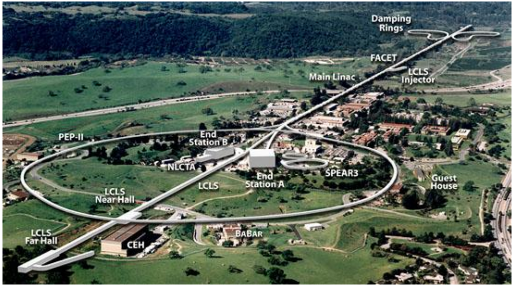
General theory of the bubble regime

Anton Golovanov,[1,2] Igor Kostyukov,[2] Alexander Pukhov,[3] Victor Malka[1]

[1]Weizmann Institute of Science, Rehovot, Israel
[2]Institute of Applied Physics RAS, Nizhny Novgorod, Russia
[3]Institut für Theoretische Physik I, Heinrich-Heine-Universität Düsseldorf, Germany

LPAW-2025
April 16, 2025

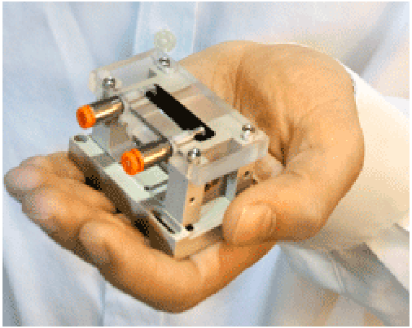
Stanford linear acceleration center (SLAC)



50 GeV at 3 km

Acceleration gradient ~100 MeV/m

Plasma accelerators

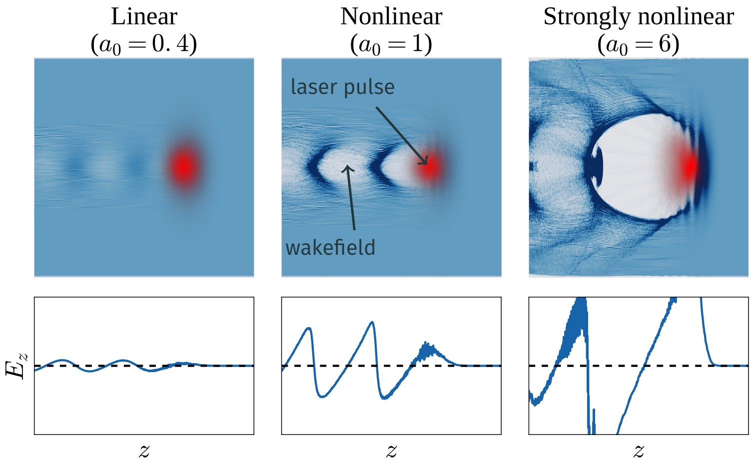


10 GeV at 30 cm

Acceleration gradient ~50 GeV/m

Picksley et al. *PRL* **133**, 255001 (2024);
Josh Stackhouse and **Ela Rockafellow**'s talks

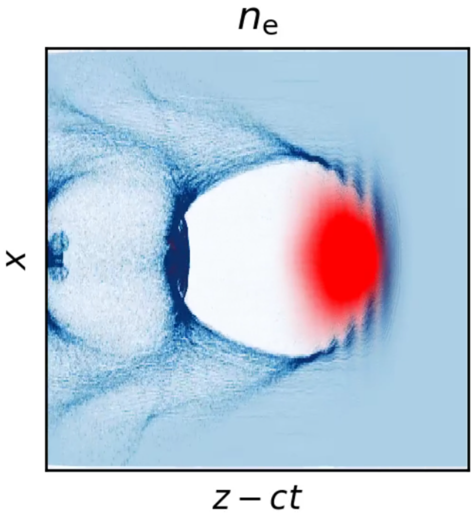
Wakefield in plasma



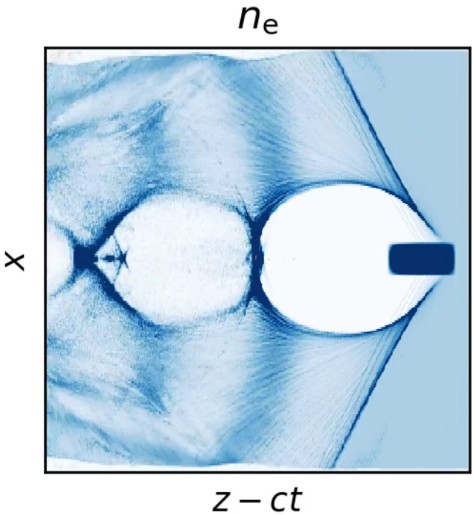
$a_0 = \frac{eE_L}{mc\omega_L}$ is the dimensionless laser amplitude

Strongly nonlinear (“bubble” or “blowout”) regime of wakefield

Laser driver ($a_0 \gg 1$)



Electron driver ($n_B \gg n_p$)



Outline

- Previous models of the bubble

Outline

- Previous models of the bubble
- Energy conservation in a plasma wake

Outline

- Previous models of the bubble
- Energy conservation in a plasma wake
- Energy conserving model of a bubble and a beam-driven case
  A. Golovanov et al. *PRL* **130**, 105001 (2023)

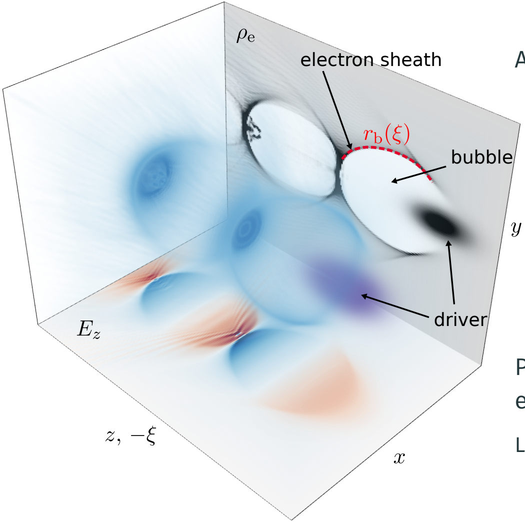
Outline

- Previous models of the bubble
- Energy conservation in a plasma wake
- Energy conserving model of a bubble and a beam-driven case

  A. Golovanov et al. *PRL* **130**, 105001 (2023)
- Laser-driven bubble

Previous models

Phenomenological models of plasma wakefield



Assumptions

- Axial symmetry, coordinates (r, z)
- Quasi-static approximation:
  $f(t, z, r) = f(\xi, r), \quad \xi = t - z.$
- The plasma bubble has a border $r_b(\xi)$. For $r < r_b$, there are no plasma electrons; $r \geq r_b$ lies the electron sheath
- Ions are immobile

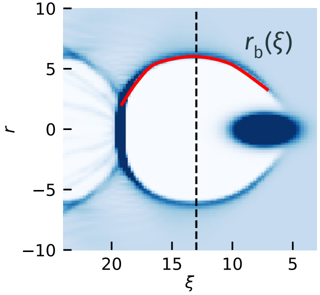
Plasma units: $t \to \omega_p t$, $\mathbf{r} \to k_p \mathbf{r}$, $\mathbf{E} \to e\mathbf{E}/mc\omega_p$, etc.

Lu et al. *Phys. Plasmas* **13**, 056709 (2006)

Plasma boundary equation

The boundary of the bubble is described by the equation

$$A(r_b)\frac{d^2 r_b}{d\xi^2} + B(r_b)\left(\frac{dr_b}{d\xi}\right)^2 + C(r_b) = \lambda(\xi, r_b),$$



Lu et al. *Phys. Plasmas* **13**, 056709 (2006); Golovanov et al. *Quantum electron.* **46**, 295 (2016).

Plasma boundary equation

The boundary of the bubble is described by the equation

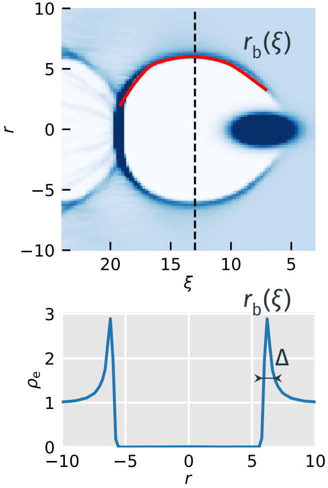
$$A(r_b)\frac{d^2 r_b}{d\xi^2} + B(r_b)\left(\frac{dr_b}{d\xi}\right)^2 + C(r_b) = \lambda(\xi, r_b),$$

where

$$A(r_b) = r_b\left(1 + \frac{r_b^2}{4} + \frac{3r_b\Delta}{4}\right)$$

$$B(r_b) = \frac{r_b^2}{2}\left(1 + \frac{\Delta}{r_b}\right)$$

$$C(r_b) = \frac{r_b^2}{4}\,\frac{1 + (1 + r_b\Delta/2)^2}{(1 + r_b\Delta/2)^2}$$



Lu et al. *Phys. Plasmas* **13**, 056709 (2006); Golovanov et al. *Quantum electron.* **46**, 295 (2016).

