKL direction in the plane  $[Q^2, x]$ ?

 $g^2$ µ=3-5GeV

 $0.1$ 

 $g^2\mu\tau$ 

 $g^2$ μτ≈0.1 fm/c

time

 $\overline{10}$ 

![](_page_31_Figure_2.jpeg)

 $\triangleright$  The unknown very early stage would not destroy our current picture, but we look for signatures to spot from this phase [~ Early Universe, inflation]

### **Impact of Glasma phase**

![](_page_32_Figure_1.jpeg)

### **Potential impact on AA observables** (starting at  $\tau = \tau_{\text{form}}$ - SU(2))

![](_page_33_Figure_1.jpeg)

![](_page_33_Figure_2.jpeg)

**• Opposite to HQ in QGP**: Dominance of diffusion-like  $\rightarrow$  initial **enhancement of R**<sub>AA</sub>( $p_T$ )!!!

• Gain in  $v_2$ : larger interaction in QGP stage needed to have same  $R_{AA}(p_T)$  [18% smaller  $D_s$ ]

High precison needed Run4, and likely alone not conclusive

### **Impact of Glasma phase**

![](_page_34_Figure_1.jpeg)

# **Glasma impact on angular**  $Q\overline{Q}$

First study of azimuthal  $Q\overline{Q}$  correlation: large decorellation in only 0.2 fm/c Significant effect of glasma on HQ!

![](_page_35_Figure_2.jpeg)

Calculation in SU(3) +longitudinal expansion

D. Avramescu et al., arXiv:2409.10.565. [hep-ph]

![](_page_35_Picture_5.jpeg)

pA collision should keep memory of it especially correlating it to  $R_{AA}$ ,  $v_n$ .

- Identify Glasma phase
- quantify in medium  $E_{loss} D_s(T)$
- solve the puzzle od  $R_{pA} \sim 1$  and  $v_2$  large

Accessible with high precsion for D and  $\Lambda_c$  from Run 5-6

Glasma

Nucleus  $\rm{B}$
### **HQ Surprise also transverse flow**



**STAR 5-40%**  $\eta_{\rm m} = 1.3$  $x10^{-2}$  $dv_1/dy|_{exp} \sim -0.0025$ Au+Au @ RHIC 200 GeV,  $b = 7$  fm

 $0.5$ 

 $\eta$ 

Would you expect charm quark to have a smaller  $v_2$ ? Or a smaller one due to its mass?

#### **Very surprising!**

 $v_1$  (HQ) ~ 30 times  $v_1$  light hadrons  $(\pi, K, ...)$ 

### **HQ Surprise also transverse flow**



## **Charm as a probe of huge B Magnetic field**





Schematic calculation: early time behavior quite uncertain theoretically (*non eq., back-reaction, glasma*…)



a current in opposite direction: delicate balance! [Cancellation at 95% level]

### **HQ best probe for v<sub>1</sub> from e.m. field:**

- $t_{\text{form}} \approx 0.08$  fm/c when By is  $\approx$  its maximum
- No contribution from neutral gluons diff. from  $\pi^{+}/\pi^{-}$ , p/ $\bar{p}$
- $\tau_{\text{th}}(c) \approx \tau_{\text{OGP}} >> \tau_{\text{e.m}}$  (keep more memory effects)



### $v_1$  transverse flow current measurement

 $0.5$ 

Oliva, Plumari, V.G., JHEP(2020)





 $-0.5$ 

 $STAR@RHIC: d(\Delta v_1)/dy|_{exp} = -0.011 \pm 0.024(stat) \pm 0.016(syst)$ 

≈ 10 times larger than charged**,** similar to S. Das et al., PLB768 (2017) but with current precision **also consistent with 0!**

**First measurement ALICE@LHC- large systematic/statistic error opposite sign & magnitude ≈ 40 times larger than predictions**

