Flavor Symmetries & Winding Modes

Michael Ratz

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Based on:
X. Li, X.-G. Liu, H.P. Nilles, M.R. & A. Stewart, arXiv:2506.12887

Apologies & disclaimers

Apologies & disclaimers:

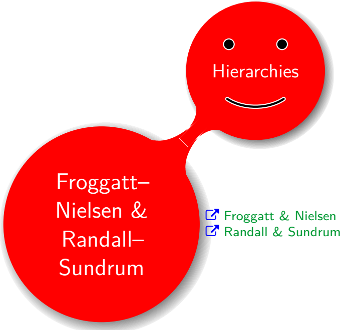
- This talk will not have extensive references to all activities that contribute to this exciting field, sorry!
- The symbols provide you with links to the respective references.
- I will suppress many details and focus on the big picture instead.

Flavor symmetries

Froggatt–Nielsen & Randall–Sundrum

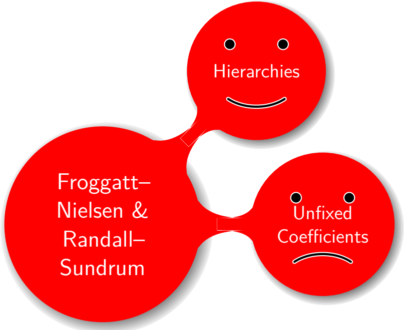
Froggatt & Nielsen [1979];. . .
Randall & Sundrum [1999];. . .

Flavor symmetries

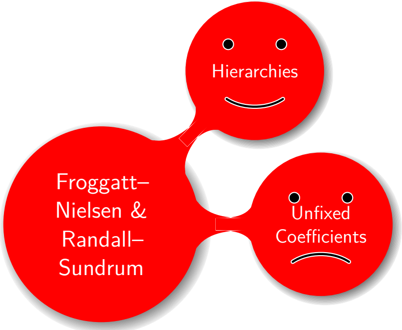


Froggatt & Nielsen [1979];...
Randall & Sundrum [1999];...

Flavor symmetries

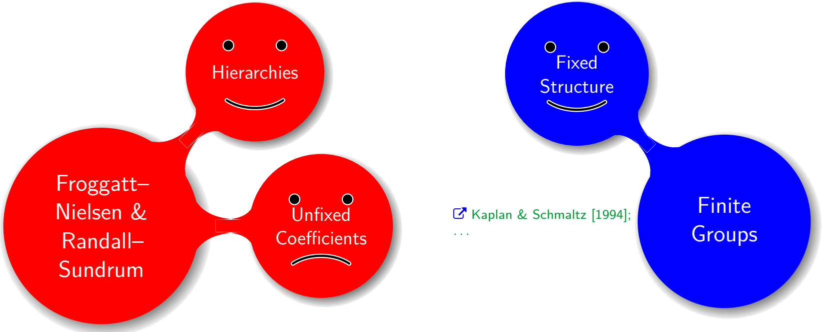


Flavor symmetries



Kaplan & Schmaltz [1994]; ...

Flavor symmetries



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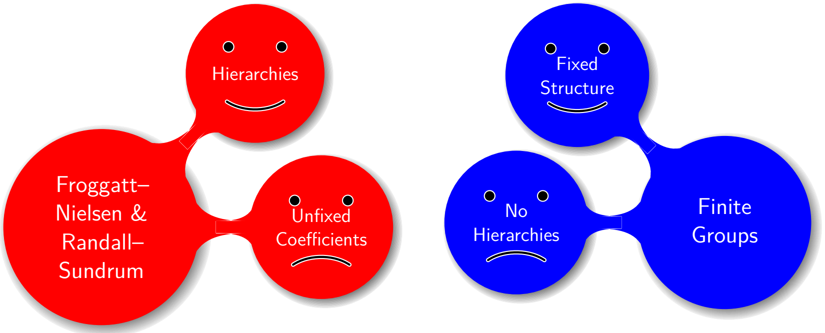
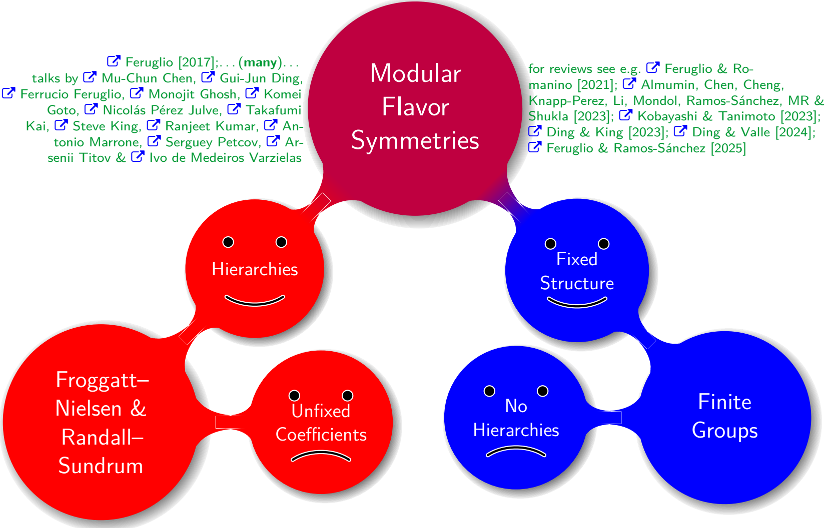
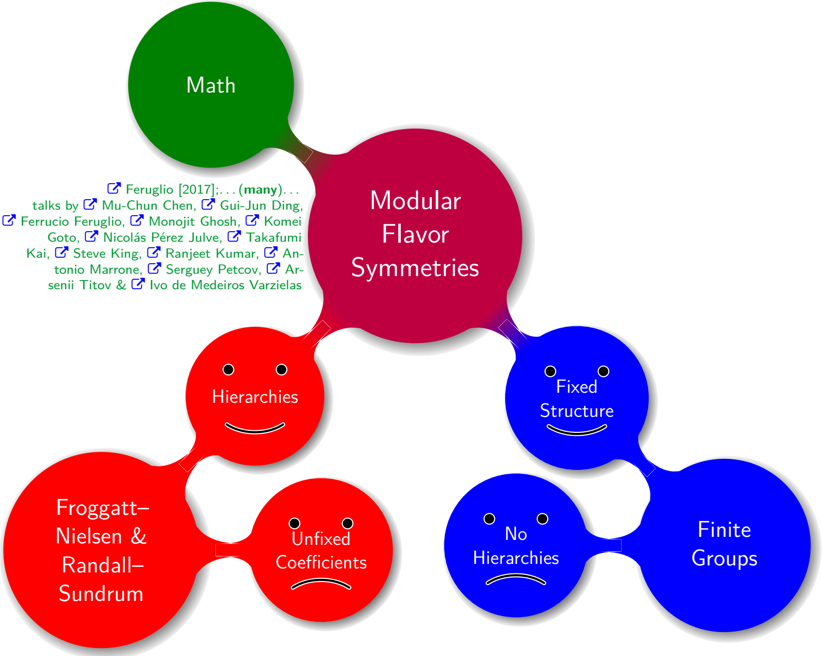


Flavor symmetries

Feruglio [2017];. . . (**many**). . .
talks by Mu-Chun Chen, Gui-Jun Ding, Ferrucio Feruglio, Monojit Ghosh, Komei Goto, Nicolás Pérez Julve, Takafumi Kai, Steve King, Ranjeet Kumar, Antonio Marrone, Serguey Petcov, Arsenii Titov & Ivo de Medeiros Varzielas

for reviews see e.g. Feruglio & Romanino [2021]; Almumin, Chen, Cheng, Knapp-Perez, Li, Mondol, Ramos-Sánchez, MR & Shukla [2023]; Kobayashi & Tanimoto [2023]; Ding & King [2023]; Ding & Valle [2024]; Feruglio & Ramos-Sánchez [2025]





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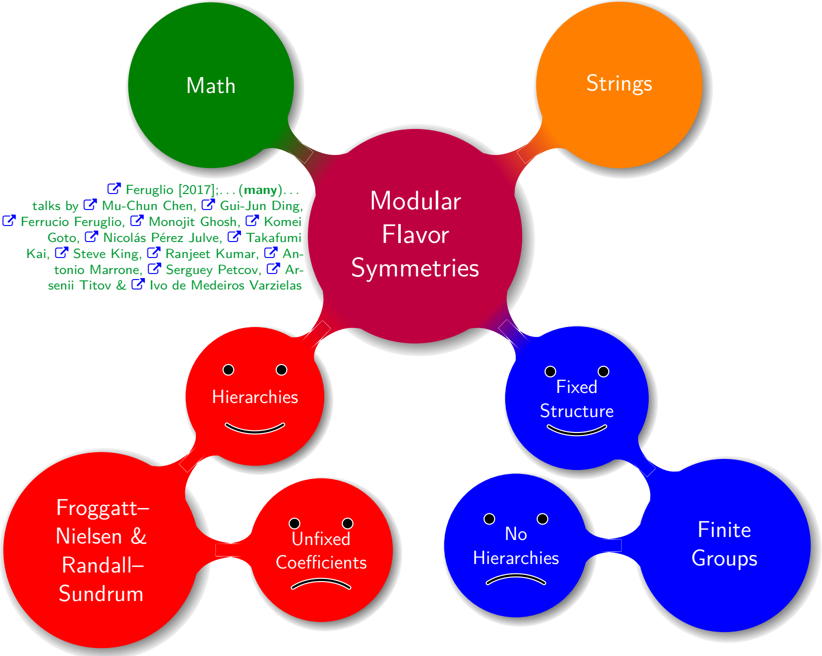

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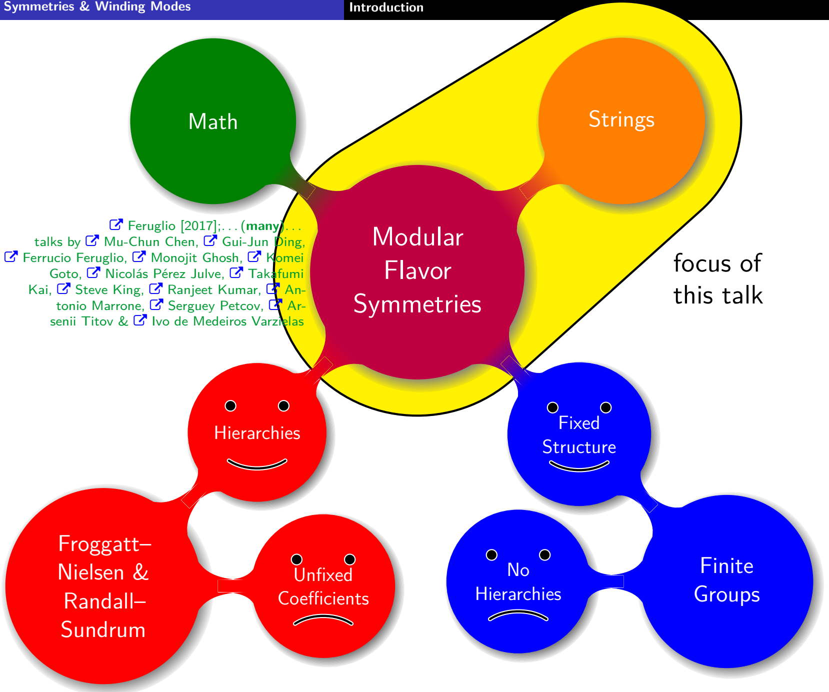
Math
Strings
Modular Flavor Symmetries
focus of this talk
Feruglio [2017];. . . (many). . . talks by Mu-Chun Chen, Gui-Jun Ding, Ferrucio Feruglio, Monojit Ghosh, Komei Goto, Nicolás Pérez Julve, Takafumi Kai, Steve King, Ranjeet Kumar, Antonio Marrone, Serguey Petcov, Arsenii Titov & Ivo de Medeiros Varzielas
Hierarchies
Fixed Structure
Froggatt–Nielsen & Randall–Sundrum
Unfixed Coefficients
No Hierarchies
Finite Groups

Modular Flavor Symmetries

Congruence subgroups of Γ & modular flavor symmetries

Feruglio [2017]

- congruence subgroups of $\Gamma := \mathrm{SL}(2,\mathbb{Z})/\mathbb{Z}_2$

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \; ; \; \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod N \right\}$$

level

$\Gamma = \mathrm{SL}(2,\mathbb{Z})/\mathbb{Z}_2$

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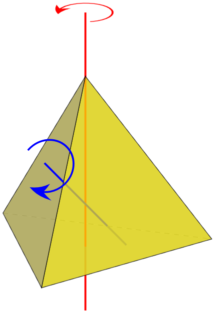
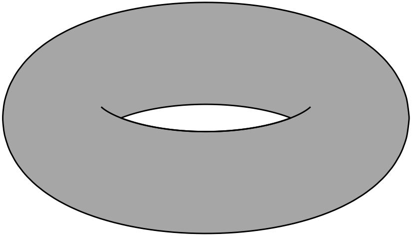
- quotients $\Gamma_N := \Gamma/\Gamma(N)$ are finite groups, e.g. $\Gamma_3 \simeq A_4$ (symmetry of tetrahedron)



How to get congruence subgroups?

