Introduction to Soft-Collinear Effective Theory and applications to LHC phenomenology

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Outline

- Basics of Soft-Collinear Effective Theory (SCET): expansion by regions, leading power (LP) Lagrangian.

  Original papers: [C.W. Bauer, S. Fleming, M.E. Luke 0005275],[C.W.Bauer, S. Fleming, D. Pirjol, I.W. Stewart 0011336],[C.W. Bauer, I.W. Stewart 0107001],[C.W. Bauer, D. Pirjol, I.W. Stewart 0109045],[M.Beneke,A.P. Chapovsky, M. Diehl, T. Feldmann 0206156],[M. Beneke, T. Feldmann 0211358], [R.J. Hill, M. Neubert 0211018]

- Applications (only a personal selection): Resummation (threshold, small q_T, N-jettiness), IR-poles of scattering amplitudes, EW resummation for DM annihilation processes, Monte Carlo event generator, NNLO calculations using N-jettiness subtraction/slicing.

- Summary & Outlook

Effective Field Theories of QCD

- Sometimes what goes under the name of Soft-Collinear Effective Theory (SCET) is a larger class of Effective Field Theories (EFT) of QCD: Heavy Quark Effective Theory (HQET), Non Relativistic QCD (NRQCD)…

- For certain problems (we will see examples later) there is an interplay among different EFTs of QCD. There are also different versions of SCET (SCET-I, SCET-II..) depending on the mode structure of the EFT

- Collider processes are typical multi-scale problems →EFTs as tool to achieve scale separation in QFT. Reduce multi-scale problems to a sequence of single-scale problems

- Scale separation is the basis of factorization formulas: crucial for separation of short-distance from long-distance physics in QCD

- Factorization + RGEs allow for systematic resummation of large logarithms of scale ratios. Particularly important in QCD, where $\alpha_s \ln(Q_1/Q_2)$ can be large if $Q_1 \gg Q_2$

Introduction to Soft-Collinear Effective Theory (SCET)

- A long-standing problem in QCD was how to systematically account for long-distance effects (including power corrections) in processes involving for example energetic light particles with momentum p^μ which has some large components, but $p^2 \approx 0$. What is there to integrate out?

- Consider $e^+e^- \to 2\,\text{jets}$, highly collimated jets of particles: large energy along jet axis, small invariant mass

$$p^\mu_{J_1} = \left(E_1, 0, 0, \sqrt{E_1^2 - m_{J_1}^2}\right), \quad p^\mu_{J_2} = \left(E_2, 0, 0, -\sqrt{E_2^2 - m_{J_2}^2}\right), \quad E_i = \sqrt{s}/2, \;\; m_{J_i}^2 \ll s$$

- One can introduce a small expansion parameter $\lambda \sim m_J/\sqrt{s} \ll 1$ and define two light-like reference vectors along the jet directions $n_-^\mu = (1,0,0,1), \;\; n_+^\mu = (1,0,0,-1)$ with $n_-^2 = 0, n_+^2 = 0, n_- \cdot n_+ = 2.$

- Decompose 4-vectors in a light-cone basis spanned by n_-^μ, n_+^μ and two perpendicular directions

$$p^\mu = (n_- \cdot p)\frac{n_+^\mu}{2} + (n_+ \cdot p)\frac{n_-^\mu}{2} + p_\perp^\mu$$

Soft-Collinear Effective Theory

- Two back to back light jets

$$n_- \cdot p_{J_1} = E_1 - \sqrt{E_1^2 - m_{J_1}^2} \simeq \frac{m_{J_1}^2}{2E_1} \simeq \frac{m_{J_1}^2}{\sqrt{s}} \sim \lambda^2\sqrt{s} \qquad n_+ \cdot p_{J_1} = E_1 + \sqrt{E^2 - m_{J_1}^2} \simeq 2E_1 \simeq 1\sqrt{s} \qquad p_{J_1}^{\perp} = 0$$

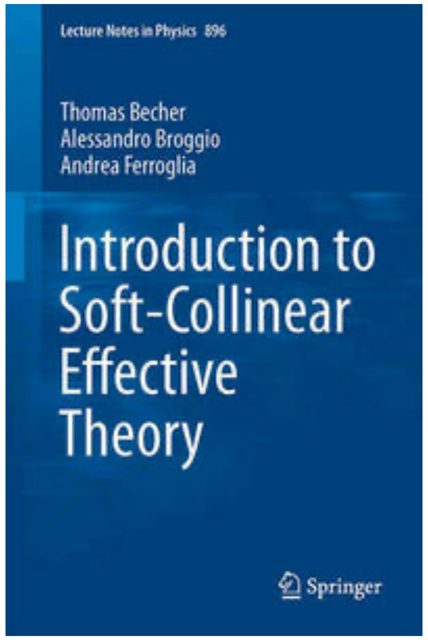
- partons inside Jet 1: $(n_- \cdot p_i, n_+ \cdot p_i, p_i^{\perp}) \sim (\lambda^2, 1, \lambda)\sqrt{s}$, collinear (or n-collinear) particles

- partons inside Jet 2: $(n_- \cdot p_i, n_+ \cdot p_i, p_i^{\perp}) \sim (1, \lambda^2, \lambda)\sqrt{s}$, anti-collinear (or $\bar{n}$-collinear) particles

- (anti-)collinear particles have virtualities much lower than the hard scale s,
  $p_i^2 = (n_- \cdot p_i)(n_+ \cdot p_i) + p_{\perp,i}^2 \sim \lambda^2\, s$

- But in virtual diagrams hard particles can also be exchanged $p_i^{\mu} \sim (1,1,1)\sqrt{s}$

- We can integrate out the hard quantum fluctuations (high-frequency modes in Fourier space) of QCD fields, but this is not the all story, soft modes are also present. Construct an EFT where the hard modes are integrated out and soft and (anti-)collinear modes are present in the theory (arXiv:1410.1892).



Lecture Notes in Physics 896

Thomas Becher
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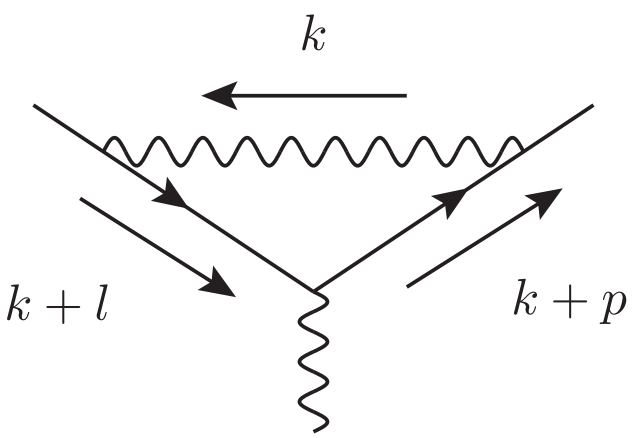
Andrea Ferroglia

Introduction to Soft-Collinear Effective Theory

Springer

Expansion by regions and SCET

Expansion by regions method [M. Beneke, V.A. Smirnov, 9711391], for example (off-shell) Sudakov form factor

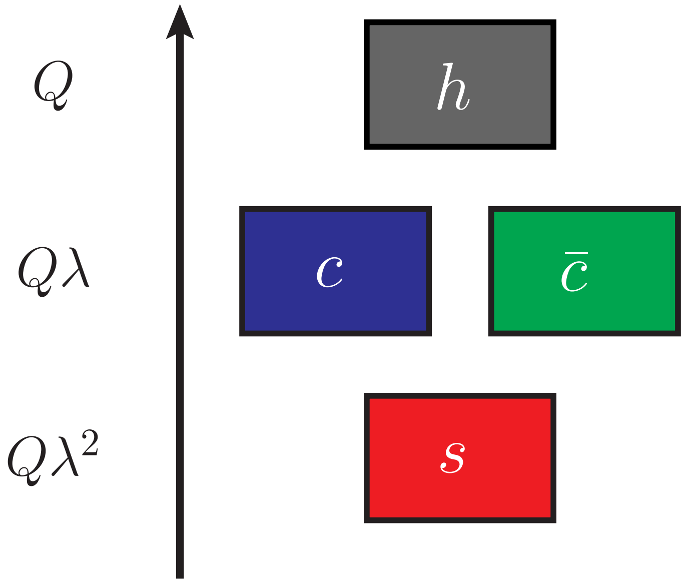


$k^\mu \sim Q(1,1,1)$

$k^\mu \sim Q(\lambda^2,1,\lambda)$

$k^\mu \sim Q(1,\lambda^2,\lambda)$

$k^\mu \sim Q(\lambda^2,\lambda^2,\lambda^2)$



Scalar integral

$$I = i\pi^{-d/2}\mu^{4-d}\int d^dk \frac{1}{(k^2+i0)\,[(k+l)^2+i0]\,[(k+p)^2+i0]}$$

$$l^\mu \sim (1,\lambda^2,\lambda)\,Q \qquad p^\mu \sim (\lambda^2,1,\lambda)\,Q$$

$$L^2 \equiv -l^2 - i0$$
$$P^2 \equiv -p^2 - i0$$
$$Q^2 \equiv -(l-p)^2 - i0$$

Expansion parameter $\lambda^2 \sim \frac{P^2}{Q^2} \sim \frac{L^2}{Q^2}$

Hard Region $k^\mu \sim Q(1,1,1)$

$$(k+l)^2 = \overbrace{k^2}^{\mathcal{O}(1)} + 2(\overbrace{k_+\cdot l_-}^{\mathcal{O}(\lambda^2)} + \overbrace{k_-\cdot l_+}^{\mathcal{O}(1)} + \overbrace{k_\perp\cdot l_\perp}^{\mathcal{O}(\lambda)}) + \overbrace{l^2}^{\mathcal{O}(\lambda^2)} = k^2 + 2k_-\cdot l_+ + \mathcal{O}(\lambda)$$

$$(k+p)^2 = k^2 + 2k_+\cdot p_- + \mathcal{O}(\lambda)$$

Corresponds to the integral with on-shell external legs
$p^2 = l^2 = 0$

Expansion by regions and SCET

Hard Region:

$$I_h = \frac{\Gamma(1+\varepsilon)}{2l_+\cdot p_-}\frac{\Gamma^2(-\varepsilon)}{\Gamma(1-2\varepsilon)}\left(\frac{\mu^2}{2l_+\cdot p_-}\right)^{\varepsilon}$$
$$= \frac{\Gamma(1+\varepsilon)}{Q^2}\left(\frac{1}{\varepsilon^2}+\frac{1}{\varepsilon}\ln\frac{\mu^2}{Q^2}+\frac{1}{2}\ln^2\frac{\mu^2}{Q^2}-\frac{\pi^2}{6}\right)+\mathcal{O}(\varepsilon)$$

- *IR divergences*
- *Result depend only on the hard scale (and μ)*

Collinear region $k^\mu \sim Q(\lambda^2, 1, \lambda)$:

$$(k+l)^2 = 2k_-\cdot l_+ + \mathcal{O}(\lambda^2)\,, \qquad (k+p)^2 = \mathcal{O}(\lambda^2)$$

$$I_c = i\pi^{-d/2}\mu^{4-d}\int d^dk\,\frac{1}{(k^2+i0)\,(2k_-\cdot l_+ + i0)\,[(k+p)^2+i0]}$$
$$= -\frac{\Gamma(1+\varepsilon)}{2l_+\cdot p_-}\frac{\Gamma^2(-\varepsilon)}{\Gamma(1-2\varepsilon)}\left(\frac{\mu^2}{P^2}\right)^{\varepsilon}$$
$$= \frac{\Gamma(1+\varepsilon)}{Q^2}\left(-\frac{1}{\varepsilon^2}-\frac{1}{\varepsilon}\ln\frac{\mu^2}{P^2}-\frac{1}{2}\ln^2\frac{\mu^2}{P^2}+\frac{\pi^2}{6}\right)+\mathcal{O}(\varepsilon)\,.$$

Anti-collinear region $k^\mu \sim Q(1, \lambda^2, \lambda)$: $\qquad P^2 \to L^2$

Soft region $k^\mu \sim Q(\lambda^2, \lambda^2, \lambda^2)$:

$$k^2 = \mathcal{O}(\lambda^4) \qquad (k+l)^2 = 2k_-\cdot l_+ + l^2 + \mathcal{O}(\lambda^3) \qquad (k+p)^2 = 2k_+\cdot p_- + p^2 + \mathcal{O}(\lambda^3)$$

$$I_s = i\pi^{-d/2}\mu^{4-d}\int d^dk\,\frac{1}{(k^2+i0)\,(2k_-\cdot l_+ + l^2 + i0)\,(2k_+\cdot p_- + p^2 + i0)}$$
$$= \frac{\Gamma(1+\varepsilon)}{Q^2}\left(\frac{1}{\varepsilon^2}+\frac{1}{\varepsilon}\ln\frac{\mu^2Q^2}{L^2P^2}+\frac{1}{2}\ln^2\boxed{\frac{\mu^2Q^2}{L^2P^2}}+\frac{\pi^2}{6}\right)+\mathcal{O}(\varepsilon)$$

Soft scale

$$\Lambda^2_{\text{soft}} \sim P^2L^2/Q^2$$

Expansion by regions and SCET

$L^2 \sim P^2 \ll Q^2$

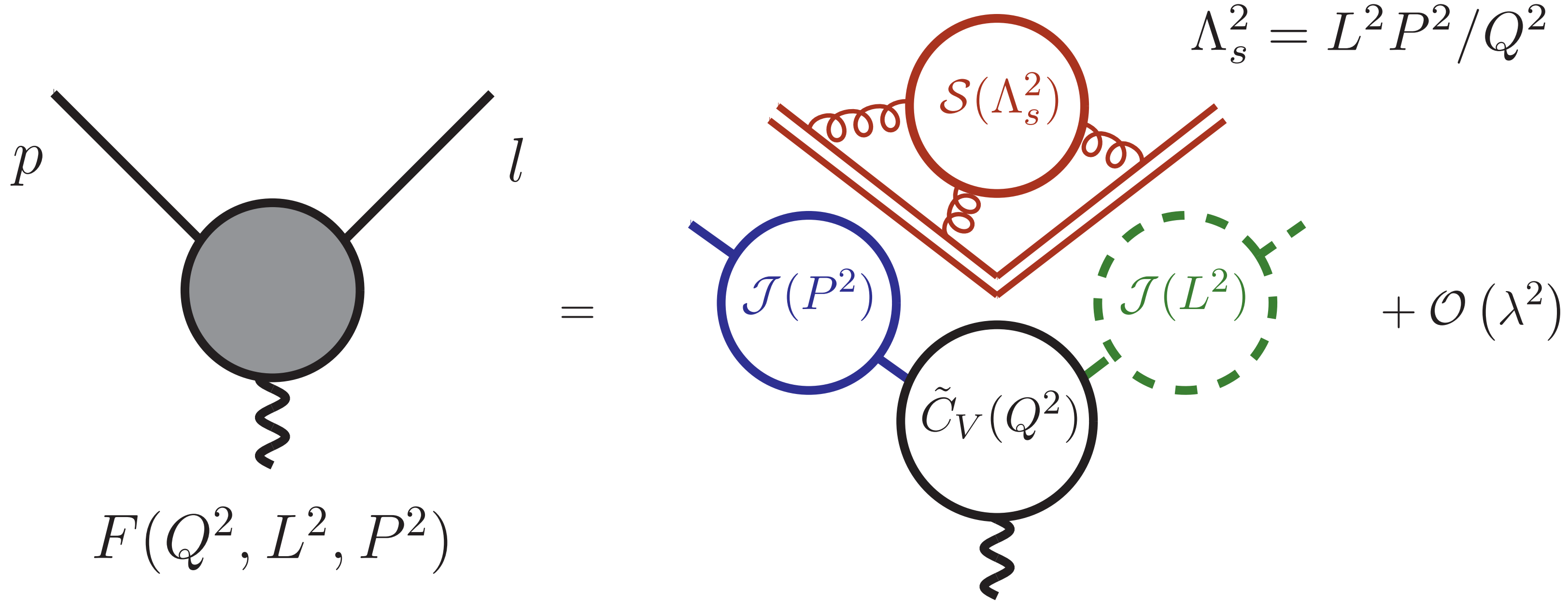
$$I \equiv I_h + I_c + I_{\bar{c}} + I_s = \frac{1}{Q^2}\left(\ln\frac{Q^2}{L^2}\ln\frac{Q^2}{P^2} + \frac{\pi^2}{3} + \mathcal{O}(\lambda)\right)$$

The IR divergences of the hard region cancel against the divergences of the soft and collinear regions (Important observation for predicting IR poles of scattering amplitudes)

For this cancellation to happen, non trivial interplay of the logarithms found in the various integrals

$$-\frac{1}{\varepsilon}\ln\frac{\mu^2}{P^2} - \frac{1}{\varepsilon}\ln\frac{\mu^2}{L^2} + \frac{1}{\varepsilon}\ln\frac{\mu^2 Q^2}{L^2 P^2} = -\frac{1}{\varepsilon}\ln\frac{\mu^2}{Q^2}$$

Collinear, anti-collinear and soft modes are kept in the effective theory while the hard modes are integrated out

