Titus Flavius Vespasianus (69-79 AD)

First emperor of Flavion Dynasty

Quark and Lepton Mass Hierarchies from Modular Flavour Symmetry with Weightons

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ROMA TRE UNIVERSITÀ DEGLI STUDI

FLASY ROME 2025

11TH WORKSHOP

Flavor Symmetries and Consequences in Accelerators and Cosmology

Standard Model

Gauge bosons

$$SU(3)_C \times SU(2)_L \times U(1)_Y$$

Fermions

$$L_e = \begin{pmatrix} \nu_{eL} \\ e_L \end{pmatrix}, \quad e_R, \quad \begin{pmatrix} u_L \\ d_L \end{pmatrix}^{r,b,g}, \quad u_R^{r,b,g}, \quad d_R^{r,b,g}$$

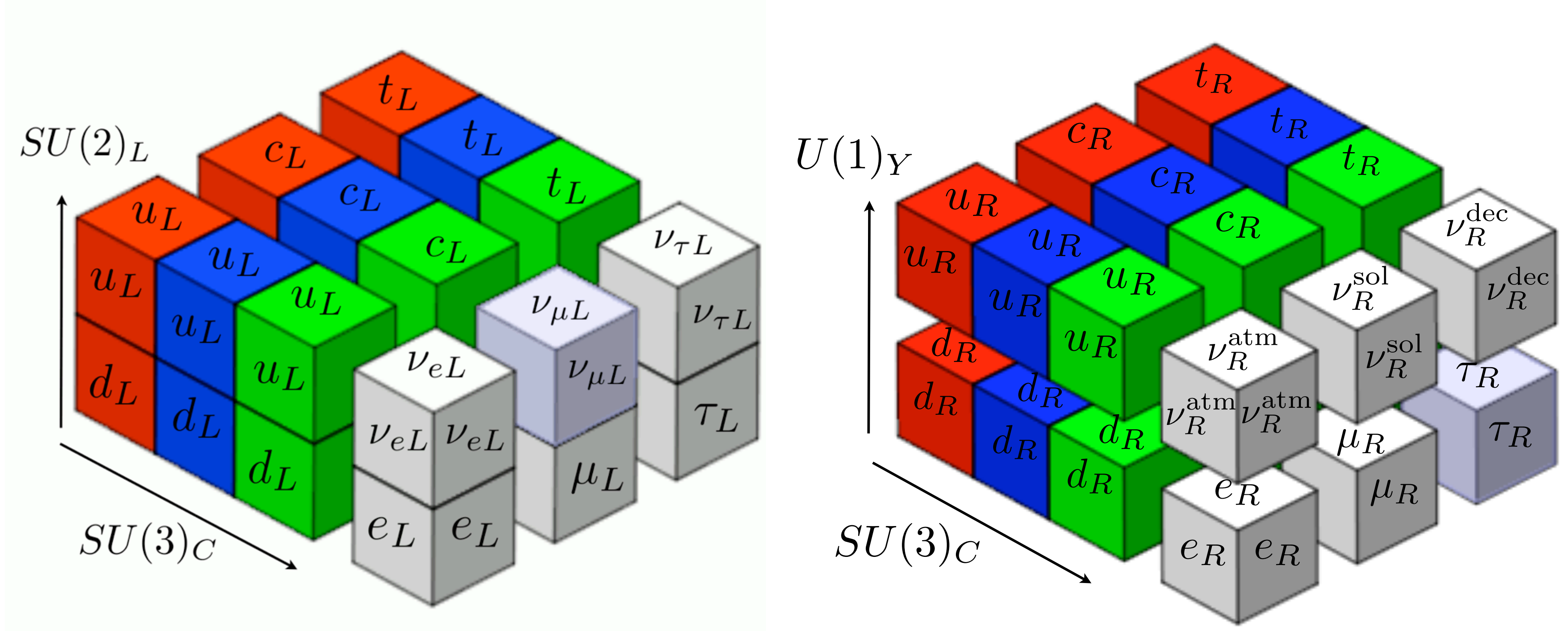
$$L_\mu = \begin{pmatrix} \nu_{\mu L} \\ \mu_L \end{pmatrix}, \quad \mu_R, \quad \begin{pmatrix} c_L \\ s_L \end{pmatrix}^{r,b,g}, \quad c_R^{r,b,g}, \quad s_R^{r,b,g}$$

$$L_\tau = \begin{pmatrix} \nu_{\tau L} \\ \tau_L \end{pmatrix}, \quad \tau_R, \quad \begin{pmatrix} t_L \\ b_L \end{pmatrix}^{r,b,g}, \quad t_R^{r,b,g}, \quad b_R^{r,b,g}$$

Hypercharges

$$Y = -\frac{1}{2}, \quad -1, \quad \frac{1}{6}, \quad \frac{2}{3}, \quad -\frac{1}{3}$$

With RHNs



Higgs

$$H = \begin{pmatrix} h^+ \\ h^0 \end{pmatrix}$$

$$\frac{1}{2}$$

EWSB

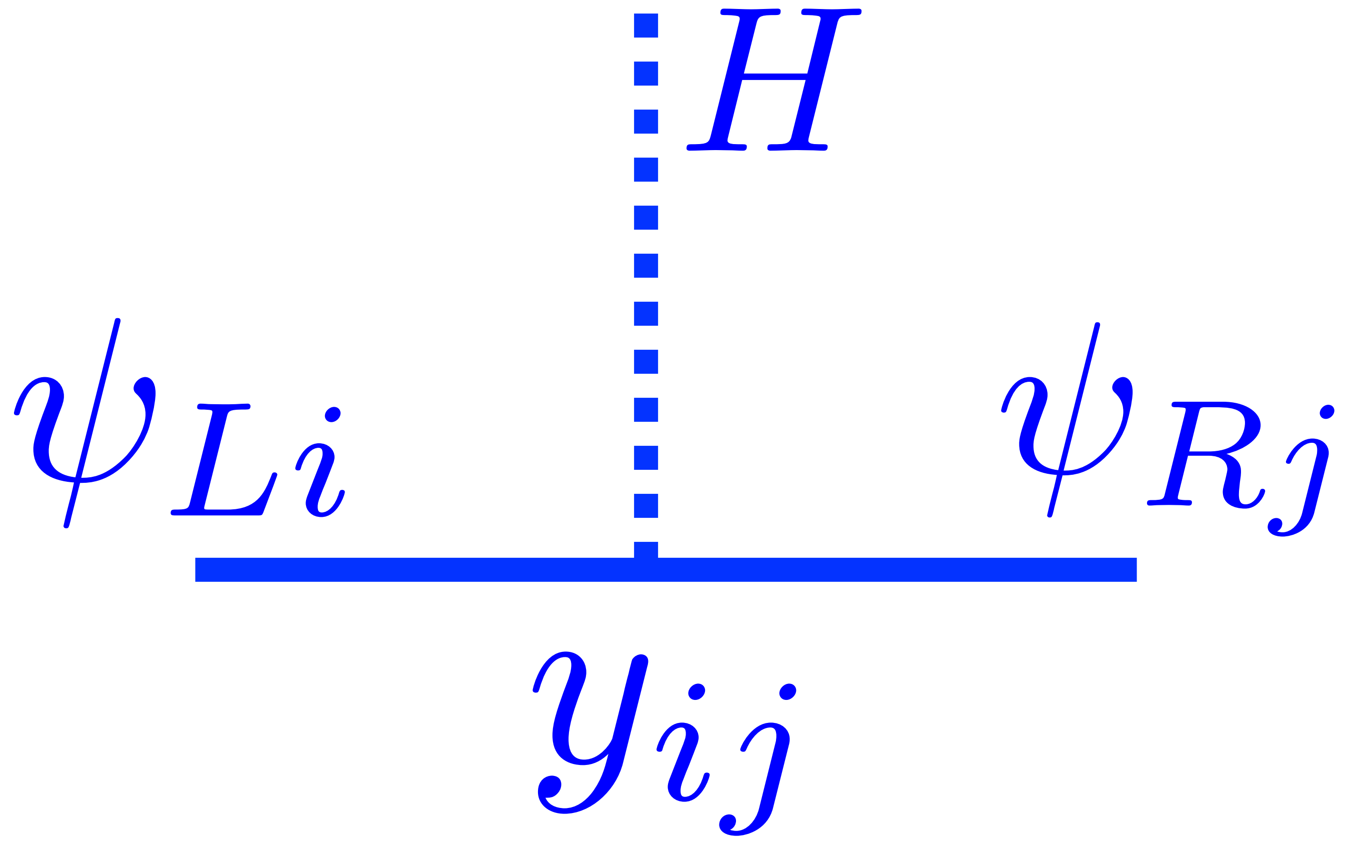
$$\langle h^0 \rangle = v/\sqrt{2}$$

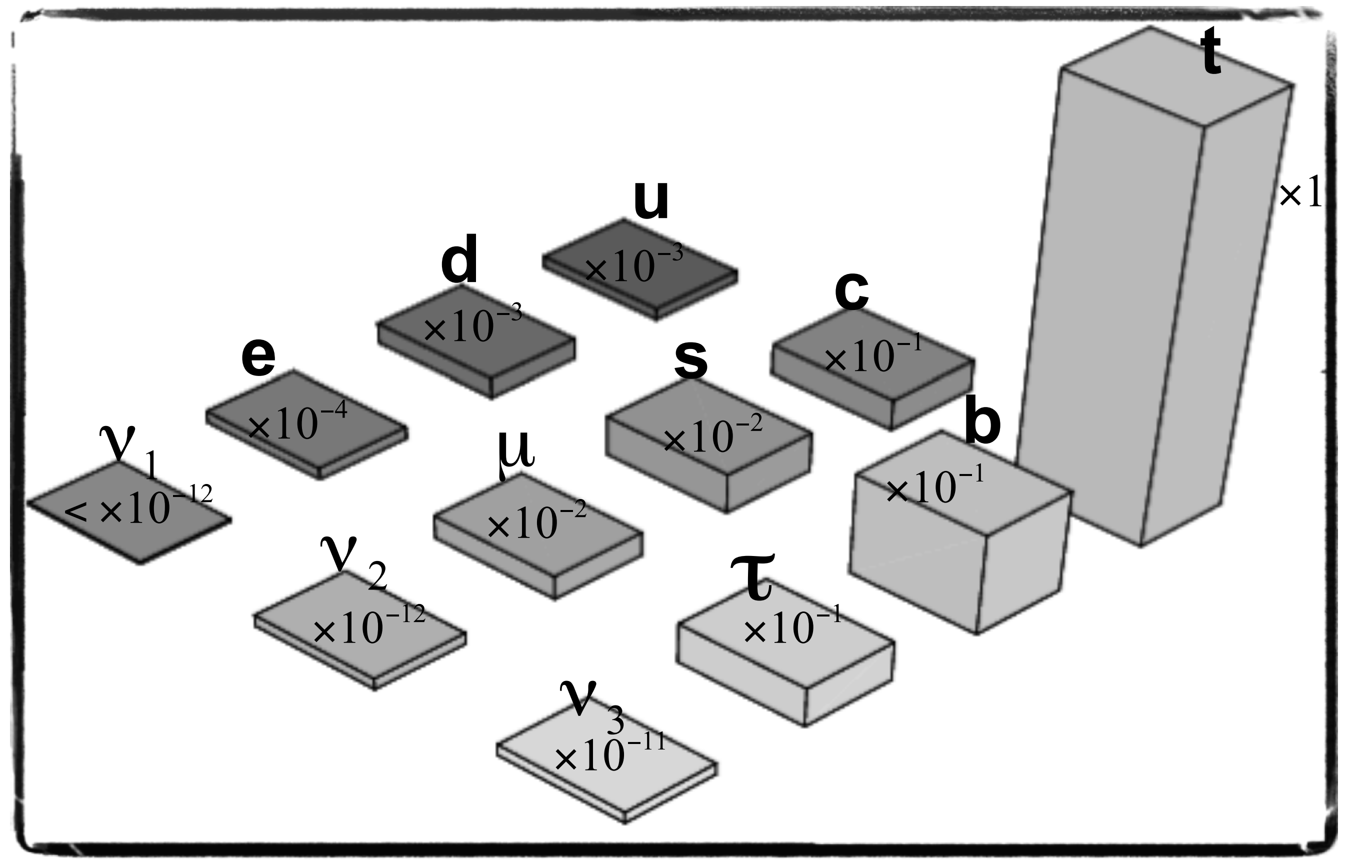
$$SU(2)_L \times U(1)_Y \to U(1)_Q$$

$$Q = T_{3L} + Y$$

Yukawa couplings

$$y_{ij} H \overline{\psi}_{Li} \psi_{Rj}$$



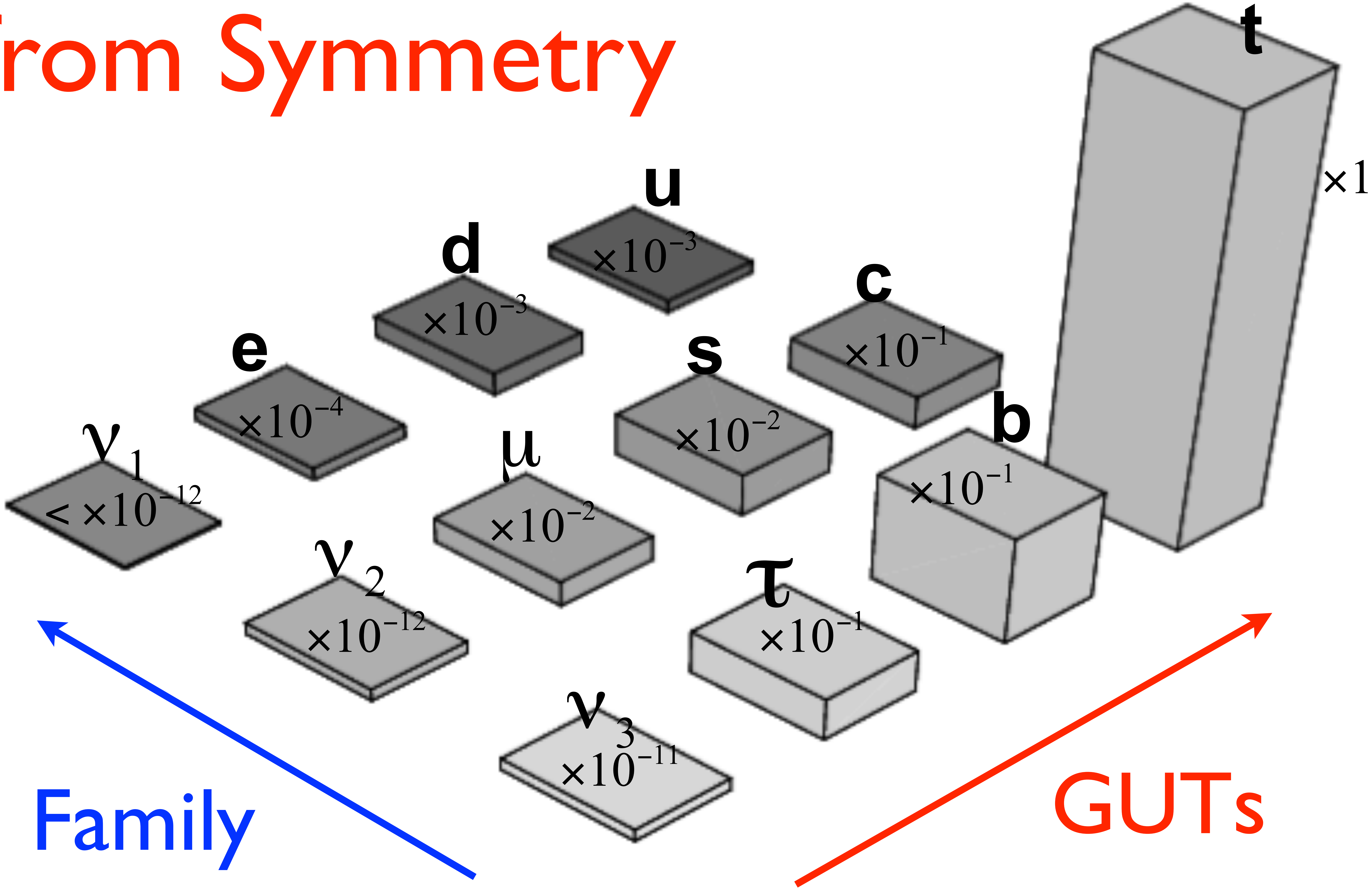


Ideally we would like all Yukawas to be of order unity, so what is going on?

Is there a flavour symmetry at work?

