An Analysis of Electromagnetic Wave Propagation in a Conducting Cylinder with a Small Aperture

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Abstract

This study investigates the propagation and reflection of electromagnetic waves in a conducting cylindrical cavity with a small aperture, analyzing the effects of boundary conditions on wave transmission and diffraction. Using a Green's function approach, we derive exact solutions for the electromagnetic potentials in the Lorentz gauge, incorporating the influence of conducting boundaries and image charges. The field distribution inside the cavity is expressed in terms of eigenmode expansions involving Bessel functions, satisfying Maxwell's equations and the cavity's boundary conditions. The presence of a small hole introduces diffraction effects, which are analyzed using Bethe's small-aperture theory and mode-matching techniques to quantify the transmitted field. For large apertures, the wave leakage is modeled through cylindrical waveguide modes, while for small apertures, the diffraction pattern follows a dipole-like radiation structure. Additionally, we examine the wakefields induced by a charged particle beam inside the cavity, illustrating their interaction with reflected and transmitted waves. The study provides a rigorous framework for understanding space-charge fields in accelerator structures and wave leakage in confined conducting environments, with applications in beam physics and electromagnetic field modeling.

Physical Model and Assumptions

- Consider a perfectly conducting cylindrical cavity of radius a and length L.
- One end cavity ($z = 0$) is a flat conducting cathode, while the other end ($z = L$) has a conducting wall with a circular hole of radius b ($\ll a$).
- A relativistic, axisymmetric electron beam with charge Q, uniform density, and radius $r_b < b$ propagates longitudinally within the cavity.
- Ansatz:
  - The beam carries only longitudinal current.
  - The hole radius b is much smaller than the cavity radius a, enable the use of small-hole perturbation theory

Beam Densities and Boundary Conditions

- We employ the Lorenz gauge condition, where the scalar and vector potentials satisfy:

$$\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = -\frac{\rho}{\varepsilon_0}$$

$$\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = -\mu_0\vec{J}$$

- where the charge and current densities for a zero-thickness beam slice locate at $z_b(t)$ are

$$\rho(r,z,t) = \frac{2Q}{\pi r_b^2}\left(1-\frac{r^2}{r_b^2}\right)\Theta(r_b - r)\delta(z - z_b(t))$$

$$J_z(r,z,t) = \rho(r,z,t)\frac{dz_b(t)}{dt}$$

- The boundary conditions for the conducting cylinder are

$$\phi(r=a,z,t) = 0,\ \phi(r,z=0,t) = 0$$
$$A_z(r=a,z,t) = 0,\ A_z(r,z=0,t) = 0$$

Green's Function Formulation

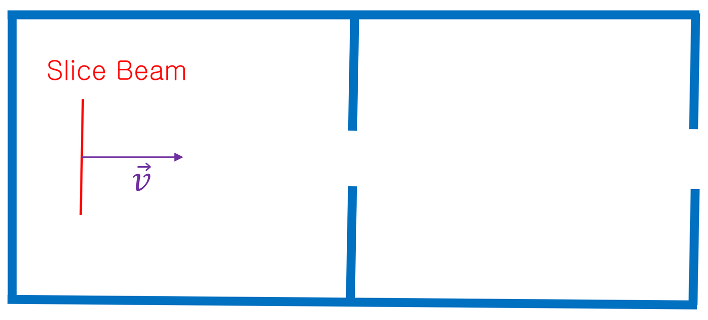
- In the absence of the hole, the Green's function for the cavity can be expanded in eigenmodes

$$G(r,z,t;r',z',t') = \frac{c}{2}\sum_{mn}\psi_{mn}(r)\psi_{mn}(r')g_{mn}(z,t;,z',t')$$

where $\psi_{mn}(r)$ are Bessel functions satisfying the radial boundary conditions, and $g_{mn}(z,t;z',t')$ accounts for the longitudinal and temporal dependencies:

$$\psi_{mn} = \frac{1}{a\sqrt{\pi}}\frac{J_m(j_{mn}\,r/a)e^{im\theta}}{|J_{m+1}(j_{mn})|},$$
$$g_{mn} = J_0(j_{mn}\lambda_{0-}/a)\Theta(\lambda_-^2) \pm J_0(j_{mn}\lambda_{0+}/a)\Theta(\lambda_+^2),$$
$$\lambda_{0\pm}^2 = c^2(t-t')^2 - (z \pm z')^2$$