Need for high precision. likely Run 4 or 5

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- $\triangleright$  if  $\Delta v_1(D^0 \overline{D}^0)$  has an e.m. origin
	- à **probe of deconfinement vs flavor**
- $\triangleright$  constraint on e.m. field  $\rightarrow$  quantitative studies of **Chiral Magnetic Effect** (by **local CP violation** at high T) + several other effects

# Magnetic field modifies  $Z^0$   $\vert$ <sup>±</sup> invariant mass and width in AA



# **Multicharm production + PbPb**  $\rightarrow$  **OO**



Understand HQ in medium hadronization: [pure recombination, no fragmentation at low  $p_T$  at least]  $\triangleright \Omega_{ccc}$  very sensitive (to cubic power) to  $(dN_{\text{charm}}/dp_T)^3$ 

A system size scanning is like looking to see  $\Delta E$  versus  $L \rightarrow dE/dx$ 

43





◆ Makes a I order of magnitude difference depending on degree of equilibirum, while very small effect on D,  $\Lambda_c \sim (dN_{\text{charm}}/dp_T)$ , also due to charm # conservation & confinement

# $\Omega_{ccc}$  **p<sub>T</sub>** evolution from PbPb to OO Minissale et al., EPJC84(2024)



Deviation from scaling  $N_c \left(\frac{N_c}{V}\right)$ ? 3 due to different final p $_\Gamma$ -charm distribution wrt PbPb

 $\Omega_{\text{ccc}}$  p<sub>T</sub> spectrum evolution with system size unveil direct information of charm  $dN_c/dp_T$  with much larger sensitivity w.r.t. D<sup>0</sup> or  $\Lambda_c$   $\rightarrow$  precise info on interaction D<sub>s</sub>(T)

Run 5-6 with ALICE3

# Summary & Perspectives

- v Open HF set up a strong connection among LQCD,NRQCD/phenomenology/exp. observables
- v HQ is a more sensitive probe of bulk QGP , but till now **has suffered from the lack of high statistic and access to exclusive observables**
- v Precision data *@low pT***|***new observables***|***extension to bottom***|***multicharm* à breakthrough toward solid determination/understanding of:
	- interaction strength at high T; agreement phenomenology with LQCD? & close to AdS/CFT? validity of NREFT/ QCD at finite T
	- understanding HQ hadronization universal/non-universal from pp@TeV to AA *[Hadronization reveals pp@TeV as a small dense medium much closer to AA than e+e-!?]*
- Open HF as novel probe of Glasma studies [especially in pA]

# Back-up Slide

# Matter under the most extreme conditions





For highest vorticity  $\omega \sim 10^{22}$  s<sup>-1</sup> F. Becattini [next talk]

### **Initial Production** - m<sub>Q</sub>>> $\Lambda$ <sub>QCD</sub>



# HQ link to lattice QCD at finite T

#### Extract the Free Energy of  $\overline{Q}Q \rightarrow NREFT/T-matrix$  $\frac{1}{2}$

 $\rightarrow$  HQ Potential F=U-TS  $q_0^2 \approx \vec{q}^4 / m_0^2 \ll \vec{q}^2$ space-like transfer momenta.  $\rightarrow V(r)$  + relat. corr. low screening into full Coulomb-like

 $\rightarrow$  Theoretical approach from T-matrix linked to LQCD and/or development of NRQCD at finite T

Scattering under a potential  $V(r,T)$  derived from IQCD Free-energy:



Van Hees, Greco, Rapp, PRL100 (2008)

Fit screened Cornell  $V(r)$ + Im. part. (pert.-like ansatz)+ relativistic corr.

$$
F_{Q\bar{Q}}(T,r) = -T \ln \left( \int_{-\infty}^{\infty} dE \frac{-1}{\pi} \frac{(V+\hat{\Sigma})_I(E)}{\left(E - (V+\hat{\Sigma})_R\right)^2 + \left(V+\hat{\Sigma}\right)_I^2(E)} e^{-\beta E} \right) \quad \text{[SYF Liu + Rapp, '15]}
$$