Plasma boundary equation

The boundary of the bubble is described by the equation

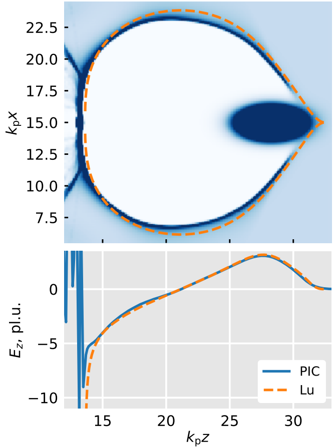
$$A(r_b)\frac{d^2 r_b}{d\xi^2} + B(r_b)\left(\frac{dr_b}{d\xi}\right)^2 + C(r_b) = \lambda(\xi, r_b),$$

where

$$A(r_b) = r_b\left(1 + \frac{r_b^2}{4} + \frac{3r_b\Delta}{4}\right)$$

$$B(r_b) = \frac{r_b^2}{2}\left(1 + \frac{\Delta}{r_b}\right)$$

$$C(r_b) = \frac{r_b^2}{4}\frac{1 + (1 + r_b\Delta/2)^2}{(1 + r_b\Delta/2)^2}$$



Lu et al. *Phys. Plasmas* **13**, 056709 (2006); Golovanov et al. *Quantum electron.* **46**, 295 (2016).

Energy conservation

Energy conservation in a plasma wakefield

EM field (wakefield)

$$\frac{\partial W_{\text{EM}}}{\partial t} + \nabla \cdot \mathbf{S}_{\text{EM}} = -\mathbf{j} \cdot \mathbf{E}$$

$W_{\text{EM}} = \frac{\mathbf{E}^2 + \mathbf{B}^2}{2}$ EM energy density

$\mathbf{S}_{\text{EM}} = \mathbf{E} \times \mathbf{B}$ Poynting vector

Energy conservation in a plasma wakefield

EM field (wakefield)

$$\frac{\partial W_{\mathrm{EM}}}{\partial t} + \nabla \cdot \mathbf{S}_{\mathrm{EM}} = -\mathbf{j} \cdot \mathbf{E}$$

$W_{\mathrm{EM}} = \dfrac{\mathbf{E}^2 + \mathbf{B}^2}{2}$ EM energy density

$\mathbf{S}_{\mathrm{EM}} = \mathbf{E} \times \mathbf{B}$ Poynting vector

Plasma electrons

$$\frac{\partial W_{\mathrm{e}}}{\partial t} + \nabla \cdot \mathbf{S}_{\mathrm{e}} = \mathbf{j}_{\mathrm{e}} \cdot \mathbf{E} + \frac{1}{2} \frac{\partial \langle \mathbf{a}^2 \rangle}{\partial t} n_{\mathrm{e}} \overline{\gamma^{-1}}$$

$W_{\mathrm{e}} = n_{\mathrm{e}} \overline{(\gamma - 1)}$ electron energy density

$\mathbf{S}_{\mathrm{e}} = n_{\mathrm{e}} \overline{\mathbf{v}(\gamma - 1)}$ energy density current

$\langle \mathbf{a}^2 \rangle$ ponderomotive potential

Energy conservation in a plasma wakefield

EM field (wakefield)

$$\frac{\partial W_{\mathrm{EM}}}{\partial t} + \nabla \cdot \mathbf{S}_{\mathrm{EM}} = -\mathbf{j} \cdot \mathbf{E}$$

$W_{\mathrm{EM}} = \dfrac{\mathbf{E}^2 + \mathbf{B}^2}{2}$ EM energy density

$\mathbf{S}_{\mathrm{EM}} = \mathbf{E} \times \mathbf{B}$ Poynting vector

Plasma electrons

$$\frac{\partial W_{\mathrm{e}}}{\partial t} + \nabla \cdot \mathbf{S}_{\mathrm{e}} = \mathbf{j}_{\mathrm{e}} \cdot \mathbf{E} + \frac{1}{2} \frac{\partial \langle \mathbf{a}^2 \rangle}{\partial t} n_{\mathrm{e}} \overline{\gamma^{-1}}$$

$W_{\mathrm{e}} = n_{\mathrm{e}} \overline{(\gamma - 1)}$ electron energy density

$\mathbf{S}_{\mathrm{e}} = n_{\mathrm{e}} \overline{\mathbf{v}(\gamma - 1)}$ energy density current

$\langle \mathbf{a}^2 \rangle$ ponderomotive potential

Total wake energy

$$\frac{\partial W}{\partial t} + \nabla \cdot \mathbf{S} = -\mathbf{j}_{\mathrm{B}} \cdot \mathbf{E} + \frac{1}{2} \frac{\partial \langle \mathbf{a}^2 \rangle}{\partial t} n_{\mathrm{e}} \overline{\gamma^{-1}}$$

$$W = W_{\mathrm{EM}} + W_{\mathrm{e}} \quad \mathbf{S} = \mathbf{S}_{\mathrm{EM}} + \mathbf{S}_{\mathrm{e}}$$

Currents $\mathbf{j} = \mathbf{j}_{\mathrm{e}} + \mathbf{j}_{\mathrm{B}}$ belong to plasma electrons ($\mathbf{j}_{\mathrm{e}}$) or to external bunches ($\mathbf{j}_{\mathrm{B}}$).

Quasistatic approximation

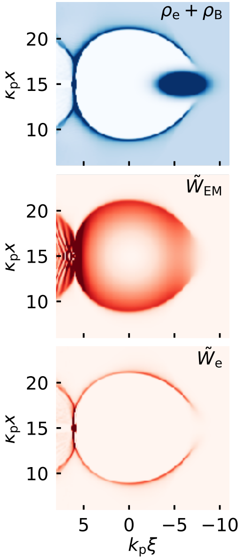
Total wake energy

$$\frac{\partial W}{\partial t} + \nabla \cdot \mathbf{S} = -\mathbf{j}_B \cdot \mathbf{E} + \frac{1}{2}\frac{\partial \langle \mathbf{a}^2 \rangle}{\partial t} n_e \overline{\gamma^{-1}}$$

Quasistatic approximation ($\xi = t - z$)

$$\frac{\partial \tilde{W}}{\partial \xi} + \nabla_\perp \cdot \mathbf{S}_\perp = -\rho_B E_z + \frac{1}{2}\frac{\partial \langle \mathbf{a}^2 \rangle}{\partial \xi} n_e \overline{\gamma^{-1}}$$

$\tilde{W} = W - S_z$ is the quasi-energy density. Ultrarelativistic bunches: $\mathbf{j}_B \approx \rho_B \mathbf{z}_0$



Total wake energy

$$\frac{\partial W}{\partial t} + \nabla \cdot \mathbf{S} = -\mathbf{j}_B \cdot \mathbf{E} + \frac{1}{2}\frac{\partial \langle \mathbf{a}^2 \rangle}{\partial t} n_e \overline{\gamma^{-1}}$$

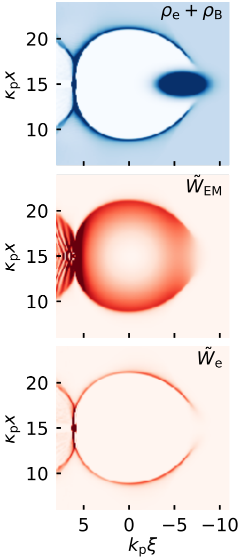
Quasistatic approximation ($\xi = t - z$)

$$\frac{\partial \tilde{W}}{\partial \xi} + \nabla_\perp \cdot \mathbf{S}_\perp = -\rho_B E_z + \frac{1}{2}\frac{\partial \langle \mathbf{a}^2 \rangle}{\partial \xi} n_e \overline{\gamma^{-1}}$$

$\tilde{W} = W - S_z$ is the quasi-energy density. Ultrarelativistic bunches: $\mathbf{j}_B \approx \rho_B \mathbf{z}_0$

$$\tilde{W}_{EM} = \frac{1}{2}\left[(\nabla\psi_w)^2 + B_z^2\right] \quad \tilde{W}_e = n_e\overline{(\gamma - 1)(1 - v_z)}$$

$\psi_w = \varphi - A_z$ is the wakefield potential.



