Non-holomorphic modular symmetry

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In collaboration with Bu-Yao Qu and Jun-Nan Lu, based on arXiv: 2406.02527, JHEP 08 (2024) 136; arXiv: 2506.19822

FLASY

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11TH WORKSHOP

Flavor Symmetries and Consequences in Accelerators and Cosmology

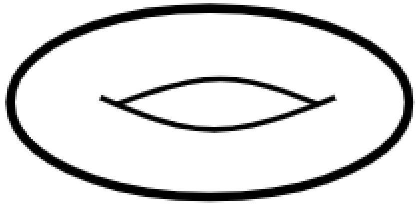
Modular invariant SUSY theory

- Modular action

$$\tau \to \gamma\tau = \frac{a\tau + b}{c\tau + d}$$

[Lauer, Mas, Nilles, 1989; Ferrara, Lust et al, 1989; Feruglio, 1706.08749]

See talks by Antonio, Ferruccio, Serguey, Steve, Michael, Mu-Chun, Morimitsu, Ivo, Andreas…



$SL(2, \mathbb{Z})$ on torus T^2

- Field **Non-linear** transformation

$$\psi \to (c\tau + d)^{-k} \rho(\gamma) \psi$$

weight $k \in \mathbb{Z}$ ρ is a unitary representation of Γ_N or Γ'_N

finite modular **groups as G_f**

- Superpotential

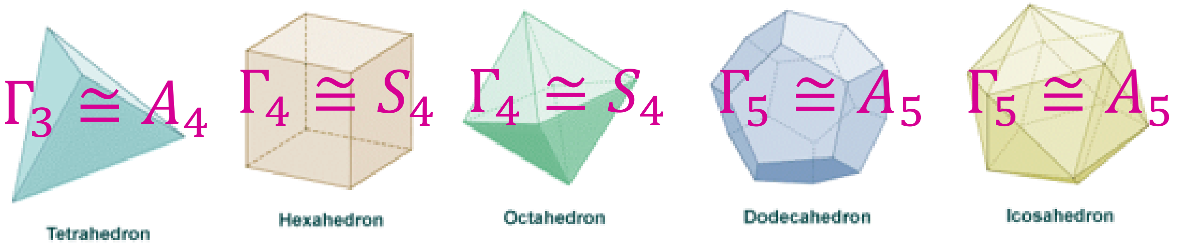
$$\mathcal{W} = \sum Y_{I_1 I_2 \ldots I_n}(\tau) \psi_{I_1} \psi_{I_2} \ldots \psi_{I_n}$$

$$G_f = \begin{cases} \Gamma_N \equiv SL(2, \mathbb{Z}) / \pm\Gamma(N) \\ \Gamma'_N \equiv SL(2, \mathbb{Z}) / \Gamma(N) \end{cases}$$

Yukawa coupling $Y_{I_1 I_2 \ldots I_n}(\tau)$ is a **holomorphic** modular form of level N:

$$\Gamma(N) = \{\gamma \in SL(2, \mathbb{Z}) \mid \gamma = 1_2 \text{ mod } N\}$$

$$Y_{I_1 I_2 \ldots I_n}(\gamma\tau) = (c\tau + d)^{k_Y} \rho_Y(\gamma) Y_{I_1 I_2 \ldots I_n}(\tau)$$



Modular invariance requires

weights balance: $k_Y = k_{I_1} + k_{I_2} + \ldots + k_{I_n}$

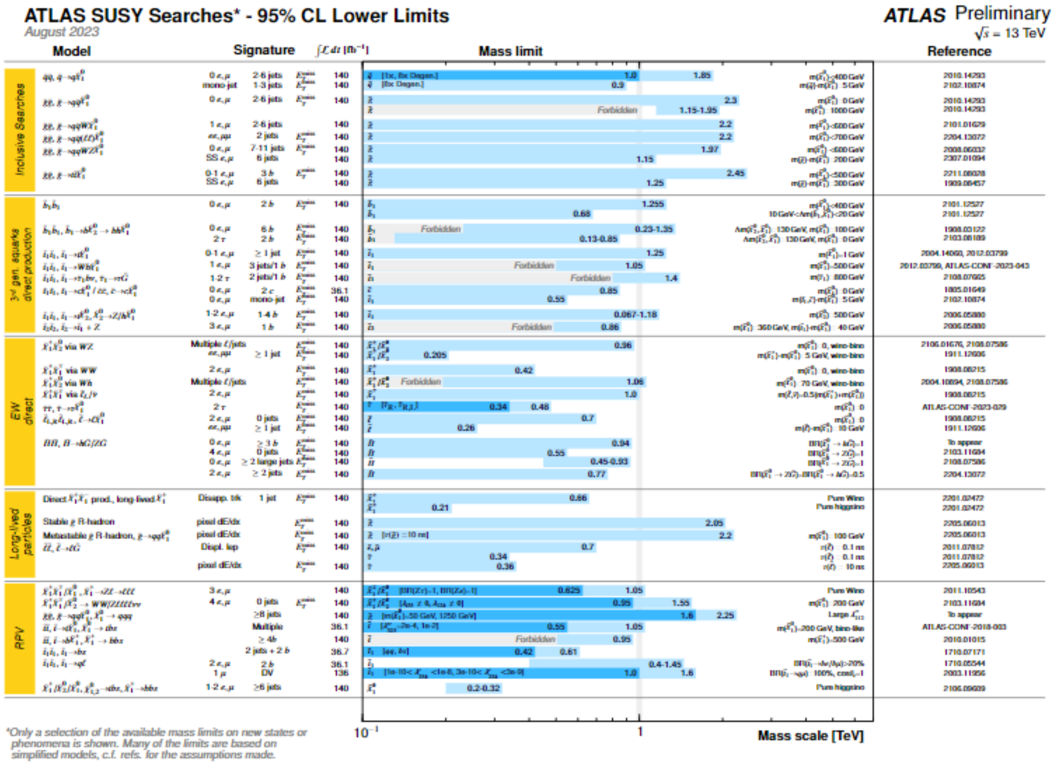
invariant singlet: $\rho_Y \otimes \rho_{I_1} \otimes \ldots\ldots \otimes \rho_{I_n} \supset \mathbf{1}$

Is SUSY unique?

- **Modular symmetry standalone is not enough!**

SUSY → holomorphicity → $Y(\tau)$: Yukawa couplings = modular form of level N and weight k_Y

A few independent modular forms at weight k_Y



[PDG,2024]

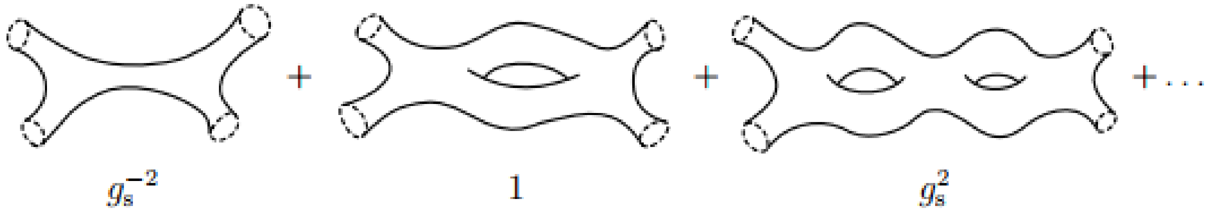
low energy SUSY is not observed so far!

Non-SUSY → **???** → $Y(\tau,\bar{\tau})$: non-holomorphic Yukawa couplings

A finite choice of $Y(\tau,\bar{\tau})$ to have prediction power

Harmonic condition replacing holomorphicity

- From top-down, the effective interactions in the low energy expansion of the four-graviton amplitude are **non-holomorphic automorphic** functions satisfying harmonic eigenvalue equations.



[Green, Gutperle, hep-th/9701093; Green, Russo, Vanhove,1001.2535; Hoker,Kaidi,2208.07242]

- Modular invariance based on automorphic forms naturally have multiple moduli **[Ding, Feruglio, Liu, 2010.07952]**
- Automorphic forms coincide with the harmonic Maaß forms for single modulus τ

- Harmonic Maaß forms of level N and weight k **[Qu, Ding, 2406.02527]**

① **modularity:** $Y(\gamma\tau) = (c\tau + d)^k Y(\tau), \ \ \gamma \in \Gamma(N)$

② **harmonic condition (eigenfunction of Casimir operator Δ_k):**

$$\Delta_k Y(\tau) = 0, \ \ \Delta_k \equiv -4y^2 \frac{\partial}{\partial\tau}\frac{\partial}{\partial\bar{\tau}} + 2iky\frac{\partial}{\partial\bar{\tau}}$$

top-down evidence of SUSY unnecessary: Almumin, Chen et al, 2102.11286

③ **growth condition:** $Y(\tau) = \mathcal{O}(y^{\alpha}), \quad \mathrm{Im}(\tau) \equiv y \rightarrow +\infty$

Modular form VS. polyharmonic Maaß form

$$\frac{\partial}{\partial\bar{\tau}} Y(\tau) = 0$$

$$\left[-4y^2 \frac{\partial}{\partial\tau}\frac{\partial}{\partial\bar{\tau}} + 2iky\frac{\partial}{\partial\bar{\tau}}\right] Y(\tau) = 0$$

$$Y(\tau) = \underbrace{\sum_{n=0}^{+\infty} a_n q_N^n}_{\text{holomorphic}}\,, \qquad q_N \equiv e^{2\pi i\tau/N}$$

$$Y(\tau) = \underbrace{\sum_{n=0}^{+\infty} c_n^+ q_N^n}_{\text{holomorphic}} + \underbrace{c_0^- y^{1-k} + \sum_{n=-1}^{-\infty} c_n^- \Gamma(1-k, -4\pi n y/N) q_N^n}_{\text{non-holomorphic}}$$

Weight: $k = 0, 1,2,3, \ldots$ $\qquad$ $k = \cdots, -3, -2 -1, 0, 1, 2, 3, \ldots$

Basis: $Y_i(\gamma\tau) = (c\tau + d)^k \rho(\gamma)_{ij} Y_j(\tau)$ $\qquad$ $Y_i(\gamma\tau) = (c\tau + d)^k \rho(\gamma)_{ij} Y_j(\tau)$

- A familiar non-holomorphic polyharmonic Maaß form

modified Eisenstein series $\quad \hat{E}_2(\tau) = E_2(\tau) - \frac{3}{\pi y} = 1 - \frac{3}{\pi y} - 24q - 72q^2 - 96q^3 + \cdots$