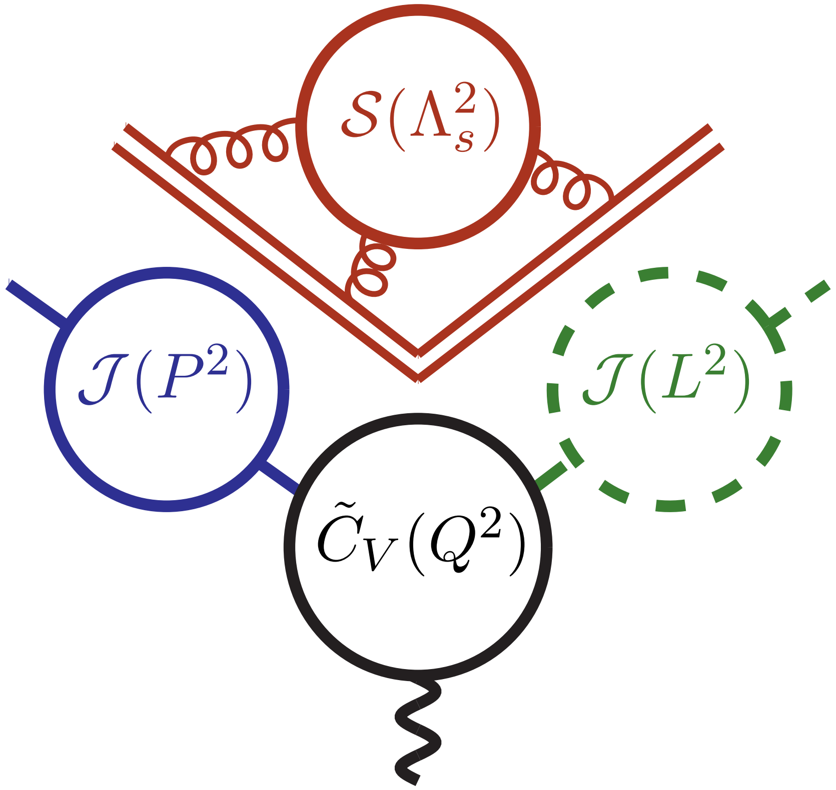


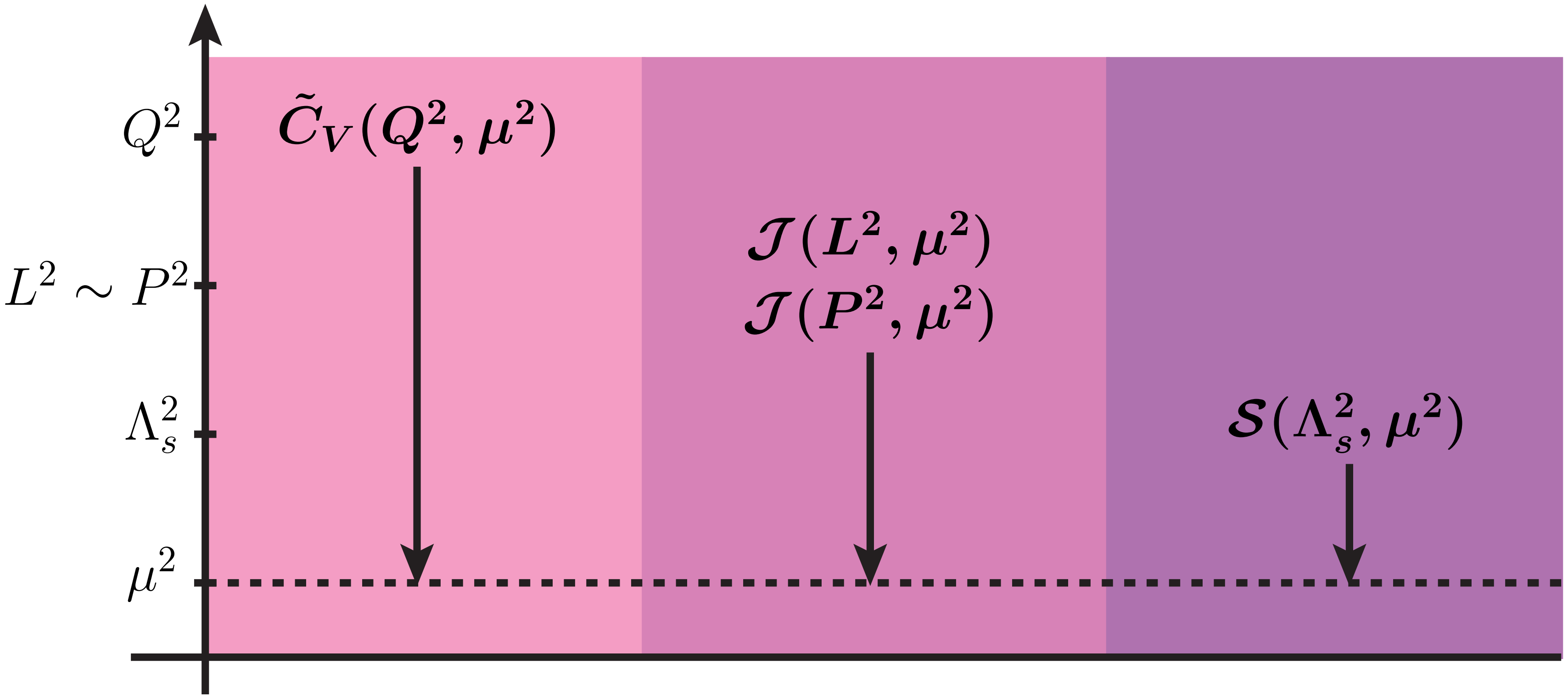
Resummation

- Resummation program in EFT schematically

$$L \equiv \ln\left(\frac{\text{“hard” scale}}{\text{“soft” scale}}\right)$$

  - separation of scales (factorization formula)
  - evaluate each single scale factor in fixed order perturbation theory at a scale for which it is free of large logs
  - use Renormalization Group (RG) equations to evolve the factors to a common scale





SCET Lagrangian

I employ position-space SCET [M. Beneke, A. Chapovsky, M. Diehl, T. Feldmann, hep-ph/0206152], [M.Beneke, T. Feldmann, hep-ph/0211358]. Lagrangian terms with only one collinear field are not possible (momentum not conserved), collinear anti-collinear interactions are forbidden

$$\psi(x) \to \underbrace{\psi_1(x) + \cdots + \psi_N(x)}_{N\ \text{collinear fermion fields}} + q(x) \qquad \mathcal{L}_{\text{SCET}} = \sum_{i=1}^{N} \mathcal{L}_{c_i} + \mathcal{L}_{\text{soft}}$$

each of the Lagrangians belonging to a collinear direction is expanded in powers of the small parameter λ

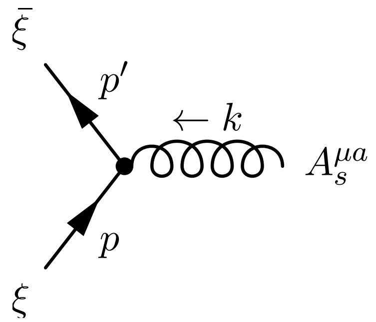
$$\mathcal{L}_{c_i} = \underbrace{\mathcal{L}_{c_i}^{(0)}}_{\text{LP}} + \underbrace{\mathcal{L}_{c_i}^{(1)}}_{\mathcal{O}(\lambda^1)} + \underbrace{\mathcal{L}_{c_i}^{(2)}}_{\mathcal{O}(\lambda^2)} + \ldots$$

Separate collinear sectors interact only through soft gluon interactions. For the leading-power (LP) term we have

$$\mathcal{L}_c^{(0)} = \bar{\xi}_c \left(i n_- D_c + g\, n_- A_s(x_-) + i \not{D}_{\perp c} \frac{1}{i n_+ D_c} i \not{D}_{\perp c} \right) \frac{\not{n}_+}{2} \xi_c + \mathcal{L}_{c,\text{YM}}^{(0)}$$

where $\quad i n_- D_c = i n_- \partial + g\, n_- A_c(x), \; x_-^\mu = (n_+ x) \dfrac{n_-^\mu}{2}$

The soft interaction with each collinear field at LP is given by the usual eikonal vertex



$$i g_s t^a \frac{\not{n}_+}{2} n_{-\mu} \qquad \mathcal{O}(\lambda^0)$$

SCET Lagrangian

The *decoupling transformation* $\xi_c \to Y_+ \xi_c^{(0)}$ and $A_c^\mu \to Y_+ A_c^{(0)\,\mu} Y_+^\dagger$ separates the soft and collinear interactions
at LP [C. Bauer, D. Pirjol, I. Stewart, hep/0109045]

$$\bar{\xi}_c\;(in_- D_c + g_s n_- A_s)\;\frac{\not{n}_+}{2}\,\xi_c = \bar{\xi}_c^{(0)}\;in_- D_c^{(0)}\;\frac{\not{n}_+}{2}\,\xi_c^{(0)}$$

where
$$Y_\pm(x) = \mathbf{P}\exp\left[ig_s\int_{-\infty}^{0} ds\, n_\mp A_s\,(x + sn_\mp)\right]$$
soft Wilson line

At subleading powers the SCET Lagrangian is more involved [M.Beneke, T. Feldmann, hep-ph/0211358]

$$\mathcal{L}_c^{(1)} = \bar{\xi}\left(x_\perp^\mu n_-^\nu\, W_c\, gF^{\mathrm{s}}_{\mu\nu} W_c^\dagger\right)\frac{\not{n}_+}{2}\,\xi + \mathcal{L}^{(1)}_{\mathrm{YM}} + \left(\bar{q}\, W_c^\dagger\, i\not{D}_\perp\,\xi + \mathrm{h.c.}\right)$$

$$\begin{aligned}\mathcal{L}_\xi^{(2)} = {} & \frac{1}{2}\,\bar{\xi}\left((n_- x)\, n_+^\mu n_-^\nu\, W_c\, gF^{\mathrm{s}}_{\mu\nu} W_c^\dagger + x_\perp^\mu x_{\perp\rho} n_-^\nu W_c\left[D_{\mathrm{s}}^\rho, gF^{\mathrm{s}}_{\mu\nu}\right] W_c^\dagger\right)\frac{\not{n}_+}{2}\,\xi \\ & + \frac{1}{2}\,\bar{\xi}\left(i\not{D}_{\perp c}\,\frac{1}{in_+ D_c}\,x_\perp^\mu \gamma_\perp^\nu\, W_c\, gF^{\mathrm{s}}_{\mu\nu} W_c^\dagger + x_\perp^\mu \gamma_\perp^\nu\, W_c\, gF^{\mathrm{s}}_{\mu\nu} W_c^\dagger\,\frac{1}{in_+ D_c}\,i\not{D}_{\perp c}\right)\frac{\not{n}_+}{2}\,\xi\end{aligned}$$

- There are no purely collinear interactions at subleading powers, in each vertex there is at least one soft field
- Coordinate space arguments appear in the Lagrangian due to multipole expansion of the soft modes interacting with collinear fields

$$\phi_s(x) = \phi_s(x_-) + \underbrace{x_\perp\cdot\partial_\perp}_{\mathcal{O}(\lambda)}\phi_s(x_-) + \underbrace{x_+\cdot\partial_-}_{\mathcal{O}(\lambda^2)}\phi_s(x_-) + \frac{1}{2}\Big(\underbrace{x_{\mu\perp}x_{\nu\perp}\partial^\mu\partial^\nu}_{\mathcal{O}(\lambda^2)}\phi_s(x_-)\Big) + \ldots$$

- Still need to add external operators constructed by multiplying collinear gauge invariant building blocks

$$\chi_i(t_i n_{i+}) \equiv W_i^\dagger \xi_i \qquad\qquad \mathcal{A}^\mu_{i\perp}(t_i n_{i+}) \equiv W_i^\dagger\left[iD^\mu_{\perp i} W_i\right]$$

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Applications to LHC physics and beyond

Classic problem: soft gluon resummation

- Threshold resummation and fixed-order expansions have been applied to many different processes at LP: Drell-Yan [Becher, Neubert, Xu `07], Higgs production [Ahrens, Becher,Neubert,Yang `09], ttbar [Ahrens,Ferroglia,Neubert,Pecjak,Yang `10,`11], ttbar+V [AB,Ferroglia,Pecjak,Ossola,Yang,Signer `15,`16,`17]

$$\frac{d\sigma}{dM^2} \sim \sum_{ab} \int_\tau^1 \frac{dz}{z} ff_{ab}(\tau/z)\hat{\sigma}_{ab}(z)$$

When real radiation is present in the final state → $\hat{s} \neq M^2$, $z = M^2/\hat{s} \to 1$

$$\hat{\sigma}_{ab}(z) = \sum_{n=0}^{\infty} \alpha_s^n \left[\underbrace{c_n \delta(1-z) + \sum_{m=0}^{2n-1} \left(c_{nm} \left[\frac{\ln^m(1-z)}{1-z} \right]_+ \right.}_{\text{LP}} + \underbrace{\left. d_{nm} \ln^m(1-z) \right)}_{\text{NLP}} + \ldots \right]$$

$(1-z)$ expansion

Power counting parameter: $\lambda = \sqrt{1-z}$

Modes, SCET-I:

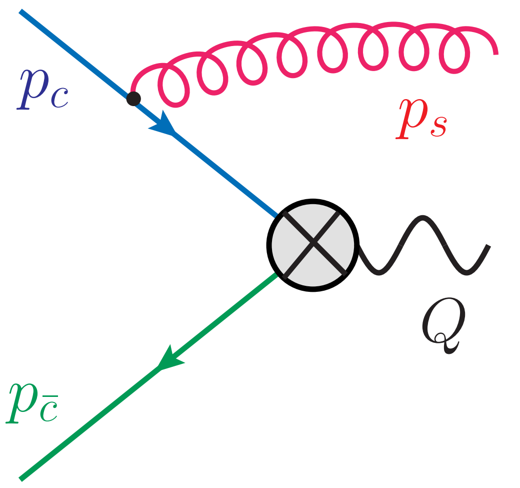
$$p_c^\mu = (n_+p_c, n_-p_c, p_{c\perp}) \sim M(1,\lambda^2,\lambda)$$

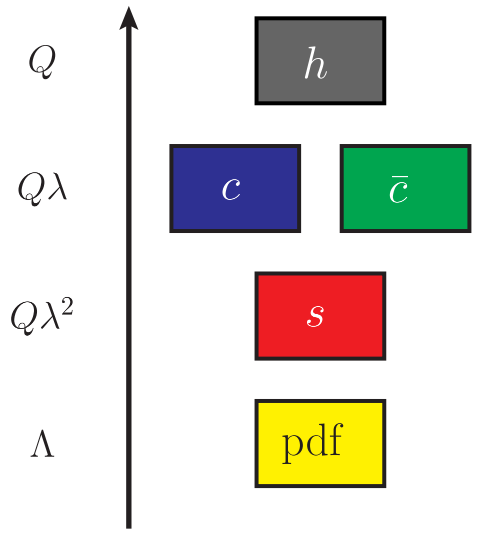
$$p_{\bar{c}}^\mu = (n_+p_{\bar{c}}, n_-p_{\bar{c}}, p_{\bar{c}\perp}) \sim M(\lambda^2,1,\lambda)$$

$$p_s^\mu = (n_+p_s, n_-p_s, p_{s\perp}) \sim M(\lambda^2,\lambda^2,\lambda^2)$$

$$p_{c-\mathrm{PDF}}^\mu \sim (M, \Lambda^2/M, \Lambda)$$

$$M^2\lambda^2 = M^2(1-z) \gg \Lambda_{\mathrm{QCD}}^2$$





Factorization and Resummation

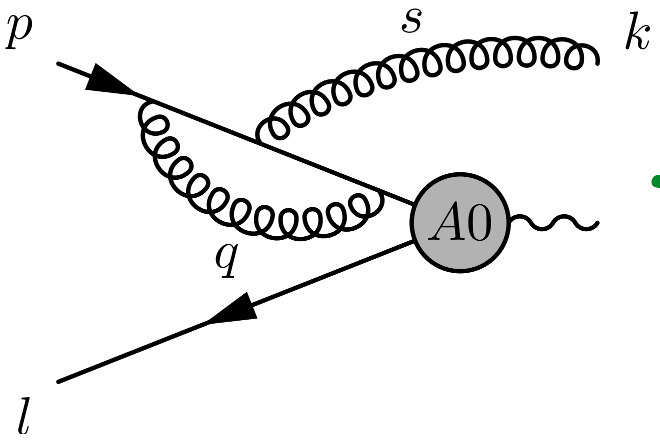
LP factorization
& Soft function

LP $$\hat{\sigma}(z) = H(M^2)\, M\, S_{\rm DY}\big(M(1-z)\big)$$

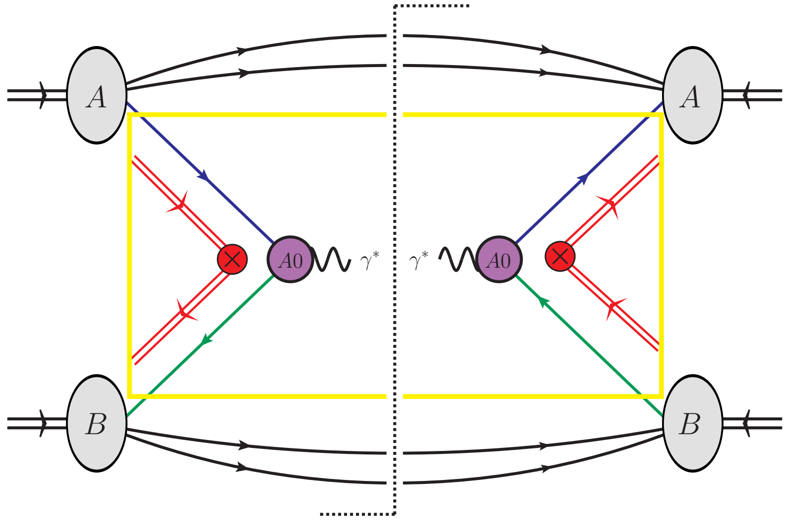
In SCET: [Becher, Neubert, Xu `07].