Total wake energy

$$\frac{\partial W}{\partial t} + \nabla \cdot \mathbf{S} = -\mathbf{j}_{\mathrm{B}} \cdot \mathbf{E} + \frac{1}{2}\frac{\partial \langle \mathbf{a}^2 \rangle}{\partial t} n_{\mathrm{e}} \overline{\gamma^{-1}}$$

Quasistatic approximation ($\xi = t - z$)

$$\frac{\partial \tilde{W}}{\partial \xi} + \nabla_{\perp} \cdot \mathbf{S}_{\perp} = -\rho_{\mathrm{B}} E_z + \frac{1}{2}\frac{\partial \langle \mathbf{a}^2 \rangle}{\partial \xi} n_{\mathrm{e}} \overline{\gamma^{-1}}$$

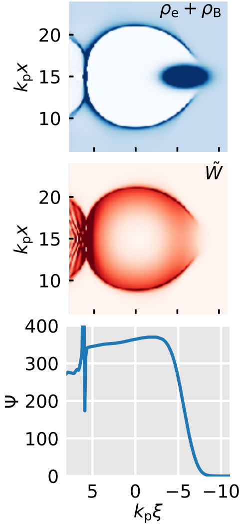
$\tilde{W} = W - S_z$ is the quasi-energy density.

Integrating over the transverse plane

$$\frac{d\Psi}{d\xi} = -\int \rho_{\mathrm{B}} E_z \, d^2\mathbf{r}_{\perp} + \frac{1}{2}\int \frac{\partial \langle \mathbf{a}^2 \rangle}{\partial \xi} n_{\mathrm{e}} \overline{\gamma^{-1}} \, d^2\mathbf{r}_{\perp}$$

$$\Psi(\xi) = \int \tilde{W} \, d^2\mathbf{r}_{\perp} \quad \text{(slice quasi-energy)}$$

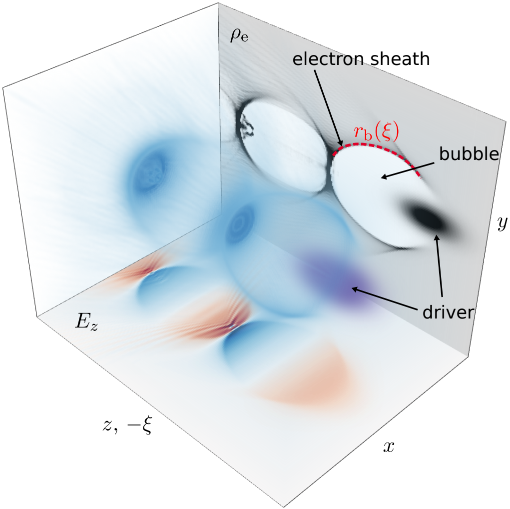
Behind the driver, no witness bunches: $\Psi(\xi) = \text{const}$.



Lotov, *Phys. Rev. E* **69**, 046405 (2004)

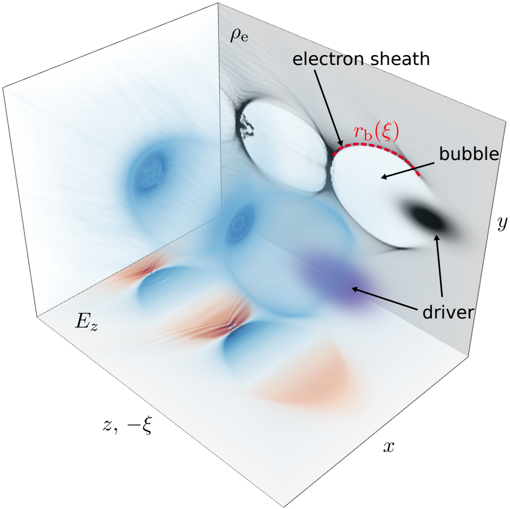
Model of the bubble

Model of the bubble



- Axial symmetry (r, z), only E_z, E_r, B_ϕ components.
- The bubble has a boundary, $r_b(\xi)$.
- Inside the boundary, no plasma electrons.
- The electron sheath on the boundary is infinitely thin.

Model of the bubble



- Axial symmetry (r, z), only E_z, E_r, B_ϕ components.
- The bubble has a boundary, $r_b(\xi)$.
- Inside the boundary, no plasma electrons.
- The electron sheath on the boundary is infinitely thin.

We need to have a model for

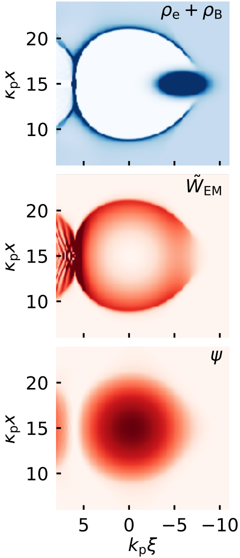
$$\Psi = \Psi_{EM} + \Psi_e$$

EM slice quasi-energy

$$\tilde{W}_{EM} = \frac{(\nabla \psi_w)^2}{2} \qquad \Psi_{EM}(\xi) = \pi \int_0^{r_b(\xi)} (\nabla \psi_w)^2 r \, dr$$

The wakefield potential inside the bubble is

$$\psi_w(\xi, r) = \frac{r_b^2(\xi) - r^2}{4}$$



EM slice quasi-energy

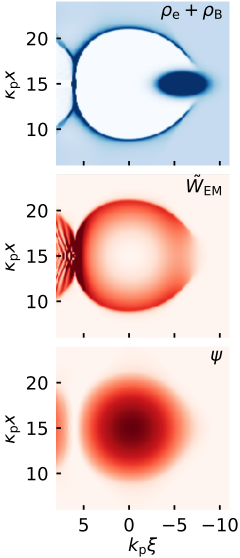
$$\tilde{W}_{\mathrm{EM}} = \frac{(\nabla\psi_{\mathrm{w}})^2}{2} \qquad \Psi_{\mathrm{EM}}(\xi) = \pi\int_0^{r_{\mathrm{b}}(\xi)} (\nabla\psi_{\mathrm{w}})^2 r\,\mathrm{d}r$$

The wakefield potential inside the bubble is

$$\psi_{\mathrm{w}}(\xi, r) = \frac{r_{\mathrm{b}}^2(\xi) - r^2}{4}$$

Then

$$\Psi_{\mathrm{EM}}(\xi) = \frac{\pi r_{\mathrm{b}}^4}{16}\left[1 + 2\left(\frac{\mathrm{d}r_{\mathrm{b}}}{\mathrm{d}\xi}\right)^2\right]$$



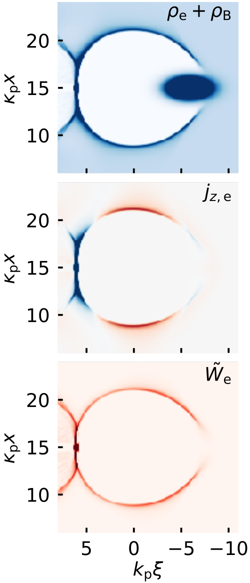
Electron slice quasi-energy

We define the electron sheath as

$$j_{z,\mathrm{e}} = j_0(\xi) r_\mathrm{b} \delta(r - r_\mathrm{b})$$

and arrive at

$$\Psi_\mathrm{e}(\xi) = \frac{\pi}{2} r_\mathrm{b}^2 \left(\frac{\mathrm{d}r_\mathrm{b}}{\mathrm{d}\xi} \right)^2$$



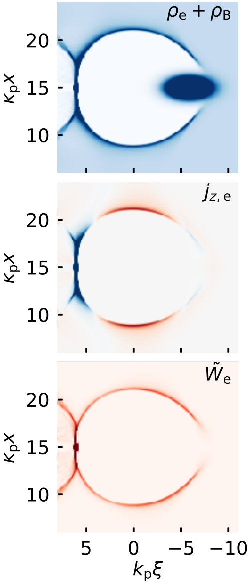
Electron slice quasi-energy

We define the electron sheath as

$$j_{z,e} = j_0(\xi) r_b \delta(r - r_b)$$

and arrive at

$$\Psi_e(\xi) = \frac{\pi}{2} r_b^2 \left(\frac{dr_b}{d\xi} \right)^2$$



Equation for the bubble boundary

Finally, the total quasienergy is

$$\Psi = \frac{\pi r_b^2}{16}\left[r_b^2 + (2r_b^2 + 8)\left(\frac{dr_b}{d\xi}\right)^2\right],$$

Red terms — EM energy, blue — plasma electrons energy.

Equation for the bubble boundary

Finally, the total quasienergy is

$$\Psi = \frac{\pi r_b^2}{16}\left[r_b^2 + (2r_b^2 + 8)\left(\frac{dr_b}{d\xi}\right)^2\right], \qquad \frac{d\Psi}{d\xi} = -\pi r_b \frac{dr_b}{d\xi}\int_0^{r_b} \rho_B r\, dr$$

Red terms — EM energy, blue — plasma electrons energy.