Formalism of non-holomorphic modular flavor symmetry

- Modular action of τ

[Qu, Ding, 2406.02527]

$$\tau \to \gamma\tau = \frac{a\tau+b}{c\tau+d}$$

τ is a scalar field not a chiral superfield

- **Non-linear** transformations of SM fields

$$\psi_i \to (c\tau+d)^{-k_\psi}\left[\rho_\psi(\gamma)\right]_{ij}\psi_j \qquad \psi_i^c \to (c\tau+d)^{-k_{\psi^c}}\left[\rho_{\psi^c}(\gamma)\right]_{ij}\psi_j^c \qquad H \to (c\tau+d)^{-k_H}\rho_H(\gamma)H$$

- Modular invariant kinetic terms

$$\mathcal{L}_K = (-i\tau+i\bar{\tau})^{-k_\psi} i\psi^\dagger\bar{\sigma}^\mu D_\mu\psi + (-i\tau+i\bar{\tau})^{-k_{\psi^c}} i\psi^{c\dagger}\bar{\sigma}^\mu D_\mu\psi^c + \cdots$$
$$\xrightarrow{\langle\tau\rangle} \langle -i\tau+i\bar{\tau}\rangle^{-k_\psi} i\psi^\dagger\bar{\sigma}^\mu\partial_\mu\psi + \langle -i\tau+i\bar{\tau}\rangle^{-k_{\psi^c}} i\psi^{c\dagger}\bar{\sigma}^\mu\partial_\mu\psi^c + \cdots$$

$$D_\mu \equiv \partial_\mu + \frac{k\pi i}{6}E_2(\tau)\partial_\mu\tau$$

- Modular invariant Yukawa interactions

$$\mathcal{L}_Y = Y^{(k_Y)}(\tau)\psi^c\psi H + \text{h.c.}$$

Thanks Joao for mail on the modular invariance of kinetic term

Yukawa couplings are harmonic Maass forms of weight k_Y and level N

$$Y^{(k_Y)}(\tau) \to Y^{(k_Y)}(\gamma\tau) = (c\tau+d)^{k_Y}\rho_Y(\gamma)Y^{(k_Y)}(\tau)$$

with
$$\begin{cases} k_Y = k_{\psi^c} + k_\psi + k_H \\ \rho_Y \otimes \rho_{\psi^c} \otimes \rho_\psi \otimes \rho_H \supset \mathbf{1} \end{cases}$$

CP and non-holomorphic modular symmetry

- A unique CP transformation up to modular transformations

$\tau \to -\tau^*$ [Baur, Nilles et al,1901.03251; Novichkov, Petcov et al, 1905.11970]

- CP action on matter field and harmonic Maaß form

matter field: $\psi(x) \xrightarrow{\mathcal{CP}} X_\psi \psi^*(x_P)$

X_ψ and X_Y are unitary matrix in flavor space

Maaß form: $Y(\tau) \xrightarrow{\mathcal{CP}} Y(-\tau^*) = X_Y Y^*(\tau)$

$u(S) = S^{-1}, \quad u(T) = T^{-1}$

- Consistency condition $\mathcal{CP} \to \gamma \to \mathcal{CP}^{-1}$

modulus: $\tau \xrightarrow{\mathcal{CP}} -\tau^* \xrightarrow{\gamma} -\frac{a\tau^*+b}{c\tau^*+d} \xrightarrow{\mathcal{CP}^{-1}} \frac{a\tau-b}{-c\tau+d}$ ⟹ $u(\gamma) = CP \circ \gamma \circ CP^{-1} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$

matter field& Maaß form : $X_r \rho_r^*(\gamma) X_r^{-1} = \rho_r(u(\gamma))$ fixing X_r for each irrep $\boldsymbol{r}$

symmetric basis: $\rho_r(S) = \rho_r^T(S), \ \rho_r(T) = \rho_r^T(T)$

See talks by Ferruccio, Serguey, Steve

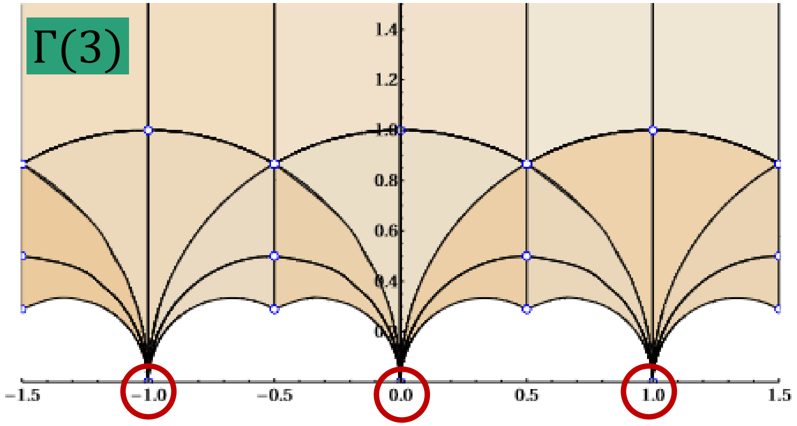
$$\begin{cases} X_r \rho_r^*(S) X_r^{-1} = \rho_r(S^{-1}) = \rho_r^*(S) \\ X_r \rho_r^*(T) X_r^{-1} = \rho_r(T^{-1}) = \rho_r^*(S) \end{cases} \Longrightarrow X_r = 1 \Longrightarrow g_i^* = g_i$$

real couplings

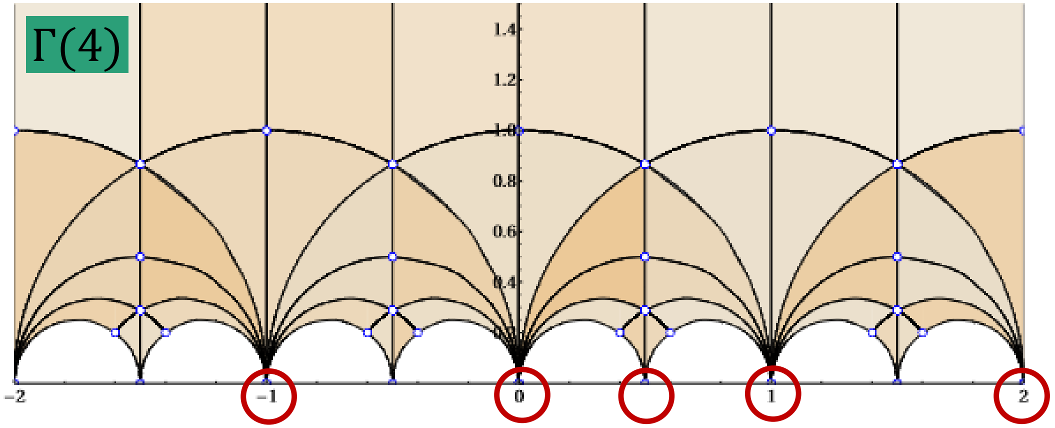
[Novichkov, Petcov et al, 1905.11970]

Non-holomorphic Eisenstein series

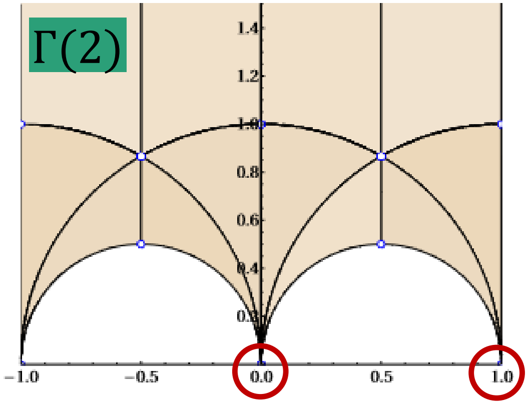
- Cusp of $\Gamma(N)$: a representative of the equivalence classes of rational numbers $\mathbb{Q} \cup \{i\infty\}$ under $\Gamma(N)$
  - Two coprime integers A and C to represent a cusp A/C
  - $\overline{A/C}$ and $\overline{A'/C'}$ describe the same cusp iff $(A, C) = \pm(A', C') \ (\mathrm{mod}\ N)$
  - $\Gamma(1) \cong SL(2,\mathbb{Z})$ has a single cusp $\overline{i\infty} = \overline{1/0}$



$\mathcal{C}(3) = \{\overline{1/0}, \overline{0/1}, \overline{1/1}, \overline{-1/1}\}$

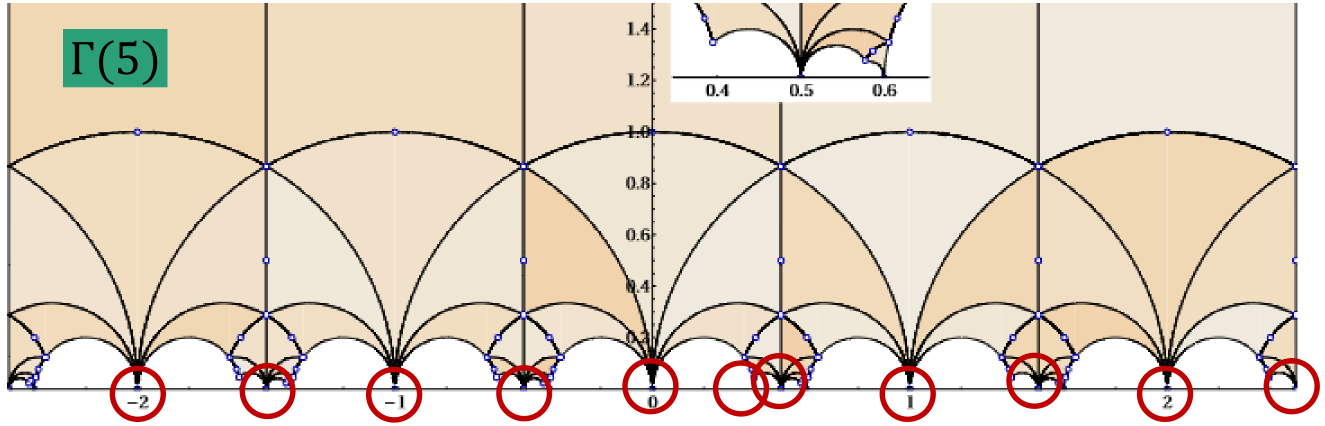


$\mathcal{C}(4) = \{\overline{1/0}, \overline{0/1}, \overline{1/1}, \overline{2/1}, \overline{1/2}, \overline{-1/1}\}$



$\mathcal{C}(2) = \{\overline{1/0}, \overline{0/1}, \overline{1/1}\}$

[figures from 2008.05329 by Ivo, Miguel,Ye-Ling]



$\mathcal{C}(5) = \{\overline{1/0}, \overline{0/1}, \overline{1/1}, \overline{2/1}, \overline{-2/1}, \overline{-1/1}, \overline{1/2}, \overline{3/2}, \overline{5/2}, \overline{-3/2}, \overline{-1/2}, \overline{2/5}\}$

➢ Non-holomorphic Eisenstein series of weight k: defined at each cusp of $\Gamma(N)$ [Qu, Lu, Ding, 2506.19822]

$$E_k(N;\tau;s;\overline{A/C}) = \sum_{\substack{(c,d)\equiv(-C,A)\ (\mathrm{mod}\ N)\\ \gcd(c,d)=1}} \frac{y^s}{(c\tau+d)^k|c\tau+d|^{2s}}$$

→ non-holomorphicity

- **Modular transformation of E_k**

$$E_k(N;\gamma\tau;s;\overline{A/C}) = (c\tau+d)^k\, E_k\left(N;\tau;s;\overline{\gamma^{-1}(A/C)}\right)$$

→ **Modularity preserved for $\gamma\in\Gamma(N)$**

- **Harmonic condition**

$$\Delta_k E_k(N;\tau;s;\overline{A/C}) = -s(s+k-1)E_k(N;\tau;s;\overline{A/C})$$

→ $s=0$ or $s=1-k$

— The linear space of weight k polyharmonic Maaß form is spanned by the Eisenstein series $E_k(N;\tau;s;\overline{A/C})\big|_{s=1-k}$, its dimension equals to the number of cusp for $k\le 0$.