$\mathrm{SL}(2,\mathbb{Z})$ and Γ

- $\mathrm{SL}(2,\mathbb{Z})$ is symmetry of torus



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- $\mathrm{SL}(2,\mathbb{Z})$ is symmetry of torus
- $\mathrm{SL}(2,\mathbb{Z})$ is generated by

$$\mathsf{T} \;:\; \tau \xmapsto{\mathsf{T}} \tau + 1$$

$$\mathsf{S} \;:\; \tau \xmapsto{\mathsf{S}} -\frac{1}{\tau}$$

modulus

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Congruence subgroups

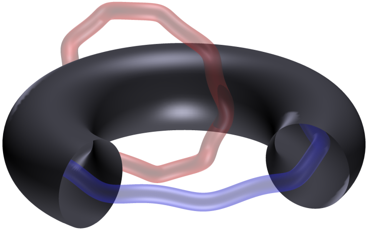
$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \; ; \; \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod N \right\}$$

- in bottom-up models based on $\Gamma(N)$ the symmetry of the action is typically $\mathrm{SL}(2,\mathbb{Z})$ (or larger)
- what is $\Gamma(N)$ a symmetry of?

Modular Symmetries

on

Tori



Modular symmetries in strings

e.g. Giveon, Porrati & Rabinovici [1994]; D'Hoker & Kaidi [2022]

☞ modular symmetries play a key role in string theory

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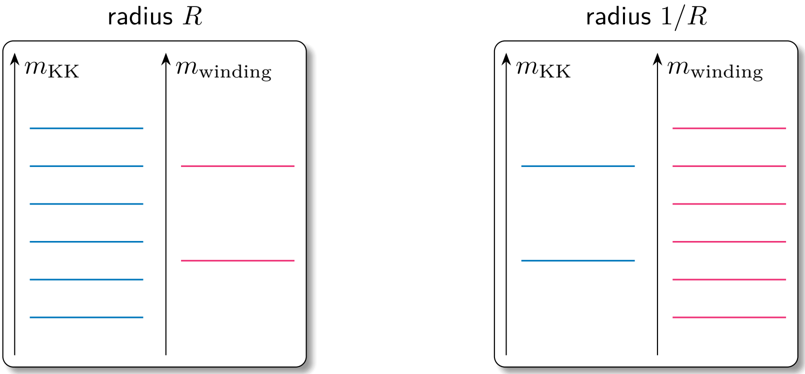
- T-duality relates Kaluza–Klein (KK) and winding modes



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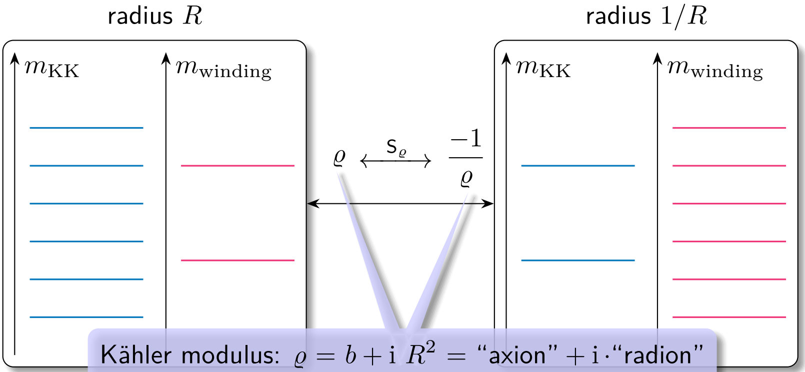
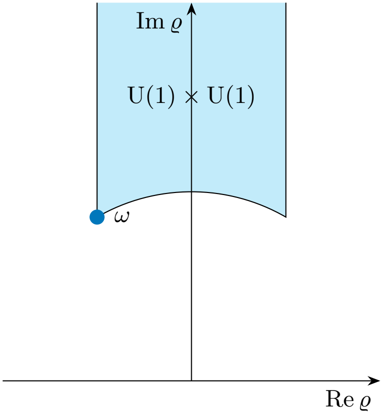


Kähler modulus: $\varrho = b + \mathrm{i}\, R^2 =$ "axion" $+\,\mathrm{i}\cdot$"radion"

Strings on a torus

- (heterotic) string on torus has two moduli: complex structure modulus τ & Kähler modulus ϱ
- at generic point in moduli space there is a $\mathrm{U}(1)\times\mathrm{U}(1)$ gauge symmetry mediated by oscillator modes



Strings on a torus

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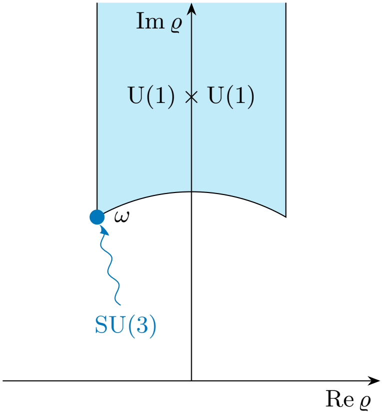
- at generic point in moduli space there is a $\mathrm{U}(1)\times\mathrm{U}(1)$ gauge symmetry mediated by oscillator modes
- if $\tau=\varrho=\omega:=\mathrm{e}^{2\pi\,\mathrm{i}\,/3}$ the gauge symmetry gets enhanced to $\mathrm{SU}(3)$ with the raising and lowering operators corresponding to winding modes
- modular symmetry is $\bigl(\mathrm{SL}(2,\mathbb{Z})_\tau\times\mathrm{SL}(2,\mathbb{Z})_\varrho\bigr)/\mathbb{Z}_2^{\mathrm{S}^2}$



Mass equation on torus

- mass of closed string on torus (Neveu–Schwarz sector)

$$m^2 = \frac{1}{\operatorname{Im}\tau\ \operatorname{Im}\varrho}\left|n_2 - n_1\,\tau - \overline{\varrho}\,(w^1 + w^2\,\tau)\right|^2 + 2\,(2N_\mathrm{R} - 1)$$

complex structure

Kaluza–Klein

winding

Kähler

oscillator

Mass equation on torus & modular symmetries

- mass of closed string on torus (Neveu–Schwarz sector)

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- modular symmetries are simultaneous transformations of the moduli τ and ϱ and the KK and winding numbers

$$\begin{pmatrix}\varrho\\ \tau\end{pmatrix}\begin{cases}\xmapsto{\mathsf{S}_\tau} (\varrho, -1/\tau)^\mathsf{T}\\ \xmapsto{\mathsf{T}_\tau} (\varrho, \tau+1)^\mathsf{T}\\ \xmapsto{\mathsf{S}_\varrho} (-1/\varrho, \tau)^\mathsf{T}\\ \xmapsto{\mathsf{T}_\varrho} (\varrho+1, \tau)^\mathsf{T}\end{cases}\qquad \begin{pmatrix}w^1\\ w^2\\ n_1\\ n_2\end{pmatrix}\begin{cases}\xmapsto{\mathsf{S}_\tau} (-w^2, w^1, -n_2, n_1)^\mathsf{T}\\ \xmapsto{\mathsf{T}_\tau} (w^1 - w^2, w^2, n_1, n_2 + n_1)^\mathsf{T}\\ \xmapsto{\mathsf{S}_\varrho} (-n_2, n_1, -w^2, w^1)^\mathsf{T}\\ \xmapsto{\mathsf{T}_\varrho} (w^1, w^2, n_1 - w^2, n_2 + w^1)^\mathsf{T}\end{cases}$$

Mass equation on torus & modular symmetries

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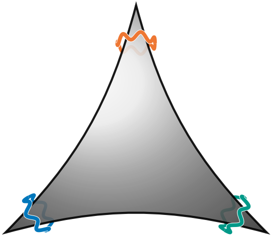
☞ however: no congruence subgroups

Modular Flavor Symmetries

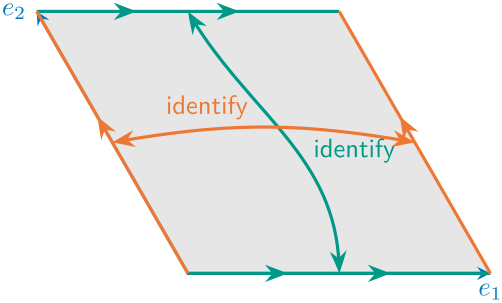
on

Orbifolds



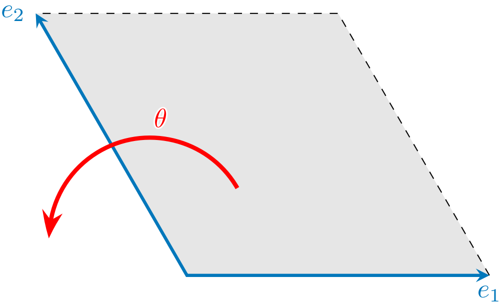
Orbifolds

start with torus with $\mathfrak{su}(3)$ lattice



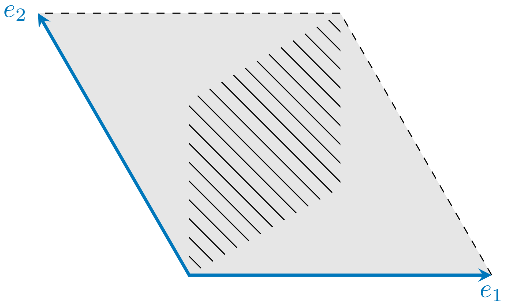
Orbifolds

divide out rotation θ by $2\pi/3$



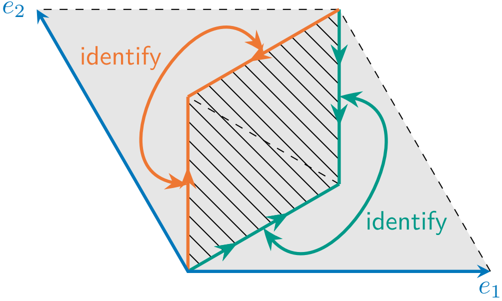
Orbifolds

(fundamental domain of torus) $= 3 \times$ (fundamental domain of orbifold)



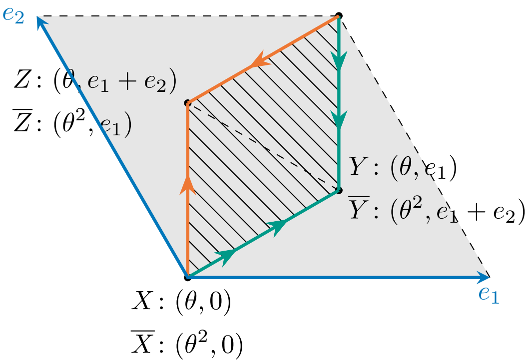
Orbifolds

opposite edges get identified



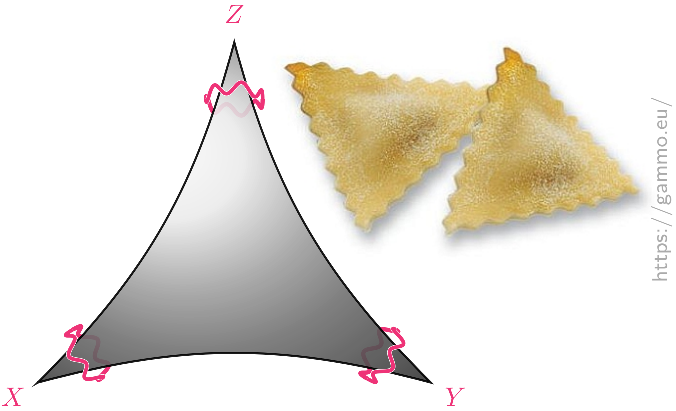
Orbifolds

chiral "twisted" states appear at fixed points (labeled by space group elements)



Orbifolds

result looks like a "ravioli" with
twisted strings sitting at the corners



Modular flavor symmetries were hiding in plain sight

☞ (Yukawa) couplings *are* modular forms in heterotic orbifolds cf. Quevedo [1996]

This is nothing but one of the $SL(2,\mathbf{Z})_{T,U}$ transformation for toroidal orbifold compactifications ($a=b=d=1, c=0$ in eq. (10)). Therefore the only conditions these symmetries impose on W is that it should transform as a modular form of a given weight ($W \to (cT+d)^{-3}W$ for the simplest toroidal orbifolds with T the overall size of the compactification space)[36]. In fact, explicit calculations for specific orbifold models show that

$$W_{tree}(T, Q^I) = \chi_{IJK}(T)\, Q^I Q^J Q^K + \cdots \tag{19}$$

with $\chi(T)$ a particular modular form of $SL(2,\mathbf{Z})$ or any other duality group and the ellipsis represent higher powers of Q, exponentially suppressed. The identification of $\chi(T)$ with modular forms was a highly nontrivial check of the explicit orbifold calculations which were preformed in refs. [37] without any relation (nor knowledge) of the underlying duality symmetry $SL(2,\mathbf{Z})$. This kind of symmetry puts also strong constraints to the higher order, nonrenormalizable, corrections to W, since each matter field Q transforms in a particular way under that symmetry ($Q \to (cT+d)^n Q$ with n the modular weight of Q). There are also other discrete symmetries, as those defined by the point group $\mathcal{P}$ and space group $\mathcal{S}$ of an orbifold which have to be respected by the superpotential W. These 'selection rules' are very important to find vanishing couplings and uncover flat directions which can be used to break the original gauge symmetries and construct quasi-realistic models.

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  cf. Quevedo [1996]

- a nontrivial modular flavor symmetry (T') has been discovered early on

  Ferrara, Lüst, Shapere & Theisen [1989]; Chun, Mas, Lauer & Nilles [1989];...