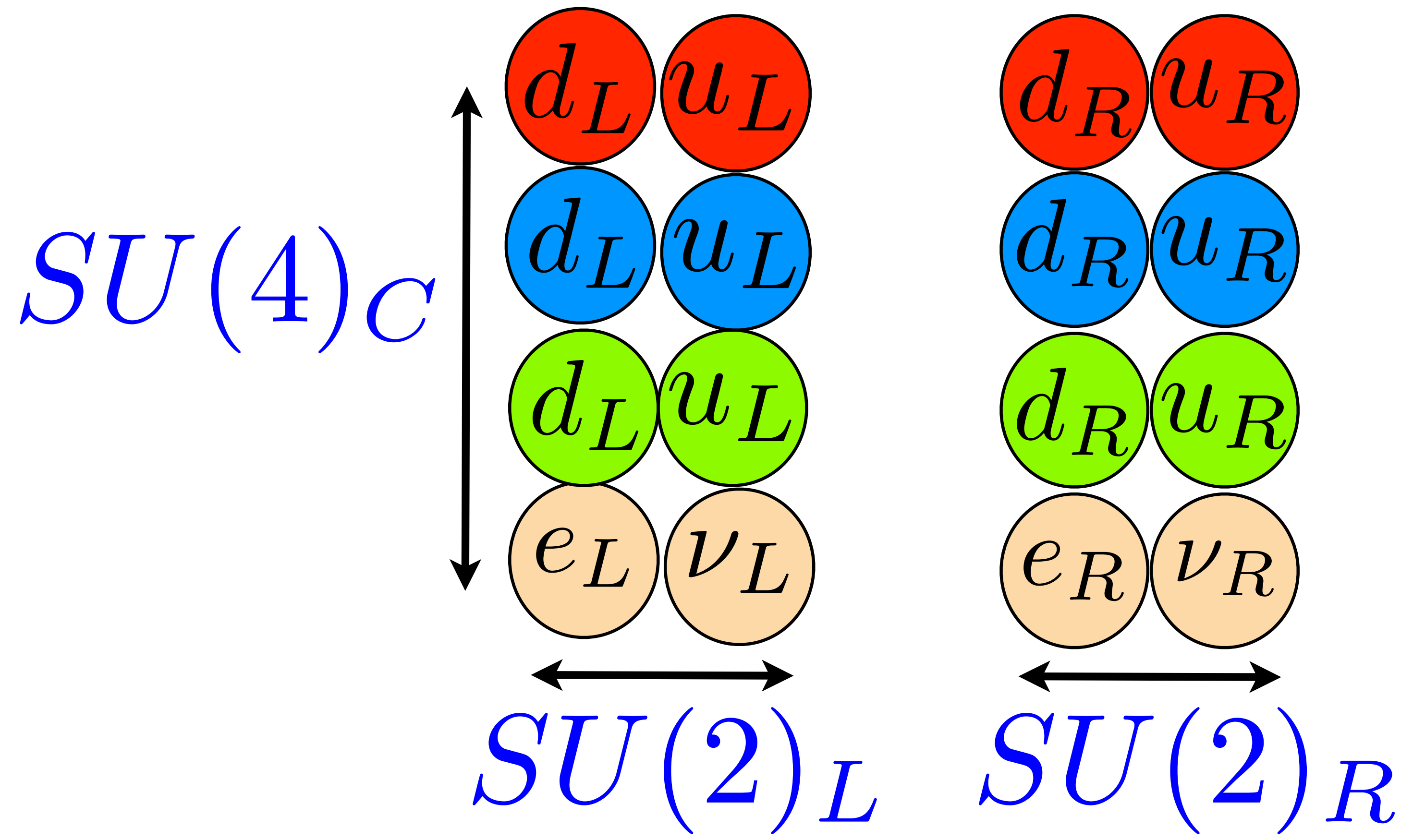
`(Maybe not! See Talk by Avelino Vicente)`

Fermion Mass Hierarchies from Symmetry

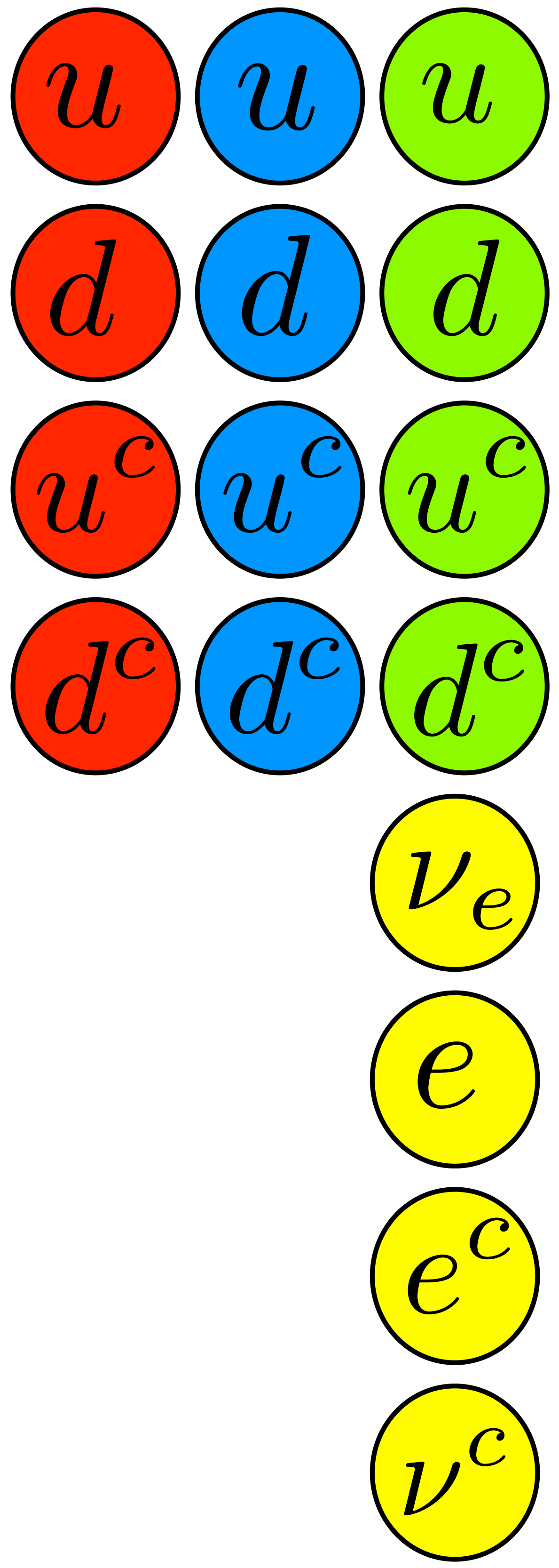


GUTs

Pati-Salam, 1974



SO(10), Fritzsch-Minkowski, 1975



$|++--+\rangle, |+-+-+\rangle, |-++-+\rangle$

$|++-+-\rangle, |+-++-\rangle, |-+++-\rangle$

$|--+++\rangle, |-+-++\rangle, |+--++\rangle$

$|--+--\rangle, |-+---\rangle, |+----\rangle$

$|----+\rangle$

$|---+-\rangle$

$|+++--\rangle$

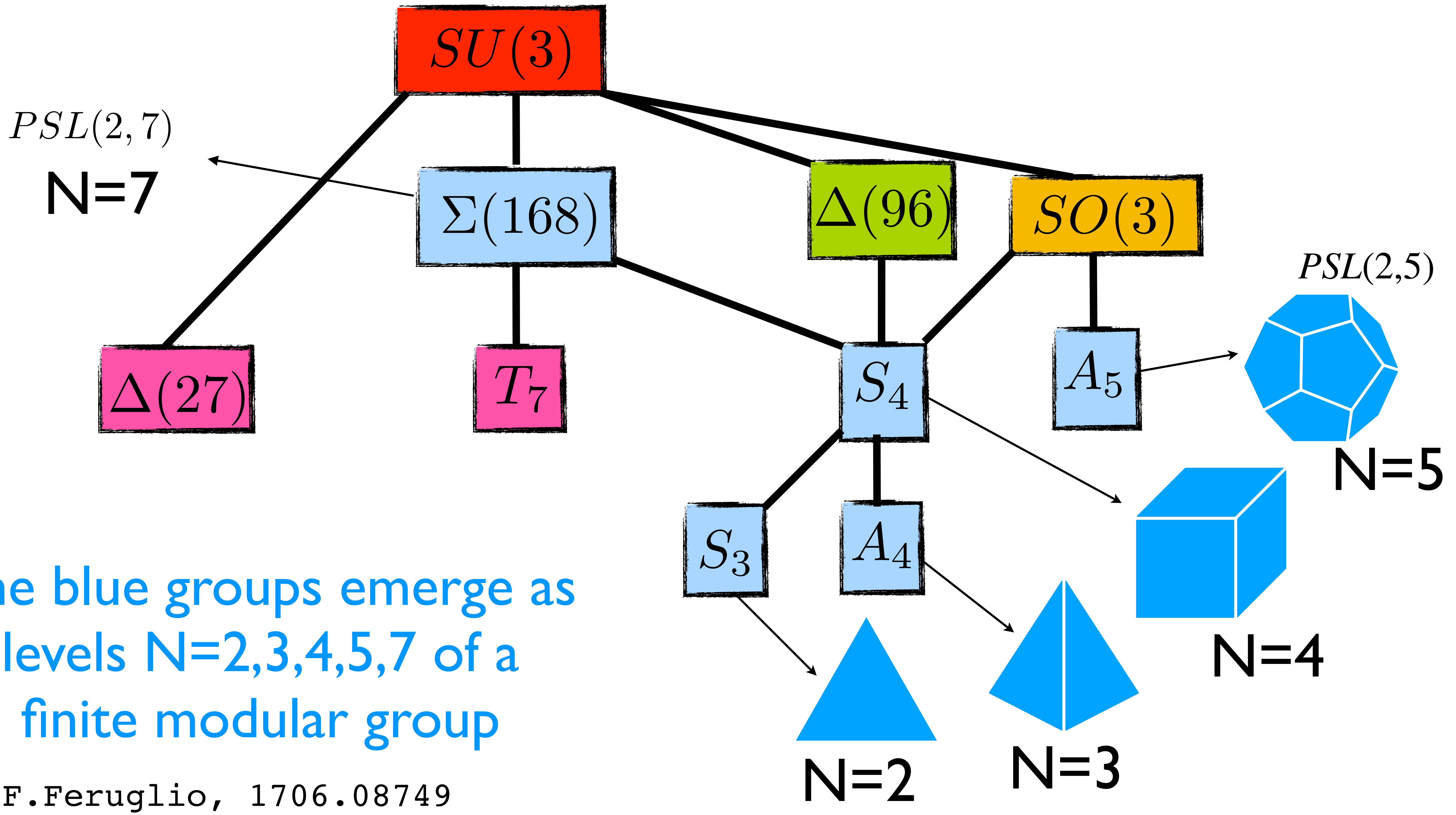
$|+++++\rangle$

SU(5), Georgi-Glashow, 1974

$$F = \begin{pmatrix} d^c_r \\ d^c_b \\ d^c_g \\ e^- \\ -\nu_e \end{pmatrix}_L , \qquad T = \begin{pmatrix} 0 & u^c_g & -u^c_b & u_r & d_r \\ . & 0 & u^c_r & u_b & d_b \\ . & . & 0 & u_g & d_g \\ . & . & . & 0 & e^c \\ . & . & . & . & 0 \end{pmatrix}_L$$

For reviews see e.g. T.Kobayashi and M.Tanimoto, 2307.03384, G.J.Ding and S.F.K., 2311.09282

Family Symmetry



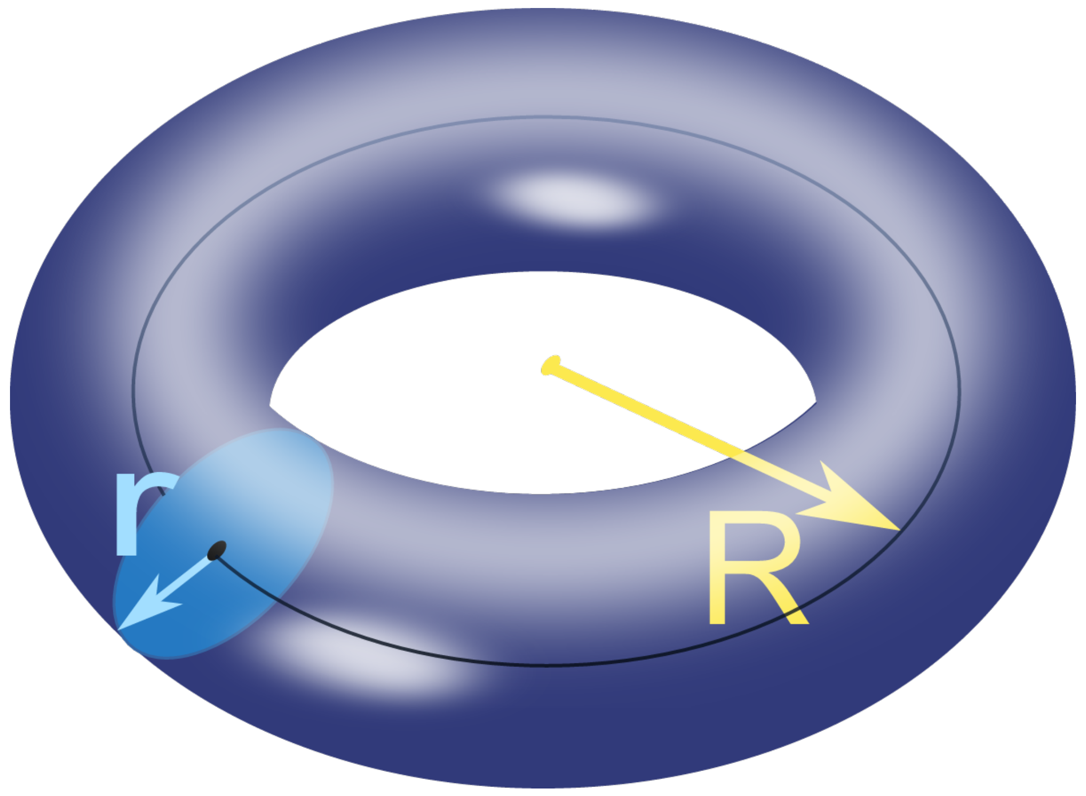
The blue groups emerge as levels N=2,3,4,5,7 of a finite modular group

F.Feruglio, 1706.08749

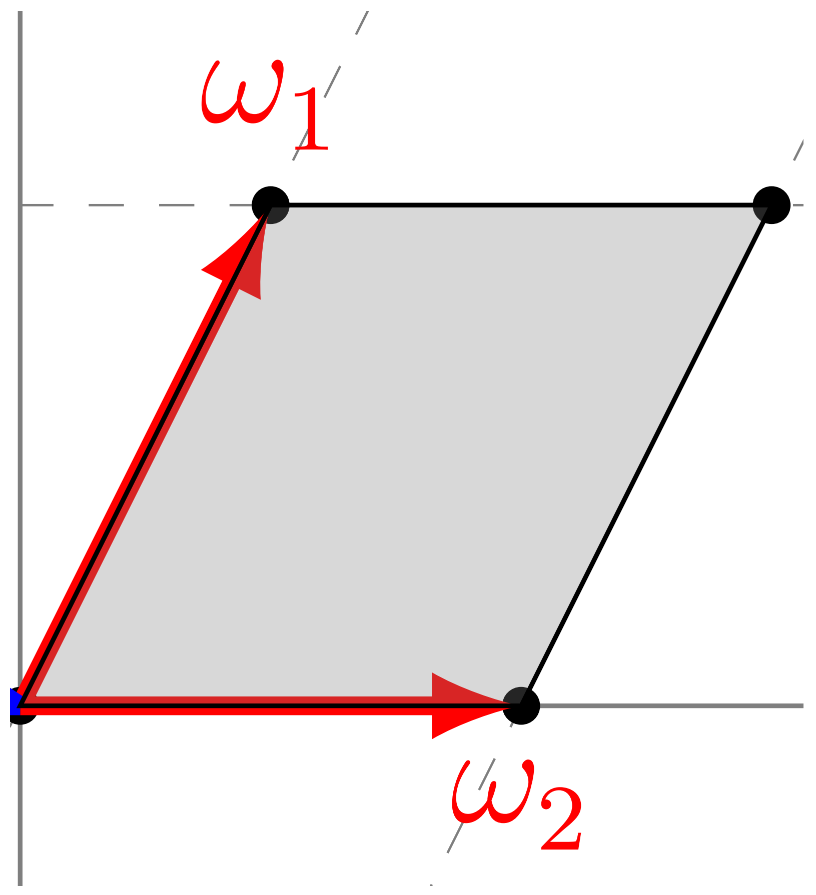
Modular Symmetry

J.Lauer, J.Mas and H.P.Nilles 1989,
S.Ferrara, D.Lust, A.D.Shapere and S.Theisen, 1989
G.Altarelli and F.Feruglio, hep-ph/0512103
R.de Adelhart Toorop, F.Feruglio and C.Hagedorn, 1112.1340