— At weight $k=1$, $E_1(N;\tau;s;\overline{A/C})\big|_{s=0}$ are holomorphic and linearly dependent. The 1st derivative

$$E_1^{(1)}(N;\tau;0;\overline{A/C}) \equiv \left.\frac{\partial E_1(N;\tau;s;\overline{A/C})}{\partial s}\right|_{s=0}$$

generates weight $k=1$ polyharmonic Maaß form.

Polyharmonic Maaß form of level N

Finite modular group $\Gamma_3' \cong T'$: $S^2 = R,\ (ST)^3 = T^3 = R^2 = 1,\ RT = TR$

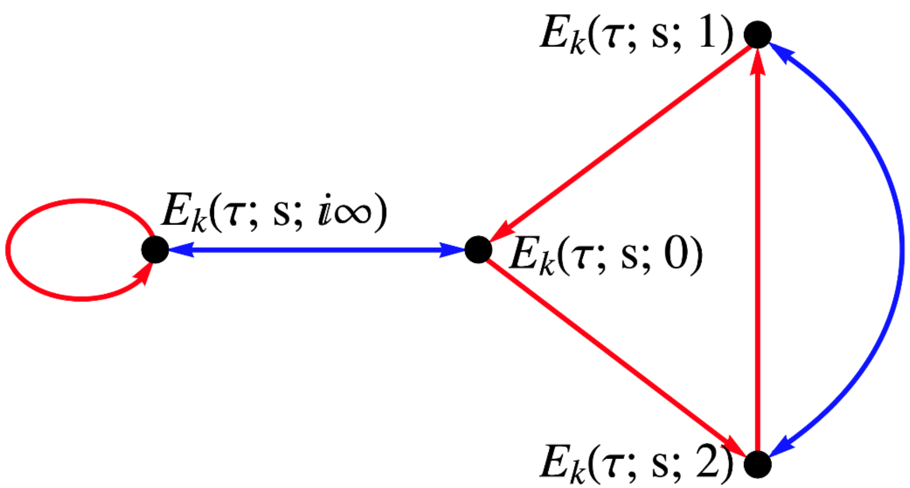
- Non-holomorphic Eisenstein series of level $N = 3$

$E_k(\tau; s; 0),\ \ E_k(\tau; s; 1),\ \ E_k(\tau; s; 2),\ \ E_k(\tau; s; i\infty)$

Cusp: $\mathcal{C}(3) = \{\overline{0/1}, \overline{1/1}, \overline{2/1}, \overline{1/0}\} \equiv \{0,1,2,i\infty\}$

$$\boldsymbol{S}:\ \begin{cases} E_k(S\tau; s; i\infty) \xmapsto{S} (-\tau)^k E_k(\tau; s; 0)\,, \\ E_k(S\tau; s; 0) \xmapsto{S} (-1)^k (-\tau)^k E_k(\tau; s; i\infty)\,, \\ E_k(S\tau; s; 1) \xmapsto{S} (-\tau)^k E_k(\tau; s; 2)\,, \\ E_k(S\tau; s; 2) \xmapsto{S} (-1)^k (-\tau)^k E_k(\tau; s; 1)\,. \end{cases}$$

$$\boldsymbol{T}:\ \begin{cases} E_k(\tau; s; i\infty) \xmapsto{T} E_k(\tau; s; i\infty)\,, \\ E_k(\tau; s; 0) \xmapsto{T} E_k(\tau; s; 2)\,, \\ E_k(\tau; s; 1) \xmapsto{T} E_k(\tau; s; 0)\,, \\ E_k(\tau; s; 2) \xmapsto{T} E_k(\tau; s; 1)\,. \end{cases}$$



S transformation distinguishes odd from even weights

[Qu, Lu, Ding, 2506.19822]

➢ Multiplets of polyharmonic Maaß forms of level N=3

Even weight 2k ($k \leq 0$):

$$Y_{\mathbf{1}}^{(2k)}(\tau) = E_{2k}(\tau;1-2k;i\infty) + E_{2k}(\tau;1-2k;0) + E_{2k}(\tau;1-2k;1) + E_{2k}(\tau;1-2k;2)\,,$$

singlet 1

$$Y_{\mathbf{3}}^{(2k)}(\tau) = \begin{pmatrix} E_{2k}(\tau;1-2k;i\infty) - \frac{1}{3}\Big(E_{2k}(\tau;1-2k;0) + E_{2k}(\tau;1-2k;1) + E_{2k}(\tau;1-2k;2)\Big) \\ \frac{2}{3}\Big(E_{2k}(\tau;1-2k;0) + e^{2\pi i/3}E_{2k}(\tau;1-2k;1) + e^{4\pi i/3}E_{2k}(\tau;1-2k;2)\Big) \\ \frac{2}{3}\Big(E_{2k}(\tau;1-2k;0) + e^{4\pi i/3}E_{2k}(\tau;1-2k;1) + e^{2\pi i/3}E_{2k}(\tau;1-2k;2)\Big) \end{pmatrix}$$

triplet 3

Odd weight 2k+1 ($k < 0$):

$$Y_{\hat{\mathbf{2}}}^{(2k+1)}(\tau) = \begin{pmatrix} i\sqrt{\frac{2}{3}}\Big(E_{2k+1}(\tau;-2k;0) + \omega E_{2k+1}(\tau;-2k;1) + \omega^2 E_{2k+1}(\tau;-2k;2)\Big) \\ E_{2k+1}(\tau;-2k;i\infty) - \frac{i}{\sqrt{3}}\Big(E_{2k+1}(\tau;-2k;0) + E_{2k+1}(\tau;-2k;1) + E_{2k+1}(\tau;-2k;2)\Big) \end{pmatrix}$$

doublet $\hat{2}$

$$Y_{\hat{\mathbf{2}}''}^{(2k+1)}(\tau) = \begin{pmatrix} E_{2k+1}(\tau;-2k;i\infty) + \frac{i}{\sqrt{3}}\Big(E_{2k+1}(\tau;-2k;0) + E_{2k+1}(\tau;-2k;1) + E_{2k+1}(\tau;-2k;2)\Big) \\ i\sqrt{\frac{2}{3}}\Big(E_{2k+1}(\tau;-2k;0) + \omega^2 E_{2k+1}(\tau;-2k;1) + \omega E_{2k+1}(\tau;-2k;2)\Big) \end{pmatrix}$$

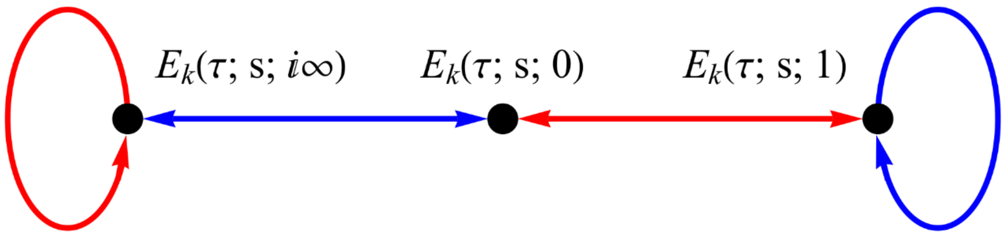
doublet $\hat{2}''$

Odd weight 1: replace E_k with 1st derivative

$$E_1(\tau;0;\overline{A/C}) \to E_1^{(1)}(\tau;0;\overline{A/C}) \equiv \left.\frac{\partial E_1(\tau;s;\overline{A/C})}{\partial s}\right|_{s=0}$$

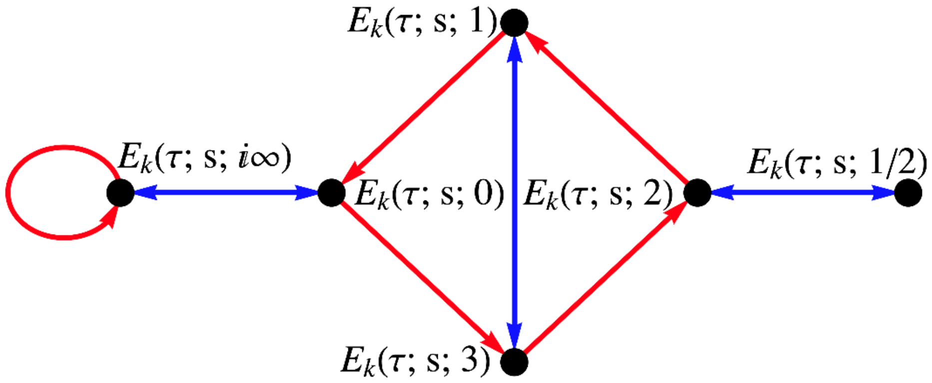
k	-3	-2	-1	0	1	2	3
T'	$\hat{2}+\hat{2}''$	$1+3$	$\hat{2}+\hat{2}''$	$1+3$	$\hat{2}+\hat{2}''$	$1+3$	$\hat{2}+\hat{2}''$

- Level $N=2$: $S^2 = T^2 = (ST)^3 = 1 \rightarrow \Gamma_2' = \Gamma_2 \cong S_3$