- vector-valued modular form couplings were obtained long ago using orbifold CFT

  Chun, Mas, Lauer & Nilles [1989]; Lauer, Mas & Nilles [1991];...

- modular symmetries can be understood neatly in the Narain formalism

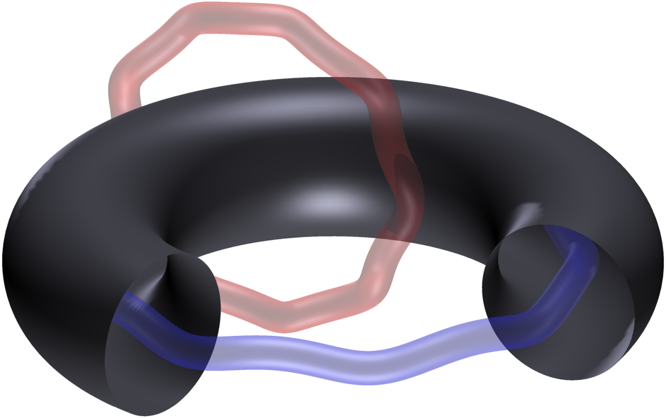
  Baur, Nilles, Trautner & Vaudrevange [2019a]; Nilles, Ramos-Sánchez & Vaudrevange [2020a]; Nilles, Ramos-Sánchez & Vaudrevange [2021]; Baur, Nilles, Ramos-Sánchez, Trautner & Vaudrevange [2024];...

- however: so far very little discussion of massive (and not so massive 😉) winding and KK modes

Massless winding modes

☞ mass of closed string on torus (NS sector)

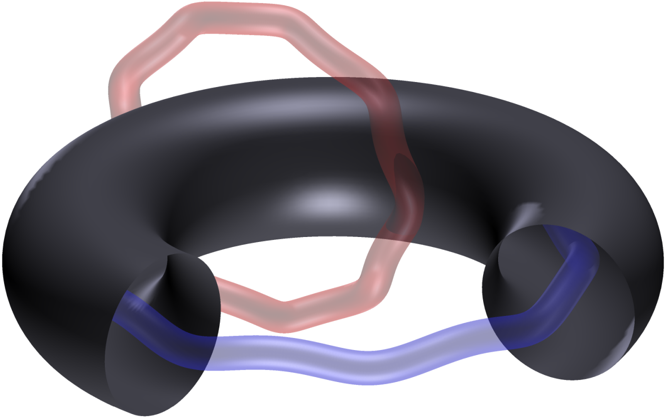
$$m^2 = \frac{1}{\operatorname{Im}\tau\ \operatorname{Im}\varrho}\left|n_2 - n_1\,\tau - \overline{\varrho}\,(w^1 + w^2\,\tau)\right|^2 + 2\,(2N_\mathrm{R} - 1)$$



Massless winding modes

☞ mass of closed string on torus (NS sector) for $\tau = \omega$

$$m^2 = \frac{2}{\sqrt{3}\,\mathrm{Im}\,\varrho}\left|n_2 - n_1\,\omega - \overline{\varrho}\,(w^1 + w^2\,\omega)\right|^2 + 2\,(2N_\mathrm{R} - 1)$$



Massless winding modes

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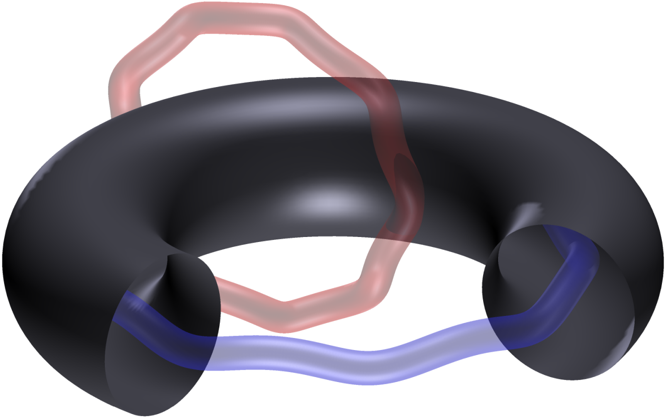
$$m^2 = \frac{2}{\sqrt{3}\,\mathrm{Im}\,\varrho}\left|n_2 - n_1\,\omega - \overline{\varrho}\,(w^1 + w^2\,\omega)\right|^2 + 2\,(1-1)$$



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- massless states if ϱ is an $\mathrm{SL}(2,\mathbb{Z})$ image of ω

▸ details

Massless winding modes at $\varrho = \omega$

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- two orbifold-invariant combinations of massless winding modes at critical point $\varrho = \omega$

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- massless states if ϱ is an $\mathrm{SL}(2,\mathbb{Z})$ image of ω ▸ details
- two orbifold-invariant combinations of massless winding modes at critical point $\varrho = \omega$
- questions:
  1. what is the interpretation of these states?
  2. what happens at other $\mathrm{SL}(2,\mathbb{Z})$ images of ω?

Gauge symmetry enhancement at $\varrho = \omega$

☞ interpretation of massless winding modes: $\mathrm{U}(1) \times \mathrm{U}(1)$ gauge fields

Ibáñez, Lerche, Lüst & Theisen [1991]; Beye, Kobayashi & Kuwakino [2014]

Gauge symmetry enhancement at $\varrho = \omega$

- interpretation of massless winding modes: $\mathrm{U}(1) \times \mathrm{U}(1)$ gauge fields
- three additional massless untwisted states $U_1^{(\omega)}$ (right-moving oscillators)
- charges of untwisted and twisted states from orbifold CFT
- untwisted charge eigenstates are linear combinations of two winding modes and one left-moving oscillator mode

Ibáñez, Lerche, Lüst & Theisen [1991]; Beye, Kobayashi & Kuwakino [2014]

state	$[\mathrm{U}(1) \times \mathrm{U}(1)]^{(\omega)}$
$U_1^{(\omega)}$	$(\sqrt{2}, 0)$
$U_2^{(\omega)}$	$(1/\sqrt{2}, \sqrt{3/2})$
$U_3^{(\omega)}$	$(-1/\sqrt{2}, -\sqrt{3/2})$
$X^{(\omega)}$	$(\sqrt{2}/3, 0)$
$Y^{(\omega)}$	$(-1/\sqrt{18}, 1/\sqrt{6})$
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Gauge symmetry enhancement at $\varrho = \omega$

- interpretation of massless winding modes: $\mathrm{U}(1) \times \mathrm{U}(1)$ gauge fields
- three additional massless untwisted states $U_1^{(\omega)}$ (right-moving oscillators)
- charges of untwisted and twisted states from orbifold CFT
- untwisted charge eigenstates are linear combinations of two winding modes and one left-moving oscillator mode
- oscillator mode can be identified with departure of ϱ from ω (*F*- & *D*-flat): $U_1^{(\omega)} + U_2^{(\omega)} + U_3^{(\omega)} \sim \varrho - \omega$

Ibáñez, Lerche, Lüst & Theisen [1991]; Beye, Kobayashi & Kuwakino [2014]

state	$[\mathrm{U}(1) \times \mathrm{U}(1)]^{(\omega)}$
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Kähler modulus

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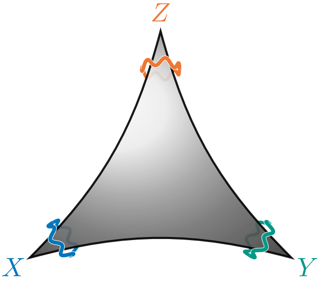
- away from $\varrho = \omega$: residual $[\mathbb{Z}_3 \times \mathbb{Z}_3]^{(\omega)}$ symmetry

Interlude: localization vs. charge eigenstates

Chun, Mas, Lauer & Nilles [1989]

☞ twisted $[\mathrm{U}(1)\times\mathrm{U}(1)]^{(\omega)}$ charge eigenstates do *not* coincide with localization eigenstates

$$\underbrace{\begin{pmatrix} X^{(\omega)} \\ Y^{(\omega)} \\ Z^{(\omega)} \end{pmatrix}}_{=:\ \Phi^{(\omega)}_{-2/3}} = \underbrace{\frac{1}{\sqrt{3}}\begin{pmatrix} \omega & 1 & 1 \\ 1 & \omega & 1 \\ 1 & 1 & \omega \end{pmatrix}}_{=:\ \mathcal{U}_{(\omega)}} \cdot \underbrace{\begin{pmatrix} X \\ Y \\ Z \end{pmatrix}}_{=:\ \Phi_{-2/3}}$$



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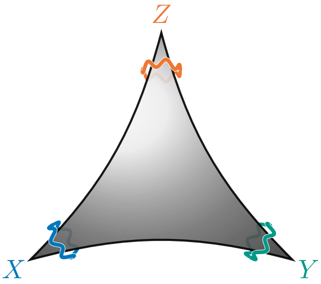
charge eigenstates

localization eigenstates



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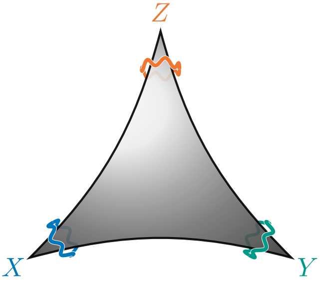
  $$\Phi^{(\omega)}_{-2/3} = \mathcal{U}_{(\omega)}\,\Phi_{-2/3}$$

Li, Liu, Nilles, MR & Stewart [2025]

- basis transformation is modular
  $$\mathcal{U}_{(\omega)} = -\,\mathrm{i}\,\omega^2\,\rho(\mathsf{T}^{-1}\,\mathsf{S}\,\mathsf{T}^{-1})$$
  with
  $$\rho(\mathsf{S}) = \frac{\mathrm{i}}{\sqrt{3}}\begin{pmatrix}1 & 1 & 1\\ 1 & \omega^2 & \omega\\ 1 & \omega & \omega^2\end{pmatrix}$$
  $$\rho(\mathsf{T}) = \mathrm{diag}(\omega^2,1,1)$$



Gauge symmmetry enhancement at $\varrho = \omega$ (cont'd)

details

☞ couplings of gauge fields $W^{\pm}_{(\omega)}$ to localization eigenstates are not diagonal

$$\begin{pmatrix} \Phi^{\dagger}_{-2/3}\, \mathsf{t}^{(\omega)}_{+}\, \Phi_{-2/3} \\ \Phi^{\dagger}_{-2/3}\, \mathsf{t}^{(\omega)}_{-}\, \Phi_{-2/3} \end{pmatrix} \cdot \begin{pmatrix} W^{-}_{(\omega)} \\ W^{+}_{(\omega)} \end{pmatrix}$$

$$= \frac{1}{\sqrt{6}} \begin{pmatrix} \omega^2\, \overline{X}\, Y + \overline{Y}\, Z + \omega\, \overline{Z}\, X \\ \omega\, \overline{Y}\, X + \overline{Z}\, Y + \omega^2\, \overline{X}\, Z \end{pmatrix} \cdot \begin{pmatrix} W^{-}_{(\omega)} \\ W^{+}_{(\omega)} \end{pmatrix}$$

U(1) generators

Gauge symmmetry enhancement at $\varrho = \omega$ (cont'd)

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$$\begin{pmatrix} \Phi^\dagger_{-2/3}\,\mathsf{t}_+^{(\omega)}\,\Phi_{-2/3} \\ \Phi^\dagger_{-2/3}\,\mathsf{t}_-^{(\omega)}\,\Phi_{-2/3} \end{pmatrix} \cdot \begin{pmatrix} W^-_{(\omega)} \\ W^+_{(\omega)} \end{pmatrix} = \frac{1}{\sqrt{6}} \begin{pmatrix} \omega^2\,\overline{X}\,Y + \overline{Y}\,Z + \omega\,\overline{Z}\,X \\ \omega\,\overline{Y}\,X + \overline{Z}\,Y + \omega^2\,\overline{X}\,Z \end{pmatrix} \cdot \begin{pmatrix} W^-_{(\omega)} \\ W^+_{(\omega)} \end{pmatrix}$$

☞ residual $[\mathbb{Z}_3 \times \mathbb{Z}_3]^{(\omega)}$ symmetries for $\varrho \neq \omega$

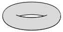
$$\Phi_{-2/3} \xmapsto{\mathbb{Z}_3^{(\omega,1)}} \omega\,\Phi_{-2/3}$$

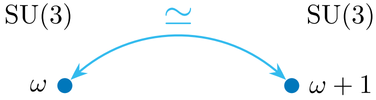
$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \xmapsto{\mathbb{Z}_3^{(\omega,2)}} \begin{pmatrix} 0 & 0 & \omega^2 \\ \omega & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} =: Z_{(\omega,2)} \cdot \Phi_{-2/3}$$

Modular transformations on torus vs. orbifold

Li, Liu, Nilles, MR & Stewart [2025]

- what happens at $\varrho = \omega + 1$?
- naively: same as $\varrho = \omega$ since both points are related by $\mathrm{SL}(2,\mathbb{Z})$





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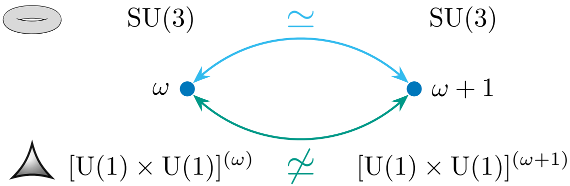

Main message of this talk:

points related by $\mathrm{SL}(2,\mathbb{Z})$ can be physically different

- fundamental domain of $\mathrm{SL}(2,\mathbb{Z})$ is too small (need $\Gamma(3)$)
- non-Abelian flavor symmetries
- special values of modular forms