Equation for the bubble boundary

Finally, the total quasienergy is

$$\Psi = \frac{\pi r_b^2}{16}\left[r_b^2 + (2r_b^2 + 8)\left(\frac{dr_b}{d\xi}\right)^2\right], \qquad \frac{d\Psi}{d\xi} = -\pi r_b \frac{dr_b}{d\xi}\int_0^{r_b} \rho_B r\, dr$$

And we get the equation

$$\left(\frac{r_b^3}{4} + r_b\right)\frac{d^2 r_b}{d\xi^2} + \left(\frac{r_b^2}{2} + 1\right)\left(\frac{dr_b}{d\xi}\right)^2 + \frac{r_b^2}{2} = \lambda(\xi, r_b), \quad \lambda(\xi, r_b) = \int_0^{r_b} \rho_B r\, dr$$

Red terms — EM energy, blue — plasma electrons energy.

Bubble equation = energy conservation

Comparison of equations

$$A(r_\mathrm{b})\frac{\mathrm{d}^2 r_\mathrm{b}}{\mathrm{d}\xi^2} + B(r_\mathrm{b})\left(\frac{\mathrm{d}r_\mathrm{b}}{\mathrm{d}\xi}\right)^2 + C(r_\mathrm{b}) = \lambda(\xi, r_\mathrm{b}), \quad \lambda(\xi, r_\mathrm{b}) = \int_0^{r_\mathrm{b}} \rho_\mathrm{B} r\,\mathrm{d}r$$

Energy conservation

$$A = \frac{r_\mathrm{b}^3}{4} + r_\mathrm{b} \qquad B = \frac{r_\mathrm{b}^2}{2} + 1$$

$$C = \frac{r_\mathrm{b}^2}{4} \qquad E_z = \frac{r_\mathrm{b}}{2}\frac{\mathrm{d}r_\mathrm{b}}{\mathrm{d}\xi}$$

Lu's model

$$A = \frac{r_\mathrm{b}^3}{4} + r_\mathrm{b} + \frac{3}{4}r_\mathrm{b}^2\Delta, \qquad B = \frac{r_\mathrm{b}^2}{2} + \frac{r_\mathrm{b}\Delta}{2},$$

$$C = \frac{r_\mathrm{b}^2}{4}\left[1 + \left(1 + \frac{r_\mathrm{b}\Delta}{2}\right)^{-2}\right], \qquad E_z = \frac{r_\mathrm{b} + \Delta}{2}\frac{\mathrm{d}r_\mathrm{b}}{\mathrm{d}\xi}.$$

Does not correspond to energy conservation

Comparison of equations

$$A(r_b)\frac{d^2 r_b}{d\xi^2} + B(r_b)\left(\frac{dr_b}{d\xi}\right)^2 + C(r_b) = \lambda(\xi, r_b), \quad \lambda(\xi, r_b) = \int_0^{r_b} \rho_B r \, dr$$

Energy conservation

$$A = \frac{r_b^3}{4} + r_b \qquad B = \frac{r_b^2}{2} + 1$$

$$C = \frac{r_b^2}{4} \qquad E_z = \frac{r_b}{2}\frac{dr_b}{d\xi}$$

Lu's model

$$A = \frac{r_b^3}{4} + r_b + \frac{3}{4}r_b^2\Delta, \qquad B = \frac{r_b^2}{2} + \frac{r_b\Delta}{2},$$

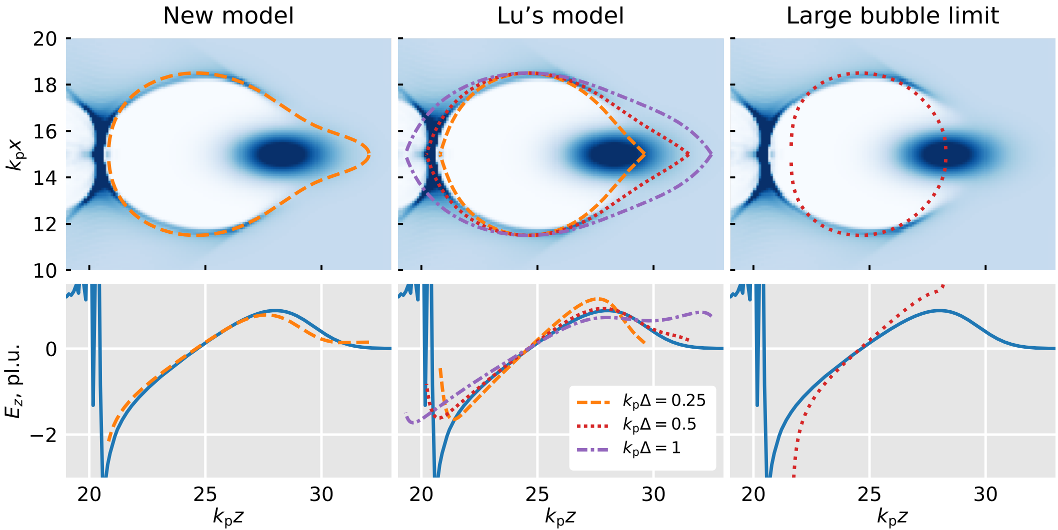
$$C = \frac{r_b^2}{4}\left[1 + \left(1 + \frac{r_b\Delta}{2}\right)^{-2}\right], \qquad E_z = \frac{r_b + \Delta}{2}\frac{dr_b}{d\xi}.$$

Does not correspond to energy conservation

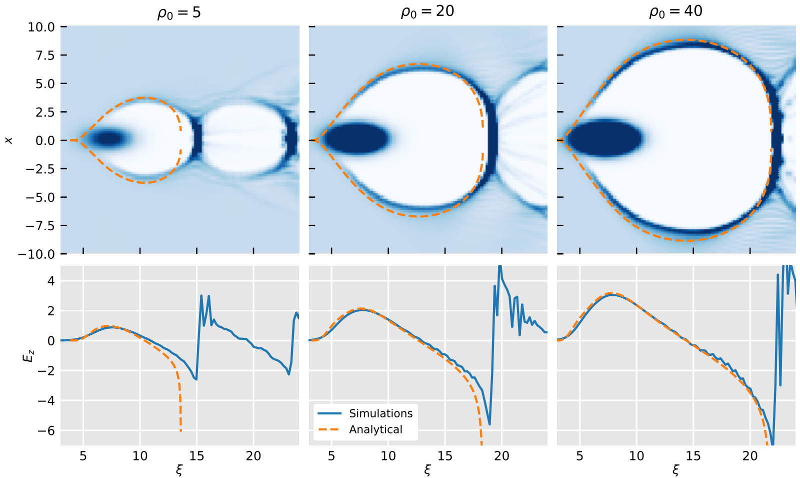
For large bubbles $r_b \gg 1$, we get the same equation,

$$r_b\frac{d^2 r_b}{d\xi^2} + 2\left(\frac{dr_b}{d\xi}\right)^2 + 1 = \frac{4\lambda}{r_b^2}$$

Comparison of models



Bubble excitation



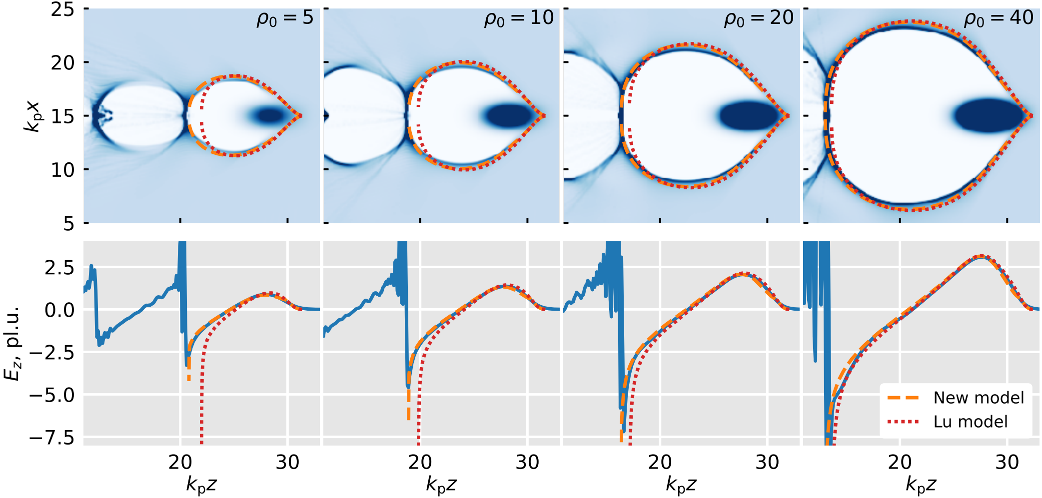
A self-consistent model of excitation of the bubble by an electron bunch based on Lu's model in the limit $r_b \gg 1$.