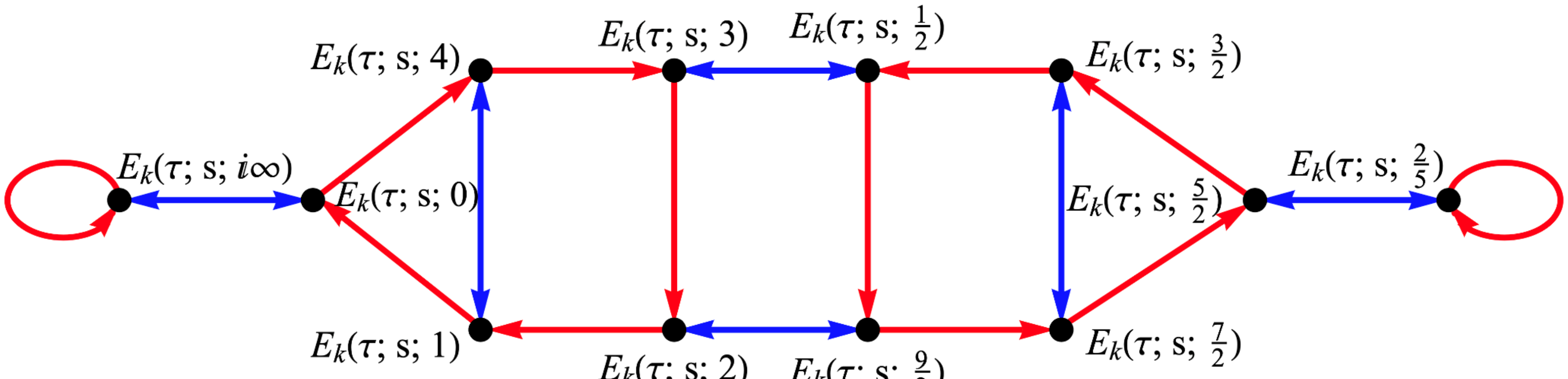
k	-3	-2	-1	0	1	2	3
S_3	--	$1+2$	--	$1+2$	--	$1+2$	--

- Level $N=4$: $S^2 = R,\ (ST)^3 = T^4 = R^2 = 1,\ RT = TR \rightarrow \Gamma_4' \cong S_4'$



k	-3	-2	-1	0	1	2	3
S_4'	$\hat{3}+\hat{3}'$	$1+2+3$	$\hat{3}+\hat{3}'$	$1+2+3$	$\hat{3}+\hat{3}'$	$1+2+3$	$\hat{1}'+\hat{3}+\hat{3}'$

- Level $N=5$: $S^2 = R,\ (ST)^3 = T^5 = R^2 = 1,\ RT = TR \rightarrow \Gamma_5' \cong A_5'$



k	-3	-2	-1	0
A_5'	$\hat{6}+\hat{6}$	$1+3+3'+5$	$\hat{6}+\hat{6}$	$1+3+3'+5$
k	1	2	3	4
A_5'	$\hat{6}+\hat{6}$	$1+3+3'+5$	$\hat{4}'+\hat{6}+\hat{6}$	$1+3+3'+4+5+5$

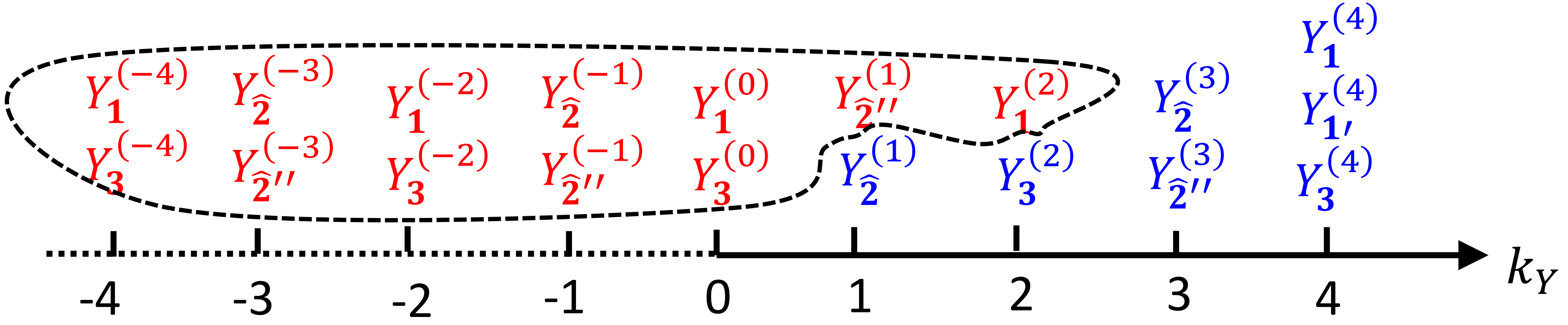
Non-holomorphic modular model based on $\Gamma_3' \cong T'$

➢ **Minimality as a guiding principle** **[Qu, Lu, Ding, 2506.19822]**

- **No flavons other than τ** are introduced
- Three generations of LH lepton doublets transform as a T' **triplet**
- RH lepton singlets transform as T' **singlets**
- Higgs field is invariant under T'
- **Minimal** type-I seesaw with **two RH neutrinos**

$$-\mathcal{L} = \sum_i \alpha_i [E_i^c L f_i(Y)]_1 H^* + g[N^c L f_N(Y)]_1 H + \Lambda[N^c N^c f_M(Y)] + \text{h.c.}$$

➢ Non-holomorphic polyharmonic Maaß forms appear at weights $k \leq 2$



[Other example models: Qu, Ding, 2406.02527; Nomura, Okada, 2408.01143; Ding,Lu,Petcov,Qu, 2408.15988; Nomura, Okada, Popov,2409.12547; Li, Lu, Ding, 2410.24103; Nomura, Okada, 2412.18095…]

Lepton sector

	L	e^c	μ^c	τ^c	N^c	H
T'	$\mathbf{3}$	$\mathbf{1}'$	$\mathbf{1}'$	$\mathbf{1}$	$\widehat{\mathbf{2}}$	$\mathbf{1}$
k_I	-1	1	3	5	2	0

gCP symmetry is included so that all couplings are real in the working basis

[Qu, Lu, Ding, 2506.19822]

➢ Charged leptons

$$-\mathcal{L}_l^Y = \alpha(e^c L Y_{\mathbf{3}}^{(0)} H^*)_{\mathbf{1}} + \beta(\mu^c L Y_{\mathbf{3}}^{(2)} H^*)_{\mathbf{1}} + \gamma(\tau^c L Y_{\mathbf{3}}^{(4)} H^*)_{\mathbf{1}} + \text{h.c.}$$

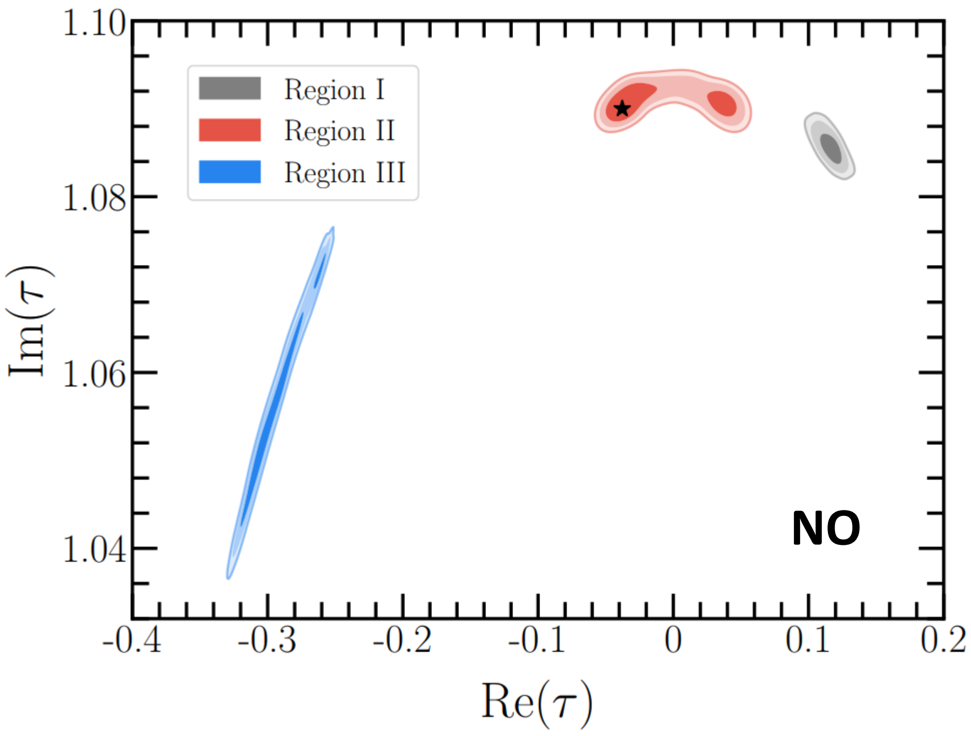
$$\Rightarrow \quad M_l = \begin{pmatrix} \alpha Y_{\mathbf{3},3}^{(0)} & \alpha Y_{\mathbf{3},2}^{(0)} & \alpha Y_{\mathbf{3},1}^{(0)} \\ \beta Y_{\mathbf{3},3}^{(2)} & \beta Y_{\mathbf{3},2}^{(2)} & \beta Y_{\mathbf{3},1}^{(2)} \\ \gamma Y_{\mathbf{3},1}^{(4)} & \gamma Y_{\mathbf{3},3}^{(4)} & \gamma Y_{\mathbf{3},2}^{(4)} \end{pmatrix} v$$

➢ Neutrino mass : seesaw mechanism with 2RHN

$$-\mathcal{L}_\nu = g_1(N^c L H Y_{\hat{\mathbf{2}}''}^{(1)})_{\mathbf{1}} + g_2(N^c L H Y_{\hat{\mathbf{2}}}^{(1)})_{\mathbf{1}} + \frac{1}{2}\Lambda\left(N^c N^c Y_{\mathbf{3}}^{(4)}\right)_{\mathbf{1}} + \text{h.c.}$$