Gauge symmmetry enhancement at $\varrho = \omega + 1$

☞ again two gauge fields yet different couplings

$$\begin{pmatrix} \omega\,\overline{X}\,Z + \omega^2\,\overline{Y}\,X + \overline{Z}\,Y \\ \omega^2\,\overline{Z}\,X + \omega\,\overline{X}\,Y + \overline{Y}\,Z \end{pmatrix} \cdot \begin{pmatrix} W^+_{(\omega+1)} \\ W^-_{(\omega+1)} \end{pmatrix}$$

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- S_ϱ^2 swaps components

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- S_ϱ^2 swaps components
- $[\mathrm{U}(1) \times \mathrm{U}(1)]^{(\omega)}$ and $[\mathrm{U}(1) \times \mathrm{U}(1)]^{(\omega+1)}$ generators do not commute

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details

$$\bigl[\mathbb{Z}_3 \times \mathbb{Z}_3\bigr]^{(\omega)} \cup \bigl[\mathbb{Z}_3 \times \mathbb{Z}_3\bigr]^{(\omega+1)} = \bigl(\mathbb{Z}_3 \times \mathbb{Z}_3\bigr) \rtimes \mathbb{Z}_3 = \Delta(27)$$

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- $\mathbb{Z}_2^{\mathsf{S}_\varrho^2}$ enhances $\mathbb{Z}_3$ to S_3 and $\Delta(27)$ to $\Delta(54)$

$$\overbrace{[\mathrm{U}(1)\times\mathrm{U}(1)]^{(\omega+1)} \rtimes S_3}^{\varrho = \omega+1} \longrightarrow \overbrace{\underbrace{\left[\mathbb{Z}_3\times\mathbb{Z}_3\right]^{(\omega+1)} \rtimes S_3}_{=\Delta(54)}}^{\varrho \neq \omega+1}$$

Gauge origin of flavor symmetries & string selection rules

Beye, Kobayashi & Kuwakino [2014]

☞ it has been noted previously that $[\mathrm{U}(1)\times\mathrm{U}(1)]^{(\omega)}$ gives rise to $\mathbb{Z}_3\times\mathbb{Z}_3$ …

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point group

space group

Hamidi & Vafa [1987]; Dixon, Friedan, Martinec & Shenker [1987];…

- string selection rules: $\mathbb{Z}_3^{\mathrm{PG}}\times\mathbb{Z}_3^{\mathrm{SG}}$

Kobayashi, Nilles, Plöger, Raby & MR [2007]

- $\Delta(54)$ has been obtained from $\mathbb{Z}_3^{\mathrm{PG}}\times\mathbb{Z}_3^{\mathrm{SG}}$

(again treating the missing $\mathbb{Z}_3$ as an accidental symmetry)

state	$\mathbb{Z}_3^{\mathrm{PG}}$	$\mathbb{Z}_3^{\mathrm{SG}}$
X	1	0
Y	1	1
Z	1	2
$\overline{X}$	2	0
$\overline{Y}$	2	2
$\overline{Z}$	2	1

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Li, Liu, Nilles, MR & Stewart [2025]

- we show explicitly that the $\mathbb{Z}_3^{\mathrm{PG}}\times\mathbb{Z}_3^{\mathrm{SG}}$ and additional $\mathbb{Z}_3$ are all gauged

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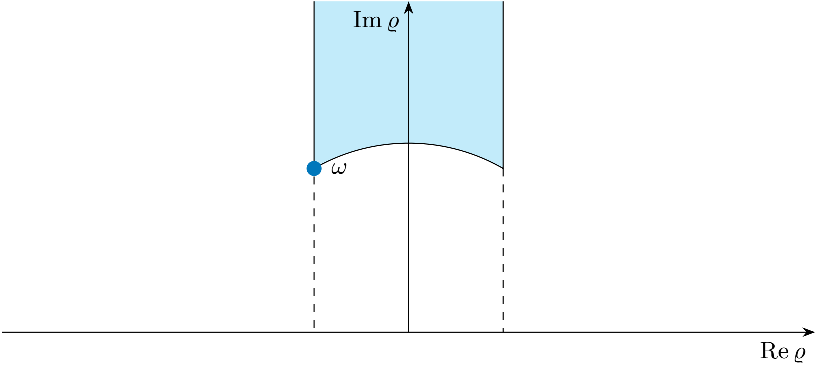
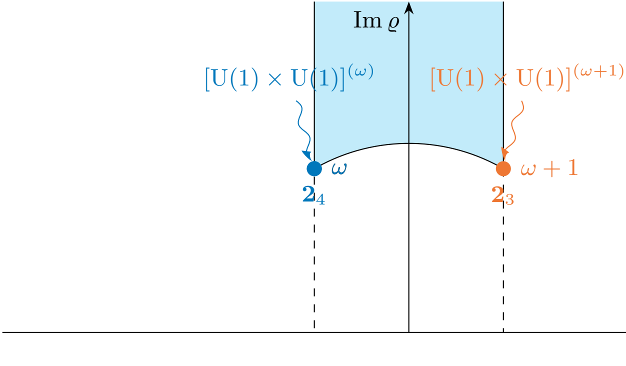
Enhanced gauge symmetries & fundamental domains

fundamental domain of $\mathrm{SL}(2,\mathbb{Z})$



Enhanced gauge symmetries & fundamental domains

however, the $\mathrm{U}(1)$ symmetries at ω and $\omega+1$ are different, so the fundamental domain of $\mathrm{SL}(2,\mathbb{Z})$ is *too small*



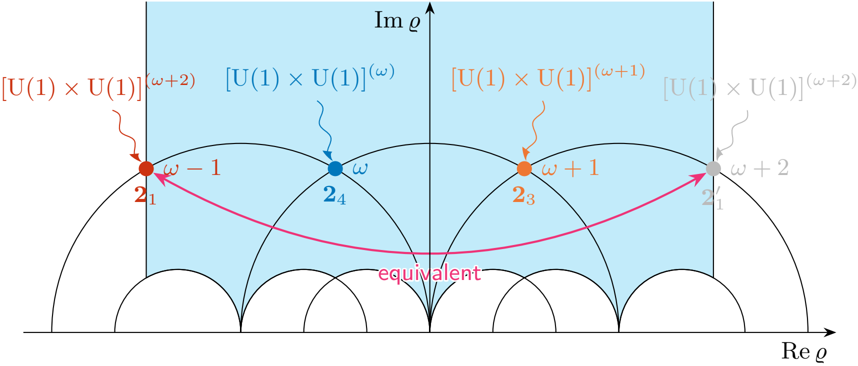
https://www.pinterest.com/

Enhanced gauge symmetries & fundamental domains

$\omega + 3n$ and ω are physically equivalent

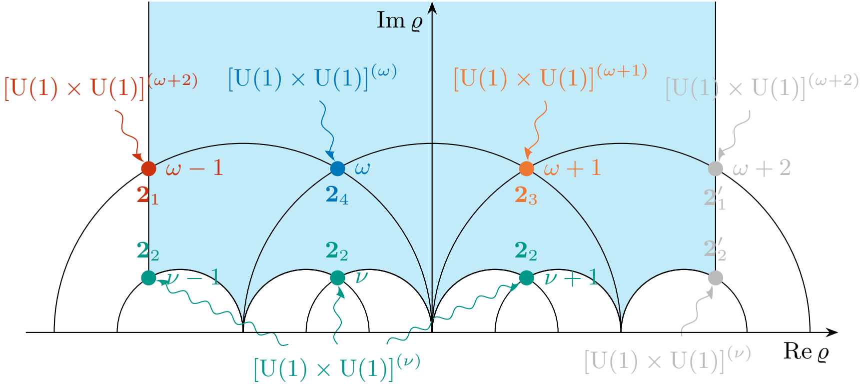
↷ need fundamental domain of $\Gamma(3)$

$$\Gamma(3) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})_\varrho;\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod 3 \right\}$$



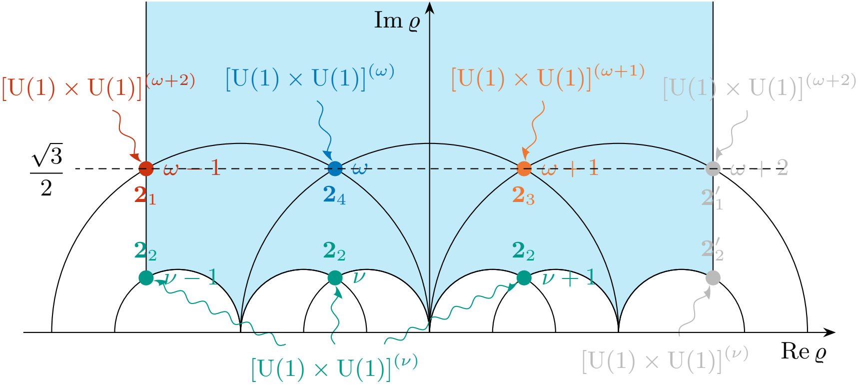
Enhanced gauge symmetries & fundamental domains

the critical points ν, $\nu+1$ & $\nu+2$ carry the 'same' $[\mathrm{U}(1)\times\mathrm{U}(1)]^{(\nu)}$ symmetries but can be shown to be physically distinct

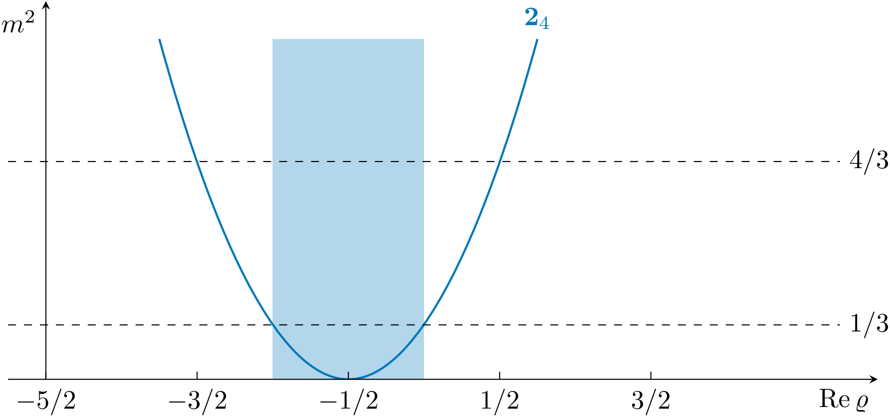


Enhanced gauge symmetries & fundamental domains

let us now look at the doublet as
functions of $\mathrm{Re}\,\varrho$ at $\mathrm{Im}\,\varrho = \sqrt{3}/2$



Massive winding modes at $\mathrm{Im}\,\varrho = \sqrt{3}/2$



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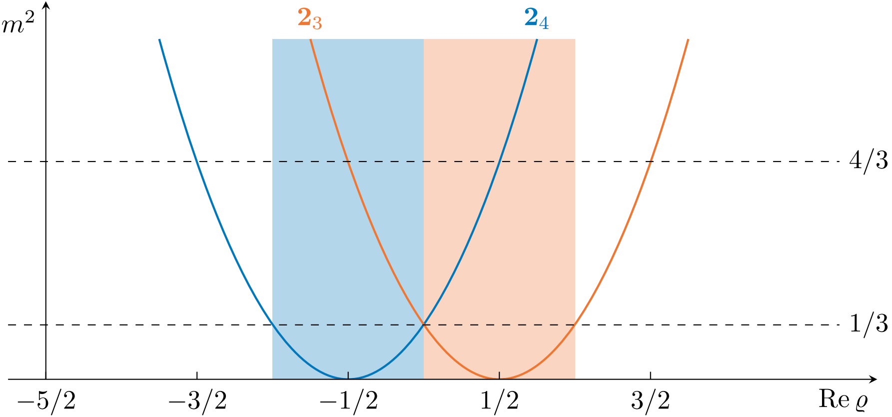
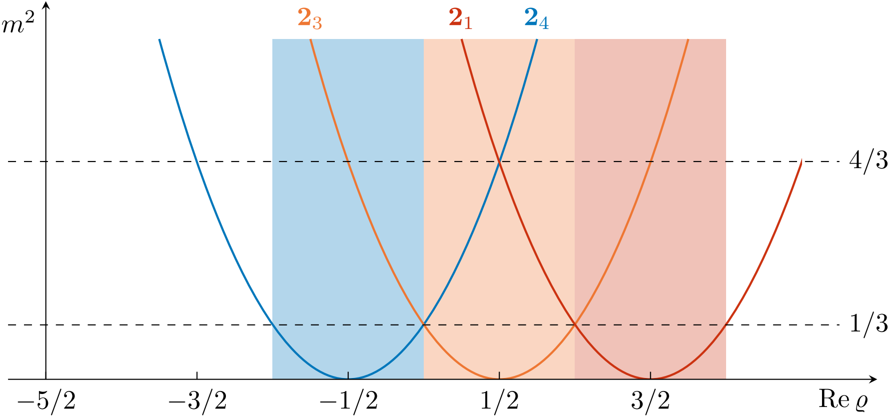
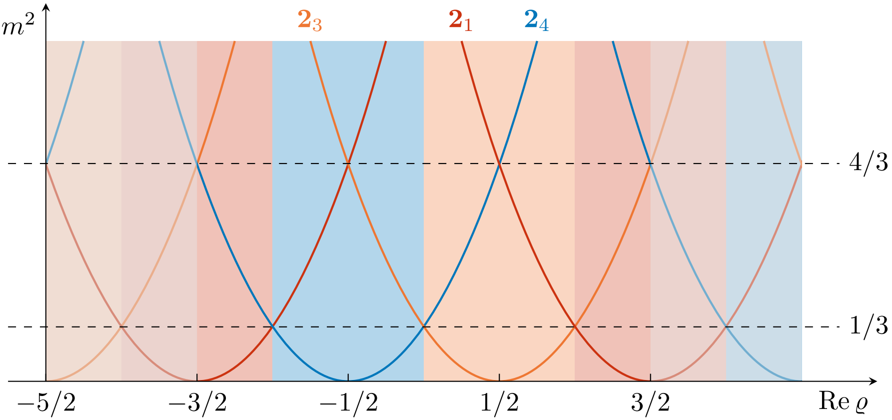