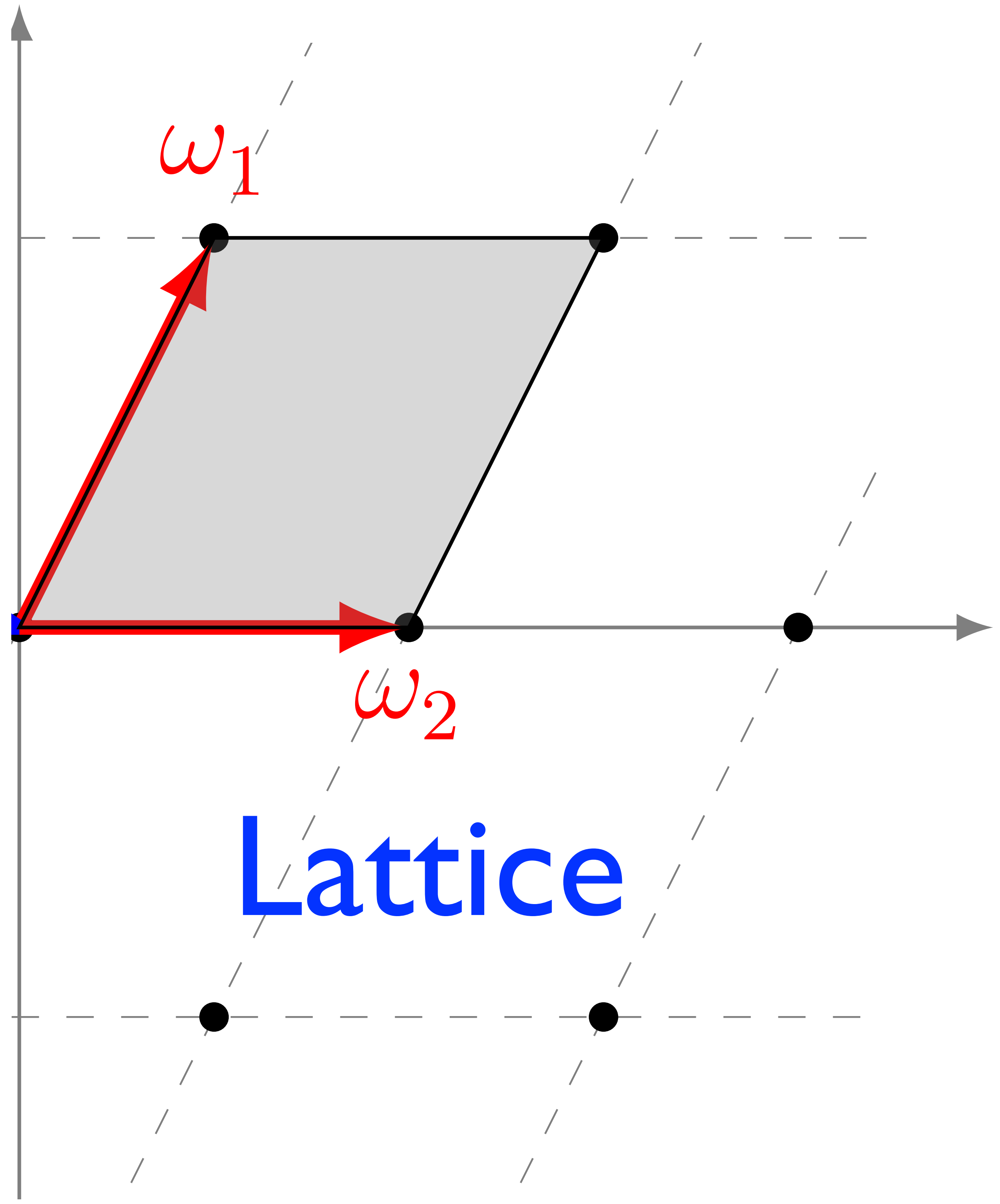
6d with 2 compact dimensions



Torus



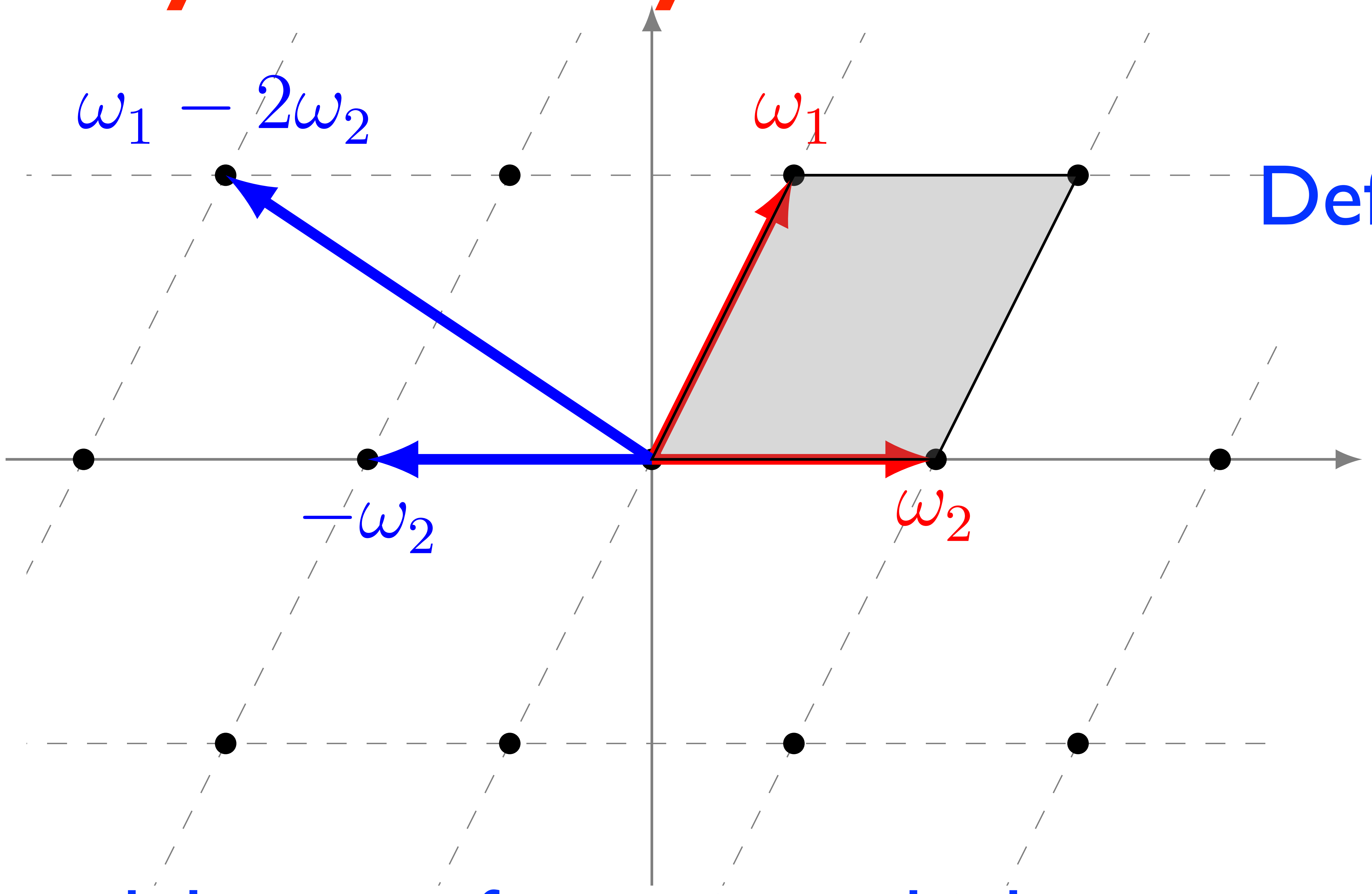
Parallelogram



Modular symmetry = "geometrical symmetry of the torus"

Modular Symmetry

J.Lauer, J.Mas and H.P.Nilles 1989,
S.Ferrara, D.Lust, A.D.Shapere and S.Theisen, 1989
G.Altarelli and F.Feruglio, hep-ph/0512103
R.de Adelhart Toorop, F.Feruglio and C.Hagedorn, 1112.1340



Define **modulus field**

$$\tau \equiv \omega_1/\omega_2$$

In general the modular transformation which preserves the torus area is

$$\tau \to \gamma\tau = \frac{a\tau + b}{c\tau + d} \qquad \Gamma \equiv SL(2, \mathbb{Z})$$

$$ad - bc = 1$$

F.Feruglio, 1706.08749

From Infinite to Finite Modular Symmetry (3 steps to heaven)

$$\Gamma \equiv SL(2,\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a,b,c,d \in \mathbb{Z}, ad - bc = 1 \right\}$$ Infinite step 1

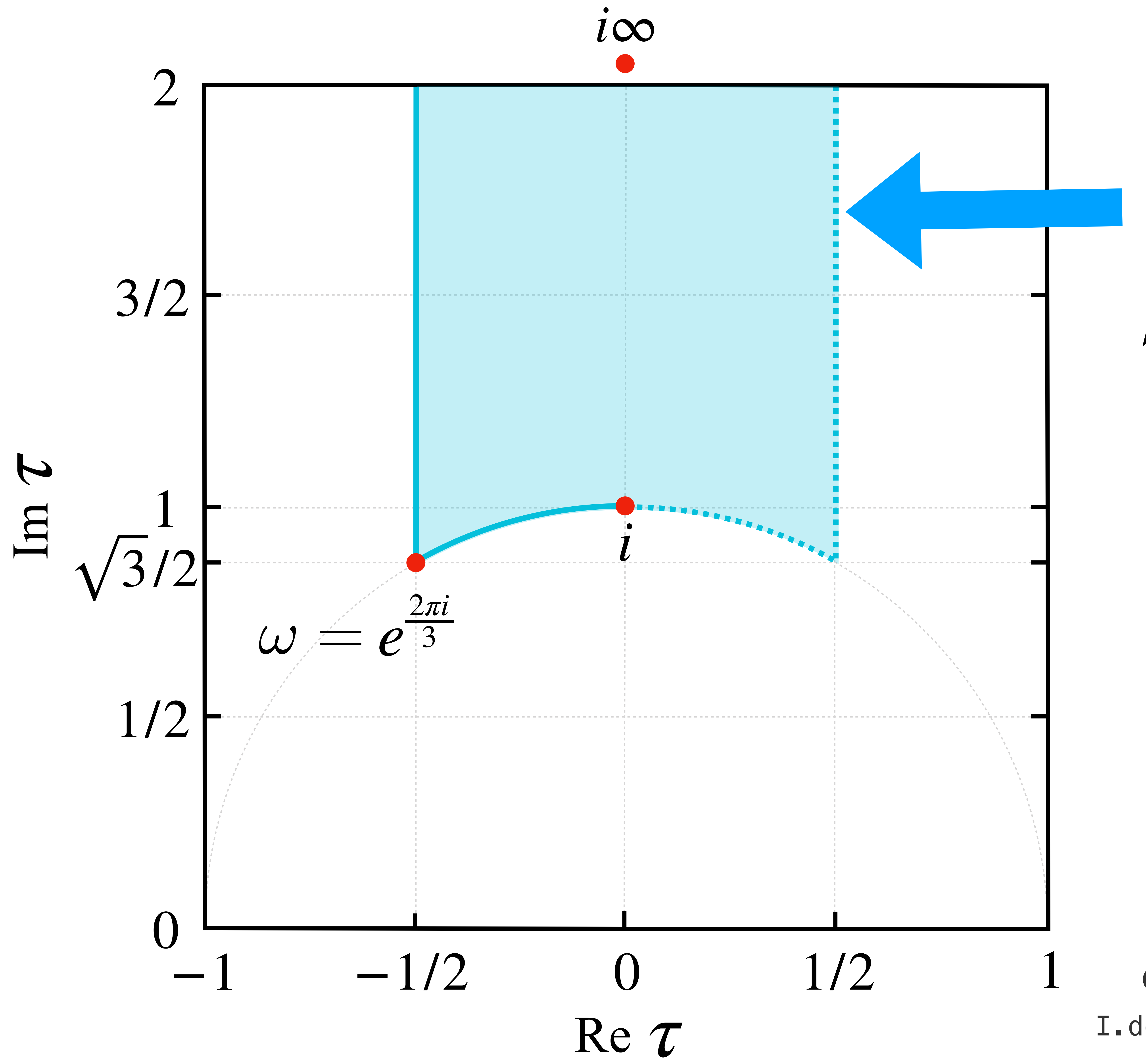
Principle congruence subgroups

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}$$ Infinite step 2

$$\Gamma'_N \equiv SL(2,\mathbb{Z})/\Gamma(N) \equiv \Gamma/\Gamma(N)$$ Finite (double cover) step 3

P.P.Novichkov, S.T.Petcov and M.Tanimoto, 1812.11289
P.P.Novichkov, J.T.Penedo, S.T.Petcov and A.V.Titov, 1811.04933

Fundamental Complex Domain of τ and Fixed Points



Any point can be moved into fundamental domain by applying

$$S:\ \tau \mapsto -\frac{1}{\tau}\,, \qquad T:\ \tau \mapsto \tau + 1$$

Fixed Points

$$\gamma_0 \tau_0 = \tau_0\,.$$

τ_0	γ_0
i	S
$e^{2\pi i/3}$	ST, S^2
$i\infty$	T, S^2

G.J.Ding, S.F.K., X.G.Liu and J.N.Lu, 1910.03460
I.de Medeiros Varzielas, M.Levy and Y.L.Zhou, 2008.05329

Examples of Finite Modular Symmetry

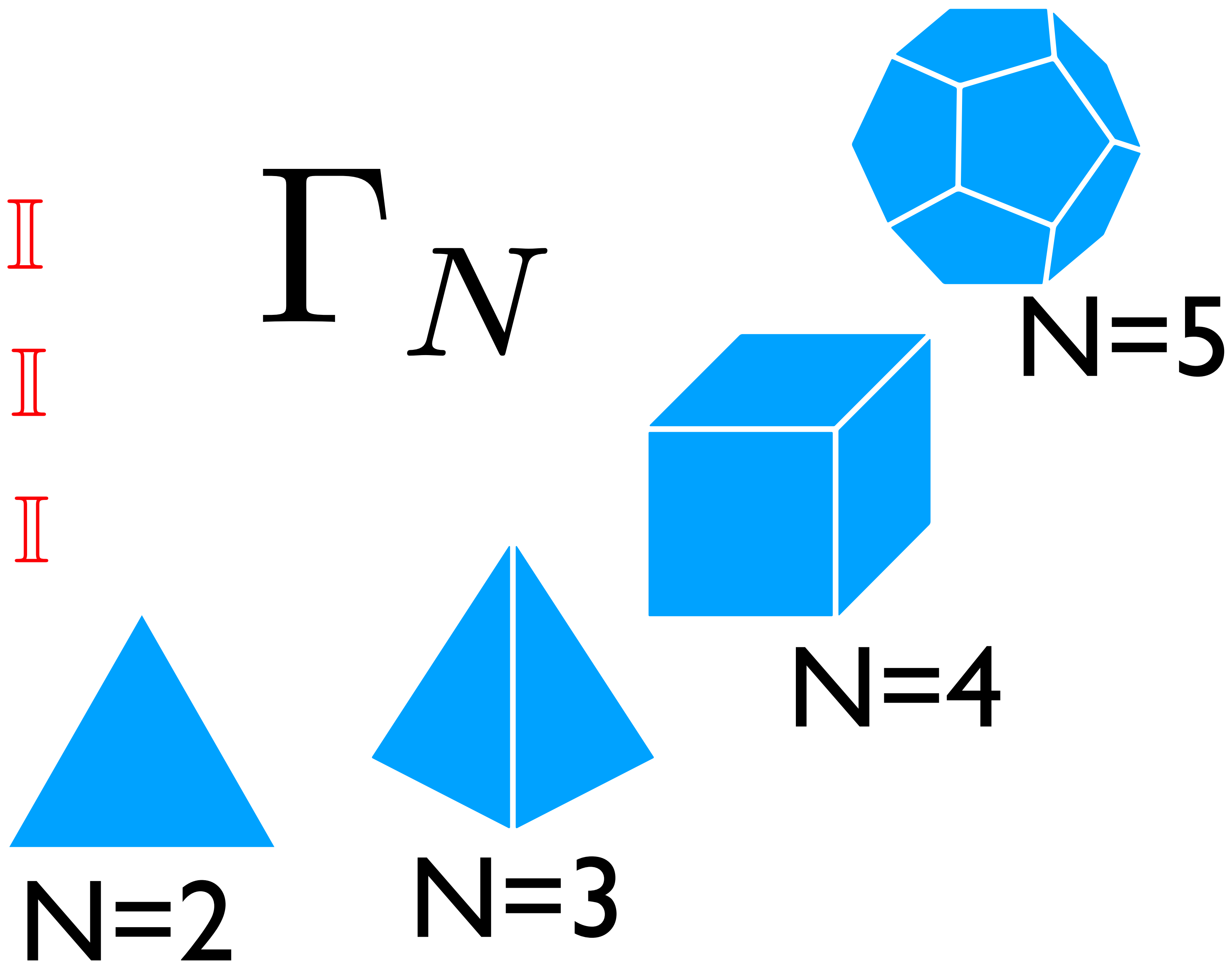
F.Feruglio, 1706.08749

Level

N	Γ_N	$\lvert\Gamma_N\rvert$	Γ'_N	$\lvert\Gamma'_N\rvert$
2	S_3	6	S_3	6
3	A_4	12	T'	24
4	S_4	24	S'_4	48
5	A_5	60	A'_5	120
6	$S_3 \times A_4$	72	$S_3 \times T'$	144
7	$PSL(2,7) \cong \Sigma(168)$	168	$SL(2,7)$	336

$$S^2 = \mathbb{I}$$
$$T^N = \mathbb{I}$$
$$(ST)^3 = \mathbb{I}$$

$$\Gamma_N$$



Yukawa coupling

$$Y(\tau)\phi_1\phi_2\phi_3$$

Fields transform under modular symmetry with weight k

$$\phi_1 \to (c\tau + d)^{k_1}\rho_1(\gamma)\phi_1$$

Weights must add up to zero so the Yukawa carries weight

$$-k_Y = k_1 + k_2 + k_3$$

Yukawa coupling is a modular form

$$Y(\tau) \to Y(\gamma\tau) = (c\tau + d)^{k_Y}\rho_{\mathbf{r}_Y}(\gamma)Y(\tau)$$

$$\rho_{\mathbf{r}_Y} \times \rho_1 \times \rho_2 \times \rho_3 = 1 + \ldots$$

E.g. Level N=3: $\Gamma_3 \sim A_4$

F.Feruglio, 1706.08749

Yukawa coupling is a modular form

$q \equiv e^{i2\pi\tau}$

A$_4$ triplet **3**
Weight k$_Y$=2

$$Y_{\mathbf{3}}^{(2)} = \begin{pmatrix} Y_1(\tau) \\ Y_2(\tau) \\ Y_3(\tau) \end{pmatrix} = \begin{pmatrix} 1 + 12q + 36q^2 + 12q^3 + 84q^4 + 72q^5 + \ldots \\ -6q^{1/3}(1 + 7q + 8q^2 + 18q^3 + 14q^4 + \ldots) \\ -18q^{2/3}(1 + 2q + 5q^2 + 4q^3 + 8q^4 + \ldots) \end{pmatrix}$$

In the fundamental domain $|q| \leq e^{-\sqrt{3}\pi} \simeq 0.0043 \ll 1$

Possibility of hierarchical Yukawa couplings

$$\begin{pmatrix} Y_1(\tau) \\ Y_2(\tau) \\ Y_3(\tau) \end{pmatrix} = \begin{pmatrix} 1 + \mathcal{O}(q) \\ -6q^{1/3} + \mathcal{O}(q) \\ -18q^{2/3} + \mathcal{O}(q) \end{pmatrix}$$

Hard to obtain both mass hierarchies and mixing...

P.P. Novichkov, J.T. Penedo and S.T. Petcov, 2102.07488

E.g. Lepton Model at level N=3

F.Feruglio, 1706.08749

	L	e_3^c	e_2^c	e_1^c	N^c	$H_{u,d}$
A_4	$\mathbf{3}$	$\mathbf{1}'$	$\mathbf{1}''$	$\mathbf{1}$	$\mathbf{3}$	$\mathbf{1}$
k_I	1	1	1	1	1	0

Weinberg operator $\frac{1}{\Lambda}(H_u H_u \; LL \; Y)$

A4 rep: **3 3 3**

Modular weights k : 1 1 2

Neutrino mass matrix

$$m_\nu = \begin{pmatrix} 2Y_1 & -Y_3 & -Y_2 \\ -Y_3 & 2Y_2 & -Y_1 \\ -Y_2 & -Y_1 & 2Y_3 \end{pmatrix} \frac{v_u^2}{\Lambda}$$

(plus seesaw)

Holomorphic superpotential for charged leptons

$$W_e = \alpha e_1^c (L Y_{\mathbf{3}}^{(2)})_{\mathbf{1}} H_d + \beta e_2^c (L Y_{\mathbf{3}}^{(2)})_{\mathbf{1}'} H_d + \gamma e_3^c (L Y_{\mathbf{3}}^{(2)})_{\mathbf{1}''} H_d$$

$$\mathbf{3} \otimes \mathbf{3} = \mathbf{1} \oplus \mathbf{1}' \oplus \mathbf{1}'' \oplus \mathbf{3}_S \oplus \mathbf{3}_A$$

$$= \alpha e_1^c (L_1 Y_1 + L_2 Y_3 + L_3 Y_2) H_d + \beta e_2^c (L_3 Y_3 + L_1 Y_2 + L_2 Y_1) H_d + \gamma e_3^c (L_2 Y_2 + L_3 Y_1 + L_1 Y_3) H_d \,.$$

Charged lepton mass matrix

$$M_e = \begin{pmatrix} \alpha Y_1(\tau) & \alpha Y_3(\tau) & \alpha Y_2(\tau) \\ \beta Y_2(\tau) & \beta Y_1(\tau) & \beta Y_3(\tau) \\ \gamma Y_3(\tau) & \gamma Y_2(\tau) & \gamma Y_1(\tau) \end{pmatrix} v_d$$

Mass hierarchies unexplained

Best Fit

$\mathrm{Re}(\tau)$	$\mathrm{Im}(\tau)$
0.049	2.264

β/α	γ/α
210.767	3581.720

S.J.D. King and S.F.K, 2002.00969

Introduce a weighton (complete singlet*)

