MMNLP:

MATHEMATICAL METHODS OF NONLINEAR PHYSICS

Part I : General information on MMNLP

Part II : The Milano unit

I. PRESENTATION OF MMNLP (based on notes by R. Vitolo)

- The project started in Rome and was initially led by Francesco Calogero, now 'affiliato' INFN and Università di Roma La Sapienza. Some members were, among others, Degasperis, Levi, Ragnisco, Santini, ... The focus was (and is currently) on the **integrable systems (≈ explicitly solvable models in Mathematical Physics., typically evolving)**.
- Substantial participation by the Lecce group: Boiti, Konopelchenko, Leo, Martina, Pempinelli, Prinari, Soliani, Solombrino, ...
- Retirements were shrinking the number of project members to a dangerous level. R. Vitolo was appointed the new national coordinator. He started new units whose staff are mostly Mathematical Physicists of various extractions (with interests in nonlinear phenomena, including possible excursions beyond the paradigm of integrability).

I. DIGRESSION ON INTEGRABLE SYSTEMS

- Integrable systems have been known for a long time in classical mechanics. Earlier examples: the motion of a particle under a central force (e.g., the Kepler problem), several types of tops.
- Most integrable systems are Hamiltonian. So there is a manifold of states, or phase space $\mathcal{U}$. There is a Poisson tensor $\mathcal{P}$, meaning that

$$\mathcal{P} : \mathcal{U} \ni u \mapsto \mathcal{P}_u \,, \qquad \mathcal{P}_u : T^*_u\mathcal{U} \to T_u\mathcal{U} \,,$$

and that one gets a Lie product $\{\ ,\ \}$ on the functions $\mathcal{U} \to \mathbb{R}$, called Poisson bracket, setting

$$\{F, G\}(u) := \langle d_u F, \mathcal{P}_u d_u G \rangle \qquad \text{for } F, G : \mathcal{U} \to \mathbb{R} \,.$$

Finally, the state $u = u(t)$ evolves with time $t \in \mathbb{R}$ according to

$$\frac{du}{dt} = \mathcal{P}_u d_u H \,, \qquad H : \mathcal{U} \to \mathbb{R} \text{ the Hamiltonian function.}$$

I. DIGRESSION ON INTEGRABLE SYSTEMS

- **Liouville-Arnol'd theorem:** if $\dim \mathcal{U} = 2n$, $\mathcal{P}$ is nondegenerate and there exist n independent constants of motion $H_k : \mathcal{U} \to \mathbb{R}$ $(k = 1, ..., n)$ in mutual involution $(\{H_k, H_\ell\} = 0$ for $k, \ell = 1, ..., n)$, then the evolution equation can be solved explicitly (by quadratures) for *arbitrary* initial data. So, we have an integrable system.
- There are deep connections between the constants of motion and the (one-parameter groups of) symmetries of a Hamiltonian system (:= transformations $\mathcal{U} \to \mathcal{U}$ preserving all structures).
- Integrable systems are rare. About the 1960-1970s, there was an explosion of this research area, with the discoverey of many new integrable systems and of algebro-geometric techniques to generate or analyze them (Lax pairs, biHamiltonian structures, Lie algebra methods, r-matrix theory,...). This was the seed for MMNLP.

I. DIGRESSION ON INTEGRABLE SYSTEMS: EXAMPLES

- The Calogero-Moser (CM) system (1969-1975) consists of n particles on a line, interacting in pairs with an inversely quadratic potential.

- CM is a Hamiltonian system. The phase space $\mathcal{U}$ is the set of pairs $u = (x, p)$, $x = (x_1, ..., x_n)$ the positions, $p = (p_1, ..., p_n)$ the momenta $(x_i, p_i \in \mathbb{R}, x_i \neq x_j)$. Time evolution reads (^{T} := transposition)

$$\frac{d}{dt}\begin{pmatrix} x^T \\ p^T \end{pmatrix} = \begin{pmatrix} \mathbf{0}_n & \mathbf{1}_n \\ -\mathbf{1}_n & \mathbf{0}_n \end{pmatrix}\begin{pmatrix} \partial H/\partial x^T \\ \partial H/\partial p^T \end{pmatrix}, H(x,p) := \sum_i \frac{p_i^2}{2} - \sum_{i<j} \frac{1}{(x_i - x_j)^2};$$

the above matrix is the Poisson tensor, H is the Hamiltonian function.

- CM admits n independent constants of motion in involution

$$H_k(x,p) := \frac{1}{2}\mathrm{tr}(L^k(x,p)) \; (k = 1, ..., n), \quad L(x,p)_{ij} := \delta_{ij}p_i + \frac{1-\delta_{ij}}{x_i - x_j}$$

($L(x,p)$ is the so-called Lax matrix; $H = H_2$).