[G. Sterman, 1987] [S. Catani, L. Trentadue, 1989] [G.P. Korchemsky G. Marchesini, 1993]
[S. Moch, A Vogt, hep-ph/0508265]

$$S_{\rm DY}(\Omega) = \int \frac{dx^0}{4\pi}\, e^{i\Omega\, x^0/2}\, \frac{1}{N_c}\, {\rm Tr}\, \langle 0|\bar{\mathbf{T}}(Y_+^\dagger(x^0)Y_-(x^0))\, \mathbf{T}(Y_-^\dagger(0)Y_+(0))|0\rangle$$



Nontrivial statement, at LP collinear virtual diagrams cancel in the sum, consequence of decoupling transformation



Hard $H(M^2) = |\,\tilde{C}_V(M^2)\,|^2$ and Soft functions satisfy RG equations:

$$\frac{d}{d\ln\mu}\tilde{C}_V(-M^2-i0^+,\mu) = \left[C_F\gamma_{\rm cusp}(\alpha_s)\left(\ln\frac{M^2}{\mu^2}-i\pi\right)+\gamma_V(\alpha_s)\right]\tilde{C}_V(-M^2-i0^+,\mu)$$

Solution:

$$\tilde{C}_V(-M^2-i0^+,\mu_f) = \exp\left[2C_F S(\mu_h,\mu_f) - A_{\gamma_V}(\mu_h,\mu_f) + i\pi C_F A_{\gamma_{\rm cusp}}(\mu_h,\mu_f)\right] \times \left(\frac{M^2}{\mu_h^2}\right)^{-C_F A_{\gamma_{\rm cusp}}(\mu_h,\mu_f)} \tilde{C}_V(-M^2,\mu_h)\,.$$

$$S(\nu,\mu) = -\int_{\alpha_s(\nu)}^{\alpha_s(\mu)} d\alpha \frac{\gamma_{\rm cusp}(\alpha)}{\beta(\alpha)} \int_{\alpha_s(\nu)}^{\alpha} \frac{d\alpha'}{\beta(\alpha')}$$

$$A_{\gamma_i}(\nu,\mu) = -\int_{\alpha_s(\nu)}^{\alpha_s(\mu)} d\alpha \frac{\gamma_i(\alpha)}{\beta(\alpha)}$$

NLP Factorization & Generalised soft functions

$q\bar{q}$ channel [Beneke,AB,Jaskiewicz,Vernazza, JHEP 07 (2020) 078], qg channel [AB,Jaskiewicz,Vernazza, JHEP 12 (2023) 028] for Drell-Yan

$$\Delta^{dyn}_{\text{NLP}}(z) = -\frac{2}{(1-\epsilon)}\, Q \left[\left(\frac{\not{n}_-}{4}\right) \gamma_{\perp\rho} \left(\frac{\not{n}_+}{4}\right) \gamma_\perp^\rho \right]_{\beta\gamma}$$

$$\times \int d(n_+p)\, C^{A0,A0}(n_+p, x_b n_- p_B)\, C^{*A0A0}(x_a n_+ p_A, x_b n_- p_B)$$

$$\times \sum_{i=1}^{5} \int \{d\omega_j\} \boxed{J_{i,\gamma\beta}(n_+p, x_a n_+ p_A; \{\omega_j\})} \boxed{S_i(\Omega; \{\omega_j\})} + \text{h.c.}$$

At NLP the power suppression is entirely coming from Lagrangian insertions in time ordered operator products

Radiative collinear functions
(contain derivative contributions),
calculated up to $\mathcal{O}(\alpha_s)$

Generalized soft functions

$$S_i(\Omega; \{\omega_j\}) = \int \frac{dx^0}{4\pi}\, e^{i\Omega x^0/2} \int \left\{\frac{dz_{j-}}{2\pi}\right\} e^{-i\omega_j z_{j-}} S_i(x_0; \{z_{j-}\})$$

- LL resummation at NLP [Beneke,AB,Garny,Jaskiewicz,Szafron,Vernazza,Wang, JHEP 03 (2019) 043].

- NNLO computation of generalised soft functions at NLP! [AB,Jaskiewicz,Vernazza JHEP 10 (2011) 061]

  Difficult computation in d-dimensions to avoid endpoint singularities. DE method and canonical basis.

- Strong test of the validity of the factorisation formula at NNLO and beyond

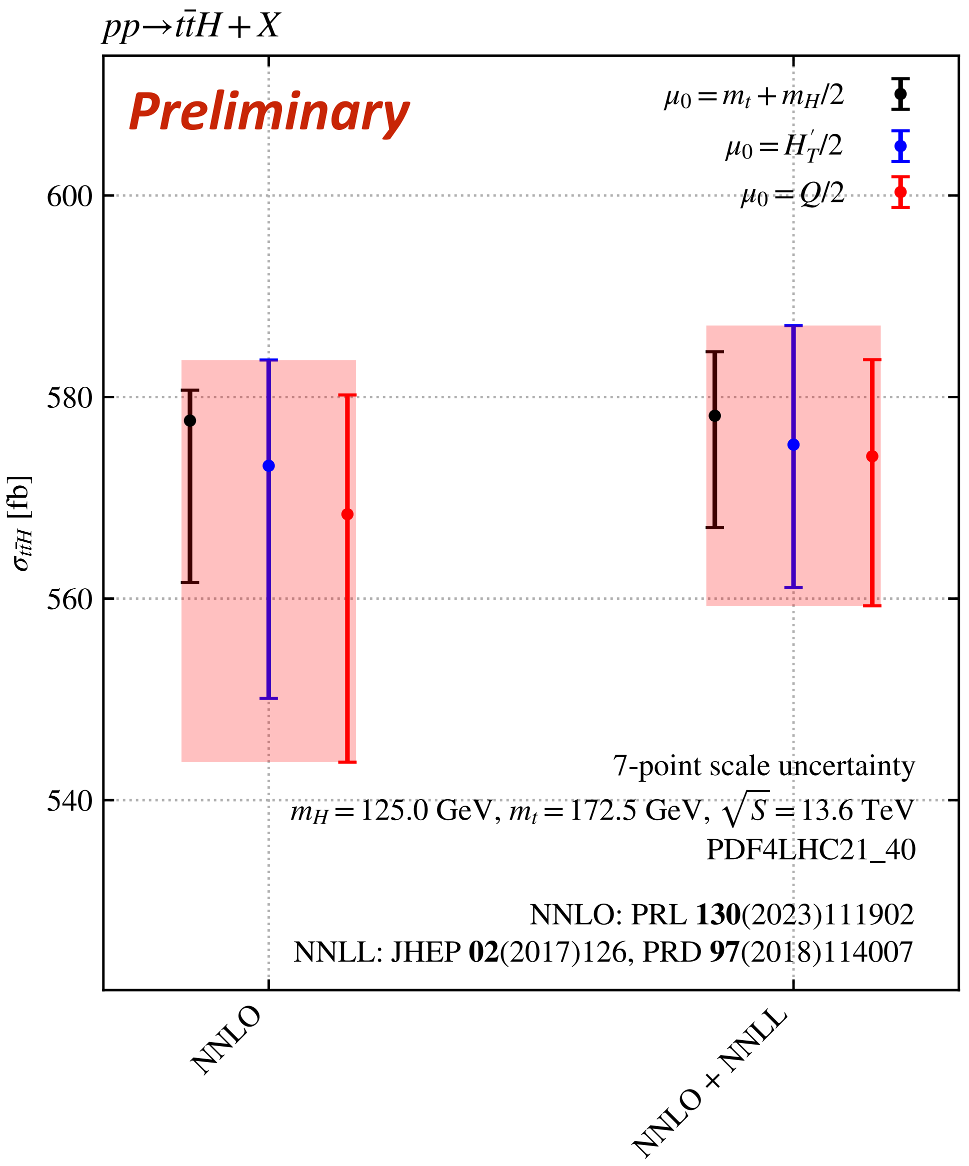
  [Beneke,AB,Jaskiewicz,Vernazza, JHEP 07 (2020) 078]

Soft gluon resummation at NNLL for $t\bar{t}H$, $t\bar{t}W^{\pm}$, $t\bar{t}Z$

Soft-gluon resummation as a way to estimate leading higher order corrections

[AB,Ferroglia,Pecjak,Signer, Yang, JHEP 03 (2016) 124], [AB,Ferroglia,Pecjak,Ossola, JHEP 09 (2016) 089], [AB,Ferroglia,Pecjak,Yang, JHEP 02 (2017) 126], [AB,Ferroglia,Pecjak,Ossola,Sameshima, JHEP 04 (2017) 105], [AB,Ferroglia,Frederix, Pagani,Pecjak,Tsinikos, JHEP 08 (2019) 039]

$t\bar{t}H$

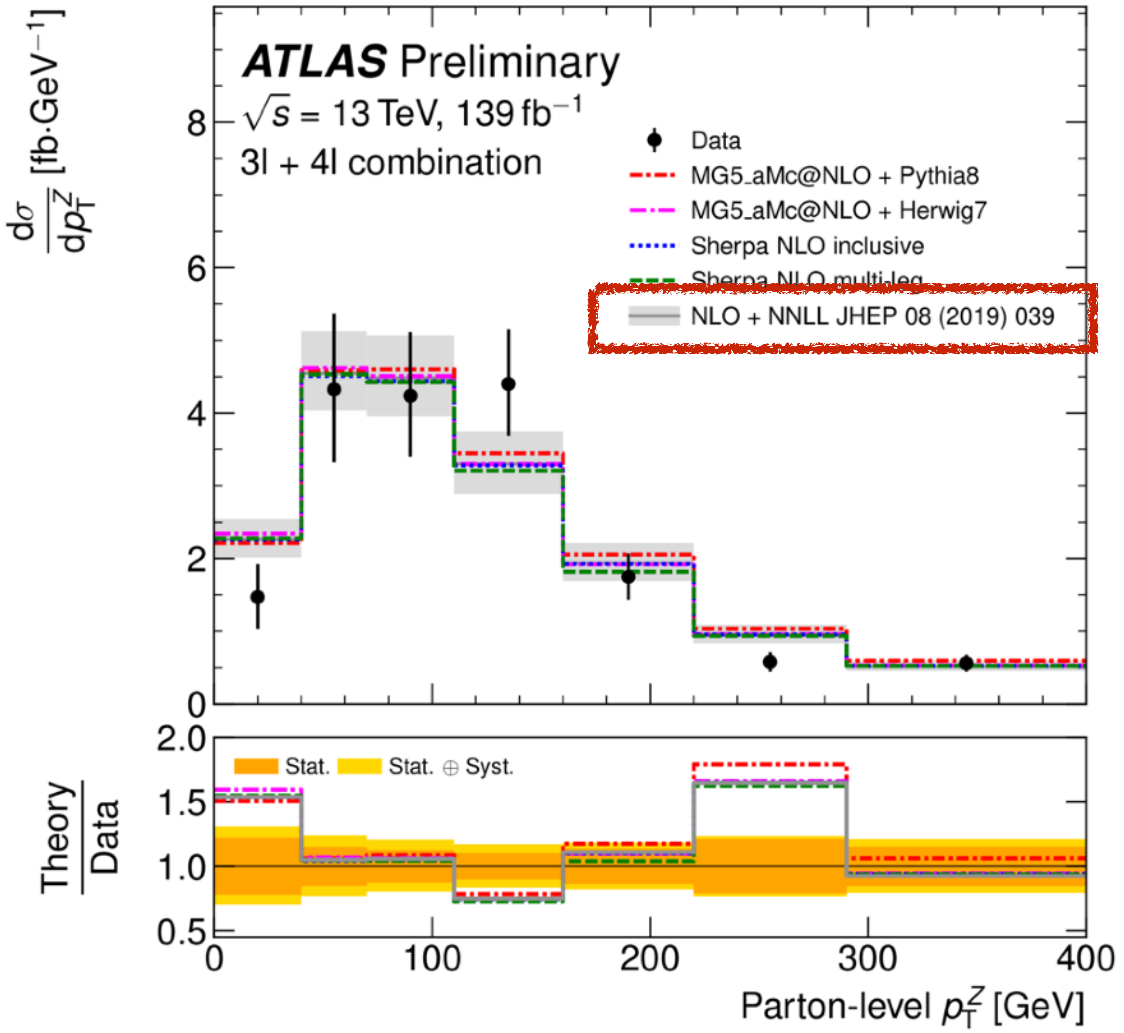


NNLL [AB,Ferroglia,Frederix, Pagani,Pecjak,Tsinikos `19]

matched to the NNLO approximation

[arXiv:2210.07846]

$t\bar{t}Z$



$t\bar{t}Z$ ATLAS comparison

IR poles of scattering amplitudes

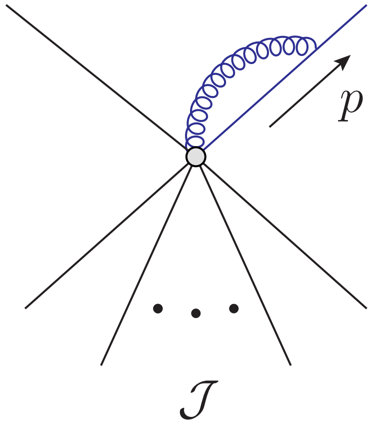
- Prediction of IR divergences of n-loops amplitudes [S. Catani 9802439] from N-jet operator renormalization in SCET [T.Becher, M. Neubert 0903.1126] [T.Becher, M. Neubert 1908.11379]

- Correspondence between IR divergences in on-shell amplitudes and UV divergences in the Wilson coefficients

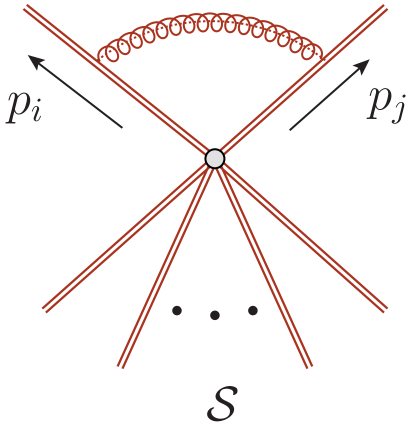
$$|\mathcal{M}_n(\varepsilon,\{\underline{p}\})\rangle = |\tilde{\mathcal{C}}_n(\varepsilon,\{\underline{p}\})\rangle \times (\text{"spinors and polarization vectors"})$$

$$\searrow \qquad\qquad \searrow \qquad\qquad \searrow$$

$$\underbrace{\frac{1}{\varepsilon_{\text{IR}}}}_{\text{on-shell amplitude}} = \underbrace{\frac{1}{\varepsilon_{\text{UV}}}}_{\text{Wilson coeff.}} + \underbrace{\left(\frac{1}{\varepsilon_{\text{IR}}} - \frac{1}{\varepsilon_{\text{UV}}}\right)}_{\text{soft and coll. loop integrals}}$$



$\mathcal{J}$



$\mathcal{S}$

$$\mathcal{J}_q(p^2,\mu) = 1 + \frac{\alpha_s}{4\pi} C_F \left(\frac{2}{\varepsilon^2} + \frac{2}{\varepsilon}\ln\frac{\mu^2}{-p^2} + \frac{3}{2\varepsilon}\right) + \mathcal{O}(\varepsilon^0)\,,$$

$$\mathcal{J}_g(p^2,\mu) = 1 + \frac{\alpha_s}{4\pi}\left[C_A\left(\frac{2}{\varepsilon^2} + \frac{2}{\varepsilon}\ln\frac{\mu^2}{-p^2}\right) + \frac{\beta_0}{2\varepsilon}\right] + \mathcal{O}(\varepsilon^0)\,,$$

$$\mathcal{S}(\{\underline{p}\},\mu) = 1 + \frac{\alpha_s}{4\pi}\sum_{(i,j)}^{n}\frac{\mathbf{T}_i\cdot\mathbf{T}_j}{2}\left(\frac{2}{\varepsilon^2} + \frac{2}{\varepsilon}\ln\frac{-s_{ij}\mu^2}{(-p_i^2)(-p_j^2)}\right) + \mathcal{O}(\varepsilon^0)$$

$$\mathcal{S}(\{\underline{p}\},\mu)\prod_{i=1}^{n}\mathcal{J}_i(p^2,\mu) = 1 - \frac{\alpha_s}{4\pi}\left[\sum_{\substack{i,j=1\\ i\neq j}}^{n}\frac{\mathbf{T}_i\cdot\mathbf{T}_j}{2}\left(\frac{2}{\varepsilon^2} + \frac{2}{\varepsilon}\ln\frac{\mu^2}{-s_{ij}}\right) + \sum_i\frac{\gamma_0^i}{2\varepsilon} + \mathcal{O}(\varepsilon^0)\right] \longrightarrow \boxed{\boldsymbol{\Gamma}(\{\underline{p}\},\mu) = \sum_{(i,j)}\frac{\mathbf{T}_i\cdot\mathbf{T}_j}{2}\gamma_{\text{cusp}}(\alpha_s)\ln\frac{\mu^2}{-s_{ij}} + \sum_{i=1}^{n}\gamma_i(\alpha_s)}$$

$$\ln\boldsymbol{Z} = \frac{\alpha_s}{4\pi}\left(\frac{\Gamma_0'}{4\varepsilon^2} + \frac{\boldsymbol{\Gamma}_0}{2\varepsilon}\right) + \left(\frac{\alpha_s}{4\pi}\right)^2\left[-\frac{3\beta_0\Gamma_0'}{16\varepsilon^3} + \frac{\Gamma_1' - 4\beta_0\boldsymbol{\Gamma}_0}{16\varepsilon^2} + \frac{\boldsymbol{\Gamma}_1}{4\varepsilon}\right]$$
$$+ \left(\frac{\alpha_s}{4\pi}\right)^3\left[\frac{11\beta_0^2\,\Gamma_0'}{72\varepsilon^4} - \frac{5\beta_0\Gamma_1' + 8\beta_1\Gamma_0' - 12\beta_0^2\,\boldsymbol{\Gamma}_0}{72\varepsilon^3} + \frac{\Gamma_2' - 6\beta_0\boldsymbol{\Gamma}_1 - 6\beta_1\boldsymbol{\Gamma}_0}{36\varepsilon^2} + \frac{\boldsymbol{\Gamma}_2}{6\varepsilon}\right] + \mathcal{O}(\alpha_s^4)$$

Reg. Scheme dependence was also studied
[AB,C.Gnendinger,A.Signer,D.Stockinger,A.Visconti JHEP 01 (2016) 078]

Extension to external massive particles ($t\bar{t}$ production) [A.Ferrroglia, M. Neubert, B.D. Pecjak, L.L Yang 0908.3676].
But also MUonE in QED [Bonciani,AB,DiVita,Ferroglia,Mandal,Mastrolia et al., PRL 128 (2022) 2]

Small q_T resummation in SCET for CS production

[T. Becher, M. Neubert 1007.4005], [T. Becher, M. Neubert, D.Wilhelm 1109.6027], [M. Echevarria, A. Idilbi, I. Scimemi, 1111.4996], [M. Echevarria, A. Idilbi, I. Scimemi, 1211.1947], [T. Becher, M. Neubert, D.Wilhelm 1212.2621], [M. Echevarria, A. Idilbi, I. Scimemi, 1402.0869], [M. Ebert, F. Tackmann 1611.08610], [M. Ebert, J. Michel, I. Stewart, F. Tackmann 2006.11382], [T. Becher, T. Neumann 2009.11437], [G. Billis, B. Dehnadi,M.Ebert,J.Michel,F. Tackmann 2102.08039]

SCET-II Modes

$p_c^\mu \sim (\lambda^2, 1, \lambda)\, M$

$p_{\bar{c}}^\mu \sim (1, \lambda^2, \lambda)\, M$

$p_s^\mu \sim (\lambda, \lambda, \lambda)\, M$

Rapidity divergences, need for analytic regulator in addition to dimensional regularization

$$\int d^d k\, \delta(k^2)\theta(k^0) \to \int d^d k \left(\frac{\nu}{k_+}\right)^{\alpha} \delta(k^2)\theta(k^0)$$

$$\frac{d^3\sigma}{dM^2\, dq_T^2\, dy} = \frac{4\pi\alpha^2}{3N_c M^2 s} \sum_q e_q^2 \sum_{i=q,g} \sum_{j=\bar{q},g} \int_{\xi_1}^1 \frac{dz_1}{z_1} \int_{\xi_2}^1 \frac{dz_2}{z_2}$$