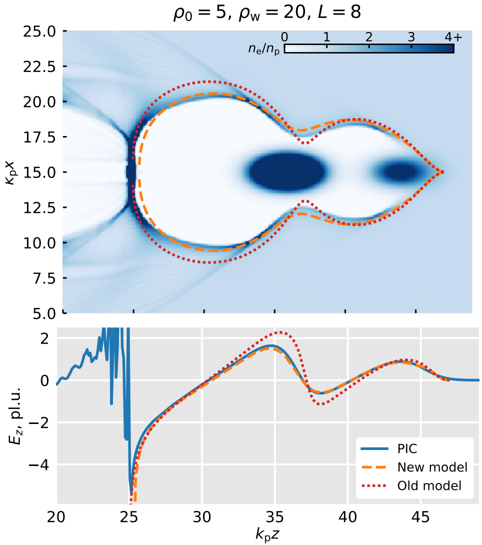
Golovanov et al. *PPCF* **63**, 085004 (2021).

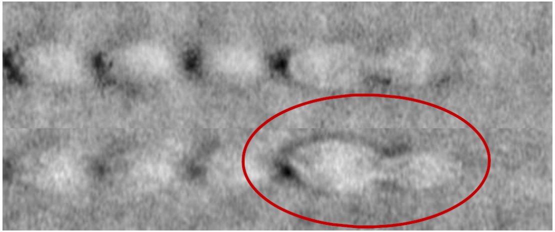
Bubble excitation

Generalization: for $r_b < 1$ old solution, for $r_b > 1$ the new (energy-conserving) one.

A. Golovanov et al. *PRL* **130**, 105001 (2023)







PWFA beam loading shadowgraphy, see talks by **Stefan Karsch** and **Moritz Foerster** on Monday.

Laser driver

Bubble equation with the laser term

The energy conservation law with a laser driver (electron bunches are neglected)

$$\frac{d\Psi}{d\xi} = \pi \int \frac{\partial \langle \mathbf{a}^2 \rangle}{\partial \xi} n_e \overline{\gamma^{-1}} r \, dr \qquad \Psi = \Psi_{EM} + \Psi_e$$

Bubble equation with the laser term

The energy conservation law with a laser driver (electron bunches are neglected)

$$\frac{d\Psi}{d\xi} = \pi \int \frac{\partial \langle \mathbf{a}^2 \rangle}{\partial \xi} n_e \overline{\gamma^{-1}} r \, dr \qquad \Psi = \Psi_{EM} + \Psi_e$$

Under the delta-layer approximation $n_e \propto \delta(r - r_b(\xi))$

$$\frac{d\Psi}{d\xi} = \frac{\pi r_b^2}{2} \frac{\partial \langle \mathbf{a}^2 \rangle}{\partial \xi},$$

Bubble equation with the laser term

The energy conservation law with a laser driver (electron bunches are neglected)

$$\frac{d\Psi}{d\xi} = \pi \int \frac{\partial \langle \mathbf{a}^2 \rangle}{\partial \xi} n_e \overline{\gamma^{-1}} r \, dr \qquad \Psi = \Psi_{EM} + \Psi_e$$

Under the delta-layer approximation $n_e \propto \delta(r - r_b(\xi))$

$$\frac{d\Psi}{d\xi} = \frac{\pi r_b^2}{2} \frac{\partial \langle \mathbf{a}^2 \rangle}{\partial \xi}, \quad \Psi_{EM} = \frac{\pi r_b^2}{16} \left[r_b^2 + 2 r_b^2 \left(\frac{dr_b}{d\xi} \right)^2 \right], \quad \Psi_e = \frac{\pi r_b^2}{2} \left[\left(\frac{dr_b}{d\xi} \right)^2 + \langle \mathbf{a}^2 \rangle (\xi, r_b) \right]$$

Bubble equation with the laser term

The energy conservation law with a laser driver (electron bunches are neglected)

$$\frac{d\Psi}{d\xi} = \pi \int \frac{\partial \langle \mathbf{a}^2 \rangle}{\partial \xi} n_e \overline{\gamma^{-1}} r \, dr \qquad \Psi = \Psi_{EM} + \Psi_e$$

Under the delta-layer approximation $n_e \propto \delta(r - r_b(\xi))$

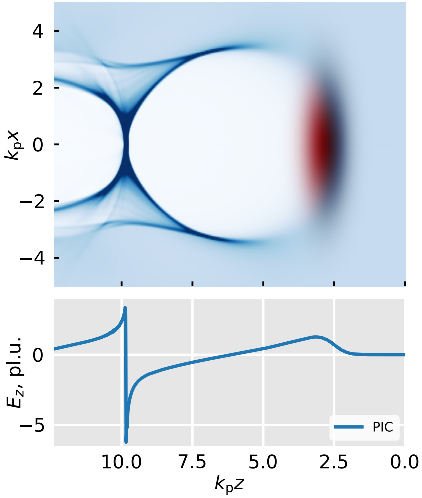
$$\frac{d\Psi}{d\xi} = \frac{\pi r_b^2}{2} \frac{\partial \langle \mathbf{a}^2 \rangle}{\partial \xi}, \quad \Psi_{EM} = \frac{\pi r_b^2}{16} \left[r_b^2 + 2 r_b^2 \left(\frac{dr_b}{d\xi} \right)^2 \right], \quad \Psi_e = \frac{\pi r_b^2}{2} \left[\left(\frac{dr_b}{d\xi} \right)^2 + \langle \mathbf{a}^2 \rangle (\xi, r_b) \right]$$

After substitution we get the equation

$$A(r_b) \frac{d^2 r_b}{d\xi^2} + B(r_b) \left(\frac{dr_b}{d\xi} \right)^2 + C(r_b) = -\frac{r_b}{2} \frac{\partial \langle \mathbf{a}^2 \rangle}{\partial r} (\xi, r_b) - \langle \mathbf{a}^2 \rangle (\xi, r_b)$$

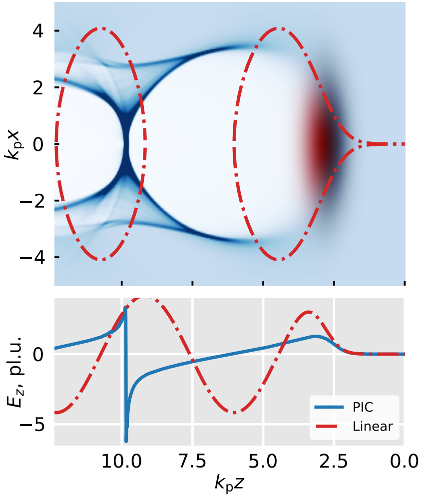
Laser driver: quasilinear solution

Quasilinear solution at the front gives us $\psi_w(\xi,0)$, $E_z(\xi,0)$.
The corresponding "bubble size" is $r_b = 2\sqrt{\psi_w(\xi,0)}$.



Laser driver: quasilinear solution

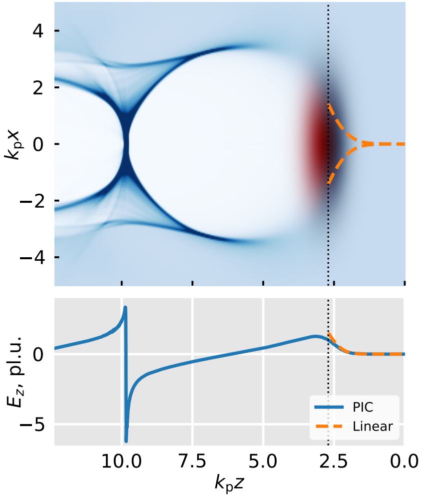
Quasilinear solution at the front gives us $\psi_{\mathrm{w}}(\xi, 0)$, $E_z(\xi, 0)$.
The corresponding "bubble size" is $r_{\mathrm{b}} = 2\sqrt{\psi_{\mathrm{w}}(\xi, 0)}$.



Laser driver: quasilinear solution

Quasilinear solution at the front gives us $\psi_\mathrm{w}(\xi, 0)$, $E_z(\xi, 0)$.
The corresponding "bubble size" is $r_\mathrm{b} = 2\sqrt{\psi_\mathrm{w}(\xi, 0)}$.

We demand continuity of $\psi_\mathrm{w}(\xi, 0)$ and $E_z(\xi, 0)$.