Compare T-matrix  $F_{O\bar{O}}(T,r)$  with lattice  $F_{O\bar{O}}(T,r)$  to extract in-medium  $V(r)$  and  $\Sigma$ 





### **In 2005-06 … first comparison to data**



## **Relativistic Boltzmann equation at finite η/s**

### **Bulk evolution**

$$
p^{\mu}\partial_{\mu}f_q(x,p) + m(x)\partial_{\mu}^{x}m(x)\partial_{p}^{\mu}f_q(x,p) = C[f_q, f_g]
$$
  

$$
p^{\mu}\partial_{\mu}f_g(x,p) + m(x)\partial_{\mu}^{x}m(x)\partial_{p}^{\mu}f_g(x,p) = C[f_q, f_g]
$$

Equivalent to viscous hydro at  $\eta/s \approx 0.1$ 

Free-streaming Field interaction Collision term  $\varepsilon - 3p \neq 0$  gauged to some  $\eta/s \neq 0$ 

### **HQ evolution**

$$
p^{\mu} \partial_{\mu} f_Q(x, p) = C[f_q, f_g, f_Q](x, p)
$$
\n
$$
c[f_Q] = \frac{1}{2E_1} \int \frac{d^3 p_2}{2E_2(2\pi)^3} \int \frac{d^3 p'_1}{2E_1(2\pi)^3}
$$
\n
$$
\times [f_Q(r'_1) f_{q,g}(r'_2) - f_Q(p_1) f_{q,g}(p_2)]
$$
\n
$$
\times (2\pi)^4 \delta^4(p_1 + p_2 - p'_1 - p'_2),
$$

Non perturbative dynamics  $\rightarrow$  M scattering matrices (q,g  $\rightarrow$  Q) evaluated by Quasi-Particle Model fit to **lQCD thermodynamics**

$$
m_g^2(T) = \frac{2N_c}{N_c^2 - 1} g^2(T) T^2
$$
  
\n
$$
m_q^2(T) = \frac{1}{N_c} g^2(T) T^2
$$
  
\n
$$
g^2(T) = \frac{48\pi^2}{(11N_c - 2N_f) \ln \left[ \lambda \left( \frac{T}{T_c} - \frac{T_s}{T_c} \right) \right]^2}
$$

Impact of off-shell dynamics: M.L. Sambataro et al., *Eur.Phys.J.C* 80 (2020) 12, 1140

## $R_{AA}$  &  $v_2$  with upscaled pQCD cross section



**It's not just a matter of pumping up pQCD elastic cross section: too low**  $R_{AA}$  **or too low**  $v_2$ 

# Ratio to D<sup>0</sup> in pp



- Ø Evidence of different "Fragmentation" Fractions in pp at LHC wrt  $e^+e^-$  &  $e^-p$ but similar to AA
- Ø Coalesc.+Fragm. very close to pp FF
- $\triangleright$  Large  $\Xi_c$ ,  $\Omega_c$  only in coalescence, lack of yield in PYTHIA, SHM,…
- Ø SHM+RQM baryon resonances would have a similar agreement (T~160-170 MeV)  $\ldots$  except for  $\Xi_c$ ,  $\Omega_c$ [Andronic et al., *JHEP* 07 (2021)]

# "Fragmentation" Fractions in pp Catania Coalescence



- Ø Evidence of different "Fragmentation" Fractions in pp at LHC wrt  $e^+e^-$  &  $e^-p$ but similar to AA
- $\triangleright$  Coalesc.+Fragm. very close to pp FF
- $\triangleright$  Large  $\Xi_c$ ,  $\Omega_c$  only in coalescence, lack of yield in PYTHIA, SHM-RQM,…