	L	e_3^c	e_2^c	e_1^c	N^c	$H_{u,d}$	ϕ
A_4	$\mathbf{3}$	$\mathbf{1}'$	$\mathbf{1}''$	$\mathbf{1}$	$\mathbf{3}$	$\mathbf{1}$	$\mathbf{1}$
k_I	1	0	-1	-3	1	0	1

$$W_{driv} = \chi(Y_{\mathbf{1}}^{(4)} \frac{\phi^4}{M_{fl}^2} - M^2)$$

$$\tilde{\phi} \equiv \frac{\langle\phi\rangle}{M_{fl}} \sim (M/M_{fl})^{1/2}$$

small parameter

$$W_e = \alpha_e e_1^c \tilde{\phi}^4 (L Y_{\mathbf{3}}^{(2)})_{\mathbf{1}} H_d + \beta_e e_2^c \tilde{\phi}^2 (L Y_{\mathbf{3}}^{(2)})_{\mathbf{1}'} H_d + \gamma_e e_3^c \tilde{\phi} (L Y_{\mathbf{3}}^{(2)})_{\mathbf{1}''} H_d$$

$$Y_e = \begin{pmatrix} \alpha_e \tilde{\phi}^4 Y_1 & \alpha_e \tilde{\phi}^4 Y_3 & \alpha_e \tilde{\phi}^4 Y_2 \\ \beta_e \tilde{\phi}^2 Y_2 & \beta_e \tilde{\phi}^2 Y_1 & \beta_e \tilde{\phi}^2 Y_3 \\ \gamma_e \tilde{\phi} Y_3 & \gamma_e \tilde{\phi} Y_2 & \gamma_e \tilde{\phi} Y_1 \end{pmatrix}$$

Natural explanation of charged lepton mass hierarchy

$$m_e : m_\mu : m_\tau = \alpha_e \tilde{\phi}^4 : \beta_e \tilde{\phi}^2 : \gamma_e \tilde{\phi}$$

By definition the weighton ϕ does not break flavour symmetry (c.f. FN flavon)

* For hierarchies from triplet flavon see J.C. Criado, F. Feruglio and S.J.D. King, 1908.11867

S.J.D. King and S.F.K, 2002.00969

Quark Sector with weighton

	Q	d_3^c	d_2^c	d_1^c	u_3^c	u_2^c	u_1^c	$H_{u,d}$	ϕ
A_4	$\mathbf{3}$	$\mathbf{1}'$	$\mathbf{1}''$	$\mathbf{1}$	$\mathbf{1}'$	$\mathbf{1}''$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
k_I	1	$0, 2, 4$	-2	-3	$5, 3, 1$	$-1, 2, 4$	-3	0	1

$$\tilde{\phi} \equiv \frac{\langle\phi\rangle}{M_{fl}} \sim (M/M_{fl})^{1/2}$$

small parameter

E.g. $k_{d^c_{3,2,1}} = 0, -2, -3$ $\quad k_{u^c_{3,2,1}} = 5, -1, -3$

$$W_d = \alpha_d d_1^c \tilde{\phi}^4 (QY_{\mathbf{3}}^{(2)})_{\mathbf{1}} H_d + \beta_d d_2^c \tilde{\phi}^3 (QY_{\mathbf{3}}^{(2)})_{\mathbf{1}'} H_d + \gamma_d d_3^c \tilde{\phi} (QY_{\mathbf{3}}^{(2)})_{\mathbf{1}''} H_d$$

$$W_u = \alpha_u u_1^c \tilde{\phi}^4 (QY_{\mathbf{3}}^{(2)})_{\mathbf{1}} H_u + \beta_u u_2^c \tilde{\phi}^2 (QY_{\mathbf{3}}^{(2)})_{\mathbf{1}'} H_u + \gamma_u^I u_3^c (QY_{\mathbf{3},I}^{(6)})_{\mathbf{1}''} H_u + \gamma_u^{II} u_3^c (QY_{\mathbf{3},II}^{(6)})_{\mathbf{1}''} H_u$$

$$m_d : m_s : m_b \sim \tilde{\phi}^4 : \tilde{\phi}^3 : \tilde{\phi}$$

$$m_u : m_c : m_t \sim \tilde{\phi}^4 : \tilde{\phi}^2 : 1$$

Natural explanation of quark mass hierarchy

Good fits to mass and mixing for certain cases

Gui-Jun Ding, S.F.K, Lu, Weng, 2505.12916

General analysis with a weighton

$$\mathcal{W} = \alpha\, \tilde{\phi}^I \left(\psi_1^c \psi Y_{\boldsymbol{r}_1}^{(k_1)} H_{u/d}\right)_{\mathbf{1}} + \beta\, \tilde{\phi}^J \left(\psi_2^c \psi Y_{\boldsymbol{r}_2}^{(k_2)} H_{u/d}\right)_{\mathbf{1}} + \gamma\, \tilde{\phi}^K \left(\psi_3^c \psi Y_{\boldsymbol{r}_3}^{(k_3)} H_{u/d}\right)_{\mathbf{1}}$$

level N=3 independent of modular weights $|q| \leq e^{-\sqrt{3}\pi} \simeq 0.0043 \ll 1$

$\boldsymbol{r}_\psi \otimes \boldsymbol{r}_{\psi^c}$	power of $\tilde{\phi}$	P_ψ	$(m_{\psi_1}, m_{\psi_2}, m_{\psi_3})$	$(\theta_{12}^\psi, \theta_{23}^\psi, \theta_{13}^\psi)$
$\mathbf{3} \otimes \mathbf{3}$	$I = J = K$	P_{132}	$(\tilde{\phi}^I, \tilde{\phi}^I, \tilde{\phi}^I)$	$(\|q\|^{\frac{1}{3}}, \|q\|^{\frac{1}{3}}, \|q\|^{\frac{1}{3}})$
$\mathbf{3} \otimes (\mathbf{1}'' \oplus \mathbf{1}' \oplus \mathbf{1})$	$I \geq J \geq K$	P_{231}	$(\tilde{\phi}^I, \tilde{\phi}^J, \tilde{\phi}^K)$	$(\|q\|^{\frac{1}{3}}, \|q\|^{\frac{1}{3}}, \|q\|^{\frac{1}{3}} \tilde{\phi}^{2(I-K)} + \|q\|^{\frac{2}{3}})$
	$J \geq K \geq I$	P_{312}	$(\tilde{\phi}^J, \tilde{\phi}^K, \tilde{\phi}^I)$	$(\|q\|^{\frac{1}{3}}, \|q\|^{\frac{1}{3}}, \|q\|^{\frac{1}{3}} \tilde{\phi}^{2(J-I)} + \|q\|^{\frac{2}{3}})$
	$K \geq I \geq J$	P_{123}	$(\tilde{\phi}^K, \tilde{\phi}^I, \tilde{\phi}^J)$	$(\|q\|^{\frac{1}{3}}, \|q\|^{\frac{1}{3}}, \|q\|^{\frac{1}{3}} \tilde{\phi}^{2(K-J)} + \|q\|^{\frac{2}{3}})$
	$J \geq I \geq K$	P_{321}	$(\tilde{\phi}^J, \tilde{\phi}^I, \tilde{\phi}^K)$	$(\|q\|^{\frac{2}{3}}, \|q\|^{\frac{1}{3}} \tilde{\phi}^{2(I-K)} + \|q\|^{\frac{2}{3}}, \|q\|^{\frac{1}{3}})$
(cont'd…)	$I \geq K \geq J$	P_{213}	$(\tilde{\phi}^I, \tilde{\phi}^K, \tilde{\phi}^J)$	$(\|q\|^{\frac{2}{3}}, \|q\|^{\frac{1}{3}} \tilde{\phi}^{2(K-J)} + \|q\|^{\frac{2}{3}}, \|q\|^{\frac{1}{3}})$
	$K \geq J \geq I$	P_{132}	$(\tilde{\phi}^K, \tilde{\phi}^J, \tilde{\phi}^I)$	$(\|q\|^{\frac{2}{3}}, \|q\|^{\frac{1}{3}} \tilde{\phi}^{2(J-I)} + \|q\|^{\frac{2}{3}}, \|q\|^{\frac{1}{3}})$

Similar tables for N=4,5

Weightons control mass hierarchy

q-expansion suppresses mixing

(N=3 cont'd...)

Gui-Jun Ding, S.F.K, Lu, Weng, 2505.12916

$\boldsymbol{r}_\psi \otimes \boldsymbol{r}_{\psi^c}$	power of $\tilde{\phi}$	P_ψ	$(m_{\psi_1}, m_{\psi_2}, m_{\psi_3})$	$(\theta_{12}^\psi, \theta_{23}^\psi, \theta_{13}^\psi)$
$\mathbf{3} \otimes (\mathbf{1}'' \oplus \mathbf{1}' \oplus \mathbf{1})$	$I \geq J \geq K$	P_{231}	$\left(\tilde{\phi}^I, \tilde{\phi}^J, \tilde{\phi}^K\right)$	$(\lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{2(I-K)} + \lvert q\rvert^{\frac{2}{3}})$
	$J \geq K \geq I$	P_{312}	$(\tilde{\phi}^J, \tilde{\phi}^K, \tilde{\phi}^I)$	$(\lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{2(J-I)} + \lvert q\rvert^{\frac{2}{3}})$
	$K \geq I \geq J$	P_{123}	$(\tilde{\phi}^K, \tilde{\phi}^I, \tilde{\phi}^J)$	$(\lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{2(K-J)} + \lvert q\rvert^{\frac{2}{3}})$
	$J \geq I \geq K$	P_{321}	$(\tilde{\phi}^J, \tilde{\phi}^I, \tilde{\phi}^K)$	$(\lvert q\rvert^{\frac{2}{3}}, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{2(I-K)} + \lvert q\rvert^{\frac{2}{3}}, \lvert q\rvert^{\frac{1}{3}})$
	$I \geq K \geq J$	P_{213}	$(\tilde{\phi}^I, \tilde{\phi}^K, \tilde{\phi}^J)$	$(\lvert q\rvert^{\frac{2}{3}}, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{2(K-J)} + \lvert q\rvert^{\frac{2}{3}}, \lvert q\rvert^{\frac{1}{3}})$
	$K \geq J \geq I$	P_{132}	$(\tilde{\phi}^K, \tilde{\phi}^J, \tilde{\phi}^I)$	$(\lvert q\rvert^{\frac{2}{3}}, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{2(J-I)} + \lvert q\rvert^{\frac{2}{3}}, \lvert q\rvert^{\frac{1}{3}})$
$\mathbf{3} \otimes (\mathbf{1}' \oplus \mathbf{1} \oplus \mathbf{1})$	$I \geq J \geq K, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^J \geq \tilde{\phi}^I$	P_{231}	$(\lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^I, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^J, \tilde{\phi}^K)$	$(\lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{2}{3}})$
	$I \geq J \geq K, \tilde{\phi}^I \geq \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^J$	P_{231}	$(\lvert q\rvert^{\frac{2}{3}}\tilde{\phi}^J, \tilde{\phi}^I, \tilde{\phi}^K)$	$(\lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{2}{3}})$
	$J \geq K \geq I$	P_{213}	$(\lvert q\rvert^{\frac{2}{3}}\tilde{\phi}^J, \tilde{\phi}^K, \tilde{\phi}^I)$	$(\lvert q\rvert^{\frac{2}{3}}, \lvert q\rvert^{\frac{2}{3}} + \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{2(K-I)}, \lvert q\rvert^{\frac{1}{3}})$
	$K \geq I \geq J$	P_{231}	$(\lvert q\rvert^{\frac{2}{3}}\tilde{\phi}^K, \tilde{\phi}^I, \tilde{\phi}^J)$	$(\lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{2}{3}})$
	$J \geq I \geq K$	P_{231}	$(\lvert q\rvert^{\frac{2}{3}}\tilde{\phi}^J, \tilde{\phi}^I, \tilde{\phi}^K)$	$(\lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{2}{3}})$
	$I \geq K \geq J,\ \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^K \geq \tilde{\phi}^I$	P_{231}	$(\lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^I, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^K, \tilde{\phi}^J)$	$(\lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{2}{3}})$
	$I \geq K \geq J,\ \tilde{\phi}^I \geq \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^K$	P_{231}	$(\lvert q\rvert^{\frac{2}{3}}\tilde{\phi}^K, \tilde{\phi}^I, \tilde{\phi}^J)$	$(\lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{2}{3}})$
	$K \geq J \geq I$	P_{213}	$(\lvert q\rvert^{\frac{2}{3}}\tilde{\phi}^K, \tilde{\phi}^J, \tilde{\phi}^I)$	$(\lvert q\rvert^{\frac{2}{3}}, \lvert q\rvert^{\frac{2}{3}} + \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{2(J-I)}, \lvert q\rvert^{\frac{1}{3}})$
$\mathbf{3} \otimes (\mathbf{1}'' \oplus \mathbf{1} \oplus \mathbf{1})$	$I \geq J \geq K,\ \tilde{\phi}^I \geq \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^J$	P_{321}	$(\lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^J, \tilde{\phi}^I, \tilde{\phi}^K)$	$(\lvert q\rvert^{\frac{2}{3}}, \lvert q\rvert^{\frac{2}{3}} + \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{2(I-K)}, \lvert q\rvert^{\frac{1}{3}})$
	$I \geq J \geq K,\ \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^J \geq \tilde{\phi}^I$	P_{231}	$(\tilde{\phi}^I, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^J, \tilde{\phi}^K)$	$(\lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{2(I-K)} + \lvert q\rvert^{\frac{2}{3}})$
	$J \geq K \geq I$	P_{312}	$(\lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^J, \tilde{\phi}^K, \tilde{\phi}^I)$	$(\lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{2}{3}})$
	$K \geq I \geq J$	P_{321}	$(\lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^K, \tilde{\phi}^I, \tilde{\phi}^J)$	$(\lvert q\rvert^{\frac{2}{3}}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}})$
	$J \geq I \geq K$	P_{321}	$(\lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^J, \tilde{\phi}^I, \tilde{\phi}^K)$	$(\lvert q\rvert^{\frac{2}{3}}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}})$
	$I \geq K \geq J,\ \tilde{\phi}^I \geq \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^K$	P_{321}	$(\lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^K, \tilde{\phi}^I, \tilde{\phi}^J)$	$(\lvert q\rvert^{\frac{2}{3}}, \lvert q\rvert^{\frac{2}{3}} + \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{2(I-J)}, \lvert q\rvert^{\frac{1}{3}})$
	$I \geq K \geq J,\ \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^K \geq \tilde{\phi}^I$	P_{231}	$(\tilde{\phi}^I, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^K, \tilde{\phi}^J)$	$(\lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{2(I-J)} + \lvert q\rvert^{\frac{2}{3}})$
	$K \geq J \geq I$	P_{312}	$(\lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^K, \tilde{\phi}^J, \tilde{\phi}^I)$	$(\lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{2}{3}})$
$\mathbf{3} \otimes (\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1})$	$I \geq J \geq K$	P_{231}	$(\lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^I, \tilde{\phi}^J, \tilde{\phi}^K)$	$(\lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{2}{3}})$
	$J \geq K \geq I$	P_{213}	$(\lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^J, \tilde{\phi}^K, \tilde{\phi}^I)$	$(\lvert q\rvert^{\frac{2}{3}}, \lvert q\rvert^{\frac{2}{3}} + \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{2(K-I)}, \lvert q\rvert^{\frac{1}{3}})$
	$K \geq I \geq J,\ \tilde{\phi}^K \geq \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^I$	P_{213}	$(\lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^I, \tilde{\phi}^K, \tilde{\phi}^J)$	$(\lvert q\rvert^{\frac{2}{3}}, \lvert q\rvert^{\frac{2}{3}} + \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{2(K-J)}, \lvert q\rvert^{\frac{1}{3}})$
	$K \geq I \geq J,\ \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^I \geq \tilde{\phi}^K$	P_{123}	$(\tilde{\phi}^K, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^I, \tilde{\phi}^J)$	$(\lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{2}{3}} + \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{2(K-J)})$
	$J \geq I \geq K$	P_{231}	$(\lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^J, \tilde{\phi}^I, \tilde{\phi}^K)$	$(\lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{2}{3}})$
	$I \geq K \geq J$	P_{213}	$(\lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^I, \tilde{\phi}^K, \tilde{\phi}^J)$	$(\lvert q\rvert^{\frac{2}{3}}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}})$
	$K \geq J \geq I,\ \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^J \geq \tilde{\phi}^K$	P_{123}	$(\tilde{\phi}^K, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^J, \tilde{\phi}^I)$	$(\lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{2}{3}} + \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{2(K-I)})$
	$K \geq J \geq I,\ \tilde{\phi}^K \geq \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^J$	P_{213}	$(\lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^J, \tilde{\phi}^K, \tilde{\phi}^I)$	$(\lvert q\rvert^{\frac{2}{3}}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}})$