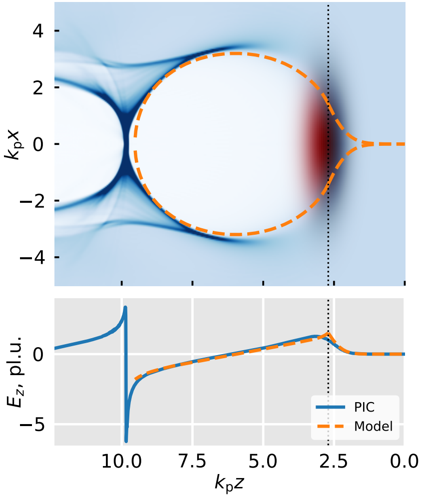
Laser driver: quasilinear solution

Quasilinear solution at the front gives us $\psi_\mathrm{w}(\xi, 0)$, $E_z(\xi, 0)$.
The corresponding "bubble size" is $r_\mathrm{b} = 2\sqrt{\psi_\mathrm{w}(\xi, 0)}$.

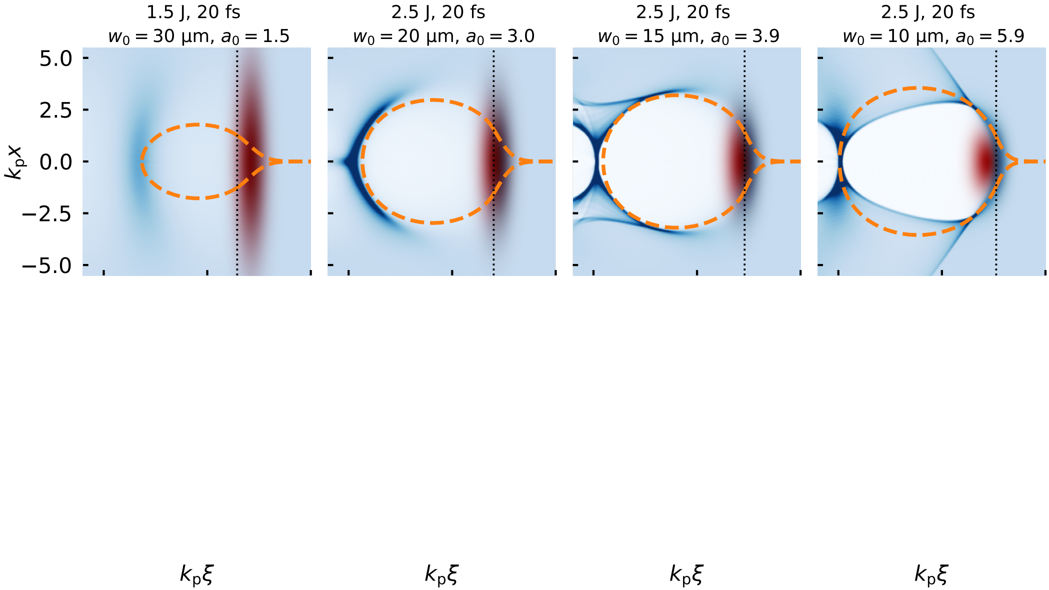
We demand continuity of $\psi_\mathrm{w}(\xi, 0)$ and $E_z(\xi, 0)$.

We arrive at the the initial conditions for the equation

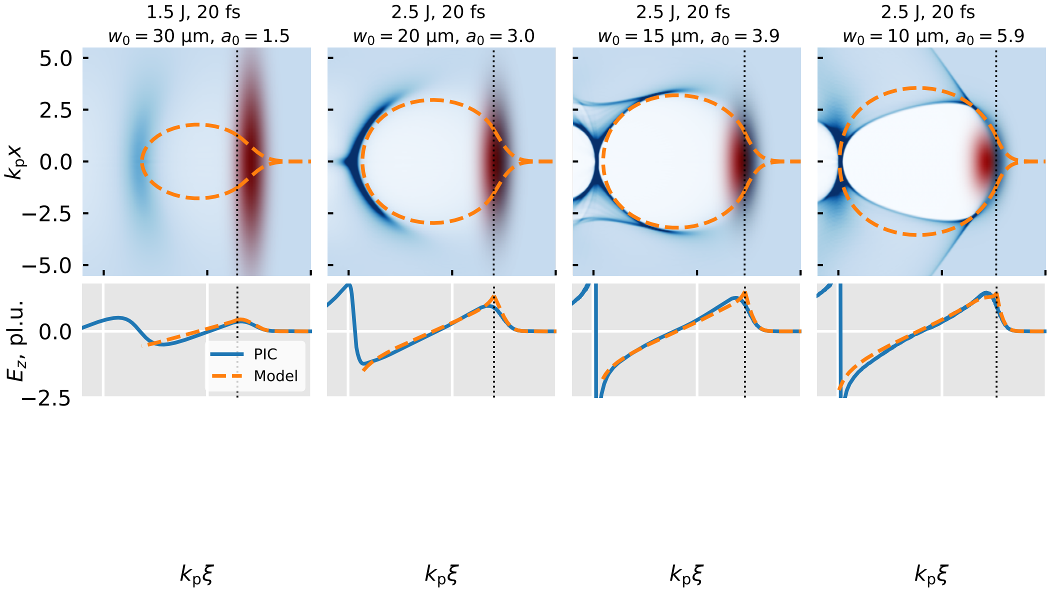
$$r_\mathrm{b}(\xi_0) = 2\sqrt{\psi_\mathrm{w}(\xi_0, 0)}$$
$$r'_\mathrm{b}(\xi_0) = \frac{E_z(\xi_0, 0)}{\sqrt{\psi_\mathrm{w}(\xi_0, 0)}}$$



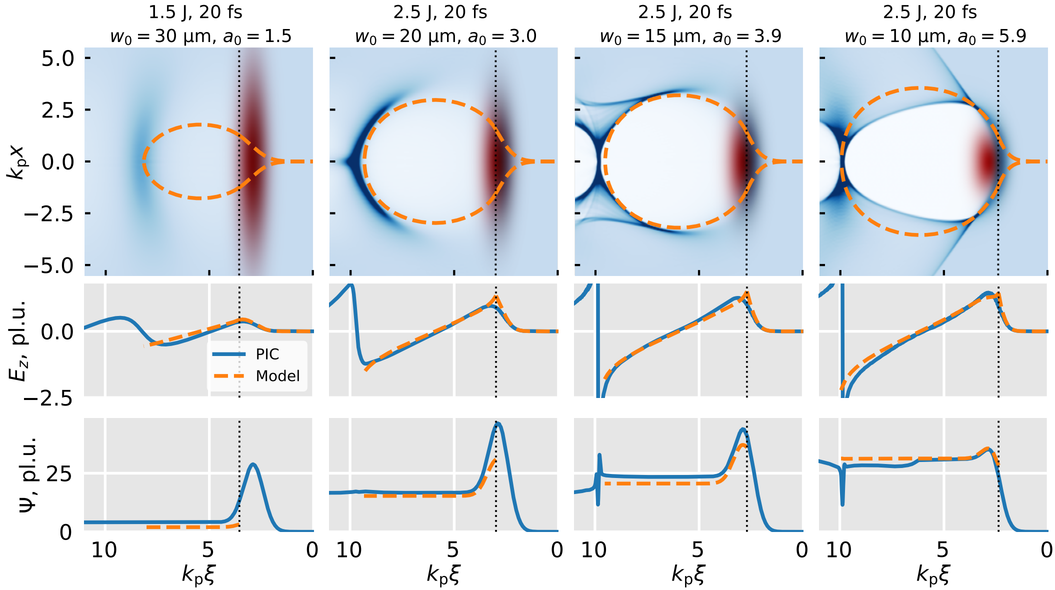
Equation result



Equation result



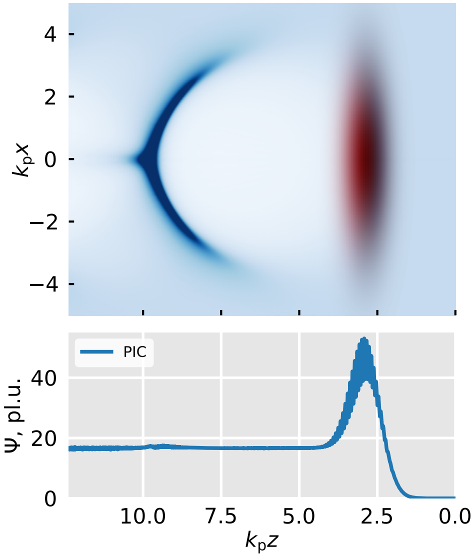
Equation result



For a general laser-driven wake,

$$\frac{d\Psi}{d\xi} = \frac{1}{2}\int \frac{\partial\langle \mathbf{a}^2\rangle}{\partial\xi} n_e \overline{\gamma^{-1}}\, d^2\mathbf{r}_\perp$$

If we know the RHS, we can calculate Ψ.