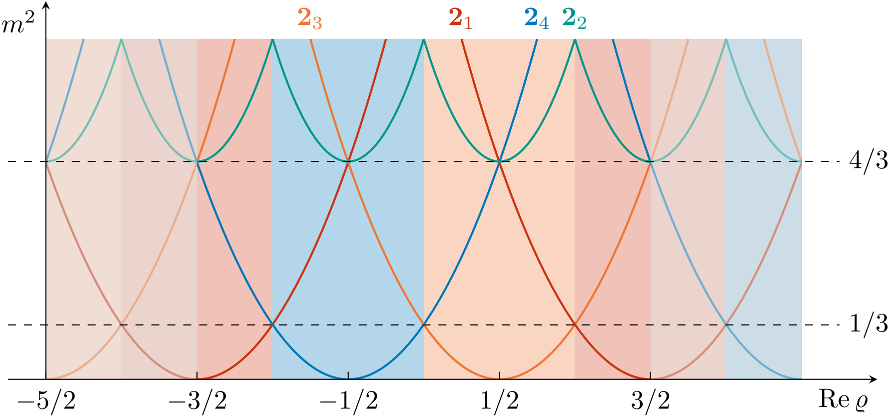
Massive winding modes at $\operatorname{Im}\varrho = \sqrt{3}/2$



Massive winding modes at $\operatorname{Im} \varrho = \sqrt{3}/2$

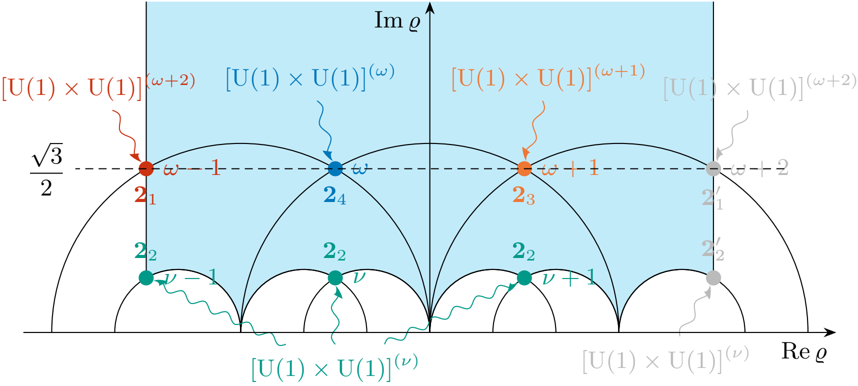


Massive winding modes at $\mathrm{Im}\,\varrho = \sqrt{3}/2$



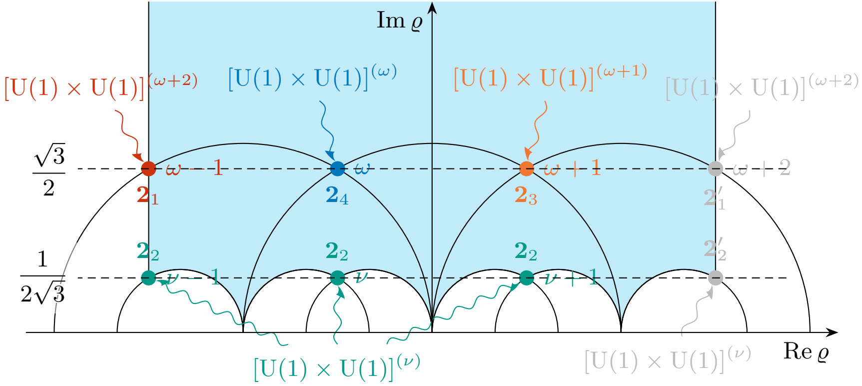
Enhanced gauge symmetries & fundamental domains

we have just seen the gauge boson masses at $\operatorname{Im}\varrho = \frac{\sqrt{3}}{2}$

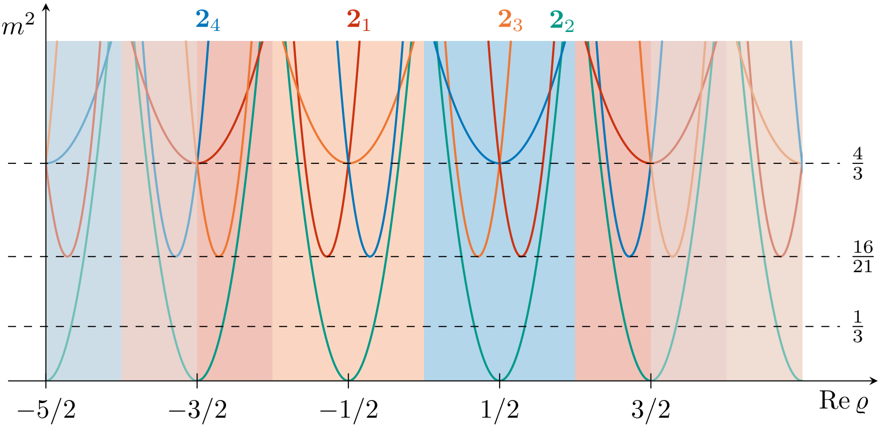


Enhanced gauge symmetries & fundamental domains

let us now look at $\mathrm{Im}\,\varrho = \dfrac{1}{2\sqrt{3}}$



Massive winding modes at $\operatorname{Im} \varrho = 1/2\sqrt{3}$



Superpotential

☞ at critical points $[\mathrm{U}(1)\times\mathrm{U}(1)]^{(\varrho_\mathrm{crit})}$ restricts superpotential to the form

$$\mathscr{W}_\mathrm{trilinear} = \mathcal{Y}(\varrho_\mathrm{crit})\, X^{(\varrho_\mathrm{crit})}\, Y^{(\varrho_\mathrm{crit})}\, Z^{(\varrho_\mathrm{crit})}$$

modular form

cartoon of
gauge eigenstates

Superpotential

- at critical points $[\mathrm{U}(1)\times\mathrm{U}(1)]^{(\varrho_\mathrm{crit})}$ restricts superpotential to the form $\mathscr{W}_\mathrm{trilinear} = \mathcal{Y}(\varrho_\mathrm{crit})\, X^{(\varrho_\mathrm{crit})}\, Y^{(\varrho_\mathrm{crit})}\, Z^{(\varrho_\mathrm{crit})}$
- in terms of localization eigenstates $(X, Y, Z)^\mathsf{T}$

ϱ	$\mathscr{W}_\mathrm{trilinear}$
ω	$(\omega\, X + Y + Z)\,(X + \omega\, Y + Z)\,(X + Y + \omega\, Z)$
$\omega + 1$	$(\omega^2\, X + Y + Z)\,(\omega\, X + \omega\, Y + Z)\,(\omega\, X + Y + \omega\, Z)$
$\omega + 2$	$(X + Y + Z)\,(\omega^2\, X + \omega\, Y + Z)\,(\omega^2\, X + Y + \omega\, Z)$
$\nu, \nu+1, \nu+2$	$X\,Y\,Z$

Superpotential

- at critical points $[\mathrm{U}(1)\times\mathrm{U}(1)]^{(\varrho_\mathrm{crit})}$ restricts superpotential to the form
  $\mathscr{W}_\mathrm{trilinear} = \mathcal{Y}(\varrho_\mathrm{crit})\, X^{(\varrho_\mathrm{crit})}\, Y^{(\varrho_\mathrm{crit})}\, Z^{(\varrho_\mathrm{crit})}$
- at generic point in ϱ moduli space
  $\mathscr{W}_\mathrm{trilinear}(\varrho; X, Y, Z) = \mathcal{Y}_2(\varrho)\,\left(X^3+Y^3+Z^3\right) - 3\sqrt{2}\,\mathcal{Y}_1(\varrho)\,X\,Y\,Z$

  cf. Liu & Ding [2022]

  with vector-valued modular form

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Superpotential

- at critical points $[\mathrm{U}(1)\times\mathrm{U}(1)]^{(\varrho_\mathrm{crit})}$ restricts superpotential to the form $\mathscr{W}_\mathrm{trilinear}=\mathcal{Y}(\varrho_\mathrm{crit})\,X^{(\varrho_\mathrm{crit})}\,Y^{(\varrho_\mathrm{crit})}\,Z^{(\varrho_\mathrm{crit})}$
- at generic point in ϱ moduli space $\mathscr{W}_\mathrm{trilinear}(\varrho;X,Y,Z)=\mathcal{Y}_2(\varrho)\,\left(X^3+Y^3+Z^3\right)-3\sqrt{2}\,\mathcal{Y}_1(\varrho)\,X\,Y\,Z$
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- normalization is known: $\mathcal{N}=2^{-1/2}3^{1/4}\dfrac{\Gamma(2/3)^2}{\Gamma(1/3)}$
  Chun, Mas, Lauer & Nilles [1989]
  Lauer, Mas & Nilles [1991]

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- the modular forms are believed to emerge from integrating out the KK and winding modes

  cf. Giveon & Porrati [1991]

Towers on orbifolds

- ☞ (n_1, n_2, w^2, w^2) on the torus: not orbifold-invariant

KK

winding

Towers on orbifolds

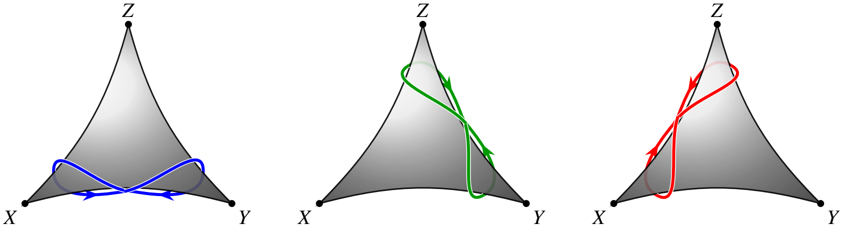
- ☞ (n_1, n_2, w^2, w^2) on the torus: not orbifold-invariant
- ☞ we can define a single orbifold-invariant KK number N and winding number W plus three auxiliary quantum numbers R, s_W & s_N

Towers on orbifolds

- (n_1, n_2, w^2, w^2) on the torus: not orbifold-invariant
- we can define a single orbifold-invariant KK number N and winding number W plus three auxiliary quantum numbers R, s_W & s_N
- ➥ masses and $\Delta(54)$ quantum numbers of massive towers

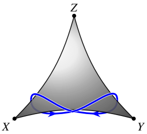
$\Delta(54)$	$\Delta(27)$	(s_W, s_N)	(W^2, N^2, R^2)	A	B	C
$\mathbf{2}_1$	$\begin{pmatrix}\mathbf{1}_{0,2}\\ \mathbf{1}_{0,1}\end{pmatrix}$	$\begin{pmatrix}(2,0)\\ (1,0)\end{pmatrix}$	$(1,3,3)$	$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$	$\begin{pmatrix}\omega^2 & 0\\ 0 & \omega\end{pmatrix}$	$\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$
$\mathbf{2}_2$	$\begin{pmatrix}\mathbf{1}_{2,0}\\ \mathbf{1}_{1,0}\end{pmatrix}$	$\begin{pmatrix}(0,1)\\ (0,2)\end{pmatrix}$	$(3,1,-3)$	$\begin{pmatrix}\omega^2 & 0\\ 0 & \omega\end{pmatrix}$	$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$	$\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$
$\mathbf{2}_3$	$\begin{pmatrix}\mathbf{1}_{2,2}\\ \mathbf{1}_{1,1}\end{pmatrix}$	$\begin{pmatrix}(2,1)\\ (1,2)\end{pmatrix}$	$(1,1,1)$	$\begin{pmatrix}\omega^2 & 0\\ 0 & \omega\end{pmatrix}$	$\begin{pmatrix}\omega^2 & 0\\ 0 & \omega\end{pmatrix}$	$\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$
$\mathbf{2}_4$	$\begin{pmatrix}\mathbf{1}_{2,1}\\ \mathbf{1}_{1,2}\end{pmatrix}$	$\begin{pmatrix}(1,1)\\ (2,2)\end{pmatrix}$	$(1,1,-1)$	$\begin{pmatrix}\omega^2 & 0\\ 0 & \omega\end{pmatrix}$	$\begin{pmatrix}\omega & 0\\ 0 & \omega^2\end{pmatrix}$	$\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$

Winding modes

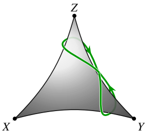


☞ doublets $\mathbf{2}_1$, $\mathbf{2}_3$ & $\mathbf{2}_4$ are linear combinations of these winding modes, e.g.