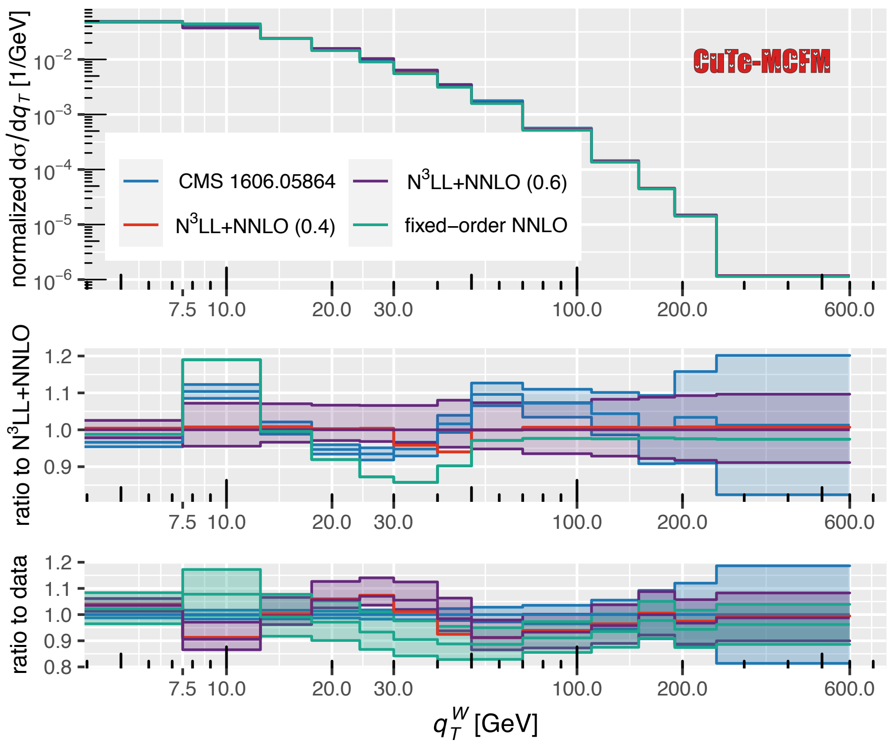
$$\times \left[C_{q\bar{q}\to ij}(z_1, z_2, q_T^2, M^2, \mu)\, f_{i/N_1}(\xi_1/z_1, \mu)\, f_{j/N_2}(\xi_2/z_2, \mu) + (q, i \leftrightarrow \bar{q}, j) \right]$$

$$C_{q\bar{q}\to ij}(z_1, z_2, q_T^2, M^2, \mu) = \left| C_V(-M^2, \mu) \right|^2 \frac{1}{4\pi} \int d^2 x_\perp\, e^{-iq_\perp \cdot x_\perp} \left(\frac{x_T^2 M^2}{4e^{-2\gamma_E}} \right)^{-F_{q\bar{q}}(x_T^2, \mu)}$$

$$\times\, I_{q\leftarrow i}(z_1, x_T^2, \mu)\, I_{\bar{q}\leftarrow j}(z_2, x_T^2, \mu)$$

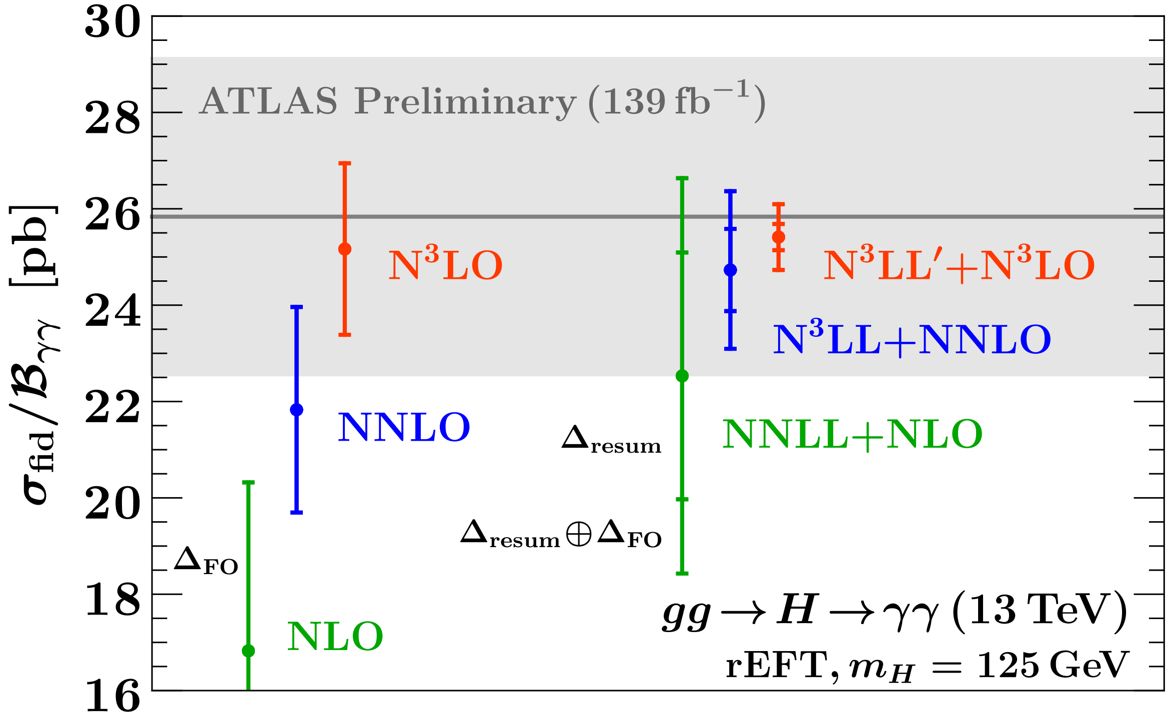
- From ν independence of the result, anomalous hard scale dependence can be completely factorized and can be shown that anomaly log exponentiate
- Other approach Rapidity Renormalization Group (RRG), uses different regulator [J.Y. Chiu, A. Jain, D. Neill and I. Z. Rothstein 1202.0814]
- Rapidity anomalous dimension known up to 4-loops [C. Duhr, B. Mistlberger, G. Vita, 2205.02242]

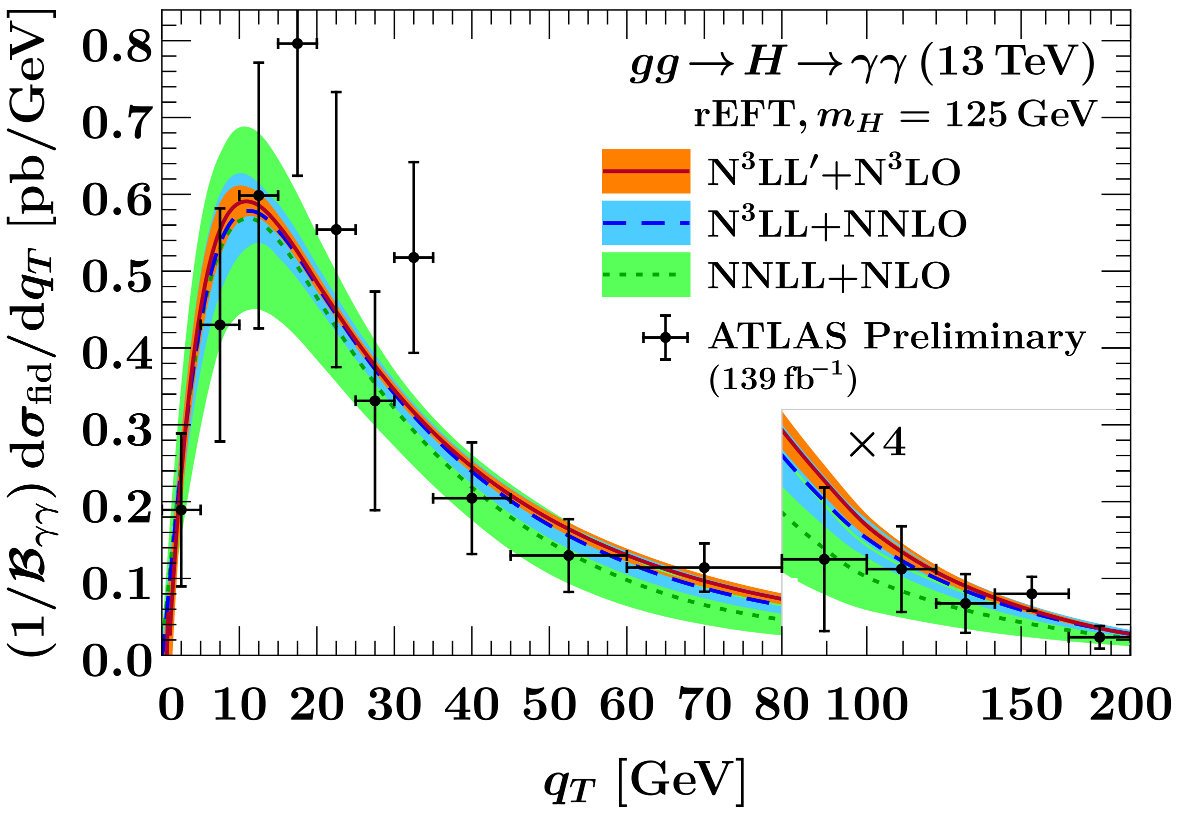
Small q_T resummation in SCET for CS production



CuTe-MCFM, N^3LL+NNLO fiducial cross sections [T. Becher, T. Neumann 2009.11437]

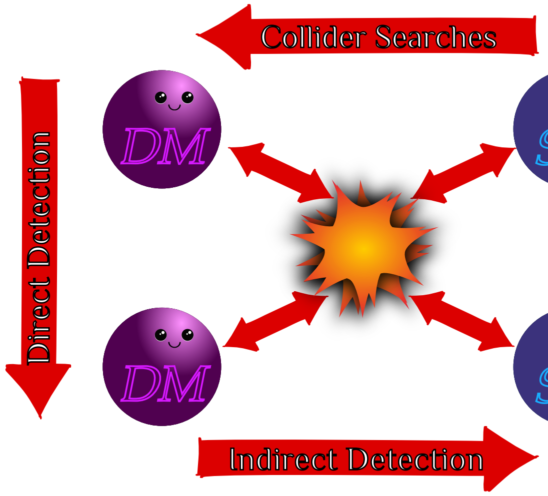
N^3LO and N^3LL' resummation obtained in [G. Billis, B. Dehnadi, M. Ebert, J. Michel, F. Tackmann 2102.08039] thanks to N^3LO calculation of transverse momentum dependent PDFs [M. Ebert, B. Mistlberger, G. Vita 2006.05329]





Sudakov resummation for WIMP Dark Matter annihilation

[Beneke, AB, Hasner, Vollmann, Phys.Lett.B 786 (2018)] and [Beneke, AB, Hasner, Urban, Vollmann, JHEP 08 (2019) 103]



Indirect searches detect the final products of dark matter annihilation in our galactic neighbourhood, using different kind of telescopes (CTA experiment)

Sommerfeld effect

$$\mathcal{O}((m_\chi \alpha_2 / m_W)^n)$$

corresponds to ladder diagrams with W, Z and photon exchange

Sudakov logarithms

$$\mathcal{O}((\alpha_2 \ln^2(m_\chi / m_W))^n)$$

- Framework for joint resummation of EW Sommerfeld and Sudakov effects up to NLL`. Renormalization Group Equations are much more involved than in QCD do to the presence of gauge boson masses and multiple couplings
- Details of resummation of EW Sudakov logs differ according to the scaling of E^γ_{res} w.r.t. m_W

Narrow: $E^\gamma_{\text{res}} \sim m_W^2 / m_\chi$
Intermediate: $E^\gamma_{\text{res}} \sim m_W$
Wide: $E^\gamma_{\text{res}} \gg m_W$

Beneke,AB,Hasner,Vollmann. [arxiv:1805.07367]

Beneke,AB,Hasner,Urban,Vollmann. [arxiv:1903.08702]

Baumgart et al. [arxiv:1808.08956]

Intermediate resolution

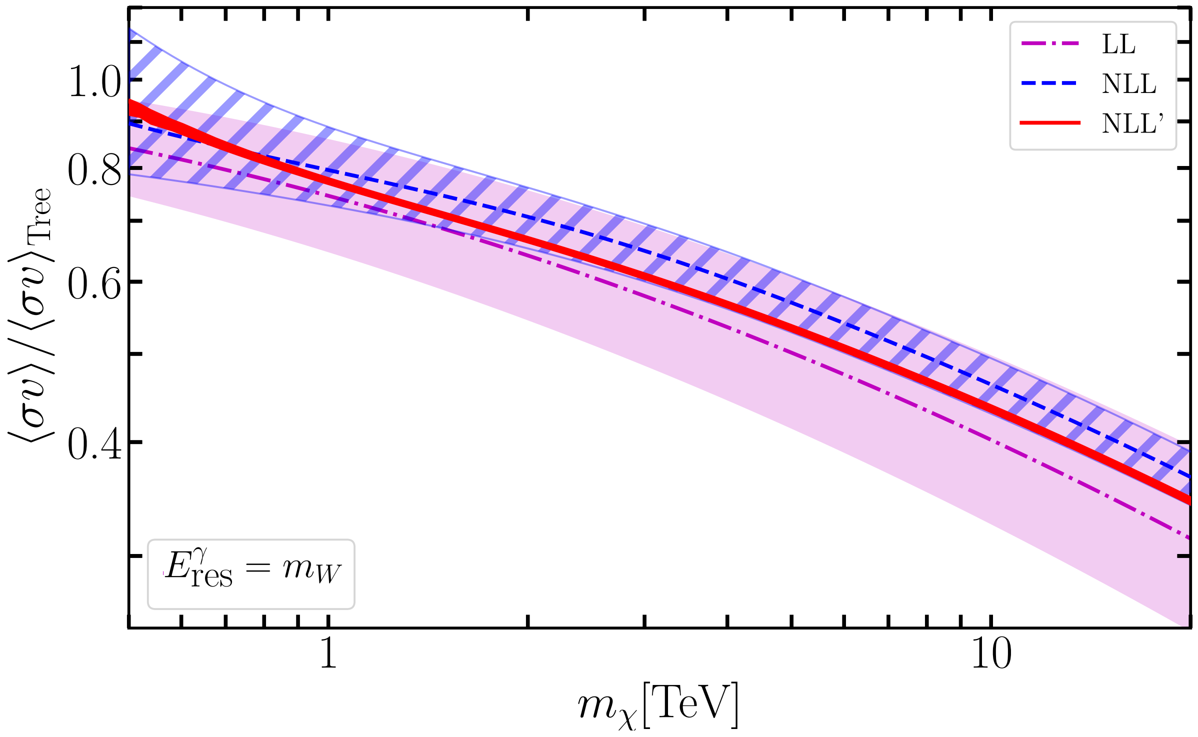
We assume that the energy resolution is parametrically of order $E^{\gamma}_{\text{res}} \sim m_W$ which implies $m_X = \sqrt{4 m_\chi E^{\gamma}_{\text{res}}}$ and the scale hierarchy $E^{\gamma}_{\text{res}} \sim m_W \ll m_X \ll m_\chi$

$$\begin{aligned}
\text{hard}\,(h): &\quad k^\mu \sim m_\chi(1,1,1) \\
\text{hard-collinear}\,(hc): &\quad k^\mu \sim m_\chi(1,\lambda,\sqrt{\lambda}) \\
\text{collinear}\,(c): &\quad k^\mu \sim m_\chi(1,\lambda^2,\lambda) \\
\text{anti-collinear}\,(\bar{c}): &\quad k^\mu \sim m_\chi(\lambda^2,1,\lambda) \\
\text{soft}\,(s): &\quad k^\mu \sim m_\chi(\lambda,\lambda,\lambda) \\
\text{potential}\,(p): &\quad k^0 \sim m_W^2/m_\chi,\ \mathbf{k} \sim m_W \\
\text{ultrasoft}\,(s): &\quad k^\mu \sim m_\chi(\lambda^2,\lambda^2,\lambda^2)
\end{aligned}$$

where $\lambda = m_w/m_\chi$

$\Big\}$ SCET-II situation, rapidity regulator needed

- The scale uncertainty reduces from 17% (LL) to 8% (NLL) to 1% (NLL’) for $m_\chi = 2$ TeV
- At $m_\chi = 2$ TeV $(10$ TeV$)$ the ratio of the resummed at NLL’ to the Sommerfeld-only rate is $0.667^{+0.007}_{-0.006}\ (0.435^{+0.005}_{-0.004})$



MC event generators and N-jettiness subtraction

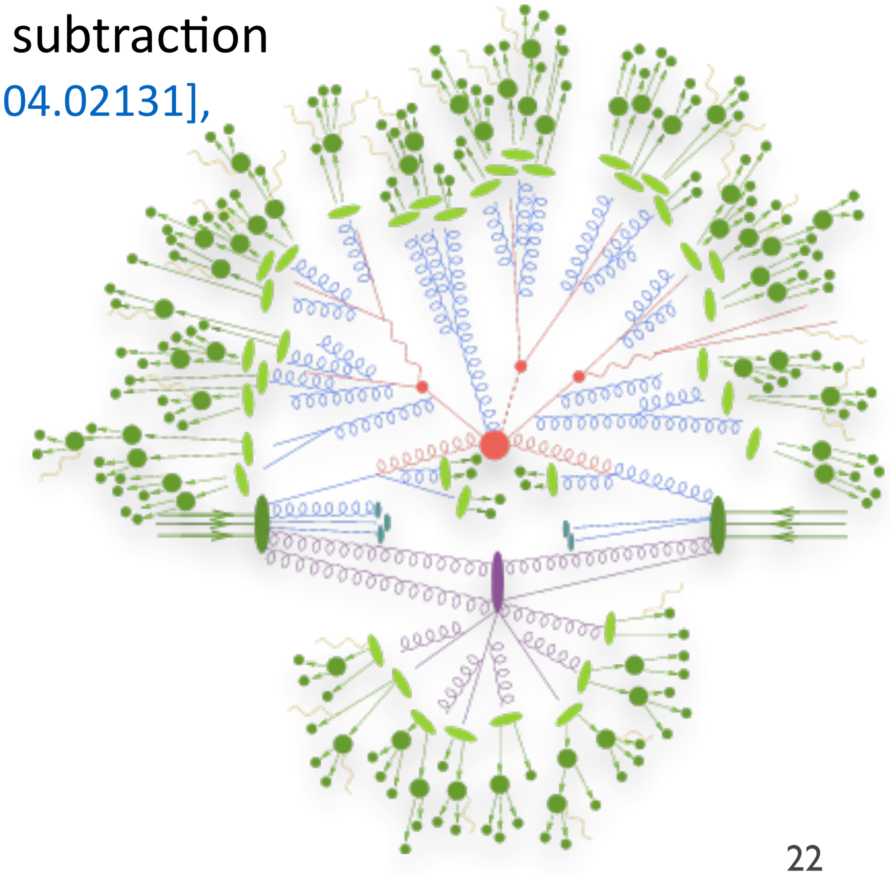
- Factorization and Resummation properties of suitable jet resolution variables such as N-jettiness are used to construct event generators and to implement IR slicing/subtraction methods for NNLO calculations