Bubble excitation by a laser

For a general laser-driven wake,

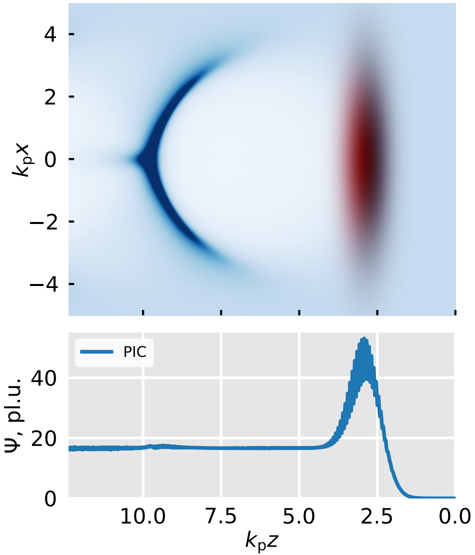
$$\frac{d\Psi}{d\xi} = \frac{1}{2}\int \frac{\partial\langle \mathbf{a}^2\rangle}{\partial\xi} n_e \overline{\gamma^{-1}}\, d^2\mathbf{r}_\perp$$

If we know the RHS, we can calculate Ψ.

We try to rely on the linear solution, but not with full linearization

$$\overline{\gamma^{-1}} = \left(1 + \langle \mathbf{a}^2\rangle(\xi, r)\right)^{-1/2}$$

$$n_e(\xi, r) = \max\left[n_{e,\mathrm{lin}}(\xi, r), 0\right]$$



Bubble excitation by a laser

For a general laser-driven wake,

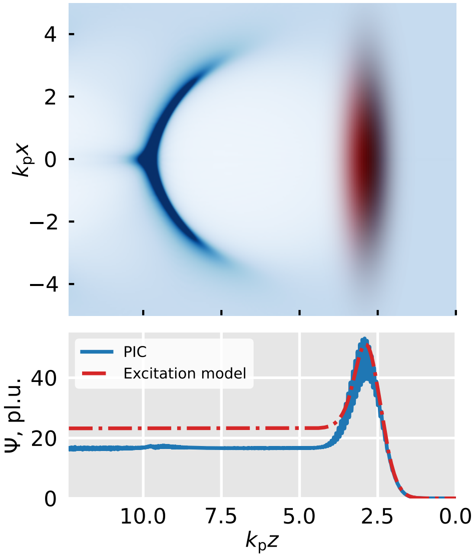
$$\frac{d\Psi}{d\xi} = \frac{1}{2}\int \frac{\partial\langle \mathbf{a}^2\rangle}{\partial\xi} n_e \overline{\gamma^{-1}}\, d^2\mathbf{r}_\perp$$

If we know the RHS, we can calculate Ψ.

We try to rely on the linear solution, but not with full linearization

$$\overline{\gamma^{-1}} = \left(1 + \langle \mathbf{a}^2\rangle(\xi, r)\right)^{-1/2}$$

$$n_e(\xi, r) = \max\left[n_{e,\mathrm{lin}}(\xi, r), 0\right]$$



For a general laser-driven wake,

$$\frac{\mathrm{d}\Psi}{\mathrm{d}\xi} = \frac{1}{2}\int \frac{\partial\langle \mathbf{a}^2\rangle}{\partial\xi} n_\mathrm{e}\overline{\gamma^{-1}}\,\mathrm{d}^2\mathbf{r}_\perp$$

If we know the RHS, we can calculate Ψ.

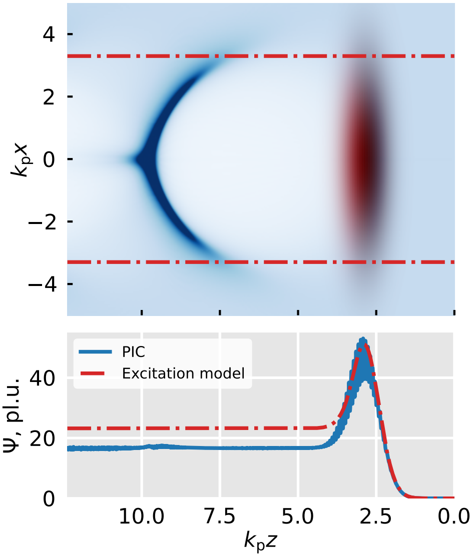
We try to rely on the linear solution, but not with full linearization

$$\overline{\gamma^{-1}} = \left(1 + \langle \mathbf{a}^2\rangle(\xi, r)\right)^{-1/2}$$

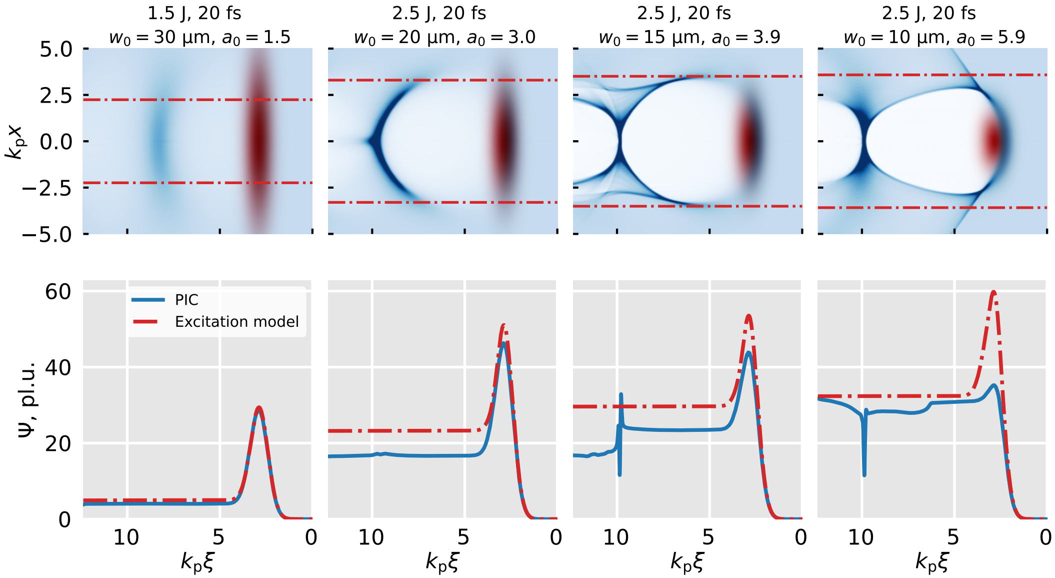
$$n_\mathrm{e}(\xi, r) = \max\left[n_{\mathrm{e,lin}}(\xi, r), 0\right]$$

The size of the bubble

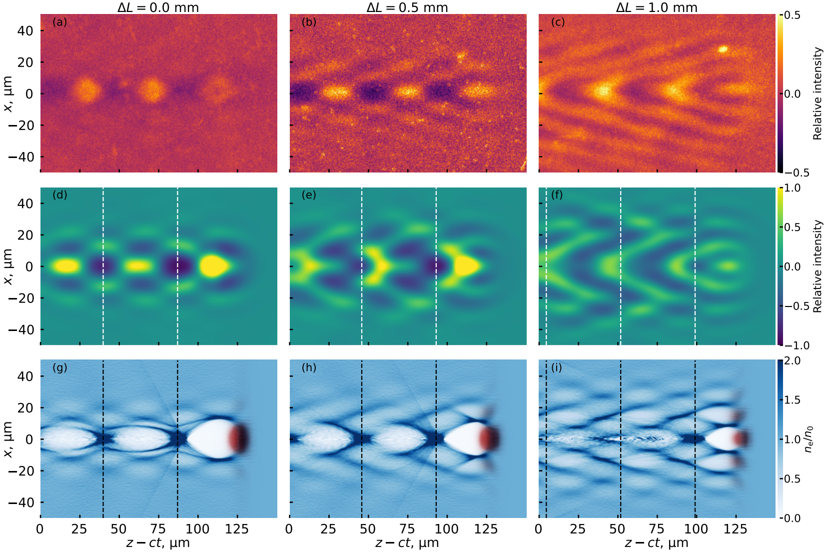
$$R_\mathrm{b} = \left(\frac{16\Psi}{\pi}\right)^{1/4}$$