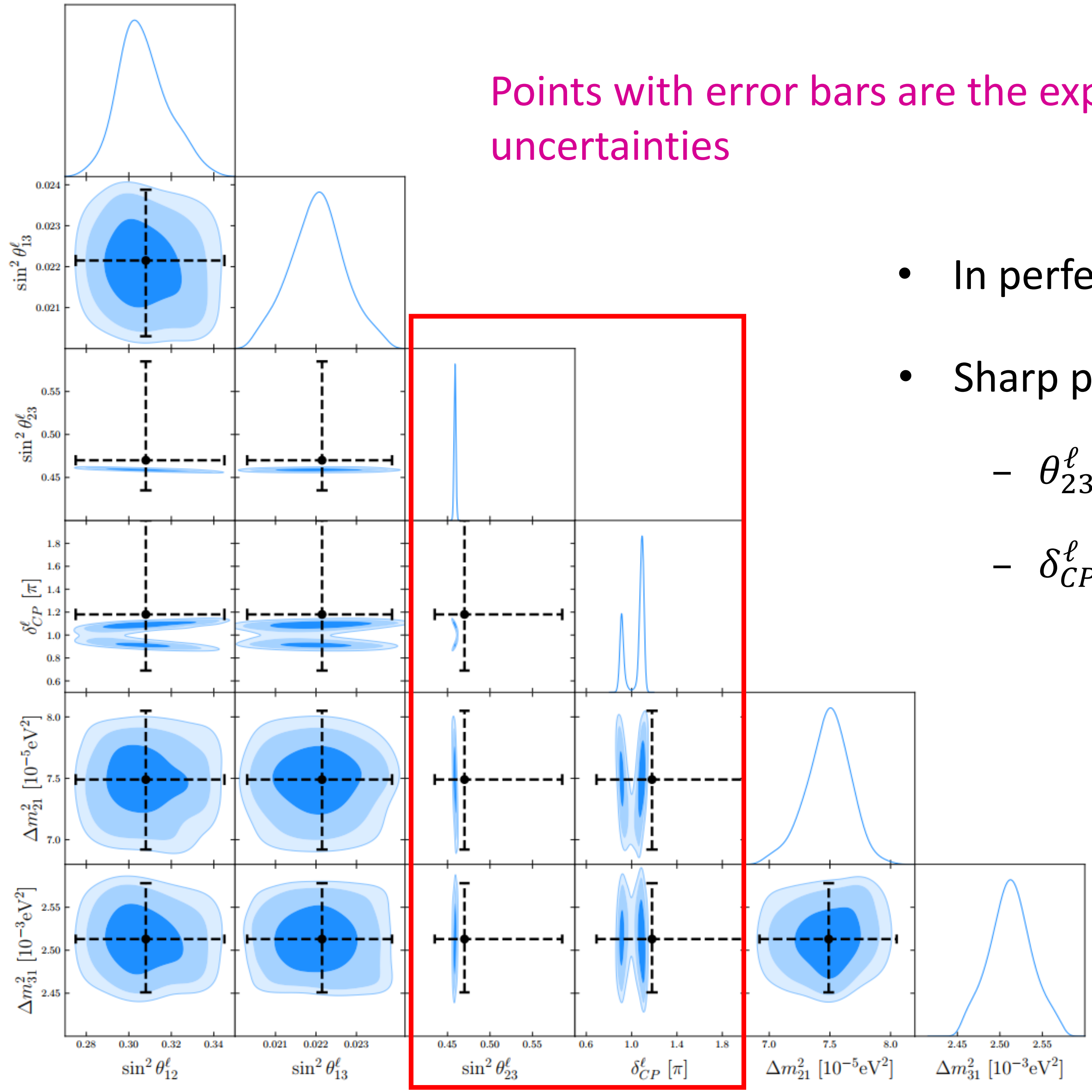
$$\Rightarrow \quad M_D = \begin{pmatrix} g_1 Y_{\hat{\mathbf{2}}'',2}^{(1)} & \sqrt{2} g_2 Y_{\hat{\mathbf{2}},1}^{(1)} & -\sqrt{2} g_1 Y_{\hat{\mathbf{2}}'',1}^{(1)} - g_2 Y_{\hat{\mathbf{2}},2}^{(1)} \\ g_1 Y_{\hat{\mathbf{2}}'',1}^{(1)} - \sqrt{2} g_2 Y_{\hat{\mathbf{2}},2}^{(1)} & \sqrt{2} g_1 Y_{\hat{\mathbf{2}}'',2}^{(1)} & -\sqrt{2} g_1 Y_{\hat{\mathbf{2}},1}^{(1)} \end{pmatrix} v\,, \quad M_N = \Lambda \begin{pmatrix} -Y_{\mathbf{3},2}^{(4)} & \frac{Y_{\mathbf{3},3}^{(4)}}{\sqrt{2}} \\ \frac{Y_{\mathbf{3},3}^{(4)}}{\sqrt{2}} & Y_{\mathbf{3},1}^{(4)} \end{pmatrix}$$

Input pars	NO	IO
$\mathrm{Re}(\tau)$	-0.03777	4.388×10^{-3}
$\mathrm{Im}(\tau)$	1.090	1.083
β/α	17.70	16.00
γ/α	284.6	257.1
g_2/g_1	0.1490	-0.08106
$\alpha v/\mathrm{GeV}$	4.696×10^{-3}	5.156×10^{-3}
$\frac{g_1^2v^2}{\Lambda}/\mathrm{meV}$	191.6	603.8



Predictions	NO	IO
$\sin^2\theta^\ell_{12}$	0.308	0.308
$\sin^2\theta^\ell_{13}$	0.02215	0.02236
$\sin^2\theta^\ell_{23}$	0.459	0.561
δ^ℓ_{CP}	1.09π	1.56π
$\Delta m^2_{21}/10^{-5}\mathrm{eV}^2$	7.48	7.49
$\Delta m^2_{31}/10^{-3}\mathrm{eV}^2$	2.513	-2.484
m_e/m_μ	0.004737	0.004737
m_μ/m_τ	0.05882	0.05882
m_e/MeV	0.469652	0.469652
ϕ	1.04π	0.35π
m_1/meV	0	49.08
m_2/meV	8.65	49.84
m_3/meV	50.13	0
$\sum_i m_i/\mathrm{meV}$	58.78	98.92
$m_{\beta\beta}/\mathrm{meV}$	1.91	42.34
m_β/meV	8.84	48.76
χ^2_ℓ	0.90	0.92

7 parameters vs 12 observables



Points with error bars are the experimental best values and the 3σ uncertainties

- In perfect agreement with neutrino data
- Sharp predictions for θ_{23}^ℓ and Dirac CP phase δ_{CP}^ℓ
  - θ_{23}^ℓ in the 1st octant: $\sin^2\theta_{23}^\ell \in [0.454, 0.463]$
  - $\delta_{CP}^\ell \in [0.84\pi, 1.16\pi]$

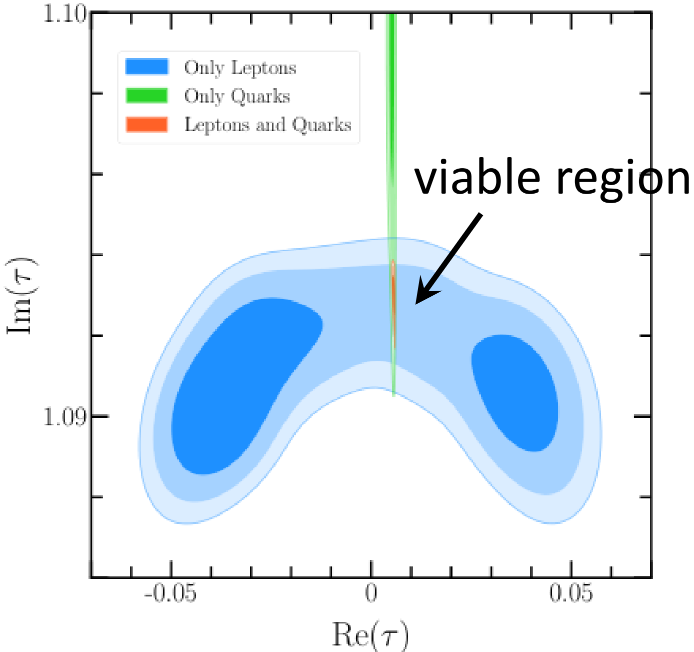
See Antonio's talk for analysis method

Extension to quark sector→ minimal modular model

	Q_L	u^c	c^c	t^c	d^c	s^c	b^c
T'	$\mathbf{3}$	$\mathbf{1}$	$\mathbf{1}'$	$\mathbf{1}''$	$\mathbf{1}$	$\mathbf{1}''$	$\mathbf{1}'$
k_I	k_Q	$-2-k_Q$	$-k_Q$	$-k_Q$	$-4-k_Q$	$-4-k_Q$	$-k_Q$

$$M_u = \begin{pmatrix} \alpha_u Y^{(-2)}_{\mathbf{3},1} & \alpha_u Y^{(-2)}_{\mathbf{3},3} & \alpha_u Y^{(-2)}_{\mathbf{3},2} \\ \beta_u Y^{(0)}_{\mathbf{3},3} & \beta_u Y^{(0)}_{\mathbf{3},2} & \beta_u Y^{(0)}_{\mathbf{3},1} \\ \gamma_u Y^{(0)}_{\mathbf{3},2} & \gamma_u Y^{(0)}_{\mathbf{3},1} & \gamma_u Y^{(0)}_{\mathbf{3},3} \end{pmatrix} v\,, \quad M_d = \begin{pmatrix} \alpha_d Y^{(-4)}_{\mathbf{3},1} & \alpha_d Y^{(-4)}_{\mathbf{3},3} & \alpha_d Y^{(-4)}_{\mathbf{3},2} \\ \beta_d Y^{(-4)}_{\mathbf{3},2} & \beta_d Y^{(-4)}_{\mathbf{3},1} & \beta_d Y^{(-4)}_{\mathbf{3},3} \\ \gamma_d Y^{(0)}_{\mathbf{3},3} & \gamma_d Y^{(0)}_{\mathbf{3},2} & \gamma_d Y^{(0)}_{\mathbf{3},1} \end{pmatrix} v$$

- **Minimal** modular invariant model for quarks and leptons: **13 parameters VS. 22 observables**



$5+6+2=$ **13** parameters
- lepton couplings: $\alpha, \beta, \gamma, g_1, g_2$
- quark couplings: $\alpha_u, \beta_u, \gamma_u, \alpha_d, \beta_d, \gamma_d$
- quark and lepton shared: $\mathrm{Re}(\tau), \mathrm{Im}(\tau)$

- The complex modulus τ common in quark and lepton sectors, is mostly fixed by the quark parameters