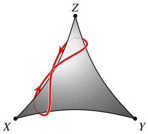
$\mathbf{2}_4 \sim \Bigg(\omega^2$



$+$



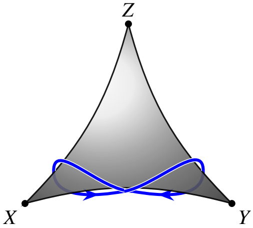
$+\,\omega$



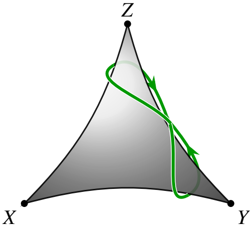
$\Bigg)$

c.c.

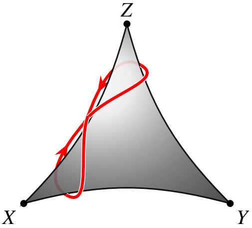
Winding modes



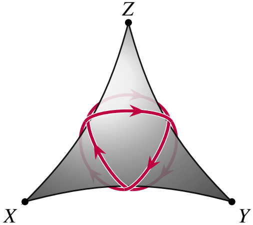
$\overline{X}Y$



$\overline{Y}Z$



$\overline{Z}X$

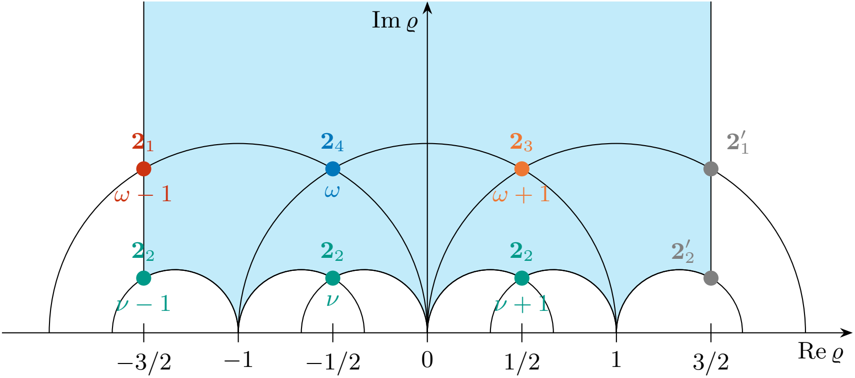


$\mathbf{2}_2$ (massless at ν, $\nu+1$ & $\nu+2$)

$\mathcal{CP}$ transformation

Li, Liu, Nilles, MR & Stewart [2025]

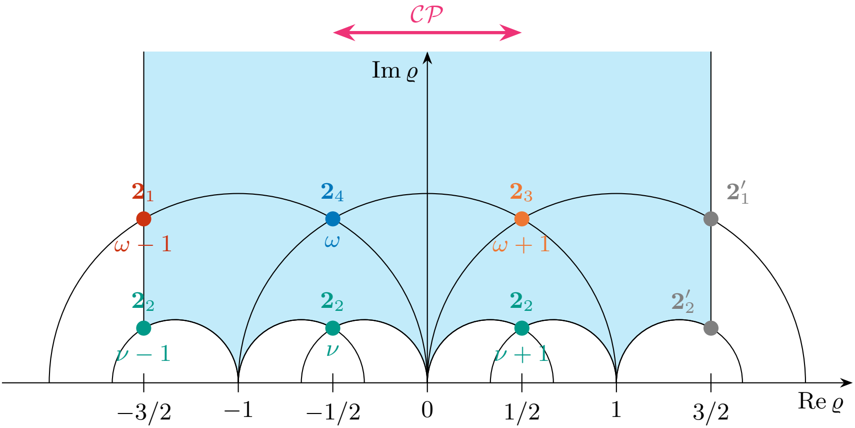
fundamental domain



$\mathcal{CP}$ transformation

Dent [2001]; Baur, Nilles, Trautner & Vaudrevange [2019b]; Novichkov, Penedo, Petcov & Titov [2019]

canonical $\mathcal{CP}$ transformation of ϱ: $\varrho \xmapsto{\mathcal{CP}} -\overline{\varrho}$



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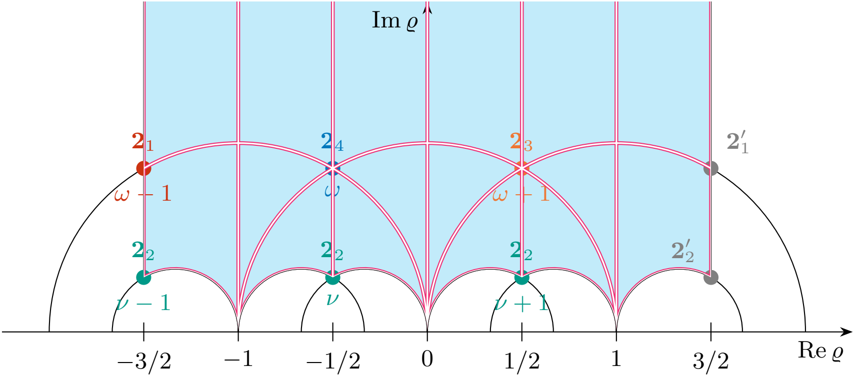
$\mathcal{CP}$ conserving curves determined by the condition that $-\overline{\varrho}_{\mathcal{CP}} \overset{!}{=} \dfrac{a\,\varrho_{\mathcal{CP}}+b}{c\,\varrho_{\mathcal{CP}}+d}$ for some $\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{Z})$



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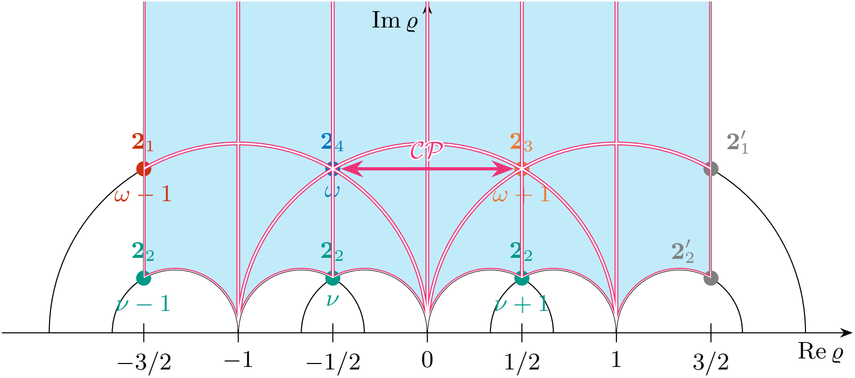
reflection at $\mathcal{CP}$ conserving curves corresponds to relations between the masses of different towers, e.g. $\mathbf{2}_4 \leftrightarrow \mathbf{2}_3$



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Li, Liu, Nilles, MR & Stewart [2025]

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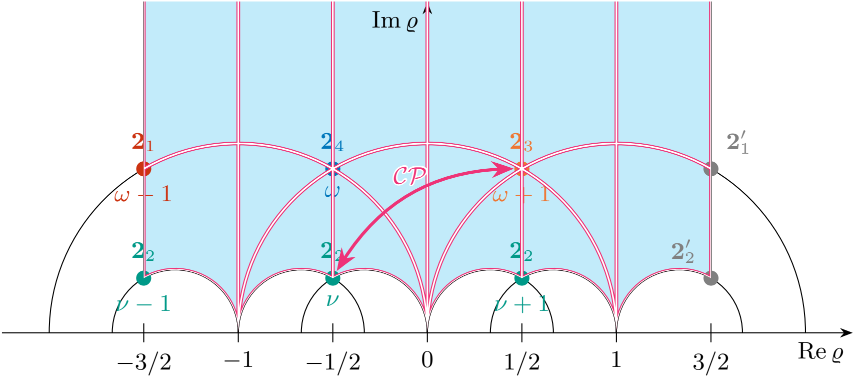

Subtleties of the $\mathcal{CP}$ transformation

☞ what is the significance of mass relations between doublet masses?

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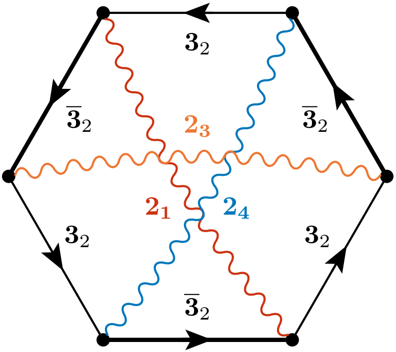
Nilles, MR, Trautner & Vaudrevange [2018]



$\mathrm{U}(1)$ generators = CG's of $\Delta(54)$

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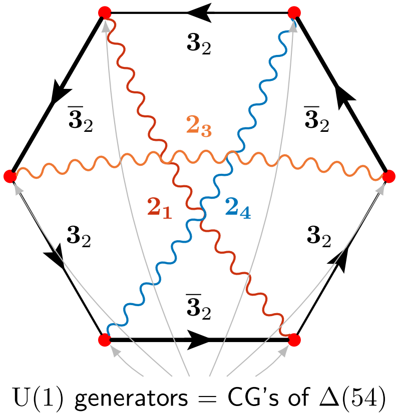

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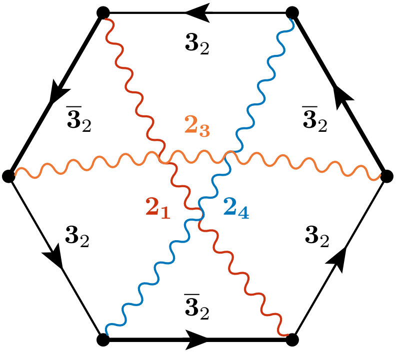
- if the doublet masses are degenerate the diagrams no longer violate $\mathcal{CP}$

Nilles, MR, Trautner & Vaudrevange [2018]



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Implications for the Kähler potential

talk by Mu-Chun Chen;
Feruglio [2023a]; Feruglio [2023b]; Petcov & Tanimoto [2023];
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Nilles, Ramos-Sánchez & Vaudrevange [2020a]; Chen, Knapp–Pérez, Ramos-Hamud, Ramos-Sánchez, MR & Shukla [2022]

- there are alternative proposals to tame the Kähler potential which require additional flavons to make the corrections parametrically small, and these VEVs of the additional flavons have to be aligned/explained

Summary

&

Outlook

Summary of symmetries

Li, Liu, Nilles, MR & Stewart [2025]

Nilles, Ramos-Sánchez & Vaudrevange [2020b]

$G^{\mathbb{Z}_3}_{\text{eclectic}}$

- combination of modular, traditional, R and $\mathcal{CP}$ symmetries has been dubbed "eclectic"

Summary of symmetries

Li, Liu, Nilles, MR & Stewart [2025]

$$G_{\text{eclectic}}^{\mathbb{Z}_3} \quad = \quad \Delta(27)$$

☞ the residual $\mathbb{Z}_3$ symmetries from relatively misaligned $\mathrm{U}(1)$ gauge symmetries that become exact at different points in ϱ moduli space combine to $\Delta(27)$

Summary of symmetries

Li, Liu, Nilles, MR & Stewart [2025]

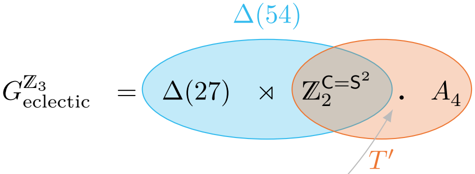
$\Delta(54)$

$$G^{\mathbb{Z}_3}_{\text{eclectic}} = \Delta(27) \rtimes \mathbb{Z}_2^{\mathsf{C}=\mathsf{S}^2}$$

☞ the S^2_ϱ transformation enhances $\Delta(27)$ to $\Delta(54)$

Summary of symmetries

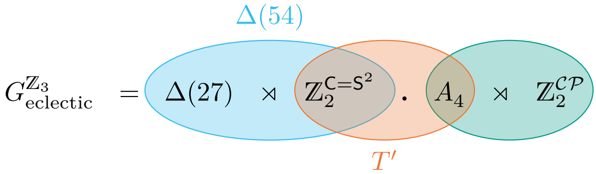
Li, Liu, Nilles, MR & Stewart [2025]



☞ the finite modular group is T', which can be written as the non-split extension of A_4

Summary of symmetries

Li, Liu, Nilles, MR & Stewart [2025]

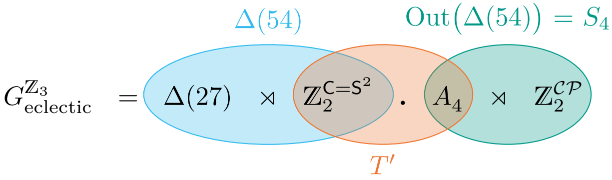


Baur, Nilles, Trautner & Vaudrevange [2019a]; Novichkov, Penedo, Petcov & Titov [2019]

- ☞ the extended modular group S_4 is obtained by adding $\mathcal{CP}$

Summary of symmetries

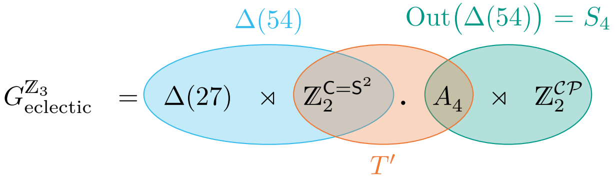
Li, Liu, Nilles, MR & Stewart [2025]



- ☞ this S_4 is the maximal outer automorphism group of $\Delta(54)$, and ϱ is faithful under this group