$\boldsymbol{r}_\psi \otimes \boldsymbol{r}_{\psi^c}$	power of $\tilde{\phi}$	P_ψ	$(m_{\psi_1}, m_{\psi_2}, m_{\psi_3})$	$(\theta_{12}^\psi, \theta_{23}^\psi, \theta_{13}^\psi)$
$\mathbf{3} \otimes (\mathbf{1}'' \oplus \mathbf{1}'' \oplus \mathbf{1})$	$I \geq J \geq K$	P_{321}	$(\lvert q\rvert^{\frac{2}{3}}\tilde{\phi}^I, \tilde{\phi}^J, \tilde{\phi}^K)$	$(\lvert q\rvert^{\frac{2}{3}}, \lvert q\rvert^{\frac{2}{3}} + \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{2(J-K)}, \lvert q\rvert^{\frac{1}{3}})$
	$J \geq K \geq I$	P_{312}	$(\lvert q\rvert^{\frac{2}{3}}\tilde{\phi}^J, \tilde{\phi}^K, \tilde{\phi}^I)$	$(\lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{2}{3}})$
	$K \geq I \geq J,\ \tilde{\phi}^K \geq \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^I$	P_{312}	$(\lvert q\rvert^{\frac{2}{3}}\tilde{\phi}^I, \tilde{\phi}^K, \tilde{\phi}^J)$	$(\lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{2}{3}})$
	$K \geq I \geq J,\ \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^I \geq \tilde{\phi}^K$	P_{312}	$(\lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^K, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^I, \tilde{\phi}^J)$	$(\lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{2}{3}})$
	$J \geq I \geq K$	P_{321}	$(\lvert q\rvert^{\frac{2}{3}}\tilde{\phi}^J, \tilde{\phi}^I, \tilde{\phi}^K)$	$(\lvert q\rvert^{\frac{2}{3}}, \lvert q\rvert^{\frac{2}{3}} + \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{2(I-K)}, \lvert q\rvert^{\frac{1}{3}})$
	$I \geq K \geq J$	P_{312}	$(\lvert q\rvert^{\frac{2}{3}}\tilde{\phi}^I, \tilde{\phi}^K, \tilde{\phi}^J)$	$(\lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{2}{3}})$
	$K \geq J \geq I,\ \tilde{\phi}^K \geq \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^J$	P_{312}	$(\lvert q\rvert^{\frac{2}{3}}\tilde{\phi}^J, \tilde{\phi}^K, \tilde{\phi}^I)$	$(\lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{2}{3}})$
	$K \geq J \geq I,\ \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^J \geq \tilde{\phi}^K$	P_{312}	$(\lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^K, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^J, \tilde{\phi}^I)$	$(\lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{2}{3}})$
$\mathbf{3} \otimes (\mathbf{1}'' \oplus \mathbf{1}' \oplus \mathbf{1}')$	$I \geq J \geq K,\ \tilde{\phi}^I \geq \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^J$	P_{123}	$(\lvert q\rvert^{\frac{2}{3}}\tilde{\phi}^J, \tilde{\phi}^I, \tilde{\phi}^K)$	$(\lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{2}{3}})$
	$I \geq J \geq K,\ \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^J \geq \tilde{\phi}^I$	P_{123}	$(\lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^I, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^J, \tilde{\phi}^K)$	$(\lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{2}{3}})$
	$J \geq K \geq I$	P_{132}	$(\lvert q\rvert^{\frac{2}{3}}\tilde{\phi}^J, \tilde{\phi}^K, \tilde{\phi}^I)$	$(\lvert q\rvert^{\frac{2}{3}}, \lvert q\rvert^{\frac{2}{3}} + \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{2(K-I)}, \lvert q\rvert^{\frac{1}{3}})$
	$K \geq I \geq J$	P_{123}	$(\lvert q\rvert^{\frac{2}{3}}\tilde{\phi}^K, \tilde{\phi}^I, \tilde{\phi}^J)$	$(\lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{2}{3}})$
	$J \geq I \geq K$	P_{123}	$(\lvert q\rvert^{\frac{2}{3}}\tilde{\phi}^J, \tilde{\phi}^I, \tilde{\phi}^K)$	$(\lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{2}{3}})$
	$I \geq K \geq J,\ \tilde{\phi}^I \geq \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^K$	P_{123}	$(\lvert q\rvert^{\frac{2}{3}}\tilde{\phi}^K, \tilde{\phi}^I, \tilde{\phi}^J)$	$(\lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{2}{3}})$
	$I \geq K \geq J,\ \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^K \geq \tilde{\phi}^I$	P_{123}	$(\lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^I, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^K, \tilde{\phi}^J)$	$(\lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{2}{3}})$
	$K \geq J \geq I$	P_{132}	$(\lvert q\rvert^{\frac{2}{3}}\tilde{\phi}^K, \tilde{\phi}^J, \tilde{\phi}^I)$	$(\lvert q\rvert^{\frac{2}{3}}, \lvert q\rvert^{\frac{2}{3}} + \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{2(J-I)}, \lvert q\rvert^{\frac{1}{3}})$
$\mathbf{3} \otimes (\mathbf{1}'' \oplus \mathbf{1}'' \oplus \mathbf{1}')$	$I \geq J \geq K$	P_{123}	$(\lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^I, \tilde{\phi}^J, \tilde{\phi}^K)$	$(\lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{2}{3}})$
	$J \geq K \geq I$	P_{132}	$(\lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^J, \tilde{\phi}^K, \tilde{\phi}^I)$	$(\lvert q\rvert^{\frac{2}{3}}, \lvert q\rvert^{\frac{2}{3}} + \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{2(K-I)}, \lvert q\rvert^{\frac{1}{3}})$
	$K \geq I \geq J,\ \tilde{\phi}^K \geq \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^I$	P_{132}	$(\lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^I, \tilde{\phi}^K, \tilde{\phi}^J)$	$(\lvert q\rvert^{\frac{2}{3}}, \lvert q\rvert^{\frac{2}{3}} + \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{2(K-J)}, \lvert q\rvert^{\frac{1}{3}})$
	$K \geq I \geq J,\ \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^I \geq \tilde{\phi}^K$	P_{312}	$(\tilde{\phi}^K, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^I, \tilde{\phi}^J)$	$(\lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{2}{3}})$
	$J \geq I \geq K$	P_{123}	$(\lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^J, \tilde{\phi}^I, \tilde{\phi}^K)$	$(\lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{2}{3}})$
	$I \geq K \geq J$	P_{132}	$(\lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^I, \tilde{\phi}^K, \tilde{\phi}^J)$	$(\lvert q\rvert^{\frac{2}{3}}, \lvert q\rvert^{\frac{2}{3}} + \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{2(K-J)}, \lvert q\rvert^{\frac{1}{3}})$
	$K \geq J \geq I,\ \tilde{\phi}^K \geq \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^J$	P_{132}	$(\lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^J, \tilde{\phi}^K, \tilde{\phi}^I)$	$(\lvert q\rvert^{\frac{2}{3}}, \lvert q\rvert^{\frac{2}{3}} + \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{2(K-I)}, \lvert q\rvert^{\frac{1}{3}})$
	$K \geq J \geq I,\ \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^J \geq \tilde{\phi}^K$	P_{312}	$(\tilde{\phi}^K, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^J, \tilde{\phi}^I)$	$(\lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{2}{3}})$
$\mathbf{3} \otimes (\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1})$	$I \geq J \geq K$	P_{231}	$(\lvert q\rvert^{\frac{2}{3}}\tilde{\phi}^I, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^J, \tilde{\phi}^K)$	$(\lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{2}{3}})$
$\mathbf{3} \otimes (\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}')$	$I \geq J \geq K$	P_{123}	$(\lvert q\rvert^{\frac{2}{3}}\tilde{\phi}^I, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^J, \tilde{\phi}^K)$	$(\lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{2}{3}})$
$\mathbf{3} \otimes (\mathbf{1}'' \oplus \mathbf{1}'' \oplus \mathbf{1}'')$	$I \geq J \geq K$	P_{312}	$(\lvert q\rvert^{\frac{2}{3}}\tilde{\phi}^I, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^J, \tilde{\phi}^K)$	$(\lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{2}{3}})$
$\mathbf{3} \otimes (\widehat{\mathbf{2}} \oplus \mathbf{1})$	$I = J \geq K$	P_{231}	$(\lvert q\rvert^{\frac{2}{3}}\tilde{\phi}^J, \tilde{\phi}^J, \tilde{\phi}^K)$	$(\lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{2}{3}})$
	$K \geq I = J$	P_{231}	$(\lvert q\rvert^{\frac{2}{3}}\tilde{\phi}^K, \tilde{\phi}^J, \tilde{\phi}^J)$	$(\lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{2}{3}})$
$\mathbf{3} \otimes (\widehat{\mathbf{2}} \oplus \mathbf{1}')$	$I = J \geq K$	P_{213}	$(\lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^J, \tilde{\phi}^J, \tilde{\phi}^K)$	$(\lvert q\rvert^{\frac{2}{3}}, \lvert q\rvert^{\frac{2}{3}} + \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{2(J-K)}, \lvert q\rvert^{\frac{1}{3}})$
	$K \geq I = J$	P_{231}	$(\lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^K, \tilde{\phi}^J, \tilde{\phi}^J)$	$(\lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{2}{3}})$
$\mathbf{3} \otimes (\widehat{\mathbf{2}} \oplus \mathbf{1}'')$	$I = J \geq K$	P_{312}	$(\tilde{\phi}^J, \tilde{\phi}^J, \tilde{\phi}^K)$	$(\lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{2}{3}} + \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{2(J-K)})$
	$K \geq I = J$	P_{231}	$(\tilde{\phi}^K, \tilde{\phi}^J, \tilde{\phi}^J)$	$(\lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{2}{3}} + \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{2(K-J)})$
$\mathbf{3} \otimes (\widehat{\mathbf{2}}' \oplus \mathbf{1})$	$I = J \geq K$	P_{321}	$(\tilde{\phi}^J, \tilde{\phi}^J, \tilde{\phi}^K)$	$(\lvert q\rvert^{\frac{2}{3}}, \lvert q\rvert^{\frac{2}{3}} + \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{2(J-K)}, \lvert q\rvert^{\frac{1}{3}})$
	$K \geq I = J$	P_{123}	$(\tilde{\phi}^K, \tilde{\phi}^J, \tilde{\phi}^J)$	$(\lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{2}{3}} + \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{2(K-J)})$
$\mathbf{3} \otimes (\widehat{\mathbf{2}}' \oplus \mathbf{1}')$	$I = J \geq K$	P_{123}	$(\lvert q\rvert^{\frac{2}{3}}\tilde{\phi}^J, \tilde{\phi}^J, \tilde{\phi}^K)$	$(\lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{2}{3}})$
	$K \geq I = J$	P_{123}	$(\lvert q\rvert^{\frac{2}{3}}\tilde{\phi}^K, \tilde{\phi}^J, \tilde{\phi}^J)$	$(\lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{2}{3}})$

(N=3 cont'd…)

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$\boldsymbol{r}_\psi \otimes \boldsymbol{r}_{\psi^c}$	power of $\tilde{\phi}$	P_ψ	$(m_{\psi_1}, m_{\psi_2}, m_{\psi_3})$	$(\theta^\psi_{12}, \theta^\psi_{23}, \theta^\psi_{13})$
$\mathbf{3} \otimes (\widehat{\mathbf{2}}' \oplus \mathbf{1}'')$	$I = J \geq K$	P_{132}	$(\lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^J, \tilde{\phi}^J, \tilde{\phi}^K)$	$(\lvert q\rvert^{\frac{2}{3}}, \lvert q\rvert^{\frac{2}{3}} + \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{2(J-K)}, \lvert q\rvert^{\frac{1}{3}})$
	$K \geq I = J$	P_{123}	$(\lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^K, \tilde{\phi}^J, \tilde{\phi}^J)$	$(\lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{2}{3}})$
$\mathbf{3} \otimes (\widehat{\mathbf{2}}'' \oplus \mathbf{1})$	$I = J \geq K$	P_{321}	$(\lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^J, \tilde{\phi}^J, \tilde{\phi}^K)$	$(\lvert q\rvert^{\frac{2}{3}}, \lvert q\rvert^{\frac{2}{3}} + \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{2(J-K)}, \lvert q\rvert^{\frac{1}{3}})$
	$K \geq I = J$	P_{321}	$(\lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^K, \tilde{\phi}^J, \tilde{\phi}^J)$	$(\lvert q\rvert^{\frac{2}{3}}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}})$
$\mathbf{3} \otimes (\widehat{\mathbf{2}}'' \oplus \mathbf{1}')$	$I = J \geq K$	P_{123}	$(\tilde{\phi}^J, \tilde{\phi}^J, \tilde{\phi}^K)$	$(\lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{2}{3}} + \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{2(J-K)})$
	$K \geq I = J$	P_{321}	$(\tilde{\phi}^K, \tilde{\phi}^J, \tilde{\phi}^J)$	$(\lvert q\rvert^{\frac{2}{3}}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}})$
$\mathbf{3} \otimes (\widehat{\mathbf{2}}'' \oplus \mathbf{1}'')$	$I = J \geq K$	P_{312}	$(\lvert q\rvert^{\frac{2}{3}}\tilde{\phi}^J, \tilde{\phi}^J, \tilde{\phi}^K)$	$(\lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{2}{3}})$
	$K \geq I = J$	P_{321}	$(\lvert q\rvert^{\frac{2}{3}}\tilde{\phi}^K, \tilde{\phi}^J, \tilde{\phi}^J)$	$(\lvert q\rvert^{\frac{2}{3}}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}})$
$(\mathbf{1}'' \oplus \mathbf{1}' \oplus \mathbf{1}) \otimes \mathbf{3}$	$I \geq J \geq K$	P_{123}	$(\tilde{\phi}^I, \tilde{\phi}^J, \tilde{\phi}^K)$	$(\lvert q\rvert^{\frac{2}{3}}\tilde{\phi}^{I-J}, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{J-K}, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{I-K})$
	$J \geq K \geq I$	P_{231}	$(\tilde{\phi}^J, \tilde{\phi}^K, \tilde{\phi}^I)$	$(\lvert q\rvert^{\frac{2}{3}}\tilde{\phi}^{J-K}, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{K-I}, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{J-I})$
	$K \geq I \geq J$	P_{312}	$(\tilde{\phi}^K, \tilde{\phi}^I, \tilde{\phi}^J)$	$(\lvert q\rvert^{\frac{2}{3}}\tilde{\phi}^{K-I}, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{I-J}, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{K-J})$
	$J \geq I \geq K$	P_{213}	$(\tilde{\phi}^J, \tilde{\phi}^I, \tilde{\phi}^K)$	$(\lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{J-I}, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{I-K}, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{J-K})$
	$I \geq K \geq J$	P_{132}	$(\tilde{\phi}^I, \tilde{\phi}^K, \tilde{\phi}^J)$	$(\lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{I-K}, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{K-J}, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{I-J})$
	$K \geq J \geq I$	P_{321}	$(\tilde{\phi}^K, \tilde{\phi}^J, \tilde{\phi}^I)$	$(\lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{K-J}, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{J-I}, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{K-I})$
$(\mathbf{1}' \oplus \mathbf{1} \oplus \mathbf{1}) \otimes \mathbf{3}$	$I \geq J \geq K,\ \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^J \geq \tilde{\phi}^I$	P_{123}	$(\lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^I, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^J, \tilde{\phi}^K)$	$(\lvert q\rvert^{-\frac{1}{3}}\tilde{\phi}^{I-J}, \tilde{\phi}^{J-K}, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{I-K})$
	$I \geq J \geq K,\ \tilde{\phi}^I \geq \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^J$	P_{213}	$(\lvert q\rvert^{\frac{2}{3}}\tilde{\phi}^J, \tilde{\phi}^I, \tilde{\phi}^K)$	$(\lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{J-I}, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{I-K}, \tilde{\phi}^{J-K})$
	$J \geq K \geq I$	P_{231}	$(\lvert q\rvert^{\frac{2}{3}}\tilde{\phi}^J, \tilde{\phi}^K, \tilde{\phi}^I)$	$(\tilde{\phi}^{J-K}, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{K-I}, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{J-I})$
	$K \geq I \geq J$	P_{312}	$(\lvert q\rvert^{\frac{2}{3}}\tilde{\phi}^K, \tilde{\phi}^I, \tilde{\phi}^J)$	$(\lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{K-I}, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{I-J}, \tilde{\phi}^{K-J})$
	$J \geq I \geq K$	P_{213}	$(\lvert q\rvert^{\frac{2}{3}}\tilde{\phi}^J, \tilde{\phi}^I, \tilde{\phi}^K)$	$(\lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{J-I}, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{I-K}, \tilde{\phi}^{J-K})$
	$I \geq K \geq J,\ \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^K \geq \tilde{\phi}^I$	P_{132}	$(\lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^I, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^K, \tilde{\phi}^J)$	$(\lvert q\rvert^{-\frac{1}{3}}\tilde{\phi}^{I-K}, \tilde{\phi}^{K-J}, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{I-J})$
	$I \geq K \geq J,\ \tilde{\phi}^I \geq \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^K$	P_{312}	$(\lvert q\rvert^{\frac{2}{3}}\tilde{\phi}^K, \tilde{\phi}^I, \tilde{\phi}^J)$	$(\lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{K-I}, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{I-J}, \tilde{\phi}^{K-J})$
	$K \geq J \geq I$	P_{321}	$(\lvert q\rvert^{\frac{2}{3}}\tilde{\phi}^K, \tilde{\phi}^J, \tilde{\phi}^I)$	$(\tilde{\phi}^{K-J}, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{J-I}, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{K-I})$
$(\mathbf{1}'' \oplus \mathbf{1} \oplus \mathbf{1}) \otimes \mathbf{3}$	$I \geq J \geq K,\ \tilde{\phi}^I \geq \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^J$	P_{213}	$(\lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^J, \tilde{\phi}^I, \tilde{\phi}^K)$	$(\lvert q\rvert^{\frac{2}{3}}\tilde{\phi}^{J-I}, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{I-K}, \tilde{\phi}^{J-K})$
	$I \geq J \geq K,\ \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^J \geq \tilde{\phi}^I$	P_{123}	$(\tilde{\phi}^I, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^J, \tilde{\phi}^K)$	$(\lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{I-J}, \tilde{\phi}^{J-K}, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{I-K})$
	$J \geq K \geq I$	P_{231}	$(\lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^J, \tilde{\phi}^K, \tilde{\phi}^I)$	$(\tilde{\phi}^{J-K}, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{K-I}, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{J-I})$
	$K \geq I \geq J$	P_{312}	$(\lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^K, \tilde{\phi}^I, \tilde{\phi}^J)$	$(\lvert q\rvert^{\frac{2}{3}}\tilde{\phi}^{K-I}, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{I-J}, \tilde{\phi}^{K-J})$
	$J \geq I \geq K$	P_{213}	$(\lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^J, \tilde{\phi}^I, \tilde{\phi}^K)$	$(\lvert q\rvert^{\frac{2}{3}}\tilde{\phi}^{J-I}, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{I-K}, \tilde{\phi}^{J-K})$
	$I \geq K \geq J,\ \tilde{\phi}^I \geq \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^K$	P_{312}	$(\lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^K, \tilde{\phi}^I, \tilde{\phi}^J)$	$(\lvert q\rvert^{\frac{2}{3}}\tilde{\phi}^{K-I}, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{I-J}, \tilde{\phi}^{K-J})$
	$I \geq K \geq J,\ \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^K \geq \tilde{\phi}^I$	P_{132}	$(\tilde{\phi}^I, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^K, \tilde{\phi}^J)$	$(\lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{I-K}, \tilde{\phi}^{K-J}, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{I-J})$
	$K \geq J \geq I$	P_{321}	$(\lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^K, \tilde{\phi}^J, \tilde{\phi}^I)$	$(\tilde{\phi}^{K-J}, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{J-I}, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{K-I})$