I. DIGRESSION ON INTEGRABLE SYSTEMS: EXAMPLES

- The Korteweg-de Vries (KdV) equation (1895) is the PDE

$$u_t = u_{xxx} + 6uu_x \qquad (u : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}, (x,t) \mapsto u(x,t))\,;$$

it describes shallow water waves in a rectilinear channel ($u(x,t) :=$ variation of the water surface height at position x and time t).

- KdV has the solitary wave solution (observed by Scott Russel in 1844)

$$u(x,t) := \frac{c}{2\cosh^2(\frac{\sqrt{c}}{2}(x+ct))} \qquad (c \in \mathbb{R}^+ \text{ arbitrary})\,.$$

- KdV is a Hamiltonian system with an infinite-dimensional phase space $\mathcal{U}$ of (regular, decaying at infinity) functions $u : \mathbb{R} \to \mathbb{R}$, $x \mapsto u(x)$. Indeed, with $\delta/\delta u$ the functional derivative, KdV reads

$$u_t = \partial_x \left(\frac{\delta H}{\delta u} \right), \qquad H : \mathcal{U} \to \mathbb{R}\,, u \mapsto H(u) := \int_{-\infty}^{+\infty} dx \left(u^3 - \frac{1}{2} u_x^2 \right)\,;$$

(H the Hamiltonian function; ∂_x represents the Poisson tensor).

I. DIGRESSION ON INTEGRABLE SYSTEMS: EXAMPLES

- KdV possesses infinitely many, independent constants of motion

$$H_k : \mathcal{U} \to \mathbb{R}\,, \quad H_k(u) := \frac{4^k}{2k+1}\mathrm{tr}(L^{k+1/2}(u)) \qquad (k = 0, 1, 2, 3, \ldots)\,,$$

$$L(u) := \partial_{xx} + u \quad \text{(Lax operator); tr the Adler trace}\,.$$

$H_0(u) = \frac{1}{2}\int dx\, u$, $H_1(u) = \frac{1}{2}\int dx\, u^2$, $H_2 = H$, $H_3(u) = \frac{1}{2}\int dx\,(5u^4 - 10uu_x^2 + u_{xx}^2), \ldots$. The H_ks are in involution with respect to the Poisson bracket $\{F, G\}(u) := \int dx\, \frac{\delta F}{\delta u}\, \partial_x \left(\frac{\delta G}{\delta u}\right)$.

An infinite-dimensional version of the Liouville-Arnol'd construction holds for this system, and the KdV equation with arbitrary initial data can be solved explicitly. (Results of the 1960-1970s by: Kruskal, Miura and Zabusky; Lax; Faddeev and Zakharov; Adler; Gelfand and Dikii...).

DIGRESSION: EXAMPLES OF INTEGRABLE SYSTEMS

- The integrability of KdV, CM and many other systems can be understood in terms of a second Hamiltonian formulation.

For example, in the KdV case, besides the first Hamiltonian formulation with Poisson tensor $\mathcal{P} = \partial_x$, there is a second formulation

$$u_t = \mathcal{P}' \, \frac{\delta H'}{\delta u} \, ,$$

$$\mathcal{P}' \equiv \mathcal{P}'_u := \partial_{xxx} + 4u\partial_x + 2u_x \, , \quad H'(u) := \frac{1}{2} \int dx \, u^2 \ (\text{i.e., } H' = H_1)$$

(Magri, 1970s. Magri is now in the Milano-Bicocca unit of MMNLP).
$\mathcal{P}, \mathcal{P}'$ are compatible (in a suitable sense), and characterize the constants of motion H_k via the recurrence relation

$$\mathcal{P}' \, \frac{\delta H_k}{\delta u} = \mathcal{P} \, \frac{\delta H_{k+1}}{\delta u} \qquad (k = 0, 1, 2, 3, \ldots).$$

- Note that KdV is a continuous system (both in space and time).

The CM system is semidiscrete (space-discrete, time-continuous).