- Resummation of 0-jettiness has been used to match NNLO calculations to parton showers for colour singlet production processes in MC event generators such as GENEVA and MINNLOPS

- Theory predictions for $\gamma^*/Z + \mathrm{jet}$ production are needed at higher precision to match the experimental accuracy of the Z boson transverse momentum spectrum

- One-jettiness [Stewart, Tackmann,Waalewijn `09,`10] is a suitable event shape for colour singlet + jet production, was used to carry out NNLO calculations using N-jettiness subtraction method [Gaunt, Stahlhofen, Tackmann, Walsh 15], [Boughezal, Focke, Liu, Petriello 1504.02131], [Boughezal, Campbell, Ellis, Focke, Giele, Liu, Petriello 1512.01291], [Boughezal, Focke, Giele, Liu, Petriello 1505.03893], [Campbell, Ellis, Seth 19]

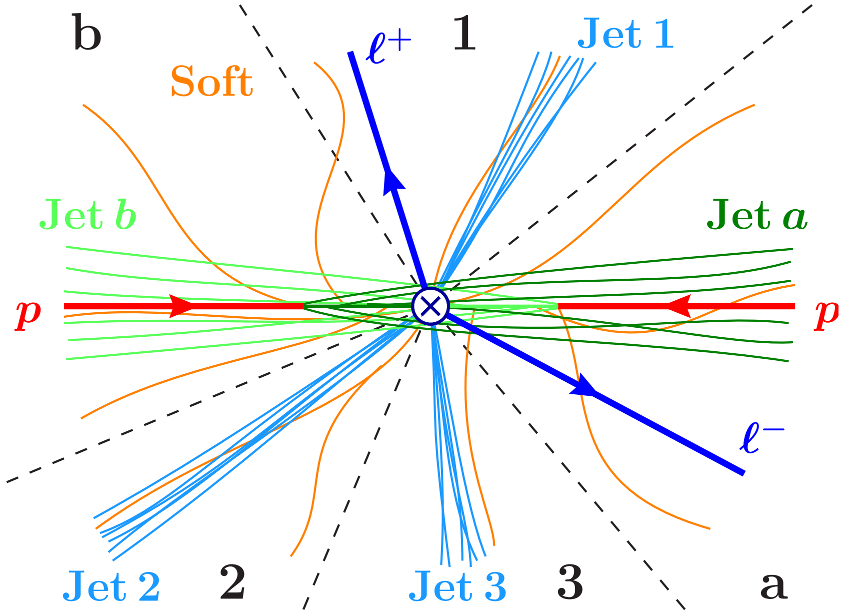
- Extend the GENEVA MC method to include processes with final state jets, first milestone is to evaluate $\mathrm{N}^3\mathrm{LL}_{\mathcal{T}_1}$+NNLO for Z+jet



N-Jettiness

- N-jettiness resolution variables: given an M-particle phase space point with $M \geq N$

$$\mathcal{T}_N(\Phi_M) = \sum_k \min\{\hat{q}_a \cdot p_k, \hat{q}_b \cdot p_k, \hat{q}_1 \cdot p_k, \ldots, \hat{q}_N \cdot p_k\}$$



- The limit $\mathcal{T}_N \to 0$ describes a N-jet event where the unresolved emissions soft or collinear to the final state jets or initial state beams

- Color singlet final state, relevant variable is ***0-jettiness*** aka "beam thrust"

$$\mathcal{T}_0 = \sum_k |\vec{p}_{kT}| \, e^{-|\eta_k - Y|}$$

- When an extra jet is present the relevant jet resolution variable is ***1-jettiness***

$$\mathcal{T}_1 = \sum_k \min\left\{\frac{2q_a \cdot p_k}{Q_a}, \frac{2q_b \cdot p_k}{Q_b}, \frac{2q_J \cdot p_k}{Q_J}\right\}$$

- We use a geometric measure $Q_i = 2\,\rho_i\,E_i$ (ρ_i dimensionless parameter), removes the dependence on the energies E_i and it only depends on the directions $\hat{q}_i$ (introduce frame dependence). We choose ρ_i to work on Color Singlet (CS) frame ($Y_V = 0$).

- Factorization formula valid in the region $\mathcal{T}_1 \ll M_{ll} \sim \sqrt{s} \sim M_{T,ll}$ [Stewart,Tackmann,Waalewijn `09,`10]

$$\frac{d\sigma}{d\Phi_1 d\mathcal{T}_1} = \sum_\kappa \boxed{H_\kappa(\Phi_1, \mu)} \int dt_a\, dt_b\, ds_J$$
$$\times \boxed{B_{\kappa_a}(t_a, x_a, \mu)\; B_{\kappa_b}(t_b, x_b, \mu)\; J_{\kappa_J}(s_J, \mu}$$
$$\times \boxed{S_\kappa\left(n_a \cdot n_J, \mathcal{T}_1 - \frac{t_a}{Q_a} - \frac{t_b}{Q_b} - \frac{s_J}{Q_J}, \mu\right)} \longrightarrow$$

Frame dependence

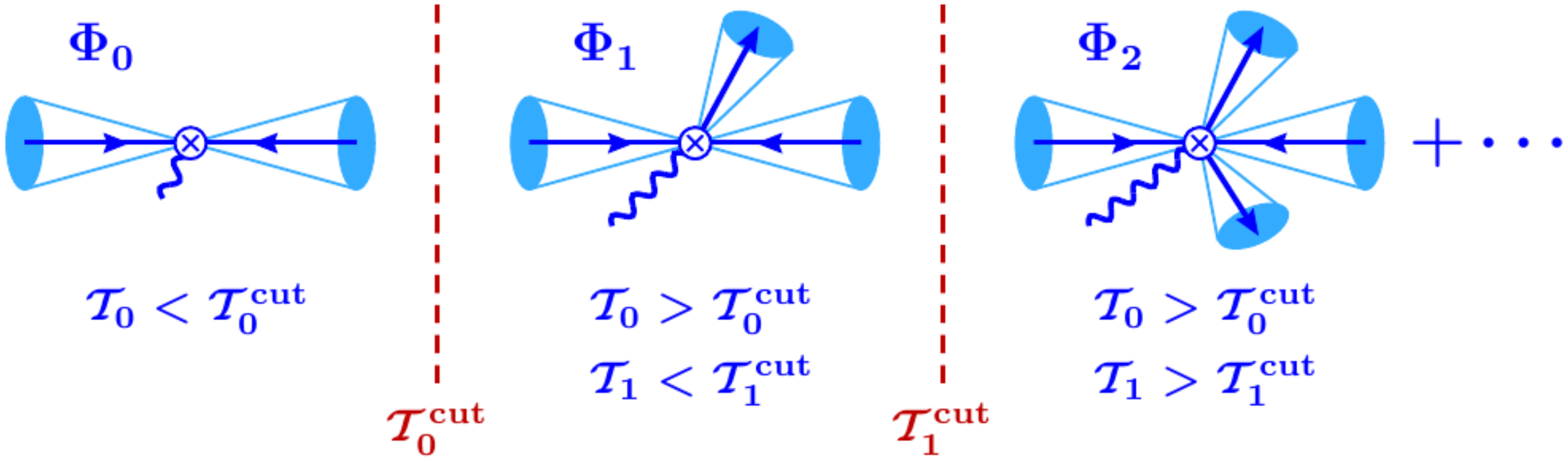
Monte Carlo implementation

- GENEVA [Alioli,Bauer,Berggren,Tackmann, Walsh `15], [Alioli,Bauer,Tackmann,Guns `16], [Alioli,Broggio,Lim, Kallweit,Rottoli `19],[Alioli,Broggio,Gavardi,Lim,Nagar,Napoletano,Kallweit,Rottoli `20-`21] combines 3 theoretical tools that are important for QCD predictions into a single framework
    - fully differential fixed-order calculations, up to NNLO via 0-jettiness or q_T subtraction
    - up to N^{3}LL resummation for 0-jettiness in SCET or N^{3}LL for q_T via RadISH for colour singlet processes
    - shower and hadronize events (PYTHIA8)
- IR-finite definition of events based on resolution parameters $\mathcal{T}_0^{\rm cut}$ and $\mathcal{T}_1^{\rm cut}$

$$\Phi_0 \text{ events:} \qquad \frac{d\sigma_0^{\rm MC}}{d\Phi_0}(\mathcal{T}_0^{\rm cut})\,,$$

$$\Phi_1 \text{ events:} \qquad \frac{d\sigma_1^{\rm MC}}{d\Phi_1}(\mathcal{T}_0 > \mathcal{T}_0^{\rm cut}; \mathcal{T}_1^{\rm cut})\,,$$

$$\Phi_2 \text{ events:} \qquad \frac{d\sigma_{\geq 2}^{\rm MC}}{d\Phi_2}(\mathcal{T}_0 > \mathcal{T}_0^{\rm cut}, \mathcal{T}_1 > \mathcal{T}_1^{\rm cut})$$



- When we take $\mathcal{T}_N^{\rm cut} \to 0$, large logarithms of $\mathcal{T}_N^{\rm cut}$, $\mathcal{T}_N$ appear and need to be resummed
- Including the higher-order resummation will improve the accuracy of the predictions across the whole spectrum

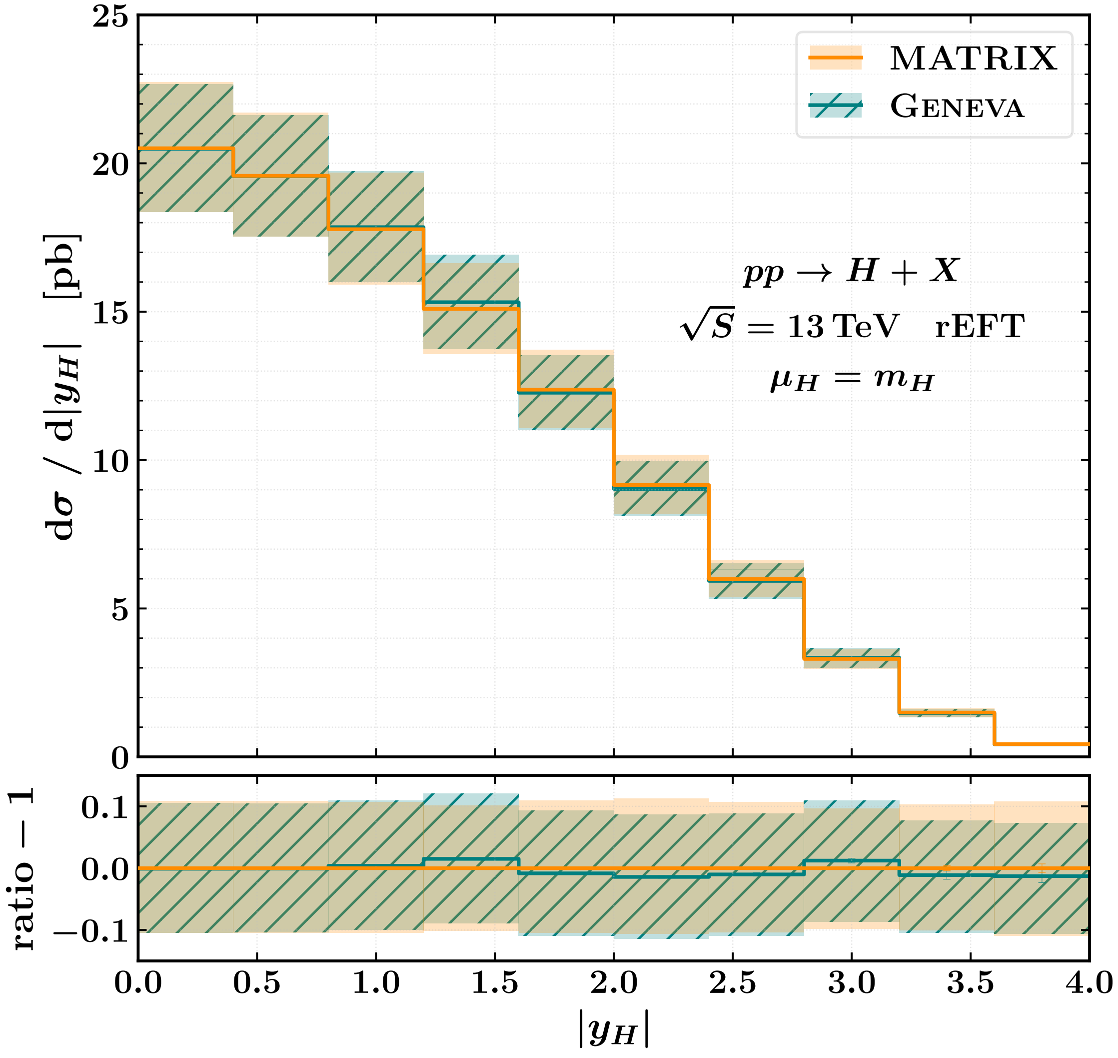
Higgs boson production via gluon fusion

[S. Alioli, AB, A. Gavardi, S. Kallweit, M.A. Lim, G. Marinelli, R. Nagar and D. Napoletano arXiv:2301.11875]

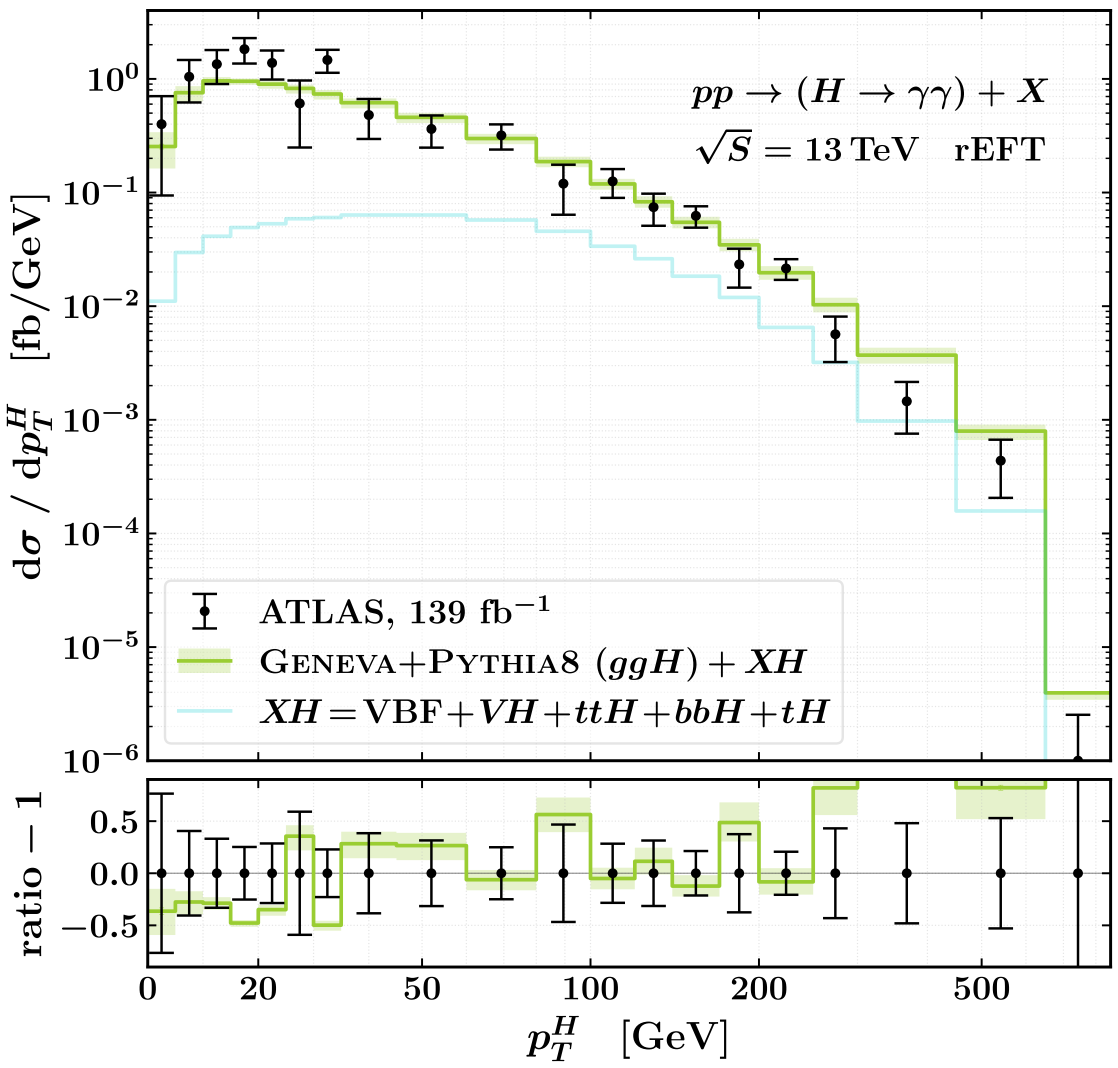
Calculation done in the Heavy Top Limit (HTL). Rescaling of HTL result by a factor equal to the ratio between the LO m_t-exact result and that obtained in pure EFT (rEFT)

NNLO validation

	GENEVA	ggHiggs	MATRIX
$\sigma^{\text{NNLO, rEFT}}_{gg\to H}$ [pb]	$42.33^{+4.39}_{-4.34}$	$42.35^{+4.55}_{-4.41}$	$42.33^{+4.54}_{-4.40}$