$\boldsymbol{r}_\psi \otimes \boldsymbol{r}_{\psi^c}$	power of $\tilde{\phi}$	P_ψ	$(m_{\psi_1}, m_{\psi_2}, m_{\psi_3})$	$(\theta^\psi_{12}, \theta^\psi_{23}, \theta^\psi_{13})$
$(\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}) \otimes \mathbf{3}$	$I \geq J \geq K$	P_{123}	$(\lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^I, \tilde{\phi}^J, \tilde{\phi}^K)$	$(\tilde{\phi}^{I-J}, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{J-K}, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{I-K})$
	$J \geq K \geq I$	P_{231}	$(\lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^J, \tilde{\phi}^K, \tilde{\phi}^I)$	$(\lvert q\rvert^{\frac{2}{3}}\tilde{\phi}^{J-K}, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{K-I}, \tilde{\phi}^{J-I})$
	$K \geq I \geq J,\ \tilde{\phi}^K \geq \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^I$	P_{132}	$(\lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^I, \tilde{\phi}^K, \tilde{\phi}^J)$	$(\lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{I-K}, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{K-J}, \tilde{\phi}^{I-J})$
	$K \geq I \geq J,\ \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^I \geq \tilde{\phi}^K$	P_{312}	$(\tilde{\phi}^K, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^I, \tilde{\phi}^J)$	$(\lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{K-I}, \tilde{\phi}^{I-J}, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{K-J})$
	$J \geq I \geq K$	P_{213}	$(\lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^J, \tilde{\phi}^I, \tilde{\phi}^K)$	$(\tilde{\phi}^{J-I}, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{I-K}, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{J-K})$
	$I \geq K \geq J$	P_{132}	$(\lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^I, \tilde{\phi}^K, \tilde{\phi}^J)$	$(\lvert q\rvert^{\frac{2}{3}}\tilde{\phi}^{I-K}, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{K-J}, \tilde{\phi}^{I-J})$
	$K \geq J \geq I,\ \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^J \geq \tilde{\phi}^K$	P_{321}	$(\tilde{\phi}^K, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^J, \tilde{\phi}^I)$	$(\lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{K-J}, \tilde{\phi}^{J-I}, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{K-I})$
	$K \geq J \geq I,\ \tilde{\phi}^K \geq \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^J$	P_{231}	$(\lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^J, \tilde{\phi}^K, \tilde{\phi}^I)$	$(\lvert q\rvert^{\frac{2}{3}}\tilde{\phi}^{J-K}, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{K-I}, \tilde{\phi}^{J-I})$
$(\mathbf{1}'' \oplus \mathbf{1}'' \oplus \mathbf{1}) \otimes \mathbf{3}$	$I \geq J \geq K$	P_{123}	$(\lvert q\rvert^{\frac{2}{3}}\tilde{\phi}^I, \tilde{\phi}^J, \tilde{\phi}^K)$	$(\tilde{\phi}^{I-J}, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{J-K}, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{I-K})$
	$J \geq K \geq I$	P_{231}	$(\lvert q\rvert^{\frac{2}{3}}\tilde{\phi}^J, \tilde{\phi}^K, \tilde{\phi}^I)$	$(\lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{J-K}, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{K-I}, \tilde{\phi}^{J-I})$
	$K \geq I \geq J,\ \tilde{\phi}^K \geq \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^I$	P_{132}	$(\lvert q\rvert^{\frac{2}{3}}\tilde{\phi}^I, \tilde{\phi}^K, \tilde{\phi}^J)$	$(\lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{I-K}, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{K-J}, \tilde{\phi}^{I-J})$
	$K \geq I \geq J,\ \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^I \geq \tilde{\phi}^K$	P_{312}	$(\lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^K, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^I, \tilde{\phi}^J)$	$(\lvert q\rvert^{-\frac{1}{3}}\tilde{\phi}^{K-I}, \tilde{\phi}^{I-J}, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{K-J})$
	$J \geq I \geq K$	P_{213}	$(\lvert q\rvert^{\frac{2}{3}}\tilde{\phi}^J, \tilde{\phi}^I, \tilde{\phi}^K)$	$(\tilde{\phi}^{J-I}, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{I-K}, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{J-K})$
	$I \geq K \geq J$	P_{132}	$(\lvert q\rvert^{\frac{2}{3}}\tilde{\phi}^I, \tilde{\phi}^K, \tilde{\phi}^J)$	$(\lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{I-K}, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{K-J}, \tilde{\phi}^{I-J})$
	$K \geq J \geq I,\ \tilde{\phi}^K \geq \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^J$	P_{231}	$(\lvert q\rvert^{\frac{2}{3}}\tilde{\phi}^J, \tilde{\phi}^K, \tilde{\phi}^I)$	$(\lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{J-K}, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{K-I}, \tilde{\phi}^{J-I})$
	$K \geq J \geq I,\ \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^J \geq \tilde{\phi}^K$	P_{321}	$(\lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^K, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^J, \tilde{\phi}^I)$	$(\lvert q\rvert^{-\frac{1}{3}}\tilde{\phi}^{K-J}, \tilde{\phi}^{J-I}, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{K-I})$
$(\mathbf{1}'' \oplus \mathbf{1}' \oplus \mathbf{1}') \otimes \mathbf{3}$	$I \geq J \geq K,\ \tilde{\phi}^I \geq^{\frac{1}{3}} \tilde{\phi}^J$	P_{213}	$(\lvert q\rvert^{\frac{2}{3}}\tilde{\phi}^J, \tilde{\phi}^I, \tilde{\phi}^K)$	$(\lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{J-I}, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{I-K}, \tilde{\phi}^{J-K})$
	$I \geq J \geq K,\ \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^J \geq \tilde{\phi}^I$	P_{123}	$(\lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^I, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^J, \tilde{\phi}^K)$	$(\lvert q\rvert^{-\frac{1}{3}}\tilde{\phi}^{I-J}, \tilde{\phi}^{J-K}, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{I-K})$
	$J \geq K \geq I$	P_{231}	$(\lvert q\rvert^{\frac{2}{3}}\tilde{\phi}^J, \tilde{\phi}^K, \tilde{\phi}^I)$	$(\tilde{\phi}^{J-K}, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{K-I}, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{J-I})$
	$K \geq I \geq J$	P_{312}	$(\lvert q\rvert^{\frac{2}{3}}\tilde{\phi}^K, \tilde{\phi}^I, \tilde{\phi}^J)$	$(\lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{K-I}, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{I-J}, \tilde{\phi}^{K-J})$
	$J \geq I \geq K$	P_{213}	$(\lvert q\rvert^{\frac{2}{3}}\tilde{\phi}^J, \tilde{\phi}^I, \tilde{\phi}^K)$	$(\lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{J-I}, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{I-K}, \tilde{\phi}^{J-K})$
	$I \geq K \geq J,\ \tilde{\phi}^I \geq \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^K$	P_{312}	$(\lvert q\rvert^{\frac{2}{3}}\tilde{\phi}^K, \tilde{\phi}^I, \tilde{\phi}^J)$	$(\lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{K-I}, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{I-J}, \tilde{\phi}^{K-J})$
	$I \geq K \geq J,\ \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^K \geq \tilde{\phi}^I$	P_{132}	$(\lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^I, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^K, \tilde{\phi}^J)$	$(\lvert q\rvert^{-\frac{1}{3}}\tilde{\phi}^{I-K}, \tilde{\phi}^{K-J}, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{I-J})$
	$K \geq J \geq I$	P_{321}	$(\lvert q\rvert^{\frac{2}{3}}\tilde{\phi}^K, \tilde{\phi}^J, \tilde{\phi}^I)$	$(\tilde{\phi}^{K-J}, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{J-I}, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{K-I})$
$(\mathbf{1}'' \oplus \mathbf{1}'' \oplus \mathbf{1}') \otimes \mathbf{3}$	$I \geq J \geq K$	P_{123}	$(\lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^I, \tilde{\phi}^J, \tilde{\phi}^K)$	$(\tilde{\phi}^{I-J}, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{J-K}, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{I-K})$
	$J \geq K \geq I$	P_{231}	$(\lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^J, \tilde{\phi}^K, \tilde{\phi}^I)$	$(\lvert q\rvert^{\frac{2}{3}}\tilde{\phi}^{J-K}, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{K-I}, \tilde{\phi}^{J-I})$
	$K \geq I \geq J,\ \tilde{\phi}^K \geq \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^I$	P_{132}	$(\lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^I, \tilde{\phi}^K, \tilde{\phi}^J)$	$(\lvert q\rvert^{\frac{2}{3}}\tilde{\phi}^{I-K}, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{K-J}, \tilde{\phi}^{I-J})$
	$K \geq I \geq J,\ \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^I \geq \tilde{\phi}^K$	P_{312}	$(\tilde{\phi}^K, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^I, \tilde{\phi}^J)$	$(\lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{K-I}, \tilde{\phi}^{I-J}, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{K-J})$
	$J \geq I \geq K$	P_{213}	$(\lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^J, \tilde{\phi}^I, \tilde{\phi}^K)$	$(\tilde{\phi}^{J-I}, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{I-K}, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{J-K})$
	$I \geq K \geq J$	P_{132}	$(\lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^I, \tilde{\phi}^K, \tilde{\phi}^J)$	$(\lvert q\rvert^{\frac{2}{3}}\tilde{\phi}^{I-K}, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{K-J}, \tilde{\phi}^{I-J})$
	$K \geq J \geq I,\ \tilde{\phi}^K \geq \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^J$	P_{231}	$(\lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^J, \tilde{\phi}^K, \tilde{\phi}^I)$	$(\lvert q\rvert^{\frac{2}{3}}\tilde{\phi}^{J-K}, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{K-I}, \tilde{\phi}^{J-I})$
	$K \geq J \geq I,\ \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^J \geq \tilde{\phi}^K$	P_{321}	$(\tilde{\phi}^K, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^J, \tilde{\phi}^I)$	$(\lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{K-J}, \tilde{\phi}^{J-I}, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{K-I})$
$(\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}) \otimes \mathbf{3}$	$I \geq J \geq K$	P_{123}	$(\lvert q\rvert^{\frac{2}{3}}\tilde{\phi}^I, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^J, \tilde{\phi}^K)$	$(\tilde{\phi}^{I-J}, \tilde{\phi}^{J-K}, \tilde{\phi}^{I-K})$
$(\mathbf{1}' \oplus \mathbf{1}' \oplus \mathbf{1}') \otimes \mathbf{3}$	$I \geq J \geq K$	P_{123}	$(\lvert q\rvert^{\frac{2}{3}}\tilde{\phi}^I, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^J, \tilde{\phi}^K)$	$(\tilde{\phi}^{I-J}, \tilde{\phi}^{J-K}, \tilde{\phi}^{I-K})$
$(\mathbf{1}'' \oplus \mathbf{1}'' \oplus \mathbf{1}'') \otimes \mathbf{3}$	$I \geq J \geq K$	P_{123}	$(\lvert q\rvert^{\frac{2}{3}}\tilde{\phi}^I, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^J, \tilde{\phi}^K)$	$(\tilde{\phi}^{I-J}, \tilde{\phi}^{J-K}, \tilde{\phi}^{I-K})$
$(\widehat{\mathbf{2}} \oplus \mathbf{1}) \otimes \mathbf{3}$	$I = J \geq K$	P_{213}	$(\lvert q\rvert^{\frac{2}{3}}\tilde{\phi}^J, \tilde{\phi}^J, \tilde{\phi}^K)$	$(\lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{J-K}, \tilde{\phi}^{J-K})$
	$K \geq I = J$	P_{312}	$(\lvert q\rvert^{\frac{2}{3}}\tilde{\phi}^K, \tilde{\phi}^J, \tilde{\phi}^J)$	$(\lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}\tilde{\phi}^{K-J}, \tilde{\phi}^{K-J})$

(N=3 cont'd…)

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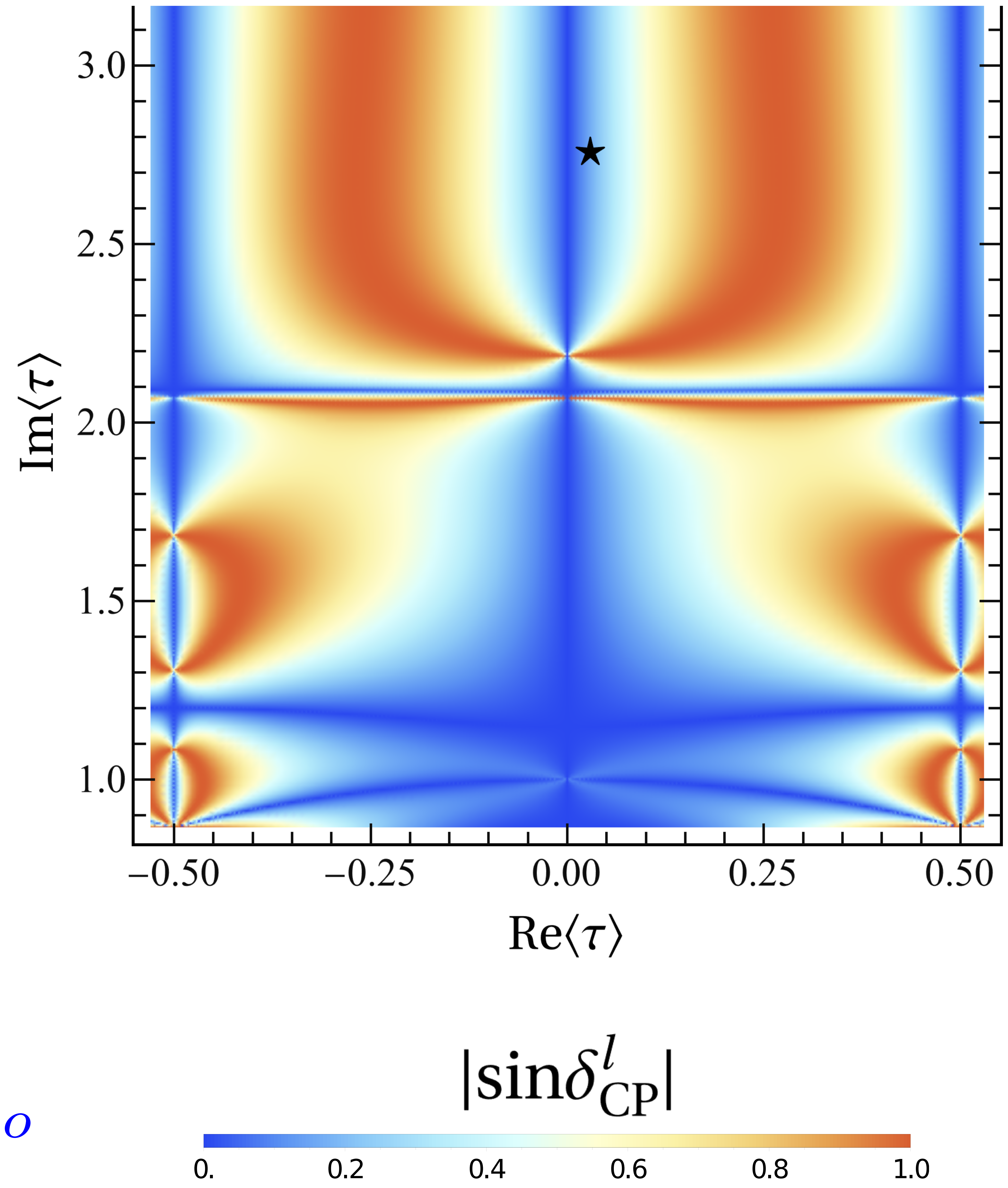
$\boldsymbol{r}_\psi \otimes \boldsymbol{r}_{\psi^c}$	power of $\tilde\phi$	P_ψ	$(m_{\psi_1}, m_{\psi_2}, m_{\psi_3})$	$(\theta^\psi_{12}, \theta^\psi_{23}, \theta^\psi_{13})$
$(\widehat{\mathbf{2}} \oplus \mathbf{1}') \otimes \mathbf{3}$	$I = J \geq K$	P_{123}	$(\lvert q\rvert^{\frac{1}{3}}\tilde\phi^J, \tilde\phi^J, \tilde\phi^K)$	$(\lvert q\rvert^{\frac{2}{3}}, \lvert q\rvert^{\frac{1}{3}}\tilde\phi^{J-K}, \tilde\phi^{J-K})$
	$K \geq I = J$	P_{312}	$(\lvert q\rvert^{\frac{1}{3}}\tilde\phi^K, \tilde\phi^J, \tilde\phi^J)$	$(\tilde\phi^{K-J}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}\tilde\phi^{K-J})$
$(\widehat{\mathbf{2}} \oplus \mathbf{1}'') \otimes \mathbf{3}$	$I = J \geq K$	P_{123}	$(\tilde\phi^J, \tilde\phi^J, \tilde\phi^K)$	$(\lvert q\rvert^{\frac{2}{3}}, \lvert q\rvert^{\frac{1}{3}}\tilde\phi^{J-K}, \lvert q\rvert^{\frac{1}{3}}\tilde\phi^{J-K})$
	$K \geq I = J$	P_{312}	$(\tilde\phi^K, \tilde\phi^J, \tilde\phi^J)$	$(\lvert q\rvert^{\frac{2}{3}}\tilde\phi^{K-J}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}\tilde\phi^{K-J})$
$(\widehat{\mathbf{2}}' \oplus \mathbf{1}) \otimes \mathbf{3}$	$I = J \geq K$	P_{213}	$(\tilde\phi^J, \tilde\phi^J, \tilde\phi^K)$	$(\lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}\tilde\phi^{J-K}, \lvert q\rvert^{\frac{1}{3}}\tilde\phi^{J-K})$
	$K \geq I = J$	P_{312}	$(\tilde\phi^K, \tilde\phi^J, \tilde\phi^J)$	$(\lvert q\rvert^{\frac{2}{3}}\tilde\phi^{K-J}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}\tilde\phi^{K-J})$
$(\widehat{\mathbf{2}}' \oplus \mathbf{1}') \otimes \mathbf{3}$	$I = J \geq K$	P_{213}	$(\lvert q\rvert^{\frac{2}{3}}\tilde\phi^J, \tilde\phi^J, \tilde\phi^K)$	$(\lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}\tilde\phi^{J-K}, \tilde\phi^{J-K})$
	$K \geq I = J$	P_{312}	$(\lvert q\rvert^{\frac{2}{3}}\tilde\phi^K, \tilde\phi^J, \tilde\phi^J)$	$(\lvert q\rvert^{\frac{1}{3}}\tilde\phi^{K-J}, \lvert q\rvert^{\frac{1}{3}}, \tilde\phi^{K-J})$
$(\widehat{\mathbf{2}}' \oplus \mathbf{1}'') \otimes \mathbf{3}$	$I = J \geq K$	P_{123}	$(\lvert q\rvert^{\frac{1}{3}}\tilde\phi^J, \tilde\phi^J, \tilde\phi^K)$	$(\lvert q\rvert^{\frac{2}{3}}, \lvert q\rvert^{\frac{1}{3}}\tilde\phi^{J-K}, \tilde\phi^{J-K})$
	$K \geq I = J$	P_{312}	$(\lvert q\rvert^{\frac{1}{3}}\tilde\phi^K, \tilde\phi^J, \tilde\phi^J)$	$(\tilde\phi^{K-J}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}\tilde\phi^{K-J})$
$(\widehat{\mathbf{2}}'' \oplus \mathbf{1}) \otimes \mathbf{3}$	$I = J \geq K$	P_{123}	$(\lvert q\rvert^{\frac{1}{3}}\tilde\phi^J, \tilde\phi^J, \tilde\phi^K)$	$(\lvert q\rvert^{\frac{2}{3}}, \lvert q\rvert^{\frac{1}{3}}\tilde\phi^{J-K}, \tilde\phi^{J-K})$
	$K \geq I = J$	P_{321}	$(\lvert q\rvert^{\frac{1}{3}}\tilde\phi^K, \tilde\phi^J, \tilde\phi^J)$	$(\lvert q\rvert^{\frac{1}{3}}\tilde\phi^{K-J}, \lvert q\rvert^{\frac{1}{3}}, \tilde\phi^{K-J})$
$(\widehat{\mathbf{2}}'' \oplus \mathbf{1}') \otimes \mathbf{3}$	$I = J \geq K$	P_{123}	$(\tilde\phi^J, \tilde\phi^J, \tilde\phi^K)$	$(\lvert q\rvert^{\frac{2}{3}}, \lvert q\rvert^{\frac{1}{3}}\tilde\phi^{J-K}, \lvert q\rvert^{\frac{1}{3}}\tilde\phi^{J-K})$
	$K \geq I = J$	P_{321}	$(\tilde\phi^K, \tilde\phi^J, \tilde\phi^J)$	$(\lvert q\rvert^{\frac{1}{3}}\tilde\phi^{K-J}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}\tilde\phi^{K-J})$
$(\widehat{\mathbf{2}}'' \oplus \mathbf{1}'') \otimes \mathbf{3}$	$I = J \geq K$	P_{213}	$(\lvert q\rvert^{\frac{2}{3}}\tilde\phi^J, \tilde\phi^J, \tilde\phi^K)$	$(\lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}\tilde\phi^{J-K}, \tilde\phi^{J-K})$
	$K \geq I = J$	P_{321}	$(\lvert q\rvert^{\frac{2}{3}}\tilde\phi^K, \tilde\phi^J, \tilde\phi^J)$	$(\tilde\phi^{K-J}, \lvert q\rvert^{\frac{1}{3}}, \lvert q\rvert^{\frac{1}{3}}\tilde\phi^{K-J})$