*Seems only hadronization models treating pp as a small QGP fireball or allowing allowing local reconnection-recombination get close to data..*

### **HF coalescence in EPOS4HQ \_**

- $\triangleright$  Advantages of implementing coal. in EPOS4:
- Full dynamical realistic dynamics from ep, pp to AA
- **Able to predict also a sizeable elliptic flows**  $\rightarrow$  more solid costraints to hadronization and the properties of the pp QCD matter created
	- $\rightarrow v_2(\Lambda_c)/v_2(D^0)$  would give more insight into coal.
- $\triangleright$  Would PYHTIA-CR predict finite  $v_2$  of D,  $\Lambda_c$  in pp? String shoving?





## Going deeper into  $A_c$  enhancement



- Catania-coal & SHM-RQM/QCM natural good description of  $\Sigma_c/D^0$  and  $A_c \leftarrow \Sigma_c$ - PYTHIA-CR too many  $\Sigma_c \rightarrow \Lambda_c/D^0$ 

## Going deeper into  $\Lambda_c$  enhancement

Altmann et al., arXiv 2405.19137



- Catania-coal & SHM-RQM/QCM natural good description of  $\Sigma_c/D^0$  and  $A_c \leftarrow \Sigma_c$ 

- PYTHIA-CR too many  $\Sigma_c \rightarrow \Lambda_c/D^0$ ; associated to a suppression of junction **diquark l=1** (set  $\sim e^+e^-$  for string di-quark). Removing it  $\rightarrow$  Agreement to data of  $\Lambda_c \leftarrow \Sigma_c$ It goes in the direction of simply recombine according to  $SU(3)$   $\sim$  simple colaescence

# HF Baryon enhancement: impact on  $R_{AA}$



 $\Lambda_c$  production was mostly neglected in the first studies of  $R_{AA}$ , but:

- Strong impact on R<sub>AA</sub> low-intermediate  $p_T \rightarrow$  affect estimates of D<sub>s</sub>
- Stronger coalescence  $\rightarrow$  smaller Ds
- $-\Lambda_c/D \sim O(1)$  already in pp@TeV: pp  $\sim AA \neq e^+e^-$ , e-p

HF Hadronization in jet shower  $-$  [S. Sadhu-this session]

## Relevance of direct Bottom measurements

Just an first example, for the more plain observable  $R_{AA}$ ....



workshop on QCD challenges from pp to AA collisions, Sept. 2024

Peak depends on the degree of b coupling to QGP medium is smeared-out in non prompt measurements

Direct B,  $L_b$  measurement at low  $p_T$  $\rightarrow$  need for Run5-6

### Early results and predictions for Bottom in pp



- Again Need CR in PYTHIA $\rightarrow$  seems too strong at forward (no rapidity dependence)
- EPOS4HQ**+coal** close to data (rapidity dependence?). At y=0 Catania results
- $SHM + RQM$  about close, less the  $p_T$  shape (Frag.-Function)
- Coal./Fragm. ratio in pp larger for B than D

Slide su importanza large rapidity coverage

Figura Lc/D a rapidita finita



Strong adavantage to see the evolution with rapidity in the same system - Disentangle size and parton density impact

# **Impact of diquark?**

**\_** QCD challenges from pp to AA, EPJC 84(2024)

q Coal. Approaches (*Catania, LBT, EPOS4HQ… RR-TAMU*)  $0.08$  $\rightarrow$  **v**<sub>2</sub>( $\Lambda$ <sub>c</sub>)  $>$  **v**<sub>2</sub>( $D$ <sup>0</sup>) at p<sub>T</sub>  $>$  2 GeV 0.06  $\int_{\mathcal{S}^N}$  0.04 because  $\Lambda_c$  gets flow from 2 light quarks, D<sup>0</sup> from 1+fragm.  $v_2^{\lambda_c}$  $0.02$ Q POWLANG assume diquark hydrodynamical flow and  $\Lambda_c = (qq) + c \Rightarrow v_2(\Lambda_c) \sim v_2(D^0)$  at intermediate  $p_T$  $-0.02$ 