Comparison to Data

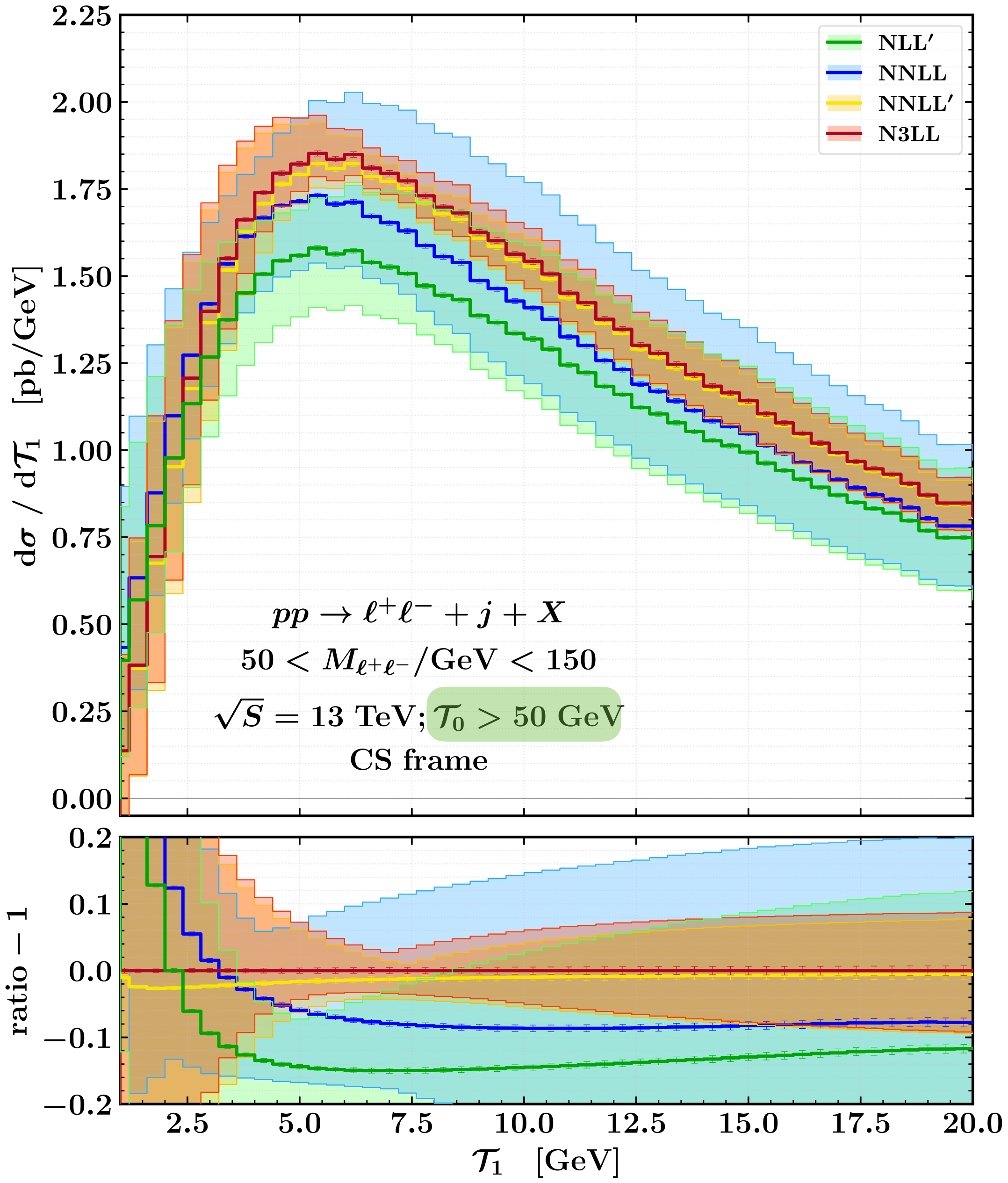


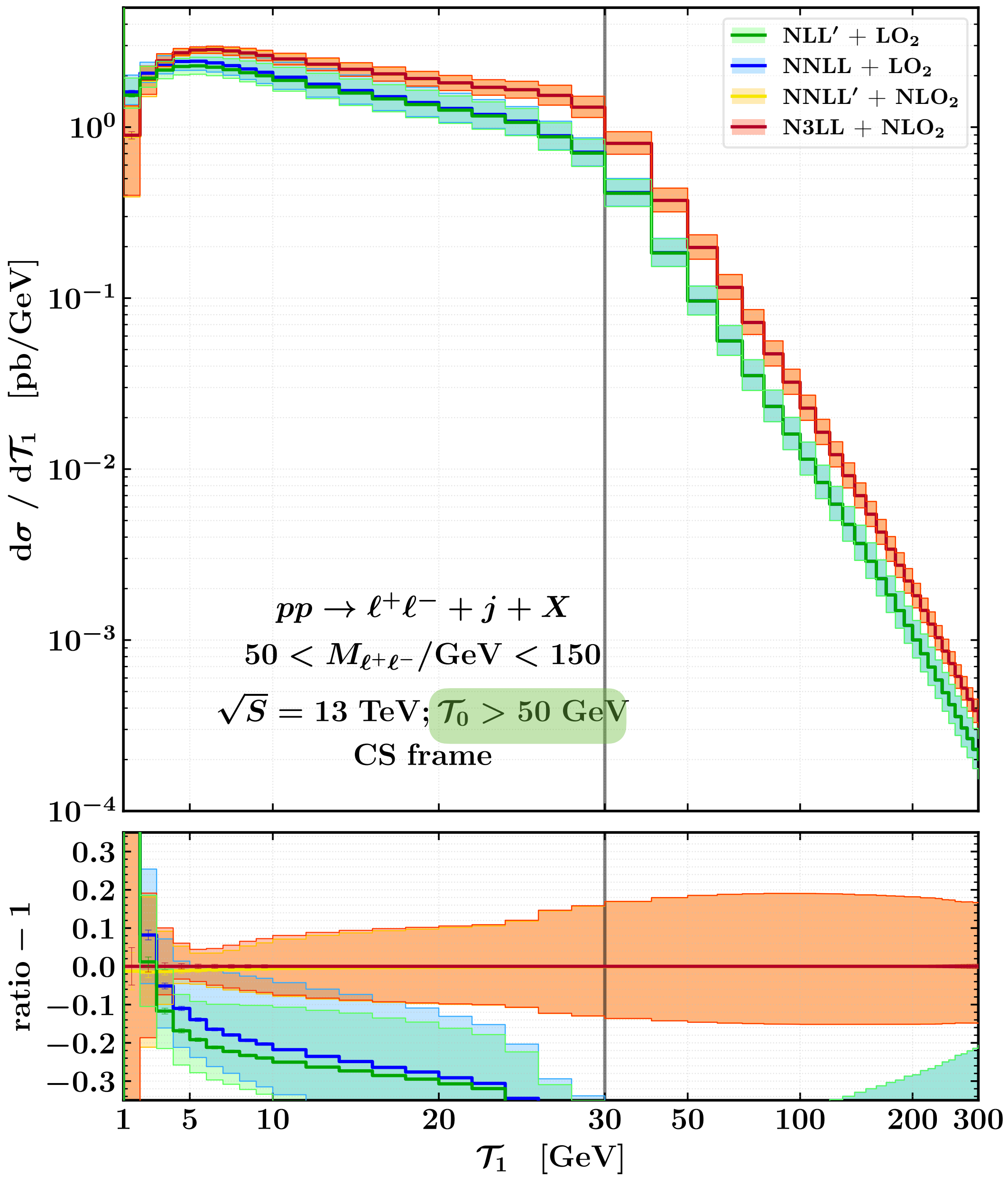
1-jettiness resummed and matched results for Z+jet

Matching Formula

$$\frac{d\sigma^{N^3LL+NLO_2}}{d\Phi_1 d\mathcal{T}_1} = \frac{d\sigma^{N^3LL}}{d\Phi_1 d\mathcal{T}_1} + \frac{d\sigma^{Nons.}}{d\Phi_1 d\mathcal{T}_1}$$

$$\frac{d\sigma^{Nons.}}{d\Phi_1 d\mathcal{T}_1} = \left(\frac{d\sigma^{NLO_2}}{d\Phi_1 d\mathcal{T}_1} - \frac{d\sigma^{N^3LL}}{d\Phi_1 d\mathcal{T}_1}\bigg|_{\mathcal{O}(\alpha_s^2)} \right) \theta(\mathcal{T}_1)$$





Uncertainties evaluated by adding in quadrature μ_{FO} variations with soft, jet, beam variations

NNLO results via 1-jettiness slicing/subtraction

Important to test the NNLO accuracy of the calculation: implementation one-jettiness slicing/subtraction compared pure $\mathcal{O}(\alpha_s^3)$ correction to NNLOJET.

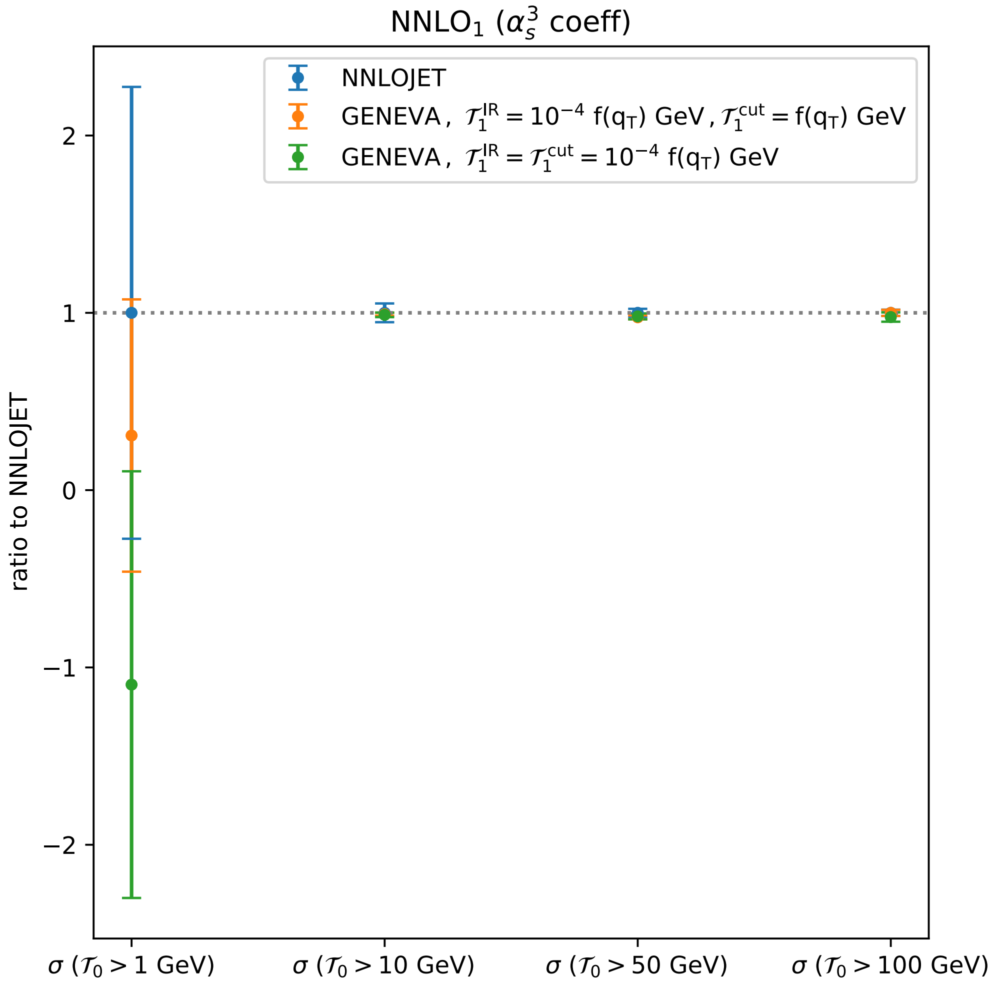
Very Preliminary!!

Slicing

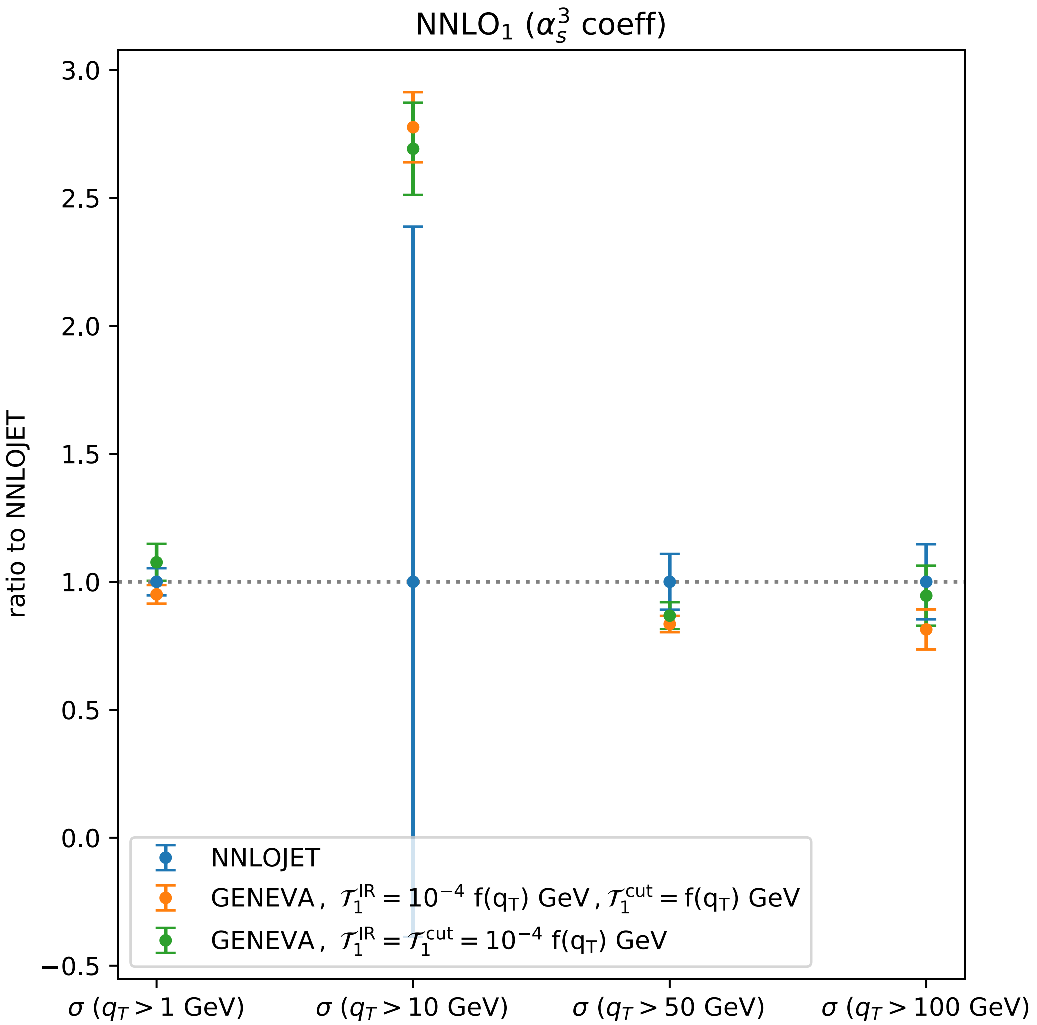
$$\mathcal{O}_{\delta\mathrm{NNLO}}(\Phi_N) = \frac{d\sigma_N^{N^3LL}(\mathcal{T}_\delta)}{d\Phi_N}\bigg|_{\mathcal{O}(\alpha_s^2)} \mathcal{O}(\Phi_N) + \int_{\mathcal{T}_\delta}^{\mathcal{T}^{\max}} \frac{d\Phi_{N+1}}{d\Phi_N}\,\frac{d\sigma_{N+1}^{\delta\mathrm{NLO}}}{d\Phi_{N+1}}\mathcal{O}(\Phi_{N+X}) + \dots$$

Non-local Subtraction $\mathcal{T}_\delta \ll \mathcal{T}_N^{\mathrm{cut}}$

$$\mathcal{O}_{\delta\mathrm{NNLO}}(\Phi_N) = \frac{d\sigma_N^{N^3LL}(\mathcal{T}_N^{\mathrm{cut}})}{d\Phi_N}\bigg|_{\mathcal{O}(\alpha_s^2)} \mathcal{O}(\Phi_N) + \int_{\mathcal{T}_\delta}^{\mathcal{T}^{\max}} \frac{d\Phi_{N+1}}{d\Phi_N}\left[\frac{d\sigma_{N+1}^{\delta\mathrm{NLO}}}{d\Phi_{N+1}}\mathcal{O}(\Phi_{N+X}) - \frac{d\sigma^{N^3LL}}{d\Phi_N\, d\mathcal{T}_N}\bigg|_{\mathcal{O}(\alpha_s^2)} \mathcal{P}\big(\Phi_{N+1}\big)\,\mathcal{O}(\Phi_N)\,\theta\big(\mathcal{T}_N \leq \mathcal{T}_N^{\mathrm{cut}}\big)\right] + \dots$$