For example…

$$L \sim (\mathbf{3}, 2)\,,\quad e^c \sim (\mathbf{1}, -1)\,,\quad \mu^c \sim (\mathbf{1}, 1)\,,\quad \tau^c \sim (\mathbf{1''}, 3)\,,\quad H_{u,d} \sim (\mathbf{1}, 0)\,,$$
$$Q_L \sim (\mathbf{3}, k_{Q_L})\,,\quad u^c \sim (\mathbf{1}, -k_{Q_L})\,,\quad c^c \sim (\mathbf{1''}, 5 - k_{Q_L})\,,\quad t^c \sim (\mathbf{1'}, 4 - k_{Q_L})\,,$$
$$D^c_D \equiv \{d^c, s^c\} \sim \left(\widehat{\mathbf{2}}', 2 - k_{Q_L}\right)\,,\quad b^c \sim (\mathbf{1}, 5 - k_{Q_L})\,,\quad \phi \sim (\mathbf{1}, 1)\,.$$

18 parameter fit to fermion mass and mixing data, predicts $\delta^l_{CP} = 190^o$

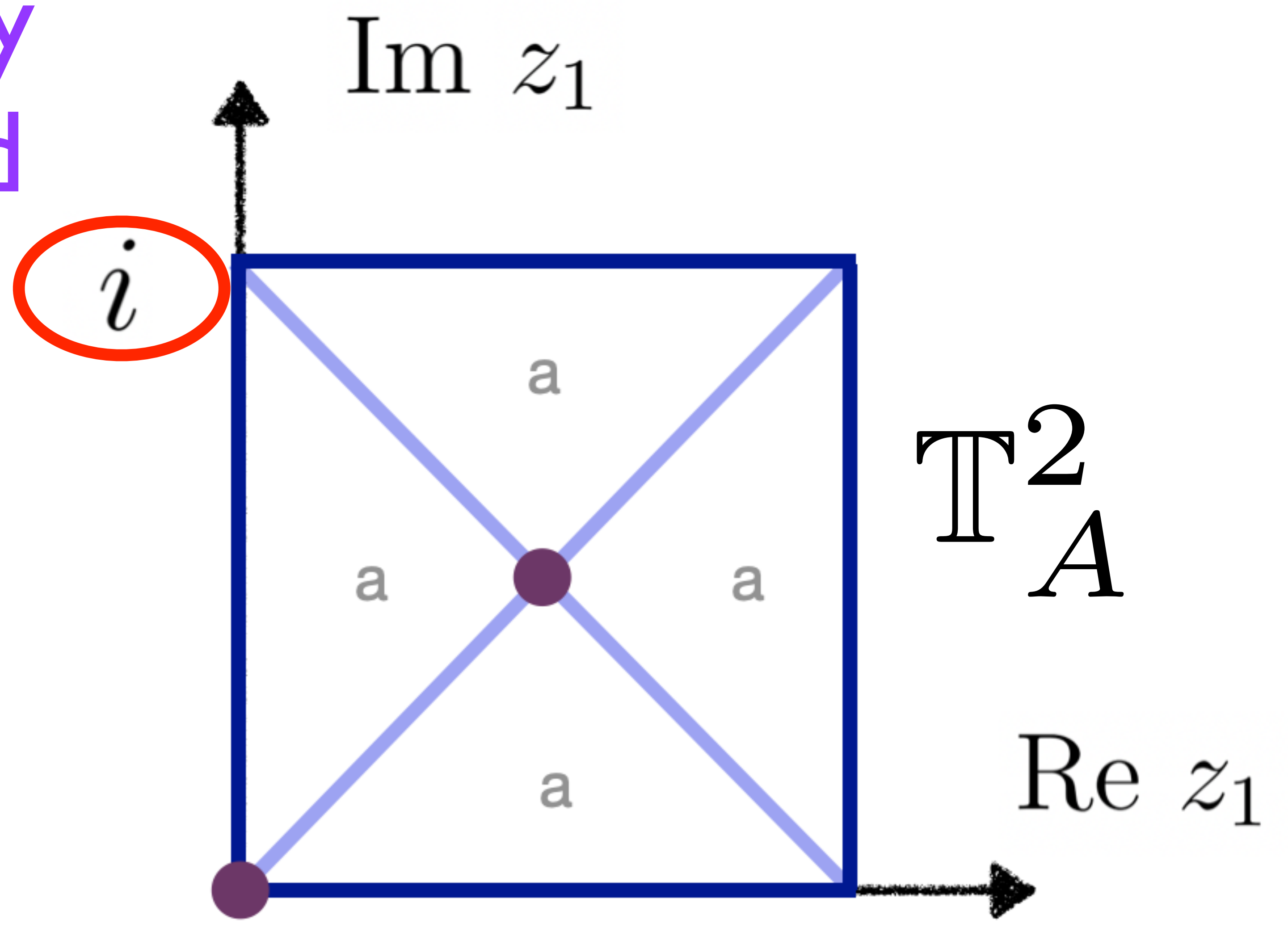
10d models $(\mathbb{T}^2)^3/(\mathbb{Z}_4 \times \mathbb{Z}_2)$