- **Q** Quark model gives (us)<sub>0</sub> large binding energy  $\rightarrow$  small mass. If V(r, T) potential at finite T with large  $m_D \sim LQCD$  Assumption:
	- Again  $(us)_0$  thermal yield flowing with the medium
	- More precise data needed to draw any conclusion  $\rightarrow$  may be Run 4
	- H. Yun, S.H. Lee et al., PLB 851(2024)



#### **Memory effect? Non-Markovian dynamics** Pooja et al., PRD108(2023)

### **Generalized Langevin equation**

$$
\frac{dp(t)}{dt} = -\int_0^t dt' \gamma(t, t') p(t') + \eta(t)
$$

$$
\langle \eta(t_1)\eta(t_2) \rangle = \frac{\mathcal{D}}{\tau} \langle h(t_1)h(t_2) \rangle,
$$

$$
\langle h(t_1)h(t_2) \rangle \cong \kappa \left(\frac{t_1}{\tau}\right)^{\beta - 1} \left(\frac{t_2}{\tau}\right)^{\beta}
$$



### **There could be correlations in the initial glasma and toward the phase transition**

 $\triangleright$  Exponential memory function t ~ 1 fm/c  $\rightarrow$  not significant final effects. In many area of physics and chemistry there power law function

M. Ruggieri et al., PRD 106(2022)

Memoryless  $\langle p_x \rangle = \langle p_{x0} \rangle e^{-t/\tau_{therm}}$  starting from FONNL checking that it leads to same  $Y_{therm} = K/K_{eq}$  for different D For memory we look at the same  $Y_{therm}$  to estimate  $\tau_{therm}$ 

For bottom even a very strong memory function leaves the estimate of  $D_s$  nearly unaffected  $\tau \ll \tau_{therm} \sim 5 - 10 fm/c$ 

Expected a smaller  $\mathsf{D}_\mathsf{s}$  to reproduce similar  $\mathsf{R}_{\mathsf{A} \mathsf{A}_\mathsf{GS}}$ 

## **Extension to higher order anisotropic flows**  $v_n(p_T)$

### **ESE tecnique and**  $v_n$  **correlations**

Selection of events with the same centrality but different initial geometry on the basis of the magnitude of the second-order harmonic reduced flow vector  $q_2$ .



#### Predictions for D mesons





## **ESE:**  $v_2$  and spectra (20% small/large  $q_2$ )



Data taken from ALICE collaboration: *Phys.Lett.B* 813 (2021) 136054

 $\triangleright$   $\upsilon$ <sub>2</sub> (large-  $q_2$  /small-  $q_2$ )  $\ge \upsilon$ <sub>2</sub> (unbiased) of about 50% in both 0-10% and 30-50% centrality

 $\triangleright$  The standard approach for R<sub>AA</sub> and  $v_2$  works for ESE observables

Y. Sun et al. in preparation

## **Going deeply into Hot QCD matter**



Possible because at LHC one starts to create about than 10,000 particle per event

### A first study of HQ in a Glasma What happens for  $0+<$  t $<$  0.3-0.5 fm/c?



$$
\langle \rho_A^a (x_T) \rho_A^b (y_T) \rangle = (g^2 \mu_A)^2 \delta^{ab} \delta^{(2)} (x_T-y_T),
$$

Inizialization by Mc-Lerran/Venugopalan model PRD49(1994)

$$
\frac{dA_i^a(x)}{dt} = E_i^a(x),\tag{16}
$$

 $\frac{dE_i^a(x)}{dt} = \sum_j \partial_j F_{ji}^a(x) - \sum_{b,c,j} f^{abc} A_j^b(x) F_{ji}^c(x)$ . (17)