Summary of symmetries

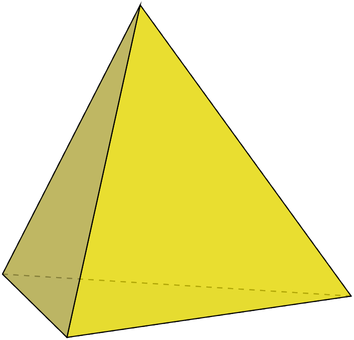
Li, Liu, Nilles, MR & Stewart [2025]



☞ the superpotential transforms in a nontrivial 1-dimensional representation of $\Delta(54)$

Full modular symmetry

- the full modular contains the symmetries of a tetrahedron acting on twisted and winding strings



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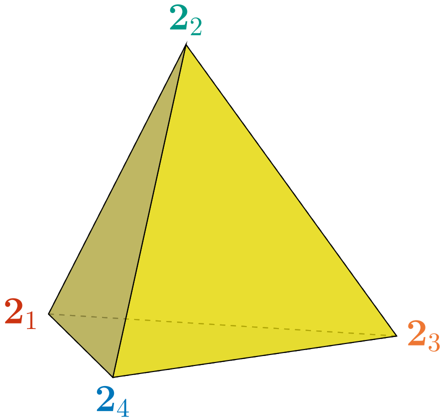

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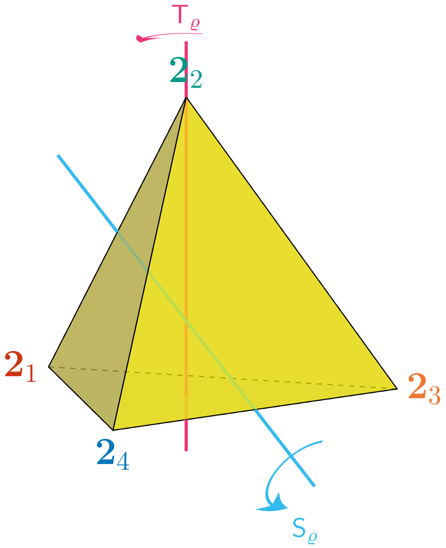

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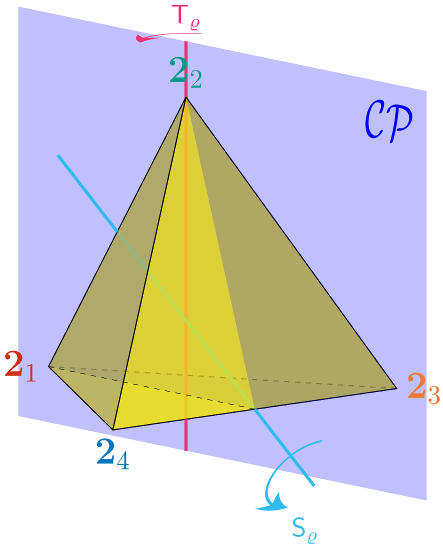
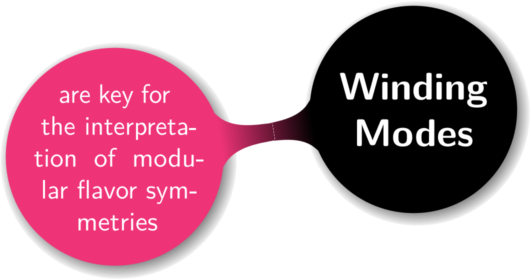
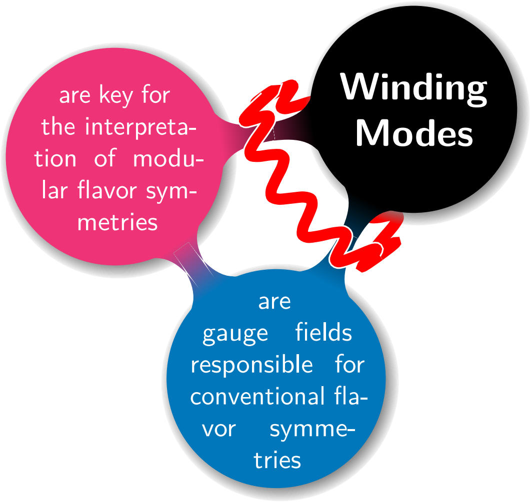
- the tetrahedron is in fact chiral because of the $\mathcal{CP}$ transformation



Summary


Winding Modes
are key for the interpretation of modular flavor symmetries

Summary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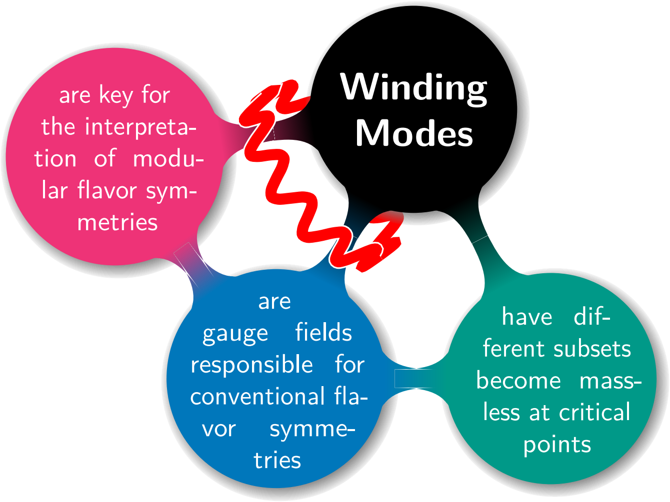
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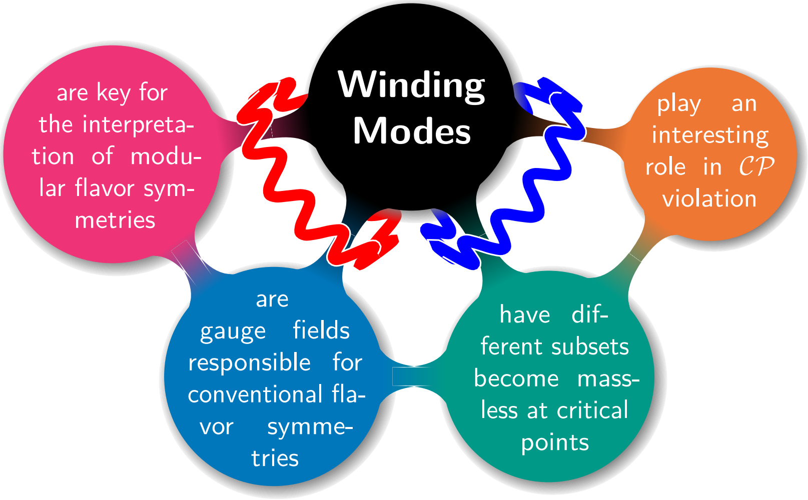
Winding Modes
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have different subsets become massless at critical points
play an interesting role in $\mathcal{CP}$ violation

Outlook

- within $\mathbb{T}^2/\mathbb{Z}_3$ orbifold: switch on Wilson lines
- explore different geometries
  (including space groups with freely acting transformations)
- combine with "misaligned supersymmetry"
  (to possibly have top-down realizations of non-holomorphic modular flavor symmetries)
- discuss winding modes in a complete string model that gives rise to the standard model

Mille Grazie!

www.wikipedia.org

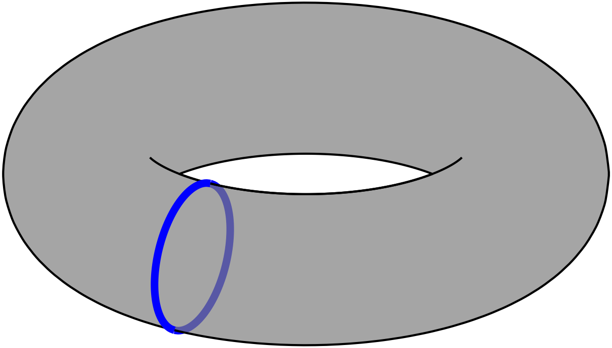
Thank You Very Much!

Backup slides

Massless winding modes

- ☞ mass of closed string on torus (NS sector)

$$m^2 = \frac{1}{\operatorname{Im}\tau\ \operatorname{Im}\varrho}\left|n_2 - n_1\,\tau - \overline{\varrho}\,(w^1 + w^2\,\tau)\right|^2 + 2\ (2N_{\mathrm{R}} - 1)$$



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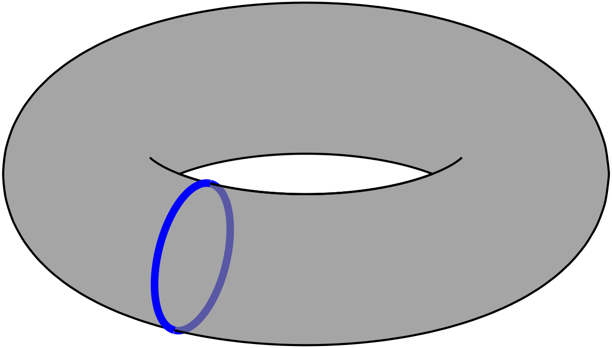

Massless winding modes

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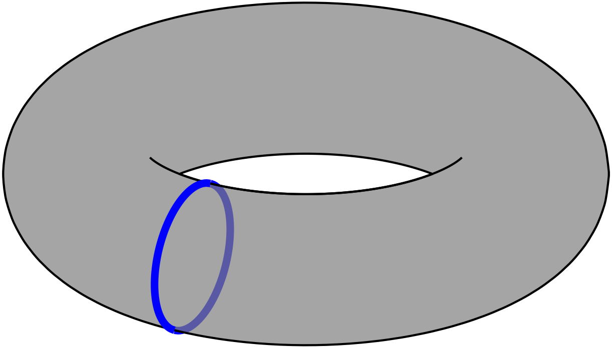
$$m^2 = \frac{1}{\operatorname{Im}\tau\ \operatorname{Im}\varrho}\left|n_2 - n_1\,\omega - \overline{\varrho}\,(w^1 + w^2\,\omega)\right|^2 + 2\,(1-1)$$



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all $+$ and $-$ states get identified in orbifold

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➥ two orbifold-invariant massless winding modes at $\varrho = \omega$

back

Gauge symmmetry enhancement at $\varrho = \omega$ (details)

☞ couplings of gauge fields $W^{\pm}_{(\omega)}$ to localization eigenstates are not diagonal

$$\mathsf{t}_1^{(\omega)} = \left(\mathcal{U}_{(\omega)}\right)^\dagger \mathsf{t}_1 \, \mathcal{U}_{(\omega)} = \frac{1}{2\sqrt{3}} \begin{pmatrix} 0 & \omega^2 & \omega^2 \\ \omega & 0 & 1 \\ \omega & 1 & 0 \end{pmatrix}$$

$$\mathsf{t}_2^{(\omega)} = \left(\mathcal{U}_{(\omega)}\right)^\dagger \mathsf{t}_2 \, \mathcal{U}_{(\omega)} = \frac{\mathrm{i}}{2\sqrt{3}} \begin{pmatrix} 0 & -\omega^2 & \omega^2 \\ \omega & 0 & -1 \\ -\omega & 1 & 0 \end{pmatrix}$$

with

$$\mathsf{t}_1 = \frac{1}{2\sqrt{3}} \operatorname{diag}(2,-1,-1) \qquad \mathsf{t}_2 = \frac{1}{2} \operatorname{diag}(0,1,-1)$$

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$$\begin{pmatrix} \Phi^\dagger_{-2/3}\,\mathsf{t}_+^{(\omega)}\,\Phi_{-2/3} \\ \Phi^\dagger_{-2/3}\,\mathsf{t}_-^{(\omega)}\,\Phi_{-2/3} \end{pmatrix} \cdot \begin{pmatrix} W^-_{(\omega)} \\ W^+_{(\omega)} \end{pmatrix}$$

$$= \frac{1}{\sqrt{6}} \begin{pmatrix} \omega^2\,\overline{X}\,Y + \overline{Y}\,Z + \omega\,\overline{Z}\,X \\ \omega\,\overline{Y}\,X + \overline{Z}\,Y + \omega^2\,\overline{X}\,Z \end{pmatrix} \cdot \begin{pmatrix} W^-_{(\omega)} \\ W^+_{(\omega)} \end{pmatrix}$$

$$\mathsf{t}_\pm^{(\omega)} := \frac{1}{\sqrt{2}}\left(\mathsf{t}_1^{(\omega)} \pm \mathrm{i}\,\mathsf{t}_2^{(\omega)}\right)$$

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☞ residual $[\mathbb{Z}_3 \times \mathbb{Z}_3]^{(\omega)}$ symmetries for $\varrho \neq \omega$

$$\Phi_{-2/3} \xmapsto{\mathbb{Z}_3^{(\omega,1)}} \omega\, \Phi_{-2/3}$$

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \xmapsto{\mathbb{Z}_3^{(\omega,2)}} \begin{pmatrix} 0 & 0 & \omega^2 \\ \omega & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} =: Z_{(\omega,2)} \cdot \Phi_{-2/3}$$

Gauge symmmetry enhancement at $\varrho = \omega + 1$

☞ again two gauge fields

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- different couplings to $[\mathrm{U}(1) \times \mathrm{U}(1)]^{(\omega+1)}$ gauge fields

$$\begin{pmatrix} \omega\, \overline{X}\, Z + \omega^2\, \overline{Y}\, X + \overline{Z}\, Y \\ \omega^2\, \overline{Z}\, X + \omega\, \overline{X}\, Y + \overline{Y}\, Z \end{pmatrix} \cdot \begin{pmatrix} W^+_{(\omega+1)} \\ W^-_{(\omega+1)} \end{pmatrix}$$