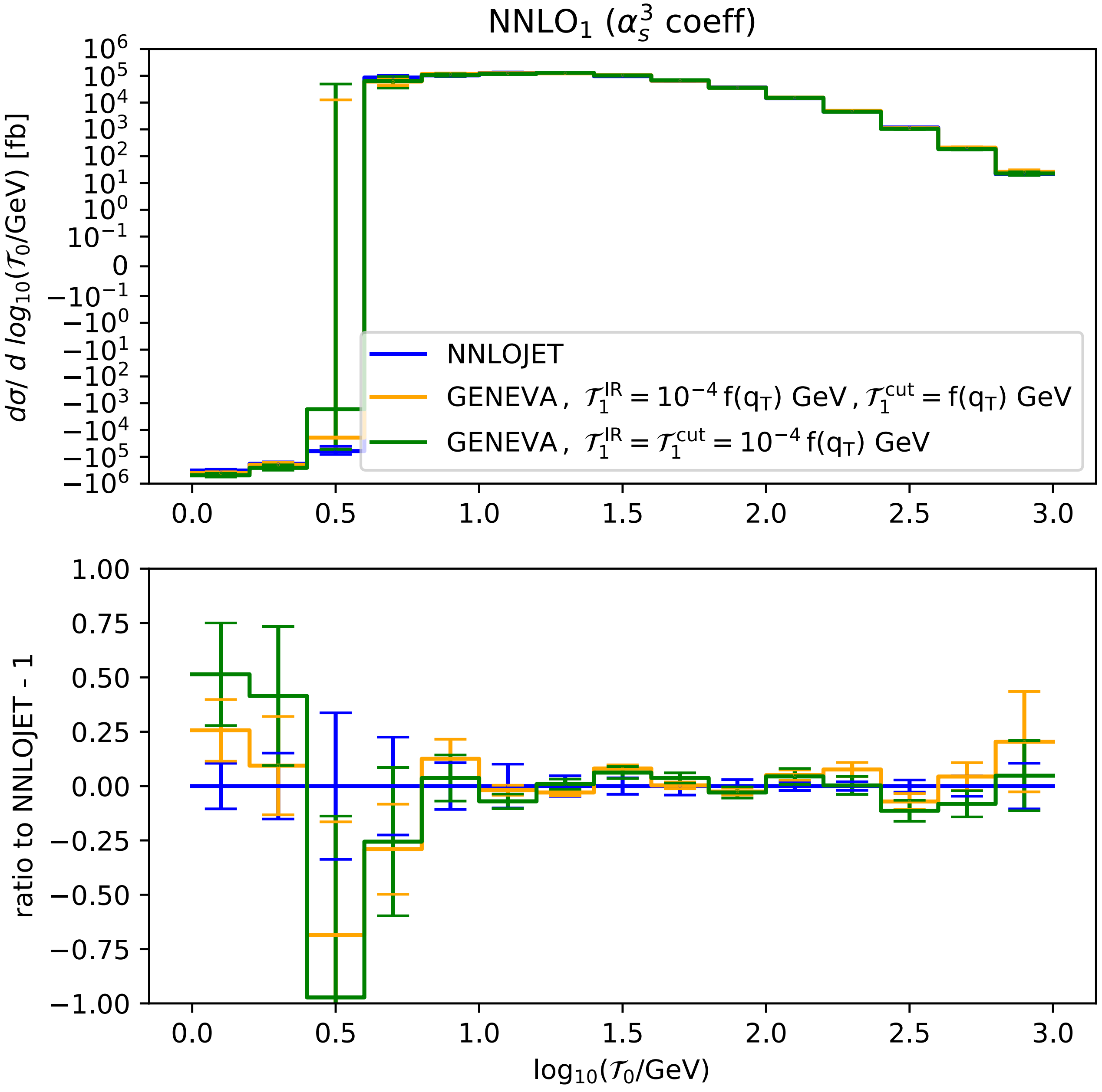
XS with $\mathcal{T}_0$ cuts



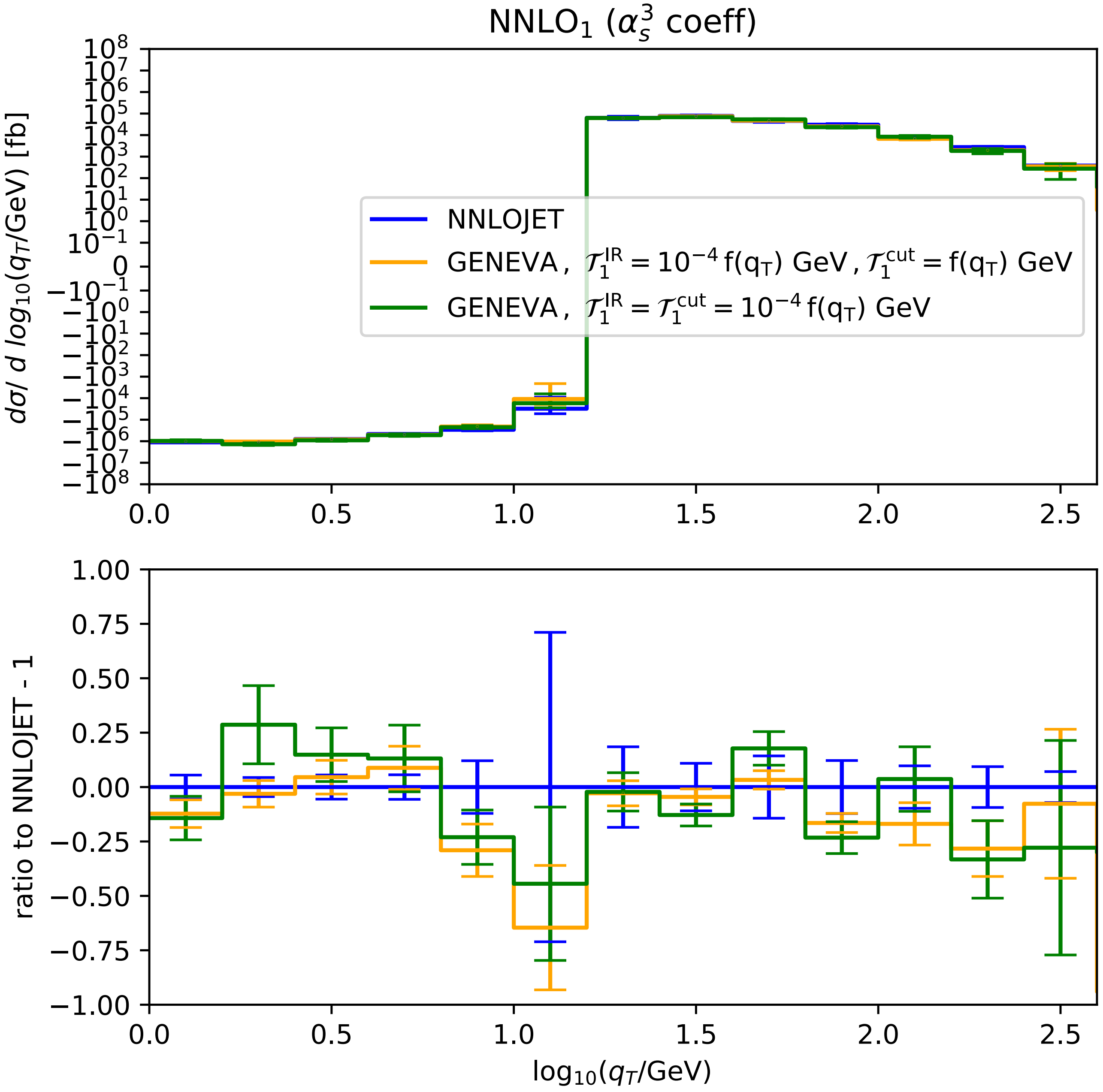
XS with q_T cuts

NNLO results differential distributions

Very Preliminary!!



$\mathscr{T}_0$ distribution



q_T distribution

Summary & Outlook

- I haven't mentioned: B-physics applications, inclusion of Glauber modes in SCET, energy correlators, resummation of event shapes, α_s extraction, multi-differential resummation, subleading power factorization and resummation, NGL resummation…
- In the last few years resummation has been achieved to very high accuracy (N^{3}LL and beyond) for important observables
- In some cases power corrections are large (compared to N^{3}LL at LP) and require resummation. At fixed-order, power corrections could improve on current LP slicing/subtraction methods.
- Extend NNLO event generators to include jet processes (with one jet at least) using N-jettiness as jet resolution variable.

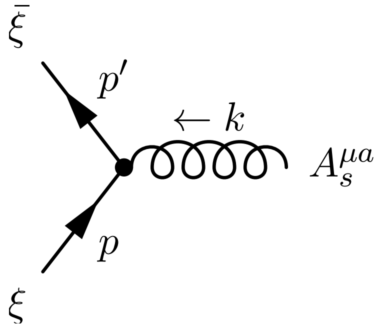
Thank you!

Backup

SCET Feynman Rules

Feynman rules up to two emissions [M. Beneke, M. Garny, R. Szafron J. Wang, 1808.04742]

Soft-Collinear interaction



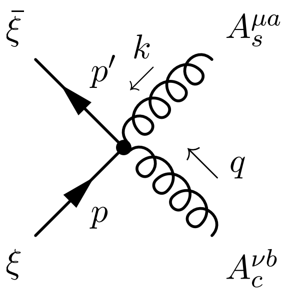
$$ig_s t^a \begin{cases} \frac{\not{n}_+}{2} n_{-\mu} & \mathcal{O}(\lambda^0) \\ \frac{\not{n}_+}{2} X_\perp^\rho n_-^\nu (k_\rho g_{\nu\mu} - k_\nu g_{\rho\mu}) & \mathcal{O}(\lambda) \\ S^{\rho\nu}(k,p,p') \frac{\not{n}_+}{2} (k_\rho g_{\nu\mu} - k_\nu g_{\rho\mu}) & \mathcal{O}(\lambda^2) \end{cases}$$

$$S^{\rho\nu}(k,p,p') \equiv \frac{1}{2}\left[(n_- X) n_+^\rho n_-^\nu + (k X_\perp) X_\perp^\rho n_-^\nu + X_\perp^\rho \left(\frac{\not{p}'_\perp}{n_+ p'} \gamma_\perp^\nu + \gamma_\perp^\nu \frac{\not{p}_\perp}{n_+ p}\right)\right]$$

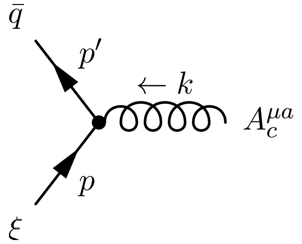
$$X^\mu \equiv \partial^\mu \Big[(2\pi)^4 \delta^{(4)} \Big(\sum p_{in} - \sum p_{out}\Big)\Big] \qquad X^\mu X^\nu \equiv \partial^\mu \partial^\nu \Big[(2\pi)^4 \delta^{(4)} \Big(\sum p_{in} - \sum p_{out}\Big)\Big]$$

After the derivative in $X_\perp^\rho$ is taken, $p'_\perp$ can be set to $p_\perp$ ($n_+ p' = n_+ p$ can be set from the start).

No surprise on the appearance of these derivatives!



$$ig_s^2 \begin{cases} 0 & \mathcal{O}(\lambda^0) \\ \frac{\not{n}_+}{2} X_\perp^\rho n_-^\sigma (k_\rho g_{\sigma\mu} - k_\sigma g_{\rho\mu}) \frac{n_{+\nu}}{n_+ q} [t^a, t^b] & \mathcal{O}(\lambda) \\ \Big[\frac{1}{2} X_\perp^\rho \left(\Gamma_\nu(p') \frac{\gamma_\perp^\sigma}{n_+(p'-q)} t^b t^a + \frac{\gamma_\perp^\sigma}{n_+(p+q)} \Gamma_\nu(p) t^a t^b\right) & \\ +S^{\rho\sigma}(k,p,p') \frac{n_{+\nu}}{n_+ q} [t^a, t^b]\Big] \frac{\not{n}_+}{2} (k_\rho g_{\sigma\mu} - k_\sigma g_{\rho\mu}) & \mathcal{O}(\lambda^2) \end{cases}$$



$$ig_s t^a \begin{cases} 0 & \mathcal{O}(\lambda^0) \\ \Gamma_\mu(p) & \mathcal{O}(\lambda) \\ \left[n_{-\mu} + \gamma_{\perp\mu} \frac{\not{p}_\perp}{n_+ p} + \frac{n_{+\mu}}{n_+ k} \frac{p^2}{n_+ p}\right] \frac{\not{n}_+}{2} - (p' X_\perp) \Gamma_\mu(p) & \mathcal{O}(\lambda^2) \end{cases}$$

$$\Gamma^\mu(p) \equiv \gamma_\perp^\mu - \frac{\not{p}_\perp}{n_+ p} n_+^\mu \, .$$

Subleading power N-jet operators

generic N-jet operator

$$J = \int \prod_{i=1}^{N} \prod_{k=1}^{n_i} dt_{i_k} \; C(\{t_{i_k}\}) \prod_{i=1}^{N} J_i(t_{i_1}, t_{i_2}, \ldots, t_{i_{n_i}})$$

[M. Beneke, M. Garny, R. Szafron J. Wang 1712.04416, 1808.04742, 1907.05463]

The J's are constructed by multiplying collinear gauge invariant building blocks in the same direction (up to $\mathcal{O}(\lambda^2)$)

$$\chi_i(t_i n_{i+}) \equiv W_i^\dagger \xi_i \qquad \mathcal{A}^\mu_{i\perp}(t_i n_{i+}) \equiv W_i^\dagger [iD^\mu_{\perp i} W_i]$$

by acting on these with derivatives $i\partial^\mu_{\perp i} \sim \lambda$ and insertions of subleading SCET Lagrangian in a time-ordered product with lower power currents. We use the notation: $J_i^{An}, J_i^{Bn}, J_i^{Cn}, J_i^{Tn}$

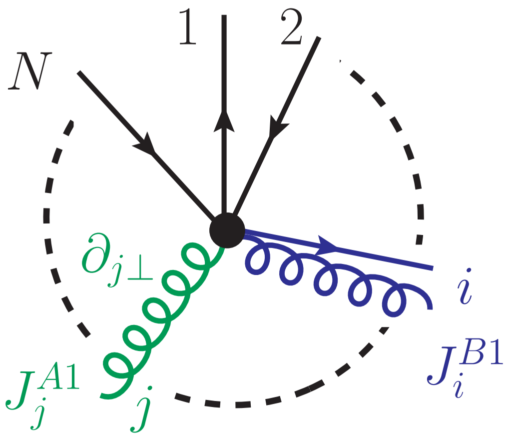
- A, B, C….refers to the number of fields in a given collinear direction
- n is the power suppression (relative to A0) in a given sector

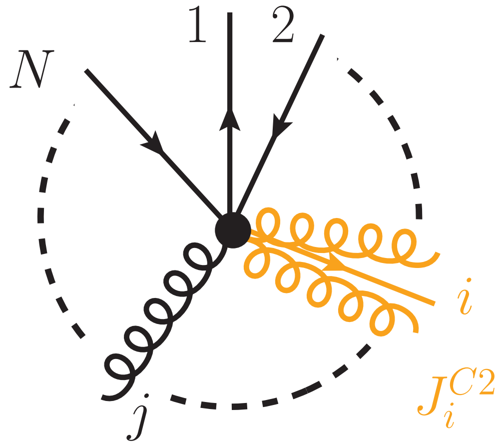
For example, up to $\mathcal{O}(\lambda^2)$ we can construct $J_i^{A1}, J_i^{B1}, J_i^{C2}, J_i^{T2}$

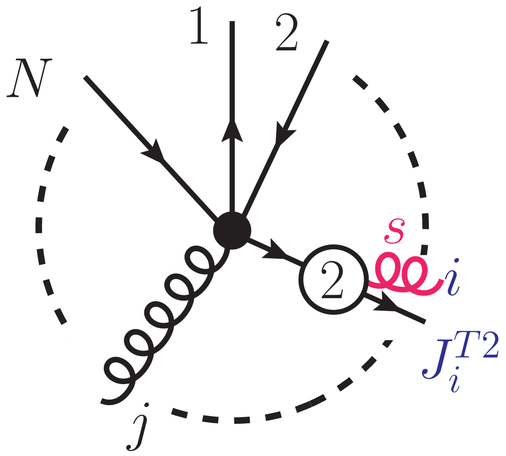
$$i\partial^\mu_{\perp i}\chi_i \qquad \chi_i(t_{i_1})\mathcal{A}^\mu_{i\perp}(t_{i_2}) \qquad \chi_i(t_{i_1})\mathcal{A}^\nu_{i\perp}(t_{i_2})\mathcal{A}^\mu_{i\perp}(t_{i_3}) \qquad i\int d^4z \, \mathrm{T}[\chi_i(t_{i_i})\mathcal{L}^{(2)}(z)]$$

$$J_1^{A0} \ldots J_i^{B1} J_j^{A1} \ldots J_N^{A0}, \qquad J_1^{A0} \ldots J_i^{C2} \ldots J_N^{A0}, \qquad J_1^{A0} \ldots J_i^{T2} \ldots J_N^{A0}, \ldots$$

At $\mathcal{O}(\lambda^2)$ N-jet operators







Operators

$$J_\rho^{A0,A1}(t,\bar{t}\,) = \bar{\chi}_{\bar{c}}(\bar{t}n_-)\, n_{+\rho}\, i\not{\partial}_\perp \chi_c(tn_+),$$

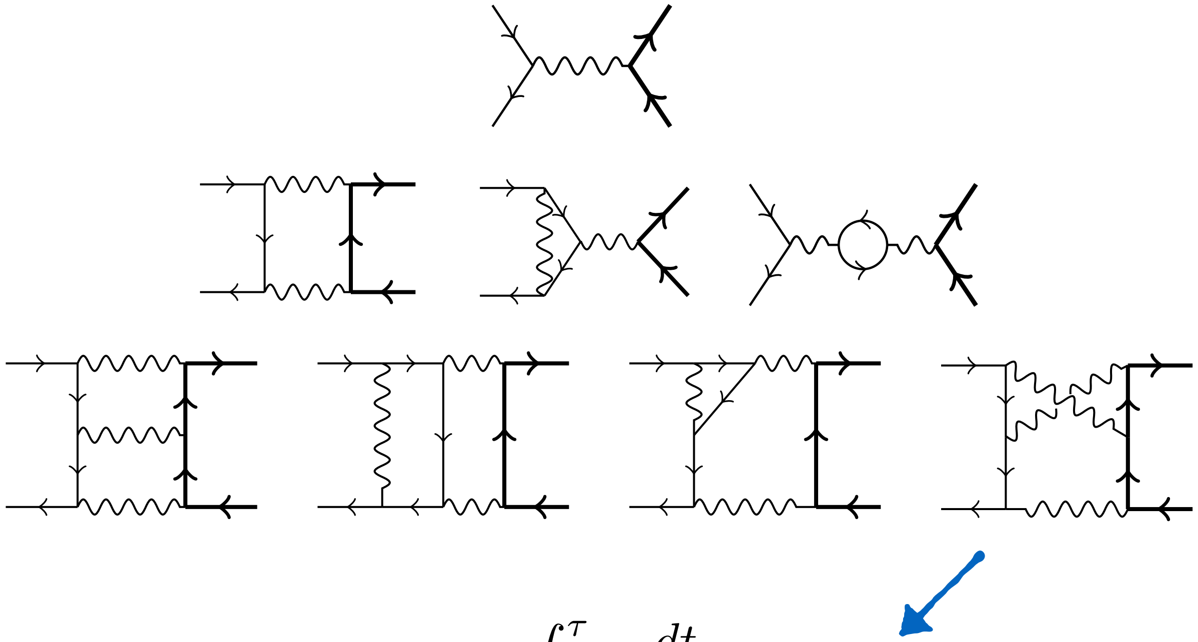
$$J_\rho^{A0,B1}(t_1,t_2,\bar{t}\,) = \bar{\chi}_{\bar{c}}(\bar{t}n_-)\, n_{\pm\rho}\, \not{\mathcal{A}}_{\perp c}(t_2 n_+)\chi_c(t_1 n_+)$$

λ suppressed operators together with $\mathscr{L}^{(1)}$ insertion on the same collinear leg for example.
The interference with the LP complex conjugate amplitude will be zero $\sim \gamma_\perp^\mu$.