Cavity Geometry with Small Apertures



Space-Charge Potentials in Closed Cavity

- The scalar and vector potentials inside the closed cavity are:

$$\phi(r,z,t) = \sum_n C_n J_0\left(j_{0n}\frac{r}{a}\right)\int_{-\infty}^{t}\left[J_0\left(j_{0n}\frac{\lambda_-}{a}\right)\Theta(\lambda_-^2) - J_0\left(j_{0n}\frac{\lambda_+}{a}\right)\Theta(\lambda_+^2)\right]$$

$$A_z(r,z,t) = \sum_n D_n J_0\left(j_{0n}\frac{r}{a}\right)\int_{-\infty}^{t}\left[J_0\left(j_{0n}\frac{\lambda_-}{a}\right)\Theta(\lambda_-^2) + J_0\left(j_{0n}\frac{\lambda_+}{a}\right)\Theta(\lambda_+^2)\right]$$

where j_{0n} is the *n*th zero of J_0, and

$$\lambda_\pm^2 = c^2(t-t')^2 - (z \pm z(t'))^2$$

- The coefficients C_n and D_n are determined by matching the beam's charge and current densities to the cavity modes:

$$C_n = \frac{4cQ}{\pi\varepsilon_0 r_b^2}\frac{1}{j_{0n}|J_1(j_{0n})|^2}\left[\frac{2a}{j_{0n}r_b}J_1\left(j_{0n}\frac{r_b}{a}\right) - J_0\left(j_{0n}\frac{r_b}{a}\right)\right]$$

$$D_n = \frac{1}{c^2}C_n$$

Electromagnetic Fields with a Small-Hole Approximation

- In the small-hole approximation, the hole at $z = L$ introduces an additional perturbation. Following Bethe's theory, this perturbation is modeled as an effective magnetic dipole at the aperture. The induced magnetic dipole moment $m_\theta(t)$ is:

$$m_\theta(t) = \frac{4b^3}{3\pi}\frac{\partial A_z(r,z,t)}{\partial r}\bigg|_{r=0,z=L}$$

- The perturbative correction to the vector potential due to the magnetic dipole is:

$$A_z^{\text{pert}}(r,z,t) = \frac{\mu_0 m_\theta(t)}{4\pi}\frac{r}{[(z-L)^2 - r^2]^{3/2}}$$

- Then, the perturbative electric and magnetic fields components are:

$$E_z^{\text{pert}}(r,z,t) = -\frac{\mu_0}{4\pi}\frac{r}{[(z-L)^2 - r^2]^{3/2}}\frac{dm_\theta(t)}{dt}$$

$$B_\theta^{\text{pert}}(r,z,t) = -\frac{\mu_0 m_\theta(t)}{4\pi}\frac{(z-L)^2 - 2r^2}{[(z-L)^2 - r^2]^{5/2}}$$

- Since the pure magnetic dipole does not directly generate a significant scalar potential, the perturbative radial electric field is negligible (no induced charges explicitly)

Electromagnetic Fields

- The electric and magnetic fields are obtained from potentials:

$$\vec{E}(r,z,t) = -\nabla\phi(r,z,t) - \frac{\partial\vec{A}(r,z,t)}{\partial t}$$
$$\vec{B}(r,z,t) = \nabla \times \vec{A}(r,z,t)$$

- Fields explicitly from perturbation:

$$E_{z,dipole}(r,z,t) \approx -\frac{\partial A_{z,dipole}}{\partial t}$$

$$B_{\theta,dipole}(r,z,t) = -\frac{\partial A_{z,dipole}}{\partial r}$$

- Total electromagnetic fields are sum of unperturbed fields and dipole-induced perturbations:

$$\vec{E}_{\text{total}}(r,z,t) = \vec{E}_0(r,z,t) + \vec{E}_{\text{pert}}(r,z,t)$$
$$\vec{B}_{\text{total}}(r,z,t) = \vec{B}_0(r,z,t) + B_{\text{pert}}(r,z,t)$$