Formation time of transverse E-B fields  $g^2 \mu \tau \approx 1 \approx \tau_{form}$ (charm) after  $\tau \cong Q_s^{-1}$ , all components are equal

### The very early stage has left some imprints?

J. Liu, S. Plumari, K. Das, M. Ruggieri, VG, Phys. Rev. C 102 (2020) 4, 044902

## Role of HQ also in the CGC/Glasma studies

- **↓** HQ dynamics starting from  $\tau_0 \approx 1/2 m_{\Omega} \approx 0.02$ -0.08 fm/c
- ❖ Relevance to HQ in pA collisions

 $\rightarrow$  Explain R<sub>pA</sub> ~ 1 and large v<sub>2</sub> of D meson  $\rightarrow$  may have a key role on D- $\overline{D}$  angular correlation

❖ May affect the determination of  $D_s(T)$ 

 $\rightarrow$  modify (improve) the relation R<sub>AA</sub> & v<sub>2</sub> toward a smaller D<sub>s</sub>

### A substantial goal for HL-LHC ...

The issue is not that the unknown early stage would destroy our current picture, but to find signatures from the early stage dynamics  $\left(\sim\right.$  for Early



### Impact of T dependent interaction on  $R_{AA} - v_2$



S. Das et al., PLB747 (2015) 260


# **Chiral Magnetic Effect and P &CP violation**



#### Reveals a **local Parity breaking in Strong Interactions**

Consider a homogeneous, strong magnetic field (Warringa, 2008):

 $\begin{picture}(130,10) \put(0,0){\line(1,0){150}} \put(15,0){\line(1,0){150}} \put(15,0){\line(1,0){150}} \put(15,0){\line(1,0){150}} \put(15,0){\line(1,0){150}} \put(15,0){\line(1,0){150}} \put(15,0){\line(1,0){150}} \put(15,0){\line(1,0){150}} \put(15,0){\line(1,0){150}} \put(15,0){\line(1,0){150}} \put(15,0){\line(1,0){150}}$ Momentum  $\pi^*$ Spin  $\pi^-$ P-odd current absent in Maxwell eq.s  $(N_L - N_R)_{+\infty} - (N_L - N_R)_{-\infty} = 2Q_W$ driven by axion field

A local axial  $\mu_5 = \mu_R$   $\mu_L$  (topological  $\mu_\theta$ ) induces an electric current  $J_v$  along  $B \rightarrow c$  charge separation No C-odd but CP-odd

> Expected exp. effect: dipole modulation of azimutal distribution

$$
\frac{dN_{\pm}}{d\phi} \sim 1 + 2\nu_1 \cos(\Delta\phi) + 2\nu_2 \cos(2\Delta\phi) + \dots + 2a_{\pm} \sin(\Delta\phi)
$$

Relaxation time of topological charge  $m_q^{-1} >> \tau_{\text{fireball}}$ 

Observed in Dirac semi-metals – Q. Li et al., *Nature Physics* 12 (2016)

## $v_1$  large sensitivity in the low  $p_T$



Observables sensitive to spatial inhomogeneity of HQ distribution, like the transverse flow  $v<sub>1</sub>$ , can provide a richer information on HF transport coefficients

## $Z^0$  mass and width modification in AA



## **E.m. field: a main source of uncertainty**



#### **Case A**

E-B fields like Gursoy et al., PRC89(2014) Medium at t<0 + eq. medium  $\sigma_{el}$ =0.023 fm<sup>-1</sup>

#### **Case B and C**



**B** an C similar  $B_v$  up to t< 1 fm/c

\* e.m. field  $\sigma_{el}$  as for RHIC

 $\rightarrow \Delta v_1(D^0)$  order magnitudes smaller than ALICE data + opposite sign

\* e.m. with  $B<sub>v</sub>(t=0)$  as in vacuum  $\rightarrow$  Large  $\Delta v_1(D^0)$  but **opposite** direction