I. THE LOCAL UNITS OF MMNLP

- **Lecce**: S. Abenda (50%, Sez. Bologna), G. Landolfi (50%), L. Martina (80%), G. Saccomandi (Sez. Perugia), R. Vitolo (national responsible), B. Konopelchenko (affiliate).
- **Milano**: G. Gaeta, G. Gubbiotti, L. Pizzocchero (local responsible, 80%), P. Vergallo (post-doc Messina).
- **Milano-Bicocca**: G. Falqui, F. Magri, P. Lorenzoni (local responsible), M. Pedroni, A. Raimondo, K. van Gemst (post-doc).
- **Roma**: S. Carillo, F. Coppini (PhD), A. De Sole (local responsible), P.M. Santini, D. Valeri, F. Zullo, F. Calogero (affiliate), M.V. Falessi (Ric. ENEA 50%), L. Casarin (PhD).
- **Torino**: M. Onorato (50%), G. Ortenzi (local responsible).
- **Trieste**: T. Grava (local responsible) D. Lewanski, D. Guzzetti, P. Rossi (Univ. Padova), G. Tondo, I.S. Jaztar Singh (PhD Padova), D. Rachenkov (PhD), Bing-Ying Liu (post-doc).

II. THE MILANO UNIT OF MMNLP

Giorgio Gubbiotti is RTD-B (and will soon be appointed Associate Professor) in Mathematical Physics at the Dept. of Mathematics of Milano. He works on many aspects of integrable systems, e.g.:

- time-discrete systems and their integrability: construction and analysis via group theory, coalgebra theory, geometrical methods [1,2];
- new notions of entropy for the above systems [2];
- time-continuous integrable Hamiltonian systems with homogenous Poisson structures: classification [3].

[1] G. Gubbiotti, D. Latini and B. K. Tapley, *Coalgebra symmetry for discrete systems*, J. Phys. A: Math. Theor. 56 (2023), 205205 (34 pp).

[2] M. Graffeo and G. Gubbiotti, *Growth and Integrability of Some Birational Maps in Dimension Three*, Ann. Henri Poincaré 25 (2024), 1733-1793.

[3] G. Gubbiotti, B. Van Geemen, P. Vergallo, *Line geometry of pairs of second-order Hamiltonian operators and quasi-linear systems*, arXiv:2403.09152v1 (2024).

II. THE MILANO UNIT OF MMNLP

Pierandrea Vergallo has a post-doc position at the University of Messina (and is associated to INFN at Milano). He also works on many aspects of (time-continuous) integrable systems, e.g.:

- application of continuum mechanics to model the onset of damage in aerospace structures [1];
- Hamiltonian formulation of the kinetic equations for dense soliton gases [2];
- integrable Hamiltonian systems with homogenous Poisson structures: classification (see Ref. [3] in the previous page).

[1] P. Vergallo, F. Nicassio, *S4: simple quasi-1D model for structural health monitoring of single lap joint software*, Eur. Phys. J. Plus (2023) 138, 1135 (14 pp).

[2] P. Vergallo, E.V. Ferantopov, *Hamiltonian aspects of the kinetic equation for soliton gas*, arXiv:2403.20162v1 [nlin.SI] (2024).

II. THE MILANO UNIT OF MMNLP

Giuseppe Gaeta is Full Professor of Mathematical Physics at the Dept. of Mathematics of Milano; he organized international and national conferences on symmetry, integrability and perturbation theory. His research interest include:

- the integrability of *stochastic* ODEs ($\simeq$ a Brownian motion superimposed to a deterministic evolution law) [1];
- mathematical models for biological evolution [2];
- solitary waves in models for DNA dynamics [3].

[1] G. Gaeta, *On the integration of Ito equations with a random or a W-symmetry*, J. Math. Phys. 64 (2023), 123504.

[2] G. Gaeta, *On some dynamical features of the complete Moran model for neutral evolution in the presence of mutations*, Open Commun. Nonlin. Math. Phys. 4 (2024), 22-43.

[3] G. Gaeta, L. Venier, *Solitary waves in twist-opening models of DNA dynamics*, Phys. Rev. E (3) 78(2008), no.1, 011901, 9 pp.

II. THE MILANO UNIT OF MMNLP

Livio Pizzocchero (local coordinator) is Associate Professor of Mathematical Physics at the Dept. of Mathematics of Milano. His research interest include:

- the exactly solvable cases in renormalization of vacuum states for a quantized scalar field (theory of the Casimir effect) [1];
- the exactly solvable Friedmann-Lemaître-Robertson-Walker cosmological models with matter and a scalar field [2];
- approximation methods in the dynamics of fluids and plasmas [3].

[1] D. Fermi, L. Pizzocchero, On the Casimir Effect with δ-Like Potentials, and a Recent Paper by K. Ziemian, Ann. Henri Poincaré 24 (2023), 2363-2400.

[2] D. Fermi, M. Gengo, and L. Pizzocchero, *Integrable scalar cosmologies with matter and curvature*, Nuclear Physics B 957 (2020), 115095 (102 pp).

[3] L. Pizzocchero, *On the global stability of smooth solutions of the Navier-Stokes equations*, Appl. Math. Letters 115 (2021), 106970 (11 pp).