$\langle\tau\rangle = 0.00506+1.093i$ close to imaginary axis

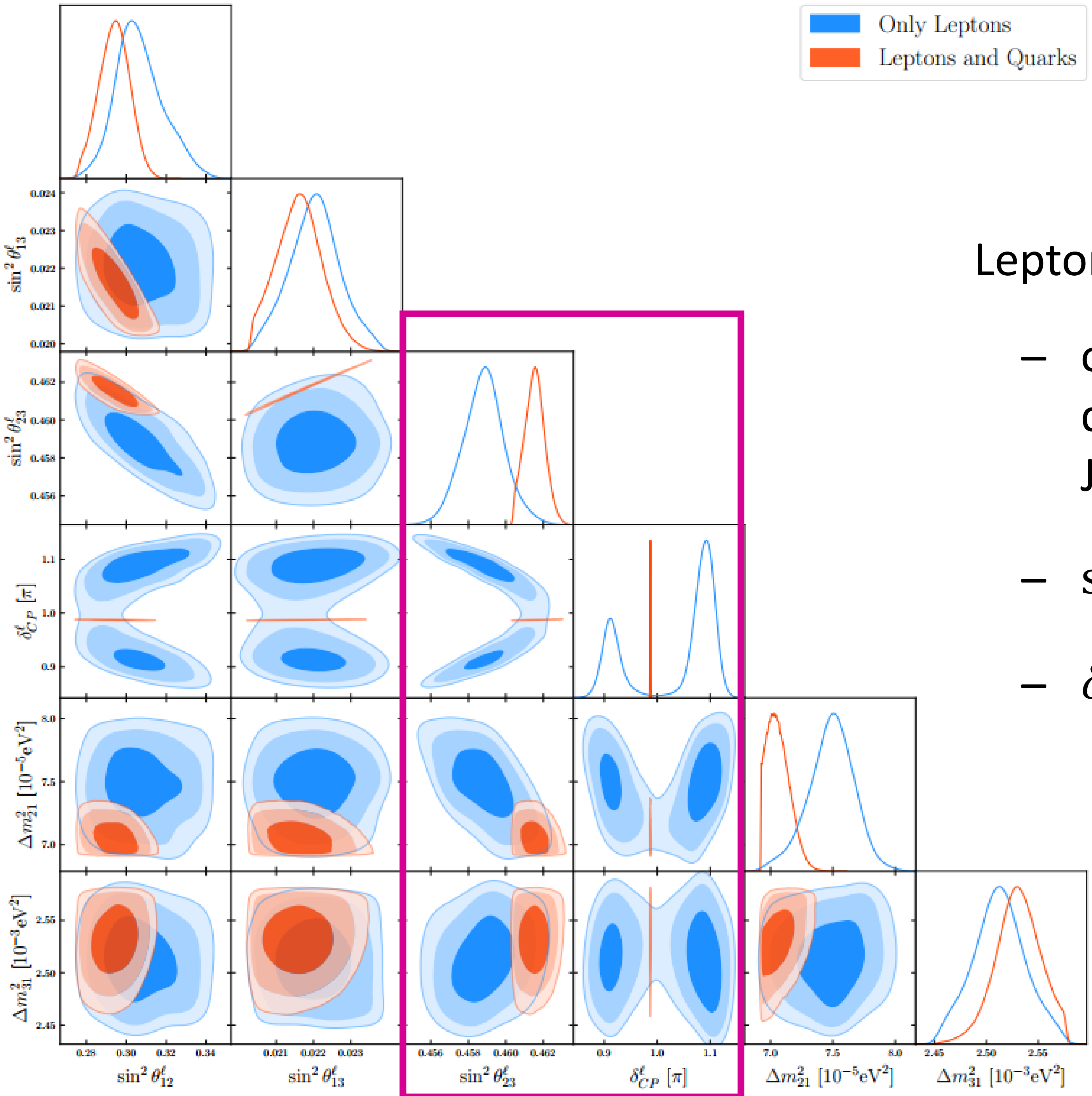
- Tiny region of τ after combining quarks & leptons

- The model can well describe the masses and mixing of both quarks and leptons: **12** masses+**6** mixing angles+**4** CP phases. The neutrino spectrum can only be NO after combined with quark sector.

	Combined		Leptons only		Quarks only	
Observable	Best fit	χ^2 breakdown	Best fit	χ^2 breakdown	Best fit	χ^2 breakdown
$\sin^2\theta^\ell_{12}$	0.291	2.385	0.308	0.001	–	–
$\sin^2\theta^\ell_{13}$	0.02158	0.974	0.02215	6.88×10^{-5}	–	–
$\sin^2\theta^\ell_{23}$	0.462	0.424	0.459	0.751	–	–
δ^ℓ_{CP}/π	0.99	0.690	1.09	0.143	–	–
$\Delta m^2_{21}/10^{-5}\text{eV}^2$	7.11	3.995	7.48	0.001	–	–
$\Delta m^2_{31}/10^{-3}\text{eV}^2$	2.53	0.387	2.51	2.28×10^{-5}	–	–
m_e/m_μ	0.004737	0.099	0.004737	0.099	–	–
m_μ/m_τ	0.05882	0.008	0.05882	0.008	–	–
$\min(\chi^2_{\text{leptons}})$	8.86		0.90		-	
θ^q_{12}	0.2263	0.033	–	–	0.2257	0.492
θ^q_{13}	0.0037	0.631	–	–	0.0037	0.449
θ^q_{23}	0.0374	12.330	–	–	0.0398	2.969
$\delta^q_{CP}/^\circ$	70.52	0.072	–	–	73.32	1.253
m_u/m_c	0.00204	1.252×10^{-8}	–	–	0.00204	2.260×10^{-7}
m_c/m_t	0.00318	7.770×10^{-8}	–	–	0.00318	1.057×10^{-7}
m_d/m_s	0.0588	0.066	–	–	0.0950	2.522
m_s/m_b	0.018	0.659	–	–	0.019	0.430
$\min(\chi^2_{\text{quarks}})$	13.79		–		8.11	
$\min(\chi^2_{\text{comb}})$	22.65		–		–	

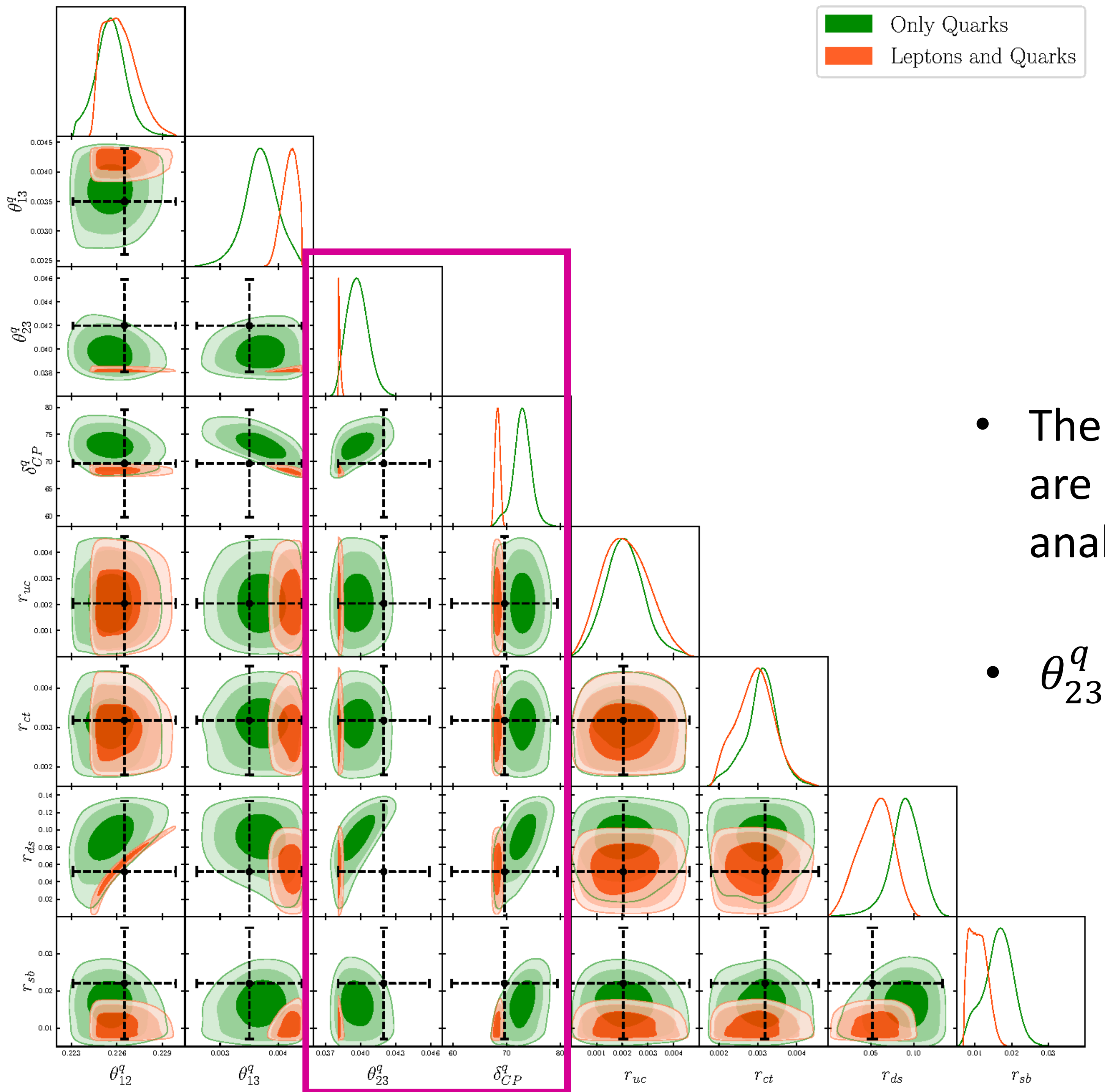
[Qu, Lu, Ding, 2506.19822]

- Slightly tension for θ^q_{23} in the combined analysis



Lepton observables:

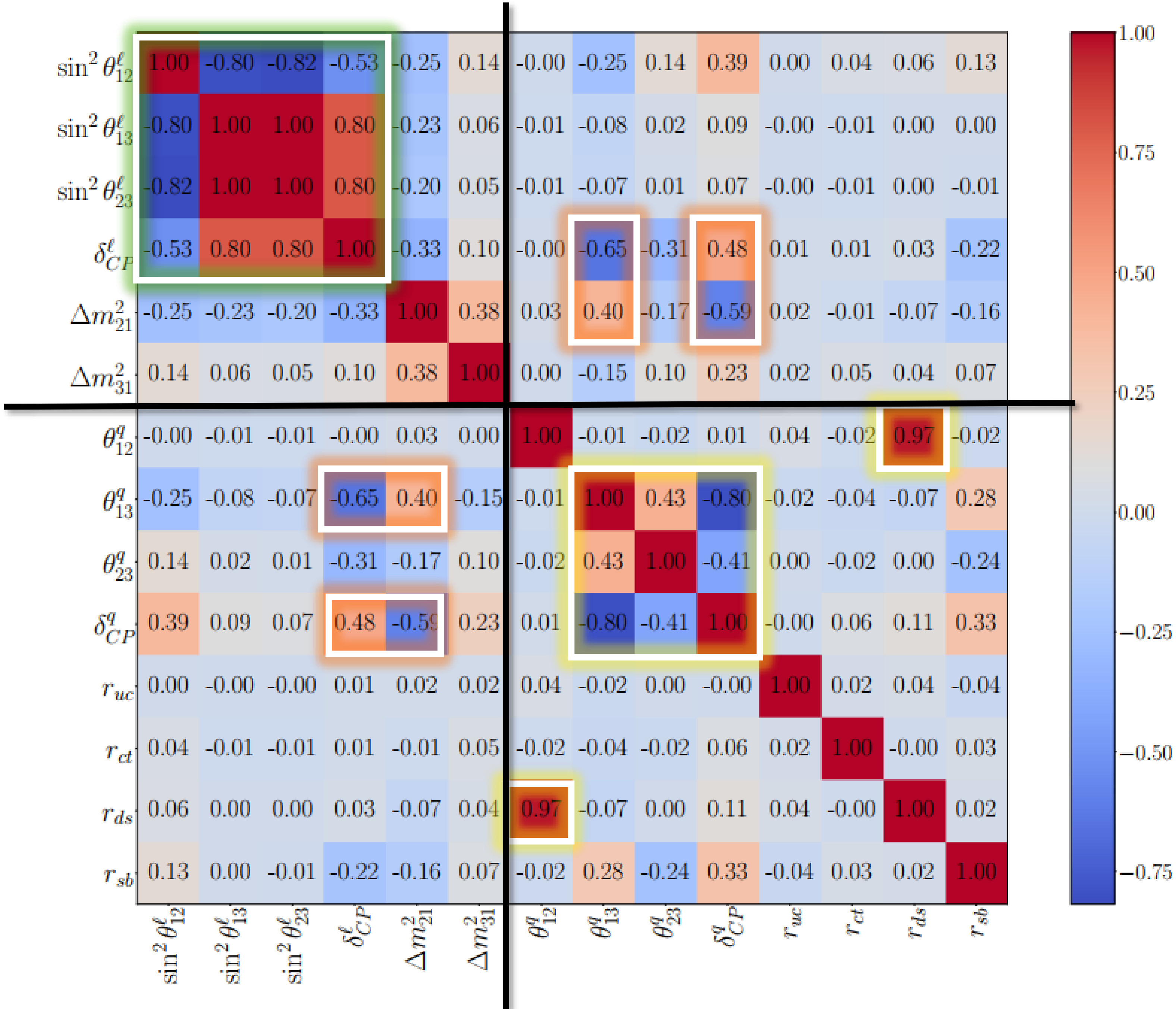
- only NO neutrino mass is viable after quark sector is combined, testable at JUNO
- $\sin^2\theta_{23}^\ell \sim 0.46$ in the 1st octant
- $\delta_{CP}^\ell \sim \pi$