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$$\mathsf{S}_\varrho^2 \curvearrowright \begin{pmatrix} \omega\,\overline{X}\,Z + \omega^2\,\overline{Y}\,X + \overline{Z}\,Y \\ \omega^2\,\overline{Z}\,X + \omega\,\overline{X}\,Y + \overline{Y}\,Z \end{pmatrix} \cdot \begin{pmatrix} W^+_{(\omega+1)} \\ W^-_{(\omega+1)} \end{pmatrix} \curvearrowleft \mathsf{S}_\varrho^2$$

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- S_ϱ^2 swaps components
- $[\mathrm{U}(1)\times\mathrm{U}(1)]^{(\omega)}$ and $[\mathrm{U}(1)\times\mathrm{U}(1)]^{(\omega+1)}$ generators do not commute

Residual discrete gauge symmetries

☞ away from $\varrho = \omega + 1$: $[\mathrm{U}(1) \times \mathrm{U}(1)]^{(\omega+1)}$ breaks to $[\mathbb{Z}_3 \times \mathbb{Z}_3]^{(\omega+1)}$

$$Z_{(\omega+1,1)} := \omega\,\mathbb{1}_3 \qquad Z_{(\omega+1,2)} := \begin{pmatrix} 0 & 0 & \omega \\ \omega^2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Residual discrete gauge symmetries & $\Delta(27)$

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- residual $\mathbb{Z}_3$ symmetries from $[\mathrm{U}(1) \times \mathrm{U}(1)]^{(\omega)}$ and $[\mathrm{U}(1) \times \mathrm{U}(1)]^{(\omega+1)}$ give rise to $\Delta(27) = \big(\mathbb{Z}_3 \times \mathbb{Z}_3\big) \rtimes \mathbb{Z}_3$

$$\left.\begin{aligned} Z^2_{(\omega,2)} \cdot Z_{(\omega+1,2)} \cdot Z^2_{(\omega,2)} &= \rho(\mathsf{A}) \\ Z_{(\omega+1,2)} \cdot Z_{(\omega,2)} \cdot Z_{(\omega+1,2)} &= \rho(\mathsf{B}) \end{aligned}\right\} \; \mathsf{A}^3 = \mathsf{B}^3 = (\mathsf{A}\,\mathsf{B})^3 = (\mathsf{A}\,\mathsf{B}^2)^3 = \mathbb{1}$$

Residual discrete gauge symmetries & $\Delta(54)$

☞ away from $\varrho=\omega+1$: $[\mathrm{U}(1)\times\mathrm{U}(1)]^{(\omega+1)}$ breaks to $[\mathbb{Z}_3\times\mathbb{Z}_3]^{(\omega+1)}$

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☞ $\mathsf{C}=\mathsf{S}_\varrho^2$ enhances $\Delta(27)$ to $\Delta(54)$

$$\begin{aligned}\Delta(54)=\langle \mathsf{A},\mathsf{B},\mathsf{C}|\mathsf{A}^3=\mathsf{B}^3=\mathsf{C}^2 \\ =(\mathsf{A}\,\mathsf{B})^3=(\mathsf{A}\,\mathsf{B}^2)^3=(\mathsf{A}\,\mathsf{C})^2=(\mathsf{B}\,\mathsf{C})^2=\mathbb{1}\rangle\end{aligned}$$

back

Group extensions

back

- group K is called an extension of G by H $:\Longleftrightarrow$ $\exists$ short exact sequence
$1 \longrightarrow G \xrightarrow{\iota} K \xrightarrow{\pi} H \longrightarrow 1$

$\operatorname{im}\iota = \ker\pi \cong G$

Group extensions

back

- group K is called an extension of G by H $:\Longleftrightarrow$ $\exists$ short exact sequence $1 \longrightarrow G \xrightarrow{\iota} K \xrightarrow{\pi} H \longrightarrow 1$
- extension is called split $:\Longleftrightarrow$ $\exists$ homomorphism $s\colon H \xrightarrow{s} K$ such that $\pi \circ s = \mathrm{id}_H$
- split extension: K contains a subgroup $Q \cong H$ such that $K = G\,Q$ and $G \cap Q = \{\mathbb{1}\}$
  $\Longleftrightarrow K = G \rtimes H$
- non-split extensions: no subgroup $Q \subseteq K$ isomorphic to H that complements G
  notation: $K = G\,.\,H$

References

References I

Yahya Almumin, Mu-Chun Chen, Murong Cheng, Victor Knapp-Perez, Yulun Li, Adreja Mondol, Saul Ramos-Sánchez, Michael Ratz, and Shreya Shukla. Neutrino Flavor Model Building and the Origins of Flavor and Violation. *Universe*, 9(12):512, 2023. doi: 10.3390/universe9120512.

Alexander Baur, Hans Peter Nilles, Andreas Trautner, and Patrick K. S. Vaudrevange. A String Theory of Flavor and $\mathcal{CP}$. *Nucl. Phys. B*, 947:114737, 2019a. doi: 10.1016/j.nuclphysb.2019.114737.

Alexander Baur, Hans Peter Nilles, Andreas Trautner, and Patrick K. S. Vaudrevange. Unification of Flavor, CP, and Modular Symmetries. *Phys. Lett. B*, 795:7–14, 2019b. doi: 10.1016/j.physletb.2019.03.066.

Alexander Baur, Hans Peter Nilles, Saul Ramos-Sánchez, Andreas Trautner, and Patrick K. S. Vaudrevange. The eclectic flavor symmetries of $\mathbb{T}^2/\mathbb{Z}_K$ orbifolds. *JHEP*, 09:159, 2024. doi: 10.1007/JHEP09(2024)159.

Florian Beye, Tatsuo Kobayashi, and Shogo Kuwakino. Gauge Origin of Discrete Flavor Symmetries in Heterotic Orbifolds. *Phys. Lett.*, B736:433–437, 2014. doi: 10.1016/j.physletb.2014.07.058.

References II

Mu-Chun Chen and K. T. Mahanthappa. Group Theoretical Origin of CP Violation. *Phys. Lett. B*, 681:444–447, 2009. doi: 10.1016/j.physletb.2009.10.059.

Mu-Chun Chen, Maximilian Fallbacher, K. T. Mahanthappa, Michael Ratz, and Andreas Trautner. CP Violation from Finite Groups. *Nucl. Phys. B*, 883: 267–305, 2014. doi: 10.1016/j.nuclphysb.2014.03.023.

Mu-Chun Chen, Víctor Knapp–Pérez, Mario Ramos-Hamud, Saúl Ramos-Sánchez, Michael Ratz, and Shreya Shukla. Quasi–eclectic modular flavor symmetries. *Phys. Lett. B*, 824:136843, 2022. doi: 10.1016/j.physletb.2021.136843.

Mu-Chun Chen, Stephen F. King, Omar Medina, and José W. F. Valle. Quark-lepton mass relations from modular flavor symmetry. *JHEP*, 02:160, 2024. doi: 10.1007/JHEP02(2024)160.

Mu-Chun Chen, Xueqi Li, Xiang-Gan Liu, and Michael Ratz. Modular Flavor Symmetries and Fermion Mass Hierarchies. 6 2025.

References III

E. J. Chun, J. Mas, J. Lauer, and Hans Peter Nilles. Duality and Landau–Ginzburg Models. *Phys. Lett. B*, 233:141–146, 1989. doi: 10.1016/0370-2693(89)90630-8.

Thomas Dent. CP violation and modular symmetries. *Phys. Rev. D*, 64:056005, 2001. doi: 10.1103/PhysRevD.64.056005.

Eric D'Hoker and Justin Kaidi. Lectures on modular forms and strings. 8 2022.

Gui-Jun Ding and Stephen F. King. Neutrino Mass and Mixing with Modular Symmetry. 11 2023.

Gui-Jun Ding and Jose W. F. Valle. The symmetry approach to quark and lepton masses and mixing. 2 2024.

Lance J. Dixon, Daniel Friedan, Emil J. Martinec, and Stephen H. Shenker. The Conformal Field Theory of Orbifolds. *Nucl. Phys.*, B282:13–73, 1987.

S. Ferrara, D. Lüst, Alfred D. Shapere, and S. Theisen. Modular Invariance in Supersymmetric Field Theories. *Phys. Lett. B*, 225:363, 1989. doi: 10.1016/0370-2693(89)90583-2.

Ferruccio Feruglio. Are neutrino masses modular forms? 2017.

References IV

Ferruccio Feruglio. Universal Predictions of Modular Invariant Flavor Models near the Self-Dual Point. *Phys. Rev. Lett.*, 130(10):101801, 2023a. doi: 10.1103/PhysRevLett.130.101801.

Ferruccio Feruglio. Fermion masses, critical behavior and universality. *JHEP*, 03:236, 2023b. doi: 10.1007/JHEP03(2023)236.

Ferruccio Feruglio and Saul Ramos-Sánchez. Quark and lepton masses. 6 2025.

Ferruccio Feruglio and Andrea Romanino. Lepton flavor symmetries. *Rev. Mod. Phys.*, 93(1):015007, 2021. doi: 10.1103/RevModPhys.93.015007.

C. D. Froggatt and Holger Bech Nielsen. Hierarchy of quark masses, cabibbo angles and cp violation. *Nucl. Phys.*, B147:277, 1979.

Amit Giveon and Massimo Porrati. Duality invariant string algebra and D $= 4$ effective actions. *Nucl. Phys. B*, 355:422–454, 1991. doi: 10.1016/0550-3213(91)90121-D.

Amit Giveon, Massimo Porrati, and Eliezer Rabinovici. Target space duality in string theory. *Phys.Rept.*, 244:77–202, 1994. doi: 10.1016/0370-1573(94)90070-1.

References V

Shahram Hamidi and Cumrun Vafa. Interactions on Orbifolds. *Nucl. Phys.*, B279:465, 1987.

Luis E. Ibáñez, W. Lerche, D. Lüst, and S. Theisen. Some Considerations About the Stringy Higgs Effect. *Nucl. Phys. B*, 352:435–450, 1991. doi: 10.1016/0550-3213(91)90450-C.

David B. Kaplan and Martin Schmaltz. Flavor unification and discrete nonAbelian symmetries. *Phys. Rev. D*, 49:3741–3750, 1994. doi: 10.1103/PhysRevD.49.3741.

Tatsuo Kobayashi and Morimitsu Tanimoto. Modular flavor symmetric models. 7 2023.

Tatsuo Kobayashi, Hans Peter Nilles, Felix Plöger, Stuart Raby, and Michael Ratz. Stringy origin of non-Abelian discrete flavor symmetries. *Nucl. Phys.*, B768:135–156, 2007.

J. Lauer, J. Mas, and Hans Peter Nilles. Twisted sector representations of discrete background symmetries for two-dimensional orbifolds. *Nucl. Phys.*, B351:353–424, 1991. doi: 10.1016/0550-3213(91)90095-F.

References VI

Xueqi Li, Xiang-Gan Liu, Hans Peter Nilles, Michael Ratz, and Alex Stewart. Flavor Symmetries and Winding Modes. 6 2025.

Xiang-Gan Liu and Gui-Jun Ding. Modular flavor symmetry and vector-valued modular forms. *JHEP*, 03:123, 2022. doi: 10.1007/JHEP03(2022)123.

Hans Peter Nilles, Michael Ratz, Andreas Trautner, and Patrick K. S. Vaudrevange. $\mathcal{CP}$ violation from string theory. *Phys. Lett. B*, 786:283–287, 2018. doi: 10.1016/j.physletb.2018.09.053.

Hans Peter Nilles, Saul Ramos-Sánchez, and Patrick K. S. Vaudrevange. Lessons from eclectic flavor symmetries. *Nucl. Phys. B*, 957:115098, 2020a. doi: 10.1016/j.nuclphysb.2020.115098.

Hans Peter Nilles, Saúl Ramos-Sánchez, and Patrick K.S. Vaudrevange. Eclectic Flavor Groups. *JHEP*, 02:045, 2020b. doi: 10.1007/JHEP02(2020)045.

Hans Peter Nilles, Saúl Ramos-Sánchez, and Patrick K. S. Vaudrevange. Eclectic flavor scheme from ten-dimensional string theory - II detailed technical analysis. *Nucl. Phys. B*, 966:115367, 2021. doi: 10.1016/j.nuclphysb.2021.115367.

References VII

P. P. Novichkov, J. T. Penedo, S. T. Petcov, and A. V. Titov. Generalised CP Symmetry in Modular-Invariant Models of Flavour. *JHEP*, 07:165, 2019. doi: 10.1007/JHEP07(2019)165.

S. T. Petcov and M. Tanimoto. A_4 modular flavour model of quark mass hierarchies close to the fixed point $\tau = i\infty$. *JHEP*, 08:086, 2023. doi: 10.1007/JHEP08(2023)086.

Fernando Quevedo. Lectures on superstring phenomenology. 1996. doi: 10.1063/1.49735. [AIP Conf. Proc.359,202(1996)].

Lisa Randall and Raman Sundrum. A Large mass hierarchy from a small extra dimension. *Phys. Rev. Lett.*, 83:3370–3373, 1999. doi: 10.1103/PhysRevLett.83.3370.