Two-loop Amplitudes and NNLO for MUonE

- Muon electron scattering amplitude in QED at two loops fundamental for MUonE experiment! [Bonciani,AB,DiVita,Ferroglia,Mandal,Mastrolia et al., PRL 128 (2022) 2] NNLO computation in [AB,Engel,Ferroglia,Mandal,Mastrolia et al., JHEP 01 (2023) 112]

- MUonE will provide an independent determination of HVP important for clarifying discrepancies between theoretical predictions and experiment for muon $g-2$



$$G(w_n, \ldots, w_1; \tau) \equiv \int_0^\tau \frac{dt}{t - w_n} G(w_{n-1}, \ldots, w_1; t)$$

$$\mathcal{M}^{(1)}\Big|_{\text{poles}} = \frac{1}{2} Z_1^{\text{IR}}\, \mathcal{M}^{(0)}\Big|_{\text{poles}}$$

$$\mathcal{M}^{(2)}\Big|_{\text{poles}} = \frac{1}{8}\Big[\Big(Z_2^{\text{IR}} - \big(Z_1^{\text{IR}}\big)^2\Big)\, \mathcal{M}^{(0)} + 2\, Z_1^{\text{IR}}\, \mathcal{M}^{(1)}\Big]\Big|_{\text{poles}}$$

- Regularisation scheme transformation rules derived for QCD amplitudes up to two-loops using SCET [AB,C.Gnendinger,A.Signer,D.Stockinger,A.Visconti JHEP 01 (2016) 078]

$$\left(Z_{\text{IR}}^{\text{FDH}}\right)^{-1} \mathcal{M}_n^{(\text{FDH})} = \left(Z_{\text{IR}}^{\text{CDR}}\right)^{-1} \mathcal{M}_n^{(\text{CDR})} + \mathcal{O}(\epsilon)$$

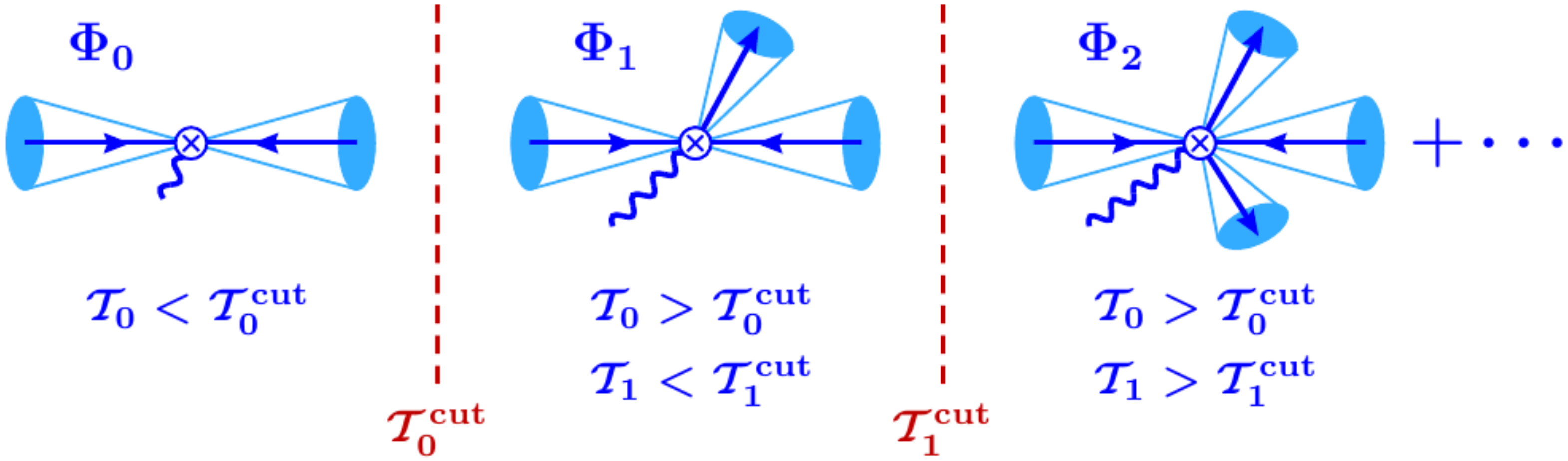
Monte Carlo implementation for CS

- GENEVA [Alioli,Bauer,Berggren,Tackmann, Walsh `15], [Alioli,Bauer,Tackmann,Guns `16], [Alioli,Broggio,Lim, Kallweit,Rottoli `19],[Alioli,Broggio,Gavardi,Lim,Nagar,Napoletano,Kallweit,Rottoli `20-`21] employs IR-finite definition of events based on resolution parameters $\mathcal{T}_0^{\rm cut}$ and $\mathcal{T}_1^{\rm cut}$ (for colour singlet production)

$$\Phi_0 \text{ events:} \quad \frac{d\sigma_0^{\rm MC}}{d\Phi_0}(\mathcal{T}_0^{\rm cut})$$

$$\Phi_1 \text{ events:} \quad \frac{d\sigma_1^{\rm MC}}{d\Phi_1}(\mathcal{T}_0 > \mathcal{T}_0^{\rm cut}; \mathcal{T}_1^{\rm cut})$$

$$\Phi_2 \text{ events:} \quad \frac{d\sigma_{\geq 2}^{\rm MC}}{d\Phi_2}(\mathcal{T}_0 > \mathcal{T}_0^{\rm cut}, \mathcal{T}_1 > \mathcal{T}_1^{\rm cut})$$



- When we take $\mathcal{T}_N^{\rm cut} \to 0$, large logarithms of $\mathcal{T}_N^{\rm cut}$, $\mathcal{T}_N$ appear and need to be resummed

- Including the higher-order resummation will improve the accuracy of the predictions across the whole spectrum

N^3LL Resummation: hard anomalous dimension

For every channel ($q\bar{q}g$, qgq,ggg,...), ***hard anomalous dimension*** has the form [T. Becher and M. Neubert 1908.11379]

$$\boldsymbol{\Gamma}_C^{\kappa}(\mu) = \Gamma_C^{\kappa}(\mu)\,\mathbf{1} = \left\{\frac{\Gamma_{\text{cusp}}(\alpha_s)}{2}\left[\left(C_c - C_a - C_b\right)\ln\frac{\mu^2}{(-s_{ab}-\text{i}0)} + \text{cyclic permutations}\right]\right.$$

$$\left. + \gamma_C^a(\alpha_s) + \gamma_C^b(\alpha_s) + \gamma_C^c(\alpha_s) + \frac{C_A^2}{8} f(\alpha_s)\left(C_a + C_b + C_c\right)\right\}\mathbf{1}$$

$$+ \sum_{(i,j)}\left[-f(\alpha_s)\mathcal{T}_{iijj} + \sum_{R=F,A} g^R(\alpha_s)\left(3\mathcal{D}^R_{iijj} + 4\mathcal{D}^R_{iiij}\right)\ln\frac{\mu^2}{(-s_{ij}-\text{i}0)}\right] + \mathcal{O}(\alpha_s^5)$$

4-loops

3-loops

$f(\alpha_s)$ and $g^R(\alpha_s)$ start at $\mathcal{O}(\alpha_s^3)$ and $\mathcal{O}(\alpha_s^4)$ computed in [Henn, Korchemsky, Mistlberger 1911.10174], [Von Manteuffel, Panzer, Schabinger 2002.04617]. Evaluated these contributions as functions of N_c using the *colour space formalism*

$$\mathcal{D}^R_{ijkl} = d_R^{abcd}\boldsymbol{T}_i^a\boldsymbol{T}_j^b\boldsymbol{T}_k^c\boldsymbol{T}_l^d \qquad \mathcal{T}_{ijkl} = f^{ade}f^{bce}(\boldsymbol{T}_i^a\boldsymbol{T}_j^b\boldsymbol{T}_k^c\boldsymbol{T}_l^d)_+$$

$$d_R^{a_1\ldots a_n} = \text{Tr}_R\left(\boldsymbol{T}^{a_1}\ldots\boldsymbol{T}^{a_n}\right)_+ \equiv \frac{1}{n!}\sum_{\pi}\text{Tr}\left(\boldsymbol{T}_R^{a_{\pi(1)}}\ldots\boldsymbol{T}_R^{a_{\pi(n)}}\right)$$

Using color conservation and symmetry properties of d_R^{abcd}, we found the following relations

$$3\left(\mathcal{D}^R_{iijj} + \mathcal{D}^R_{jjii}\right) + 4\left(\mathcal{D}^R_{iiij} + \mathcal{D}^R_{jjji}\right) = \left(D_{kR} - D_{iR} - D_{jR}\right)\mathbf{1} \qquad i \neq j \neq k$$

Quartic Casimirs

Similarity to the quadratic case $\quad \boldsymbol{T}_a\cdot\boldsymbol{T}_b = [\boldsymbol{T}_c^2 - \boldsymbol{T}_a^2 - \boldsymbol{T}_b^2]/2$

$$C_4(R_i, R) = \frac{d_{R_i}^{abcd}d_R^{abcd}}{N_{R_i}} \equiv D_{iR}$$

N^{3}LL Resummation: hard anomalous dimension

Hard anomalous dimension can be rewritten as

$$\Gamma_C^\kappa(\mu) = \left[-\bar{c}^\kappa \Gamma_{\rm cusp}(\alpha_s) + \sum_{R=F,A} \bar{c}_4^{\kappa,\,R}\, g^R(\alpha_s) \right] \ln\frac{Q^2}{\mu^2}$$

$$+ \sum_{i=a,b,c} \gamma_C^i(\alpha_s) + f(\alpha_s)\, c_f^\kappa - \bar{c}_L^\kappa \Gamma_{\rm cusp}(\alpha_s)$$

$$+ \sum_{R=F,A} g^R(\alpha_s)\, \bar{c}_{4,\,L}^{\kappa,\,R}$$

$$c_f^\kappa = -\left[\frac{C_A^2}{4}\bar{c}^\kappa + \sum_{i\neq j} \frac{\langle \mathcal{M}|\boldsymbol{\mathcal{T}}_{iijj}|\mathcal{M}\rangle}{\langle\mathcal{M}|\mathcal{M}\rangle} \right]$$

$$\bar{c}^\kappa = c_s^\kappa + c_u^\kappa + c_t^\kappa = -(C_a + C_b + C_c)/2 \quad \longleftrightarrow \quad \bar{c}_4^{\kappa,\,R} = D_{aR} + D_{bR} + D_{cR}$$

$$\bar{c}_L^\kappa = c_s^\kappa L_s + c_u^\kappa L_u + c_t^\kappa L_t \qquad \bar{c}_{4,\,L}^{\kappa,\,R} \equiv c_{4,\,s}^{\kappa,\,R} L_s + c_{4,\,u}^{\kappa,\,R} L_u + c_{4,\,t}^{\kappa,\,R} L_t$$

$$c_s^\kappa = \boldsymbol{T}_a\cdot\boldsymbol{T}_b\,, \quad c_u^\kappa = \boldsymbol{T}_b\cdot\boldsymbol{T}_c\,, \quad c_t^\kappa = \boldsymbol{T}_a\cdot\boldsymbol{T}_c \quad \longleftrightarrow \quad c_{4,\,s}^{\kappa,\,R} = D_{aR} + D_{bR} - D_{cR}$$

$$c_{4,\,t}^{\kappa,\,R} = D_{aR} + D_{cR} - D_{bR}$$

$$c_{4,\,u}^{\kappa,\,R} = D_{bR} + D_{cR} - D_{aR}$$

Kinematic dependent logs

$$L_s = \ln\frac{-s_{ab} - \mathrm{i}0}{Q^2} = \ln\frac{s_{ab}}{Q^2} - \mathrm{i}\pi$$

$$L_u = \ln\frac{s_{bc}}{Q^2} \qquad L_t = \ln\frac{s_{ac}}{Q^2}$$

Beam, Jet and Soft RGEs

Beam and Jet functions in ***Laplace space:***

$$\mu \frac{\mathrm{d}}{\mathrm{d}\mu} \ln \tilde{B}_a(\varsigma_B, x, \mu) = -2\Big[C_a\, \Gamma_{\mathrm{cusp}}(\alpha_s) + 2 \sum_{R=F,A} D_{aR}\; g^R(\alpha_s)\Big] \ln\Big(\frac{Q_a \varsigma_B}{\mu^2}\Big) + \gamma_B^a(\alpha_s)$$

$$\mu \frac{\mathrm{d}}{\mathrm{d}\mu} \ln \tilde{J}_c(\varsigma_J, \mu) = -2\Big[C_c\, \Gamma_{\mathrm{cusp}}(\alpha_s) + 2 \sum_{R=F,A} D_{cR}\; g^R(\alpha_s)\Big] \ln\Big(\frac{Q_J \varsigma_J}{\mu^2}\Big) + \gamma_J^c(\alpha_s)$$

The soft functions depend on $\hat{s}_{ij} = \dfrac{2\, q_i \cdot q_j}{Q_i\, Q_j}$ which are frame dependent

$$\hat{s}_{aJ}^{\mathrm{LAB}} = \frac{n_a \cdot n_J}{2} = \rho_a \rho_J\, \hat{s}_{aJ}^{\mathrm{CS}}$$

→

Moderately sized $\hat{s}_{aJ}^{\mathrm{CS}}$ may require
to evaluate the LAB-frame soft function
at very small values of $\hat{s}_{aJ}^{\mathrm{LAB}}$ depending on
the boost factor $\rho_a \rho_J$

Soft functions in ***Laplace space:***

$$\mu \frac{\mathrm{d}}{\mathrm{d}\mu} \ln \tilde{S}^\kappa(\varsigma_S, \mu) = 2\left[-\bar{c}^\kappa \Gamma_{\mathrm{cusp}}(\alpha_s) + \sum_{R=F,A} \bar{c}_4^{\kappa,R} g^R(\alpha_s)\right] \ln\left(\frac{\varsigma_S^2}{\mu^2}\right)$$

$$+ \Big[\gamma^\kappa_{S_{N=1}}(\alpha_s) + 2\Gamma_{\mathrm{cusp}}(\alpha_s)\big(c_s^\kappa L_{ab} + c_t^\kappa L_{ac} + c_u^\kappa L_{bc}\big)$$

$$- 2 \sum_{R=F,A} g^R(\alpha_s)\big(c_{4,s}^{\kappa,R} L_{ab} + c_{4,t}^{\kappa,R} L_{bc} + c_{4,u}^{\kappa,R} L_{bc}\big)\Big]$$

N^{3}LL formula

Combine the solutions to the RG equations for the hard, soft, beam and jet functions to obtain

$$
\begin{aligned}
\frac{\mathrm{d}\sigma^{\mathrm{N^3LL}}}{\mathrm{d}\Phi_1\mathrm{d}\mathcal{T}_1} = \sum_{\kappa} \exp\Big\{ & 4(C_a+C_b)K_{\Gamma_{\mathrm{cusp}}}(\mu_B,\mu_H) + 4C_cK_{\Gamma_{\mathrm{cusp}}}(\mu_J,\mu_H) - 2(C_a+C_b+C_c)K_{\Gamma_{\mathrm{cusp}}}(\mu_S,\mu_H) \\
& - 2C_cL_J\,\eta_{\Gamma_{\mathrm{cusp}}}(\mu_J,\mu_H) - 2(C_aL_B+C_bL'_B)\eta_{\Gamma_{\mathrm{cusp}}}(\mu_B,\mu_H) + K_{\gamma_{\mathrm{tot}}} \\
& + \left[C_a\ln\left(\frac{Q_a^2u}{st}\right) + C_b\ln\left(\frac{Q_b^2t}{su}\right) + C_{\kappa_j}\ln\left(\frac{Q_J^2s}{tu}\right) + (C_a+C_b+C_c)L_S\right]\eta_{\Gamma_{\mathrm{cusp}}}(\mu_S,\mu_H) \\
& + \sum_{R=F,A}\Big[8\,(D_{aR}+D_{bR})K_{g^R}(\mu_B,\mu_H) + 8\,D_{cR}K_{g^R}(\mu_J,\mu_H) \\
& - 4\,(D_{aR}+D_{bR}+D_{cR})K_{g^R}(\mu_S,\mu_H) - 4D_{cR}L_J\eta_{g^R}(\mu_J,\mu_H) - 4\,(D_{aR}L_B+D_{bR}L'_B)\eta_{g^R}(\mu_B,\mu_H) \\
& + 2\left[D_{aR}\ln\left(\frac{Q_a^2u}{st}\right) + D_{bR}\ln\left(\frac{Q_b^2t}{su}\right) + D_{cR}\ln\left(\frac{Q_J^2s}{tu}\right) + (D_{aR}+D_{bR}+D_{cR})L_S\right]\eta_{g^R}(\mu_S,\mu_H)\Big]\Big\} \\
& \times H_\kappa(\Phi_1,\mu_H)\tilde{S}^{\kappa}(\partial_{\eta_S}+L_S,\mu_S)\tilde{B}_{\kappa_a}(\partial_{\eta_B}+L_B,x_a,\mu_B)\tilde{B}_{\kappa_b}(\partial_{\eta'_B}+L'_B,x_b,\mu_B)\,\tilde{J}_{\kappa_J}(\partial_{\eta_J}+L_J,\mu_J) \\
& \times \frac{Q^{-\eta_{\mathrm{tot}}}}{\mathcal{T}_1{}^{1-\eta_{\mathrm{tot}}}}\frac{\eta_{\mathrm{tot}}\ e^{-\gamma_E\eta_{\mathrm{tot}}}}{\Gamma(1+\eta_{\mathrm{tot}})}
\end{aligned}
$$

Up to NNLL′

where we defined $\eta_{\mathrm{tot}} = \eta_B + \eta'_B + \eta_J + 2\eta_S$

$$L_H = \ln\left(\frac{Q^2}{\mu_H^2}\right)\,,\quad L_B = \ln\left(\frac{Q_aQ}{\mu_B^2}\right)\,,\quad L'_B = \ln\left(\frac{Q_bQ}{\mu_B^2}\right)$$

$$L_J = \ln\left(\frac{Q_JQ}{\mu_J^2}\right)\,,\quad L_S = \ln\left(\frac{Q^2}{\mu_S^2}\right)$$

$$K_{g^R}(\mu_H,\mu) \equiv \int_{\alpha_s(\mu_H)}^{\alpha_s(\mu)}\frac{\mathrm{d}\alpha_s}{\beta(\alpha_s)}g^R(\alpha_s)\int_{\alpha_s(\mu_H)}^{\alpha_s}\frac{\mathrm{d}\alpha'_s}{\beta[\alpha'_s]}$$

$$\eta_{g^R}(\mu_H,\mu) \equiv \int_{\alpha_s(\mu_H)}^{\alpha_s(\mu)}\frac{\mathrm{d}\alpha_s}{\beta(\alpha_s)}g^R(\alpha_s)$$

$$K_f(\mu_H,\mu) \equiv \int_{\alpha_s(\mu_H)}^{\alpha_s(\mu)}\frac{\mathrm{d}\alpha_s}{\beta(\alpha_s)}f(\alpha_s)$$