\* e.m. with  $B_v(t=0)$  as in vacuum,  $E_x \approx 0.5$   $B_v$  (t=0.5-1 fm/c)  $\rightarrow \Delta v_1(D^0) \approx$  ALICE Data (1/t ideal MHD)

Time derivative of B<sub>y</sub>(t) even more relevant than absolute values"<sup>76</sup>

**If**  $\Delta$ **v**<sub>1</sub>=**v**<sub>1</sub>( $D$ <sup>0</sup>) • **v**<sub>1</sub>( $D$ <sup>0</sup>) is of electromagnetic origin  $\rightarrow$  we'd have a proof of the formation of the QGP Is there some complementary way of proving it?

> Is there a further way to pin down the e.m field strength? Such a large splitting (in ALICE) has an electromagnetic origin?

**Probing the electromagnetic fields in ultra-relativistic collisions with leptons from** *Z***<sup>0</sup> decay and charmed mesons**

 $\tau_{Z^0} = 1/2m_{Z^0} = 0.0011$  fm/c



#### What one expects?

- No damping from medium interaction
- Massless more easily to drag
- Charge 1.5 times larger

One expects same sign and  $\Delta v_1(l^+, l^-) > \Delta v_1(D^0, \underline{D}^0)$  ?!

- Leptons from  $Z^0$  decay are separable by other sources
- $\tau_{decay}(Z^0)$  =  $\tau_{form}(charm)$  = 0.08 fm/c: they go through the e.m. fields at the same time  $\rightarrow$  meanfigul look at the correlation  $\Delta v_1(D^0, D^0)$  and  $\Delta v_1(I^+, I^-)$

# $V_1$  splitting for  $D^0$ - $\underline{D}^0$  and  $I$ <sup>+</sup>-  $I$ <sup>-</sup> from  $Z^0$  decay and



- No medium strong interaction

 $\tau_{\text{decay}}(Z^0) = \tau_{\text{form}}(\text{charm}) = 0.08 \text{ fm/c}$ 

- Massless more easily to drag

- Charge 1.5 times larger

#### Surprises:

1)  $\Delta v_1(l^+, l^-) < \Delta v_1(D^0, \underline{D}^0)$  even if  $\Delta p_X(l) \approx 2^* \Delta p_X(D)$ 

2) even the sign of  $\Delta v_1$  ( $|$ <sup>+</sup>, $|$ <sup>-</sup>) can be opposite!? not because wins electric field



 $\Delta p_x$  is always positive: ≈ 0.3 GeV for D charm  $\approx$  0.7 GeV for leptons with a weak  $p_T$  dependence



## **Bottom R<sub>AA</sub>: Boltzmann = Langevin**



In bottom case Langevin approximation ≈ Boltzmann But Larger M<sub>b</sub>/T ( $\approx$  10) the better Langevin approximation works

# Strangeness in pp for HF sector



- Catania Coalesc.+Frag. quite ok, but it is large the fragmentation contribution
- POWLANG/LCN too high, but the approach has only recombination also for mesons
- PYTHIA-CR seems to have a lack of strangeness [see also  $E_c$ ]

## Coalescence in pp@5 ATeV



V. Minissale, Plumari, VG, PLB 821 (2021)

Large uncertainty in the exisiting  $\Omega_c$  resonances

Seems to work from pp to PbPb  $\rightarrow$  multi-charm production from pp to PbPb Error band correspond to  $\langle r^2 \rangle$  uncertainty in quark model



 $\triangleright$  D,  $\Lambda_c$  yields constrained by charm # conseervation because they dominate the yield  $\triangleright$  Instead  $\Omega_{ccc}$  is also very sensitive to wave function -  $\langle r^2 \rangle$ 