- The allowed regions of quark observables are significantly reduced in the combined analysis!

- θ_{23}^q in marginal agreement with data

- Correlations between flavor observables, τ is the portal connecting quarks and leptons



	$\sin^2\theta_{12}^\ell$	$\sin^2\theta_{13}^\ell$	$\sin^2\theta_{23}^\ell$	δ_{CP}^ℓ	Δm_{21}^2	Δm_{31}^2	θ_{12}^q	θ_{13}^q	θ_{23}^q	δ_{CP}^q	r_{uc}	r_{ct}	r_{ds}	r_{sb}
$\sin^2\theta_{12}^\ell$	1.00	-0.80	-0.82	-0.53	-0.25	0.14	-0.00	-0.25	0.14	0.39	0.00	0.04	0.06	0.13
$\sin^2\theta_{13}^\ell$	-0.80	1.00	1.00	0.80	-0.23	0.06	-0.01	-0.08	0.02	0.09	-0.00	-0.01	0.00	0.00
$\sin^2\theta_{23}^\ell$	-0.82	1.00	1.00	0.80	-0.20	0.05	-0.01	-0.07	0.01	0.07	-0.00	-0.01	0.00	-0.01
δ_{CP}^ℓ	-0.53	0.80	0.80	1.00	-0.33	0.10	-0.00	-0.65	-0.31	0.48	0.01	0.01	0.03	-0.22
Δm_{21}^2	-0.25	-0.23	-0.20	-0.33	1.00	0.38	0.03	0.40	-0.17	-0.59	0.02	-0.01	-0.07	-0.16
Δm_{31}^2	0.14	0.06	0.05	0.10	0.38	1.00	0.00	-0.15	0.10	0.23	0.02	0.05	0.04	0.07
θ_{12}^q	-0.00	-0.01	-0.01	-0.00	0.03	0.00	1.00	-0.01	-0.02	0.01	0.04	-0.02	0.97	-0.02
θ_{13}^q	-0.25	-0.08	-0.07	-0.65	0.40	-0.15	-0.01	1.00	0.43	-0.80	-0.02	-0.04	-0.07	0.28
θ_{23}^q	0.14	0.02	0.01	-0.31	-0.17	0.10	-0.02	0.43	1.00	-0.41	0.00	-0.02	0.00	-0.24
δ_{CP}^q	0.39	0.09	0.07	0.48	-0.59	0.23	0.01	-0.80	-0.41	1.00	-0.00	0.06	0.11	0.33
r_{uc}	0.00	-0.00	-0.00	0.01	0.02	0.02	0.04	-0.02	0.00	-0.00	1.00	0.02	0.04	-0.04
r_{ct}	0.04	-0.01	-0.01	0.01	-0.01	0.05	-0.02	-0.04	-0.02	0.06	0.02	1.00	-0.00	0.03
r_{ds}	0.06	0.00	0.00	0.03	-0.07	0.04	0.97	-0.07	0.00	0.11	0.04	-0.00	1.00	0.02
r_{sb}	0.13	0.00	-0.01	-0.22	-0.16	0.07	-0.02	0.28	-0.24	0.33	-0.04	0.03	0.02	1.00

- **Lepton observables**: strong correlations among $\theta_{12}^\ell, \theta_{13}^\ell, \theta_{23}^\ell$ and δ_{CP}^ℓ
- **Quark observables**: strong correlations among $\theta_{13}^q, \theta_{23}^q$ and δ_{CP}^q, strong correlation between θ_{12}^q and m_d/m_s (GST relation?)
- **Quark & lepton observables**: strong correlations between θ_{13}^q, δ_{CP}^q and δ_{CP}^ℓ, Δm_{21}^2

Summary

- Formalism of non-holomorphic modular flavor symmetry: modular form→polyharmonic Maaß form, supersymmetry is unnecessary!
- Polyharmonic Maaß forms of weight $k \leq 1$ can be constructed from non-holomorphic Eisenstein series, the dimension equals the number of cusp.
- **Minimal** modular invariant model for quarks and leptons: 13 real parameters describe 22 observables.
- Testable predictions: NO neutrino mass, $\sin^2\theta_{23}^{\ell} \sim 0.46$, $\delta_{CP}^{\ell} \sim \pi$....
- Open questions:
  - origin of harmonic condition for polyharmonic Maaß forms ?
  - quantum corrections? (M. Ratz)
  - possible top-down connection?

……

Backup

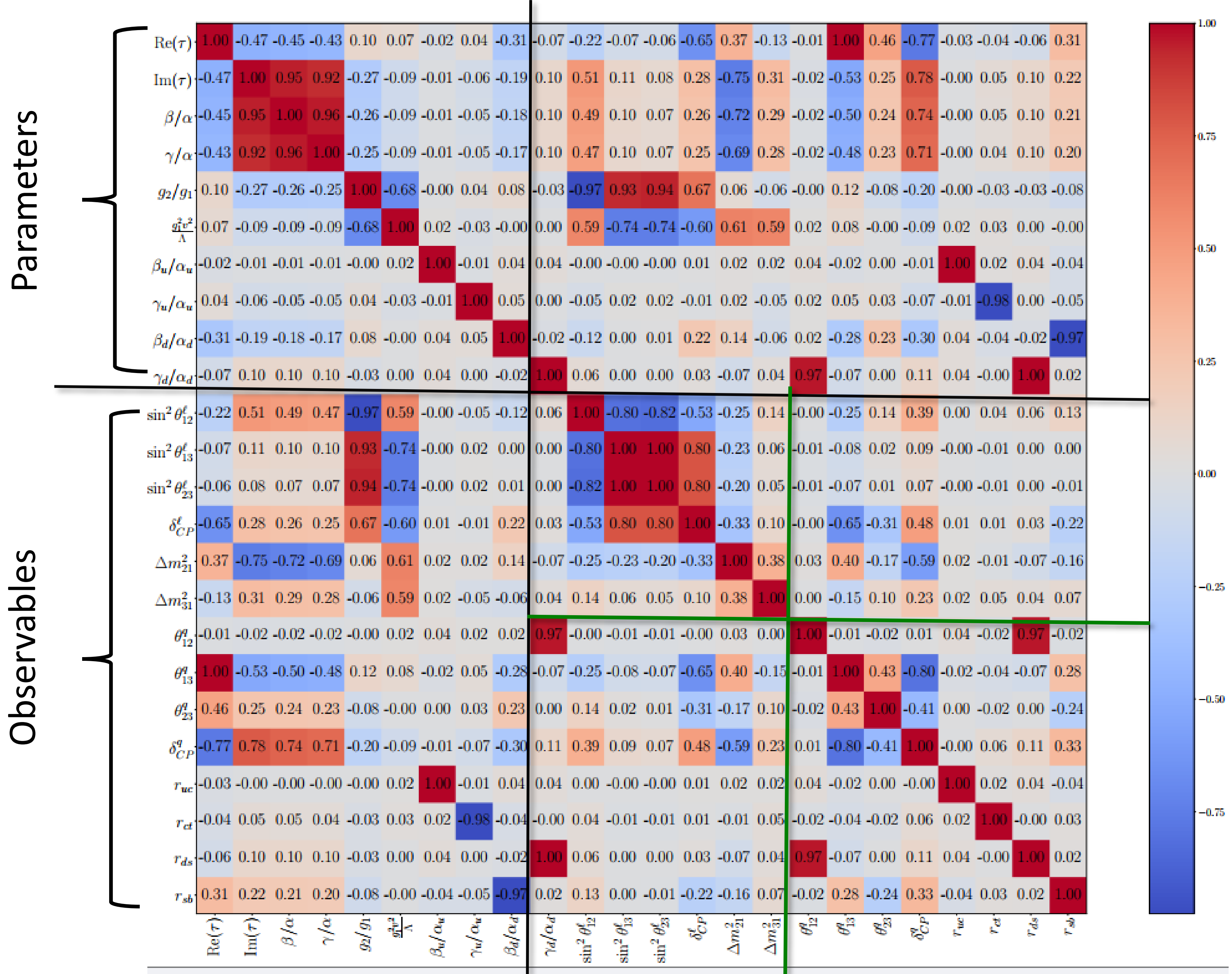
Polyharmonic Maaß forms of level N=2,3,4,5

k	-4	-3	-2	-1	0	1	2	3	4
$\Gamma_2 = S_3$	$1+2$	--	$1+2$	--	$1+2$	--	$1+2$	--	$1+2$
$\Gamma'_3 = T'$	$1+3$	$\hat{2}+\hat{2}''$	$1+3$	$\hat{2}+\hat{2}''$	$1+3$	$\hat{2}+\hat{2}''$	$1+3$	$\hat{2}+\hat{2}''$	$1+1'+3$
$\Gamma'_4 = S'_4$	$1+2+3$	$\hat{3}+\hat{3}'$	$1+2+3$	$\hat{3}+\hat{3}'$	$1+2+3$	$\widehat{3}+\hat{3}'$	$1+2+3$	$\hat{1}'+\hat{3}+\hat{3}'$	$1+2+3+3'$
$\Gamma'_5 = A'_5$	$1+3+3'+5$	$\hat{6}+\hat{6}$	$1+3+3'+5$	$\hat{6}+\hat{6}$	$1+3+3'+5$	$\hat{6}+\hat{6}$	$1+3+3'+5$	$\hat{4}'+\hat{6}+\hat{6}$	$1+3+3'+4+5+5$