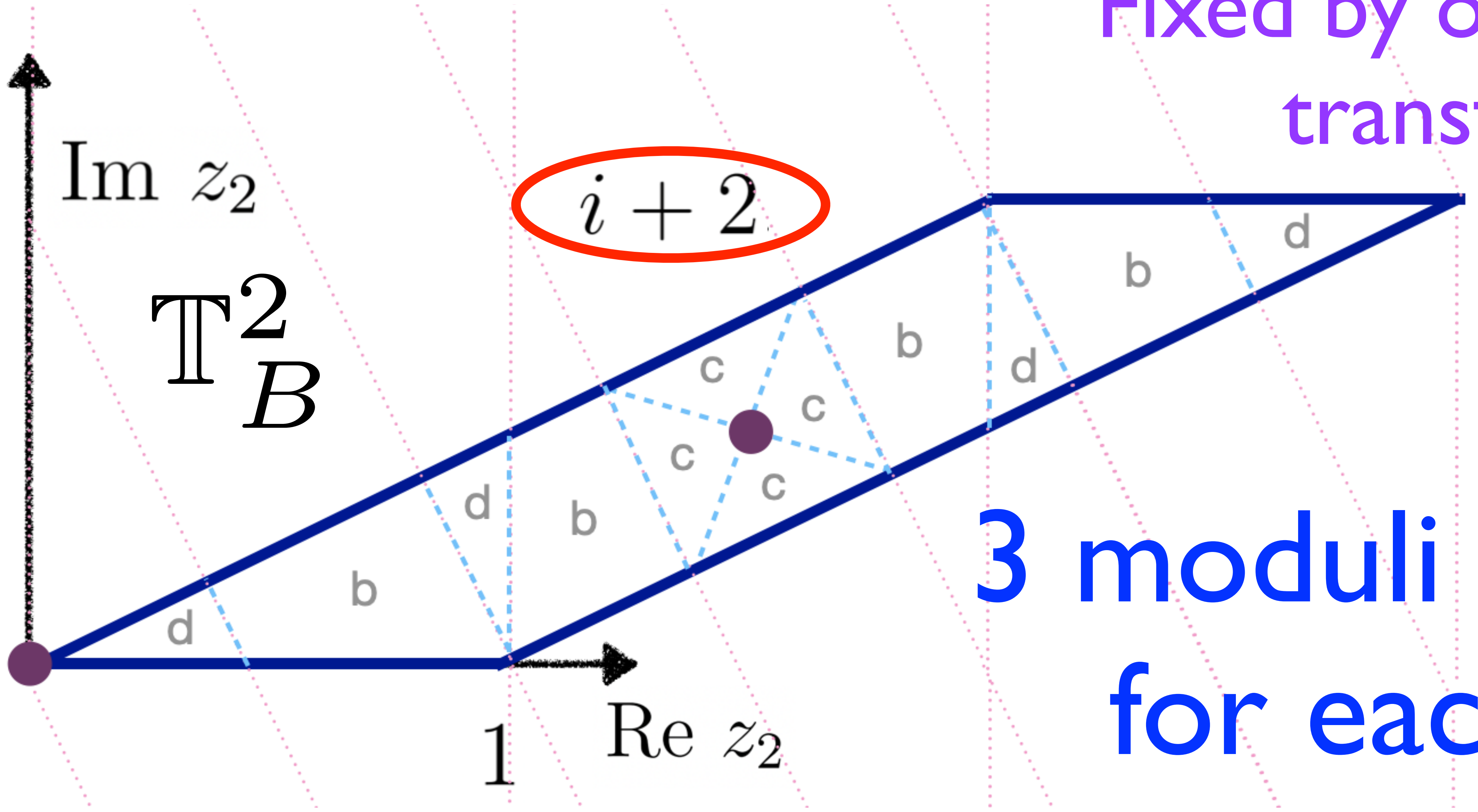
De Anda, S.F.K. 2312.09010, 2304.05958

M. Fischer, M. Ratz, J. Torrado and P. K. S. Vaudrevange 1209.3906

3 factorable tori, SUSY preserving

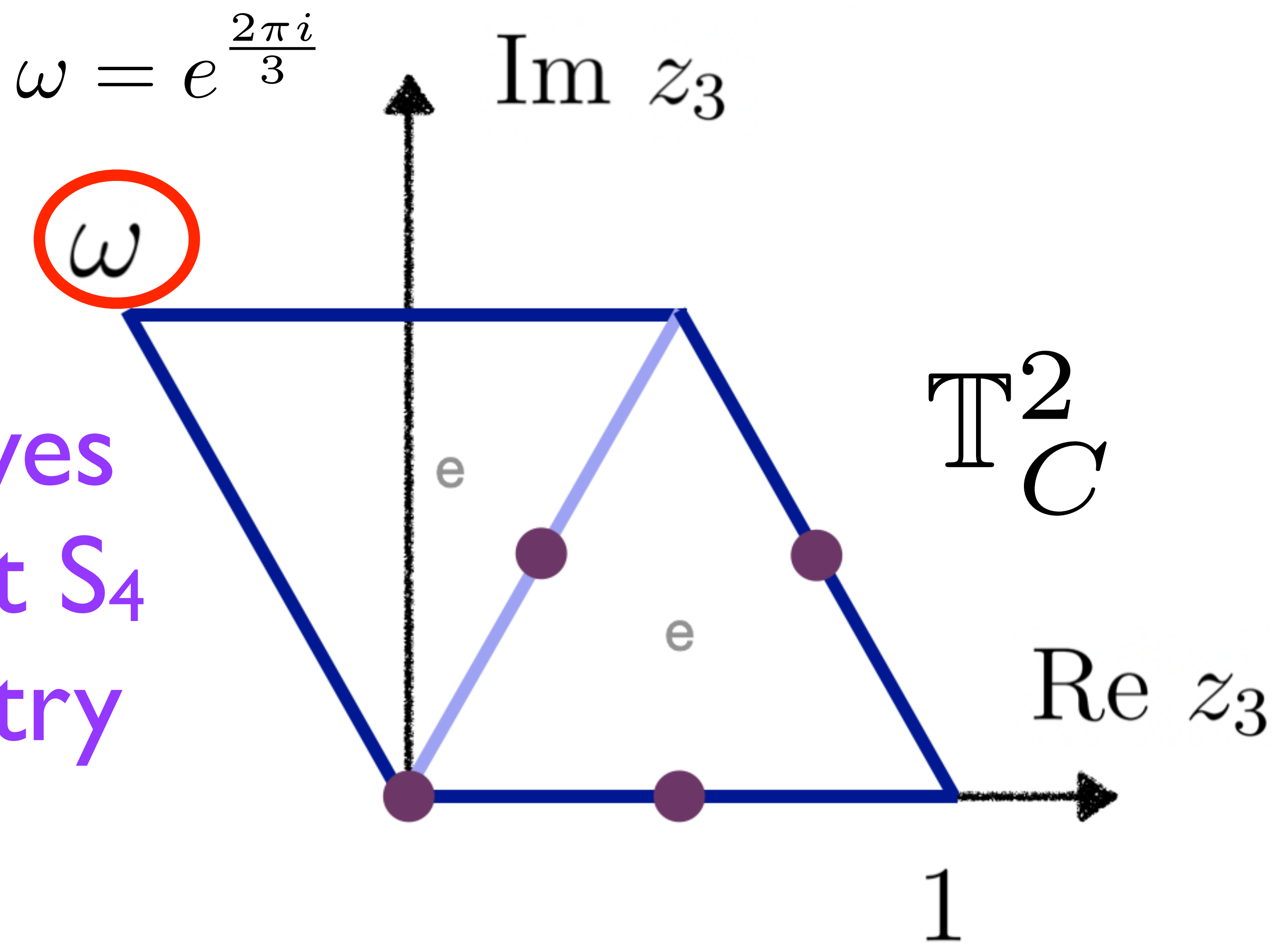
Fixed by orbifold





Fixed by orbifold up to T transformation

3 moduli fields, one for each torus



Preserves remnant S_4 symmetry

Lattice vectors for each torus are $(1, \tau_i)$

$$\boxed{\tau_1 = i, \;\; \tau_2 = i+2, \;\; \tau_3 = \omega}$$

Three moduli at stable fixed points…

S.F.K. and X.Wang,2310.10369

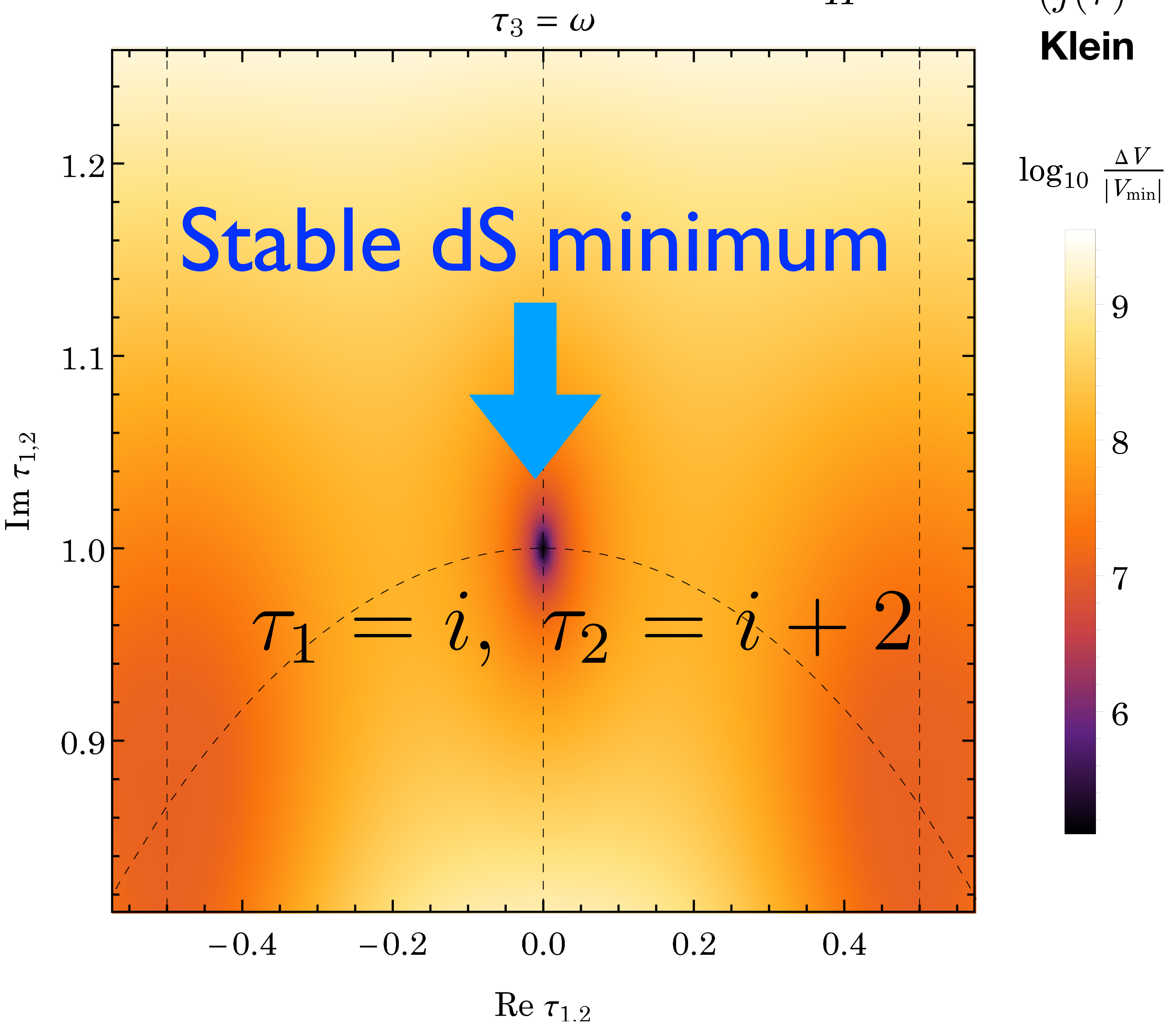
Three moduli at stable fixed points

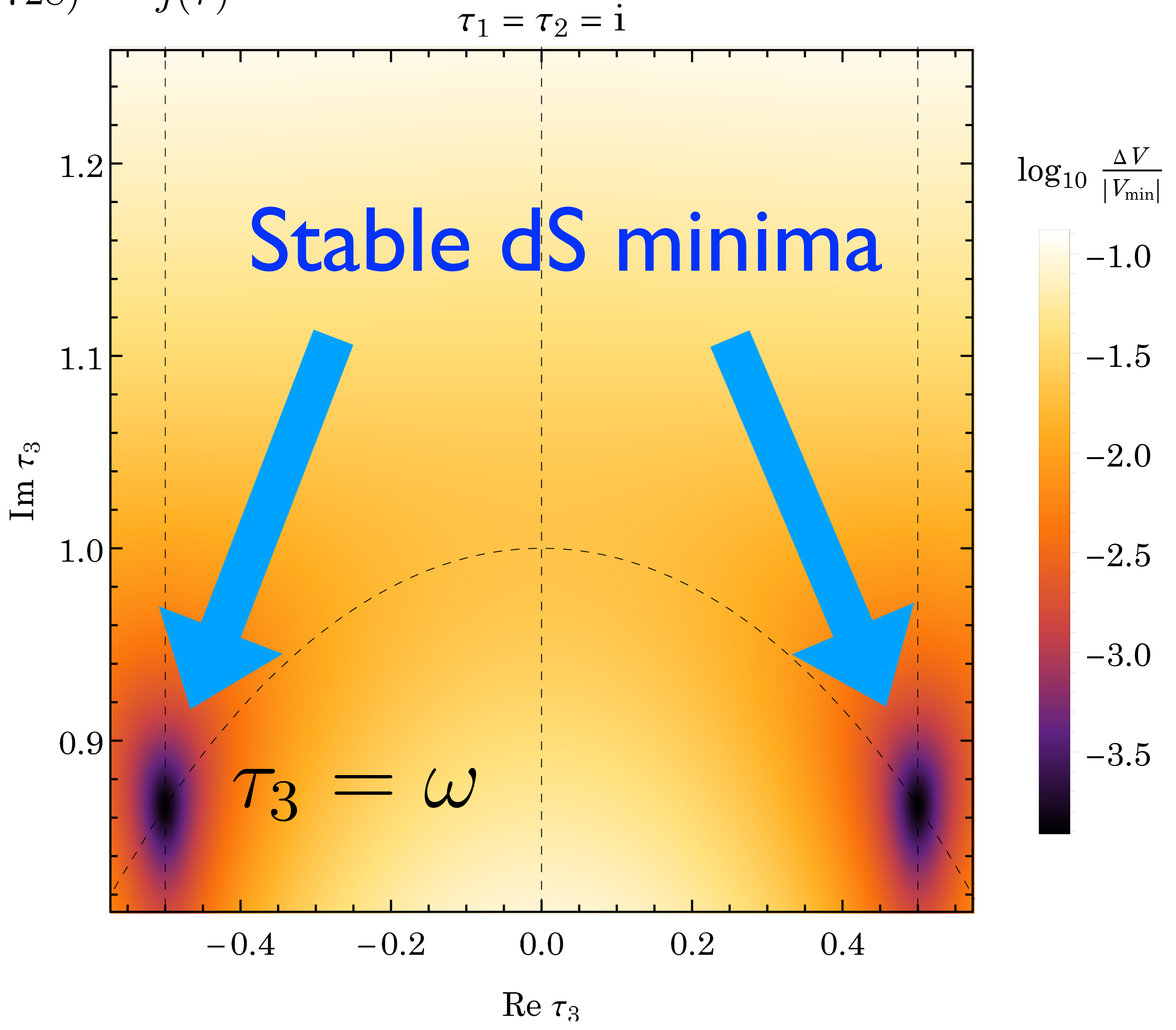
$$\mathcal{K}(\tau_i,\overline{\tau}_i,S,\overline{S}) = \Lambda_K^2\{K(S,\overline{S}) - \log[(2\,\mathrm{Im}\,\tau_1)(2\,\mathrm{Im}\,\tau_2)(2\,\mathrm{Im}\,\tau_3)]\}$$

$$\mathcal{W}(\tau_i,S) = \frac{\Lambda_W^3\Omega(S)[H^{(m_1,n_1)}(\tau_1)+H^{(m_2,n_2)}(\tau_2)+H^{(m_3,n_3)}(\tau_3)]}{\eta^2(\tau_1)\eta^2(\tau_2)\eta^2(\tau_3)}$$

$$H^{(m_i,n_i)} = (j(\tau)-1728)^{m_i/2}j(\tau)^{n_i/3}$$

Klein





De Anda, S.F.K. 2312.09010, 2304.05958

SU(5) orbifold GUT in 10d

$(\mathbb{T}^2)^3/(\mathbb{Z}_4 \times \mathbb{Z}_2)$

Field	$SU(5)$	S_4^A	S_4^B	S_4^C	k_A	k_B	k_C	Loc
F	$\mathbf{5}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{3}$	0	0	0	$\mathbb{T}^2_C$
T_1	$\mathbf{10}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	0	0	1	$\mathbb{T}^2_C$
T_2	$\mathbf{10}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	0	0	$1/2$	$\mathbb{T}^2_C$
T_3	$\mathbf{10}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	0	0	0	$\mathbb{T}^2_C$
N_a^c	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	0	-4	0	$\mathbb{T}^2_B$
N_s^c	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	-2	0	0	$\mathbb{T}^2_A$
H_u	$\mathbf{5}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	0	0	0	Bulk
H_d	$\overline{\mathbf{5}}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	0	0	$1/2$	Bulk
H_{45}	$\mathbf{45}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	0	0	$1/2$	Bulk
$H_{\overline{45}}$	$\overline{\mathbf{45}}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	0	0	0	Bulk
Φ_{BC}	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{3}$	$\mathbf{3}$	0	0	0	Bulk
Φ_{AC}	$\mathbf{1}$	$\mathbf{3}$	$\mathbf{1}$	$\mathbf{3}$	0	0	0	Bulk
ξ_F	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	0	0	$-5/2$	$\mathbb{T}^2_C$
ξ_T	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	0	0	$-1/2$	$\mathbb{T}^2_C$

Weightons

10d Lagrangian

$$
\begin{aligned}
\mathcal{L}^{(0)}_{10d} = & \left(\frac{y^u_{33}}{\Lambda^5} T_3 T_3 + \frac{y^u_{23}}{\Lambda^7} \xi_T T_2 T_3\right) H_u \delta^6(z) \\
& + \left(\frac{y^u_{22}}{\Lambda^9} \xi_T^2 T_2 T_2 + \frac{y^u_{13}}{\Lambda^9} \xi_T^2 T_1 T_3 + \frac{y^u_{12}}{\Lambda^{11}} (\xi_T^3 + \xi_F^3) T_1 T_2\right) H_u \delta^6(z) \\
& + \left(\frac{Y_a}{\Lambda^9} F N_a^c \Phi_{BC} + \frac{Y_s}{\Lambda^9} F N_s^c \Phi_{AC}\right) H_u \delta^6(z) \\
& + \left(\frac{Y_{5\tau}}{\Lambda^7} \xi_F F T_3 + \frac{Y'_{5\tau}}{\Lambda^9} \xi_F \xi_T F T_2 + \frac{Y''_{5\tau}}{\Lambda^{11}} \xi_F \xi_T^2 F T_1\right) H_{5d} \delta^6(z) \\
& + \left(\frac{Y_{5\mu}}{\Lambda^9} \xi_F^2 F T_2 + \frac{Y'_{5\mu}}{\Lambda^{11}} \xi_F^2 \xi_T F T_1 + \frac{Y_{5e}}{\Lambda^{11}} \xi_F^3 F T_1\right) H_{5d} \delta^6(z) \\
& + (H^{(0)}_{5d} \to H^{(0)}_{45d} \text{ terms}) \\
& + \frac{M_a}{2} N_a^c N_a^c \delta^2(z_1) \delta^2(z_3) + \frac{M_s}{2} N_s^c N_s^c \delta^2(z_2) \delta^2(z_3),
\end{aligned}
$$

Yuk/Mass	S_4^A	S_4^B	S_4^C	$2k_A$	$2k_B$	$2k_C$
$Y_e(\tau_3)$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{3}$	0	0	6
$Y_\mu(\tau_3)$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{3}$	0	0	4
$Y_\tau(\tau_3)$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{3}$	0	0	2
$Y_a(\tau_2)$	$\mathbf{1}$	$\mathbf{3}$	$\mathbf{1}$	0	4	0
$Y_s(\tau_1)$	$\mathbf{3}$	$\mathbf{1}$	$\mathbf{1}$	2	0	0
$M_a(\tau_2)$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	0	8	0
$M_s(\tau_1)$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	4	0	0

$$
\begin{aligned}
Y_a &= (0, 1, -1)^T, \\
Y_s &= (1, 1+\sqrt{6}, 1-\sqrt{6})^T, \\
Y_\tau &= (0, 1, 0)^T, \\
Y_\mu &= (0, 0, 1)^T, \\
Y_e &= (1, 0, 0)^T.
\end{aligned}
$$

Fixed Points

I.de Medeiros Varzielas, S.F.K., M.Levy, 2211.00654, 2309.15901

(See Talk by Ivo)

De Anda, S.F.K. 2312.09010, 2304.05958

$$M_u = \begin{pmatrix} 0 & y^u_{12}\tilde{\xi}^3_{T,F}e^{i\phi_{u1}} & y^u_{13}\tilde{\xi}^2_T \\ y^u_{12}\tilde{\xi}^3_{T,F}e^{i\phi_{u1}} & y^u_{22}\tilde{\xi}^2_T & y^u_{23}\tilde{\xi}_T e^{i\phi_{u2}} \\ y^u_{13}\tilde{\xi}^2_T & y^u_{23}\tilde{\xi}_T e^{i\phi_{u2}} & y^u_{33} \end{pmatrix} v_u$$

$$m_u \sim \tilde{\xi}^4_{T,F} v_u,\ m_c \sim \tilde{\xi}^2_T v_u,\ m_t \sim v_u$$

$$M_d = \begin{pmatrix} y_{d11}\tilde{\xi}^3_F & y_{d12}\tilde{\xi}^2_F\tilde{\xi}_T & y_{d13}\tilde{\xi}_F\tilde{\xi}^2_T \\ 0 & y_{d22}\tilde{\xi}^2_F & y_{d23}\tilde{\xi}_F\tilde{\xi}_T e^{i\phi_{d2}} \\ 0 & 0 & y_{d33}\tilde{\xi}_F \end{pmatrix} v_d,$$

LR convention

CKM mixing from M_d, M_u

$$M_e = \begin{pmatrix} y_{e11}\tilde{\xi}^3_F & 0 & 0 \\ y_{e21}\tilde{\xi}^2_F\tilde{\xi}_T & y_{e22}\tilde{\xi}^2_F & 0 \\ y_{e31}\tilde{\xi}_F\tilde{\xi}^2_T & y_{e32}\tilde{\xi}_F\tilde{\xi}_T e^{i\phi_{d1}} & y_{e33}\tilde{\xi}_F \end{pmatrix} v_d,$$

Triangular form
No PMNS
mixing from M_e

$$m_d \sim m_e \sim \tilde{\xi}^3_F v_d$$

$$m_s \sim m_\mu \sim \tilde{\xi}^2_F v_d$$

$$m_b \sim m_\tau \sim \tilde{\xi}_F v_d$$

Dirac neutrino matrix

$$M_D = \begin{pmatrix} 0 & y_s\tilde{\Phi}_{AC} \\ y_a\tilde{\Phi}_{BC} & y_s\tilde{\Phi}_{AC}(1-\sqrt{6}) \\ -y_a\tilde{\Phi}_{BC} & y_s\tilde{\Phi}_{AC}(1+\sqrt{6}) \end{pmatrix} v_u, \quad M_N = \begin{pmatrix} M_a & 0 \\ 0 & M_s \end{pmatrix}$$

Heavy Majorana neutrino matrix $\Rightarrow$ Seesaw mechanism

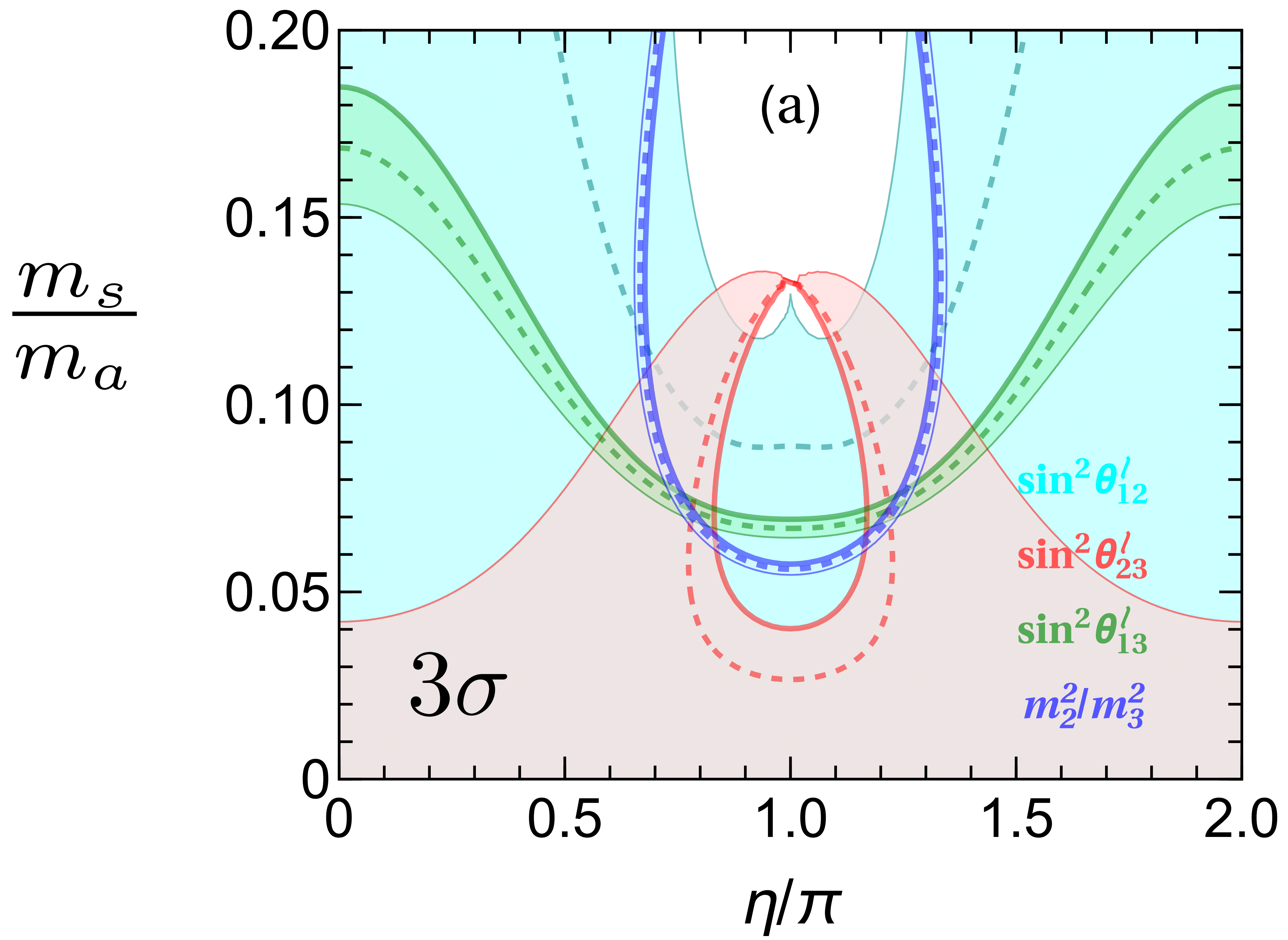
Seesaw mechanism gives 3 parameter neutrino mass matrix

$$m_\nu = m_a \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} + m_s e^{i\eta} \begin{pmatrix} 1 & 1-\sqrt{6} & 1+\sqrt{6} \\ 1-\sqrt{6} & 7-2\sqrt{6} & -5 \\ 1+\sqrt{6} & -5 & 7+2\sqrt{6} \end{pmatrix}$$

"Littlest Seesaw"

G.J.Ding, S.F.K, X.G.Liu and J.N.Lu,1910.03460

G.J.Ding, S.F.K. and C.Y.Yao, 2103.16311



De Anda, SFK 2304.05958

	without SK atmospheric data	
	NuFit $\pm 1\sigma$	Model
$\theta_{12}/^\circ$	$33.41^{+0.75}_{-0.72}$	34.34
$\theta_{23}/^\circ$	$49.1^{+1.0}_{-1.3}$	48.31
$\theta_{13}/^\circ$	$8.54^{+0.11}_{-0.12}$	8.54
$\delta/^\circ$	197^{+42}_{-25}	284
$\frac{\Delta m^2_{21}}{10^{-5}\ \mathrm{eV}^2}$	$7.41^{+0.21}_{-0.20}$	7.42
$\frac{\Delta m^2_{3\ell}}{10^{-3}\ \mathrm{eV}^2}$	$+2.511^{+0.028}_{-0.021}$	2.510
$\frac{m_a}{10^{-3}\ \mathrm{eV}}$		31.47
$\frac{m_b}{10^{-3}\ \mathrm{eV}}$	$\frac{m_{\beta\beta}}{10^{-3}\ eV}$	2.28
η/π		1.24
χ^2	Three parameter fit	6.3

6 measured observables

3 input parameters

See talk by A.Trautner, 2505.00099

[Chen, Ramon-Sanchez, Ratz 1909.06910]

Unconstrained Kahler potential

$$\mathcal{K} = \boxed{(-i\tau + i\bar{\tau})^{-k_\psi} \left(\psi^\dagger \psi\right)_{\mathbf{1}}} + \boxed{\sum_{n,\boldsymbol{r}_1,\boldsymbol{r}_2} c^{(n,\boldsymbol{r}_1,\boldsymbol{r}_2)}(-i\tau + i\bar{\tau})^{-k_\psi+n} \left(\psi^\dagger Y^{(n)\dagger}_{\boldsymbol{r}_1} Y^{(n)}_{\boldsymbol{r}_2} \psi\right)_{\mathbf{1}}}$$

minimal Kähler potential

non-canonical terms
unsuppressed since τ is dimensionless

Canonical normalisation can lead to sizeable corrections

One solution is to introduce a flavour symmetry as well as modular symmetry, leading to so called "Eclectic Flavour Symmetry"

[Nilles et al, 2001.01736; 2004.05200]

In general the resolution to this may come from the top-down approach

Conclusions

- Flavour problem of Standard Model remains
- Symmetry may guide us - GUTs and Flavour Symmetry
- Modular Family Symmetry motivated by String theory
- Modulus field τ represents a minimal “flavon” scenario
- Yukawa matrices expressed in terms of the modulus field
- Introduced “weighton” to explain fermion mass hierarchies
- 10d model gives 3 moduli fields stabilised at fixed points
- SU(5) Orbifold GUT in 10d leads to predictive LS model
- Unconstrained Kahler potential calls for new ingredients