## How HQ interact with the medium  $[$ low-medium  $p_T]$

## v **3 kinds of approaches:**

### a) **pQCD inspired + HTL**

 **[***Nantes(+rad.) …Torino, LBL-Duke*] LO diagrams, propagator with reduced IR regulator  $q^2 - \kappa m_d^2(T) \Big)^{-1}$ match **soft scale** resummed in **HTL** 

 b) **Quasi Particle Model + tree level diagrams [***Catania, Frankfurt-PHSD, QLBT o CoLBT,…***] g(T) from a fit to lQCD-EoS screened propagators with**  $m_D \sim gT$ 



c) **T-matrix:** scattering under V(r,T) deduced from lQCD (*TAMU*)





### HQ momentum diffusion: lattice-QCD

From the non-relativistic limit of the Langevin equation one gets

$$
\frac{dp^{i}}{dt} = -\eta_{D}p^{i} + \xi^{i}(t), \quad \text{with} \quad \langle \xi^{i}(t)\xi^{j}(t') \rangle = \delta^{ij}\delta(t - t')\kappa
$$
\n
$$
\text{hence} \quad \kappa = \frac{1}{3} \int_{-\infty}^{+\infty} dt \langle \xi^{i}(t)\xi^{i}(0) \rangle_{\text{HQ}} = \frac{1}{3} \int_{-\infty}^{+\infty} dt \underbrace{\langle F^{i}(t)F^{i}(0) \rangle_{\text{HQ}}}_{\equiv D^{>} (t)}
$$

Lattice-QCD simulations provide Euclidean ( $t = -i\tau$ ) electric-field ( $M = \infty$ ) correlator

$$
D_{E}(\tau)=-\frac{\langle\mathrm{Re}\,\mathrm{Tr}[U(\beta,\tau)gE^{i}(\tau,\mathbf{0})U(\tau,0)gE^{i}(0,\mathbf{0})]\rangle}{\langle\mathrm{Re}\,\mathrm{Tr}[U(\beta,0)]\rangle}
$$

How to proceed?  $\kappa$  comes from the  $\omega \to 0$  limit of the FT of  $D^>$ . In a thermal ensemble  $\sigma(\omega) \equiv D^{>}(\omega) - D^{<}(\omega) = (1 - e^{-\beta \omega})D^{>}(\omega)$ , so that

$$
\kappa \equiv \lim_{\omega \to 0} \frac{D^>(\omega)}{3} = \lim_{\omega \to 0} \frac{1}{3} \frac{\sigma(\omega)}{1 - e^{-\beta \omega}} \underset{\omega \to 0}{\sim} \frac{1}{3} \frac{T}{\omega} \sigma(\omega)
$$

From  $D_{\mathsf{E}}(\tau)$  one extracts the spectral density according to

$$
D_E(\tau) = \int_0^{+\infty} \frac{d\omega}{2\pi} \frac{\cosh(\tau - \beta/2)}{\sinh(\beta\omega/2)} \sigma(\omega)
$$

8 D > 8 O > 8 D > 8 D >

# **Two Main Observables in HIC**

### ◆ Nuclear Modification factor **A**

$$
R_{AA}(p_T) = \frac{d^2 N^{AA} / dp_T d\eta}{N_{coll} d^2 N^{NN} / dp_T d\eta}
$$

- Modification respect to pp
- Decrease with increasing partonic interaction

### $\triangle$  Anisotropy p-space: Elliptic Flow  $v_2$





## $v_1$  of D mesons: quantitative study







$$
W(x_{\perp}, \eta_s) = 2\left(N_A(x_{\perp})f_{-}(\eta_s) + N_B(x_{\perp})f_{+}(\eta_s)\right)
$$

$$
f_+(\eta_s)=f_-(-\eta_s)=\left\{\begin{array}{ll} 0 & \eta_s<-\eta_m\\ \dfrac{\eta_s+\eta_m}{2\eta_m} & -\eta_m\leq\eta_s\leq\eta_m\\ 1 & \eta_s>\eta_m \end{array}\right.
$$