- The best fit values of free parameters and flavor observables

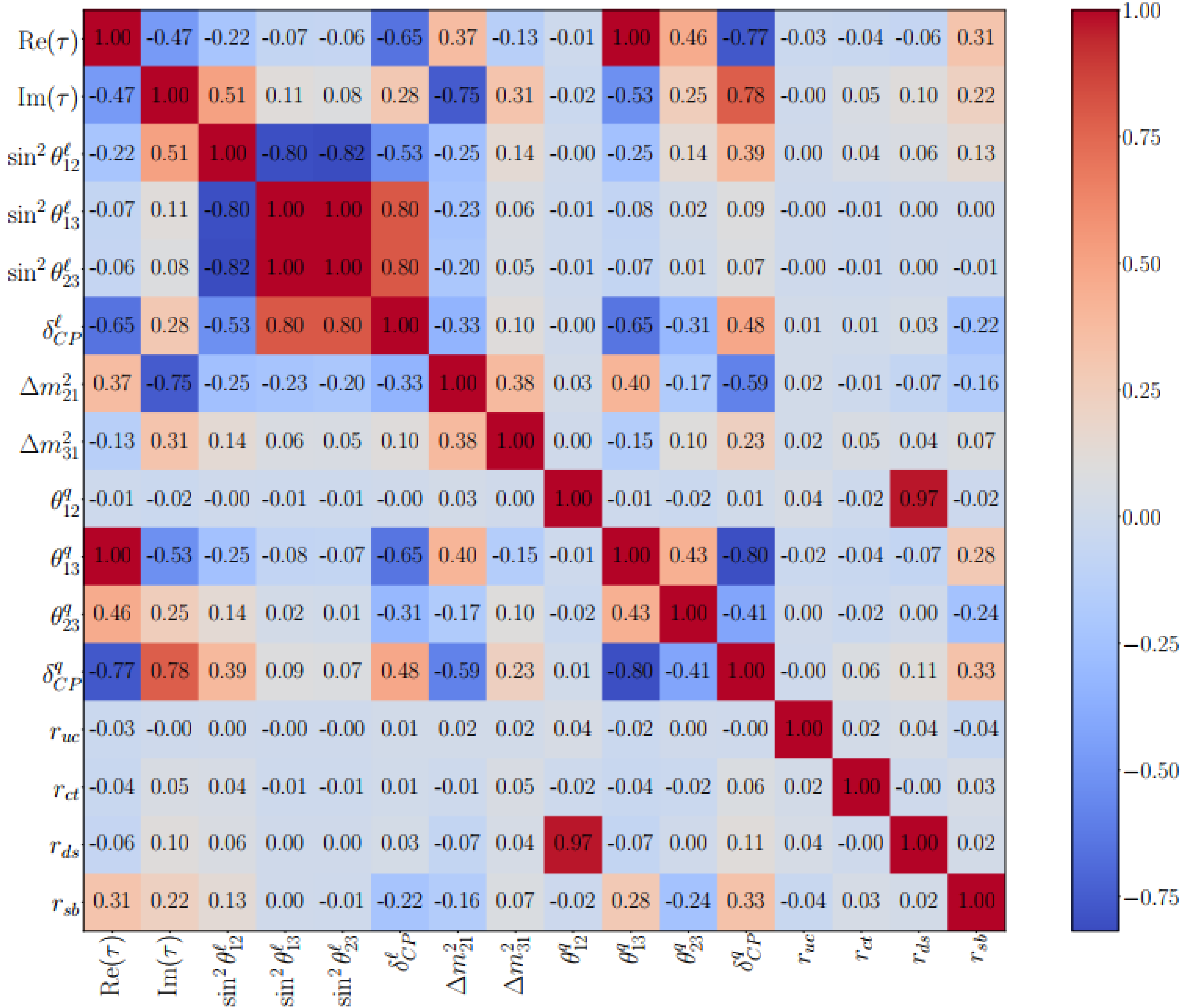
Inputs	Without gCP	With gCP
$\mathrm{Re}(\tau)$	0.005315	0.005056
$\mathrm{Im}(\tau)$	1.104	1.093
β/α	5.084	17.77
γ/α	0.004373	286.8
$\lvert g_2/g_1\rvert$	0.1647	0.1535
$\arg(g_2/g_1)$	5.799	0
$\alpha v/\mathrm{GeV}$	0.2789	0.004679
$\frac{g_1^2 v^2}{\Lambda}/\mathrm{meV}$	164.7	185.3
β_u/α_u	0.006165	0.006747
γ_u/α_u	497.3	496.2
β_d/α_d	30.11	29.12
γ_d/α_d	0.1131	0.06111
$\alpha_u v/\mathrm{GeV}$	0.3806	0.3869
$\alpha_d v/\mathrm{GeV}$	0.09457	0.1018

Observables	Without gCP	With gCP
$\sin^2\theta_{12}^\ell$	0.308	0.291
$\sin^2\theta_{13}^\ell$	0.02205	0.02158
$\sin^2\theta_{23}^\ell$	0.443	0.462
δ_{CP}^ℓ/π	1.09	0.99
$\Delta m_{21}^2/10^{-5}\mathrm{eV}^2$	7.57	7.11
$\Delta m_{31}^2/10^{-3}\mathrm{eV}^2$	2.51	2.53
m_e/m_μ	0.004737	0.004737
m_μ/m_τ	0.05882	0.05882
ϕ/π	1.10	0.99
m_1/meV	0	0
m_2/meV	8.70	8.43
m_3/meV	50.11	50.26
$\sum m_i/\mathrm{meV}$	58.81	58.69
$m_{\beta\beta}/\mathrm{meV}$	2.13	1.32
m_β/meV	8.84	8.65
θ_{12}^q	0.2256	0.2263
θ_{13}^q	0.0037	0.0037
θ_{23}^q	0.0406	0.0374
$\delta_{CP}^q/^\circ$	74.33	70.52
m_u/m_c	0.00204	0.00204
m_c/m_t	0.00318	0.00318
m_d/m_s	0.1066	0.0588
m_s/m_b	0.019	0.018
χ_ℓ^2	4.57	8.86
χ_q^2	8.77	13.79
χ_{tot}^2	13.33	22.65

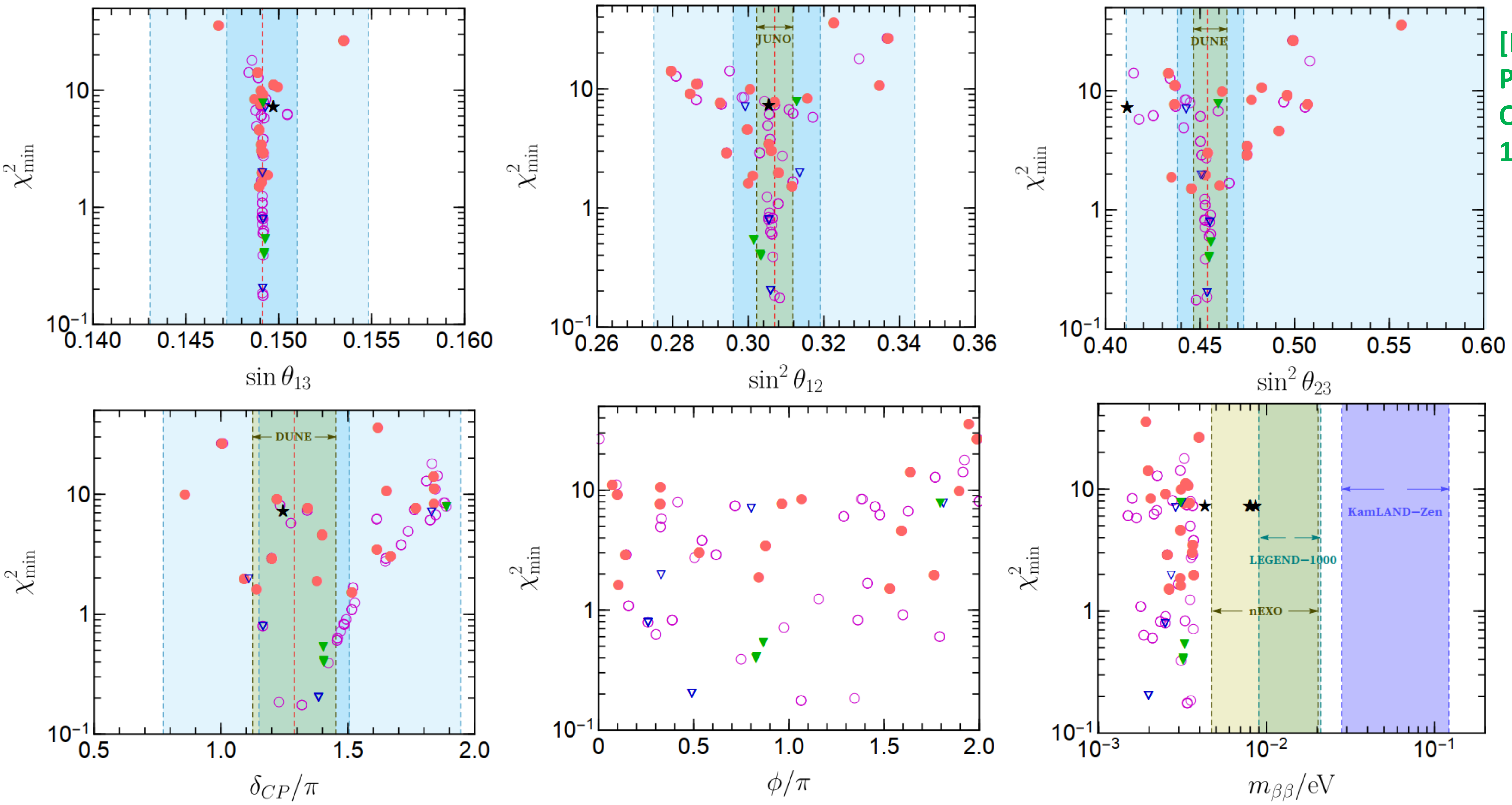


- Correlations among the free parameters and flavor observables

- Correlations among flavor observables and Re (τ), $\mathrm{Im}(\tau)$



Confronting non-holomorphic S4 models with experiments



[Ding,Lu, Petcov, Qu,2408.15988]