Bubble excitation by a laser: result



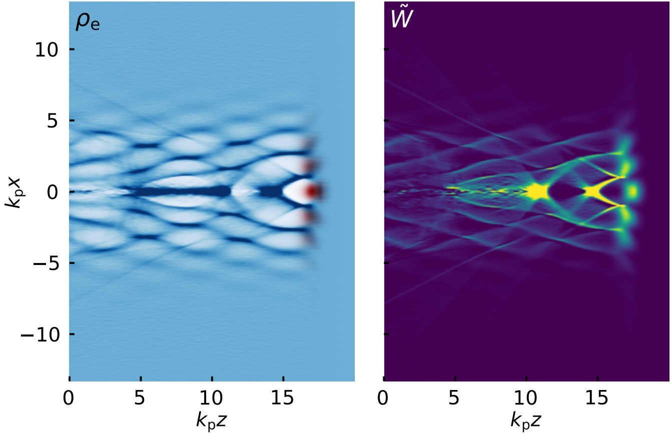
Axiparabola-driven wake



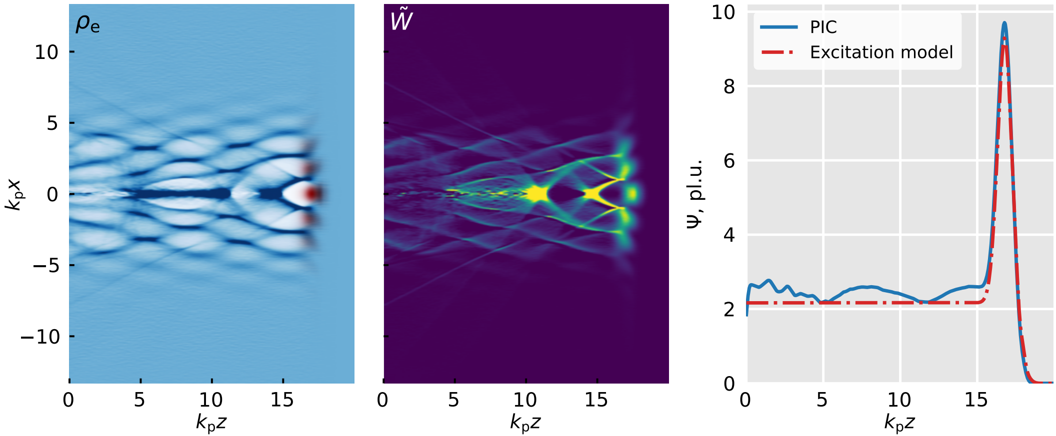
First direct observation through femtosecond relativistic electron microscopy (FREM).

A. Liberman,
A. Golovanov et al.
arXiv:2503.01516
(2025); submitted

Axiparabola-driven wake



Axiparabola-driven wake

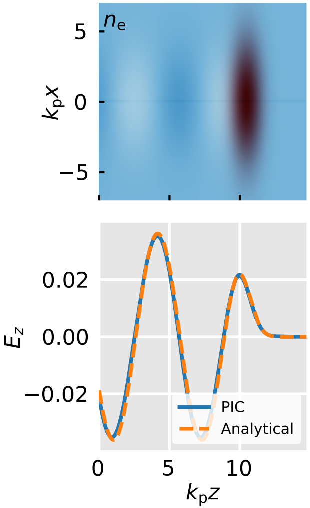


Summary

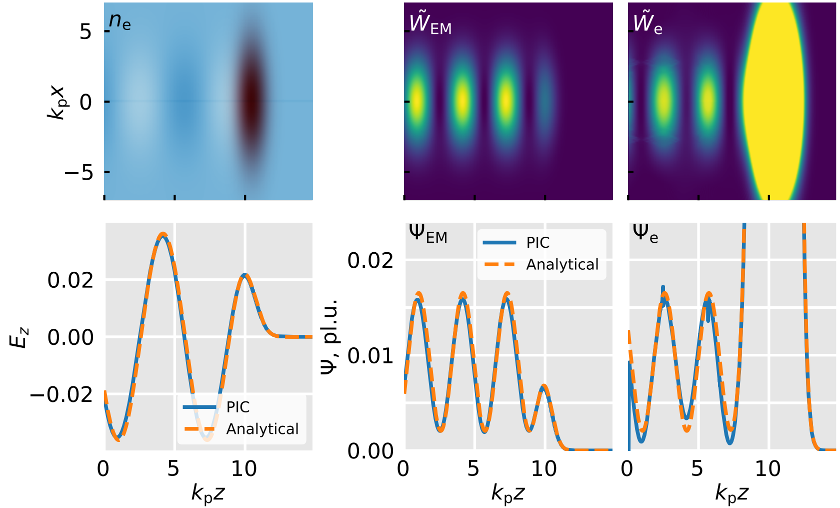
- A model of the bubble based on energy conservation was developed
- The model does not contain external parameters
- The model describes laser- and beam-driven bubbles
- The energy conservation approach is general and powerful

Thank you for your attention!

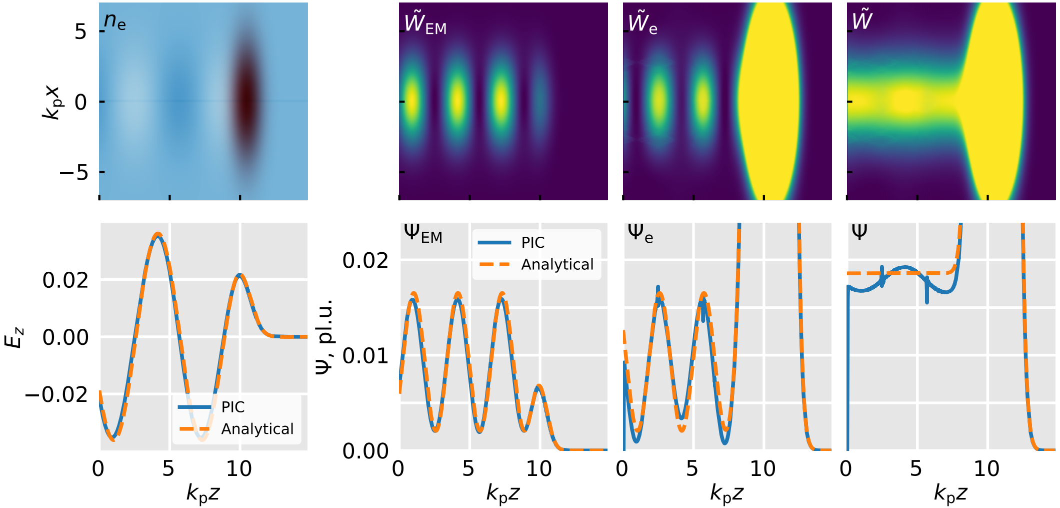
Energy in a quasi-linear wakefield



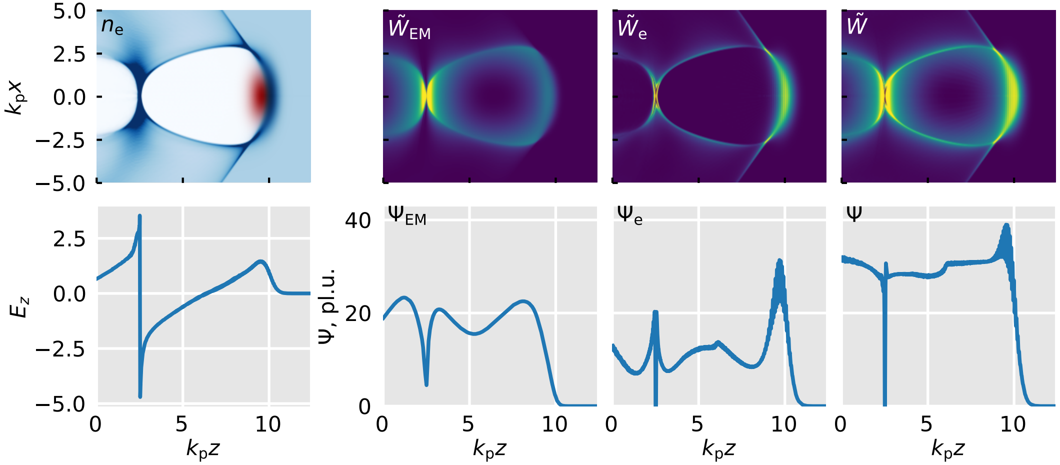
Energy in a quasi-linear wakefield



Energy in a quasi-linear wakefield



Nonlinear wake (bubble)



Starting from the quasilinear solution

As an alternative, we can use the quasilinear solution at the front and assume continuity of $\psi_w(\xi, 0)$, $E_z(\xi, 0)$.

In the bubble regime:

$$\psi_w(\xi, 0) = \frac{r_b^2}{4}, \quad E_z = \frac{r_b}{2}\frac{dr_b}{d\xi}$$

Deriving r_b и r_b', we get the initial conditions

$$r_b = 2\sqrt{\psi_w(\xi, 0)}$$
$$r_b' = \frac{E_z(\xi, 0)}{\sqrt{\psi_w(\xi, 0)}}$$