Modular-Symmetry Protected Seesaw Mechanism

S. T. Petcov

INFN/SISSA, Trieste, Italy, and
Kavli IPMU, University of Tokyo, Japan

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The Flavour Problem

Understanding the origins of flavour in both quark and lepton sectors, i.e., of the patterns of quark masses and mixing, and of the charged lepton and neutrino masses and of neutrino mixing and of CP violation in the quark and lepton sectors, is one of the most challenging still unresolved fundamental problem in contemporary particle physics.

"Asked what single mystery, if he could choose, he would like to see solved in his lifetime, Weinberg doesnt have to think for long: he wants to be able to explain the observed pattern of quark and lepton masses."

From Model Physicist, CERN Courier, 13 October 2017.

The renewed attempts to seek new better solutions of the flavour problem than those already proposed were stimulated primarily by the remarkable progress made in the studies of neutrino oscillations, which began 25 years ago with the discovery of oscillations of atmospheric ν_μ and $\bar{\nu}_\mu$ by SuperKamiokande experiment. This lead, in particular, to the determination of the pattern of the 3-neutrino mixing, which turn out to consist of two large and one small mixing angles.

In what follows we will discuss a new approach to the flavour problem within the three family framework.

The Lepton Flavour Problem

Consists of three basic elements (sub-problems), namely, understanding:

- Why $m_{\nu_j} <<< m_{e,\mu,\tau}, m_q$, $q = u, c, t, d, s, b$ ($m_{\nu_j} \lesssim 0.5$ eV, $m_l \geq 0.511$ MeV, $m_q \gtrsim 2$ MeV);

- The origins of the patterns of
i) neutrino mixing of 2 large and 1 small angles ($\theta^l_{12} = 33.65^\circ$, $\theta^l_{23} = 47.1^\circ$, $\theta^l_{13} = 8.49^\circ$), and of ii) Δm^2_{ij}, i.e., of $\Delta m^2_{21} \ll |\Delta m^2_{31}|$, $\Delta m^2_{21}/|\Delta m^2_{31}| \cong 1/30$.

- The origin of the hierarchical pattern of charged lepton masses:
$m_e \ll m_\mu \ll m_\tau$, $m_e/m_\mu \cong 1/200$, $m_\mu/m_\tau \cong 1/17$.

The first two added new important aspects to the flavour problem.

$m_{\nu_j} <<< m_{e,\mu,\tau}, m_q$, $q = u, c, t, d, s, b$:

seesaw mechanism(s), Weinberg operator, radiative ν mass generation, extra dimensions. However, additional input (symmetries) needed to explain the pattern of lepton mixing and to get specific testable predictions.

The most natural explanation of

- $m_{\nu_j} <<< m_{e,\mu,\tau}, m_q$, $q = u,c,t,d,s,b$:

is arguably provided by the seesaw mechanism of neutrino mass generation in its simplest version - the type I seesaw.
However, additional input, typically in the form of symmetries needed to explain the hierarchical pattern of charged lepton masses and the peculiar pattern of neutrino mixing and to get specific testable predictions.

In the present talk: the additional symmetry - modular invariance.

The talk is based on the following study:
"Modular-Symmetry Protected Seesaw Mechanism", A. Granelli, D. Meloni, M. Parriciatu, J.T. Penedo, S.T.P., arXiv:2505.21405.

The type I seesaw mechanism: SM + 2 or 3 RH νs, $\nu_{\kappa R}$, $\kappa = 1, \ldots, n_R$.
The Lagrangian we will use has the form:

$$\mathcal{L} \supset -(Y_e^*)_{\alpha\beta}\,\overline{\ell_{\alpha L}}\,\tilde{H}_d\,e_{\beta R} - (Y_D^*)_{\alpha\kappa}\,\overline{\ell_{\alpha L}}\,\tilde{H}_u\,\nu_{\kappa R} - \frac{1}{2}(M_R^*)_{\kappa\rho}\,\overline{\nu_{\kappa L}^c}\nu_{\rho R}$$

where we have anticipated the supersymmetric origin of these terms,
$\ell_{\alpha L} \equiv (\nu_{\alpha L}^T, e_{\alpha L}^T)^T$, $\tilde{H}_{u,d} \equiv i\sigma_2 H_{u,d}^*$; **the Higgs doublets take VEVs** $< H_u >= (0, v_u)^T$ **and** $< H_d >= (v_d, 0)^T$, $v_u = v\sin\beta = 0.98v$, $v_d = v\cos\beta = 0.20v$ **(**$\tan\beta = 5$**)**, $v = 174$ **GeV;** $\nu_{\kappa L}^c = C(\overline{\nu_{\kappa R}})^T$, $M_R^T = M_R$. **After the EWSB, the following well known neutrino mass terms are generated:**

$$-\mathcal{L}_\nu = \frac{1}{2}\overline{\mathcal{V}_R^c}\mathcal{M}\mathcal{V}_L + \textbf{h.c.} \equiv \frac{1}{2}\overline{\mathcal{V}_R^c}\begin{pmatrix} 0 & m_D \\ m_D^T & M_R \end{pmatrix}\mathcal{V}_L + \text{h.c.}\,,$$

$\mathcal{V}_L \equiv (\nu_{\alpha L}^T, (\nu_{\kappa L}^c)^T)^T$; m_D **and** M_R **denote the complex Dirac-type and Majorana-type neutrino mass matrices, respectively, with** $(m_D)_{\alpha\kappa} = (Y_D)_{\alpha\kappa}\,v_u$.

The diagonalisation of $\mathcal{L}_\nu$: 3 light massive neutrinos ν_i, $m_i \lesssim 0.5$ eV, $i = 1, 2, 3$ + 2 or 3 Heavy Majorana Neutrinos (Neutral Heavy Leptons (NHL)) N_j, M_j, $j = 1, 2$ (3).

The effective Majorana mass term for the active flavour neutrinos is given by:

$$m_\nu \cong -\, m_D\, M_R^{-1}\, m_D^T = -\, v_u^2\, Y_D\, M_R^{-1}\, Y_D^T\,.$$

The couplings of the NHL N_is in weak CC and NC interaction Lagrangian:

$$R \sim m_D\, M_R^{-1} = v_u\, Y_D\, M_R^{-1}\,, \qquad M_j \sim M_R\,.$$

$|m_\nu| \sim 0.1$ eV, $|Y_D| \sim 1$: $|M_R| \sim 10^{14}$ GeV: high scale seesaw inspired by GUTs; very difficult to test.

$|m_\nu| \sim 0.1$ eV, $|M_R| \sim 10$ GeV:$|Y_D| \sim 10^{-7}$: low scale seesaw; with $|Y_D| \sim 10^{-7}$ very difficult to test.

Low-scale seesaw with enhanced $|Y_D|$: seesaw mechanism "protected" by approximate conservation of a non-standard lepton charge L' (Wyler, Wolfenstein 1982 (see also Leung, STP 1983); Mohapatra, Valle,1986; Malinsky et al., 2005; Shaposhnikov, 2006; Kersten, Smirnov, 2007; Gavela et al., 2009; Ibarra et al., 2010; Antusch et al., 2015): inverse, linear,..., seesaw scenarios.
Relies on a "small" breaking of L'.

Assume $L' = L_e + L_\mu + L_\tau + L_1 - L_2$ is conserved, $L_{1(2)}(\nu_{1(2)R}) = 1$, 0 for other fields; $L'(\nu_{1R}) = 1$, $L'(\nu_{2R}) = -1$, $L'(\nu_{3R}) = 0$-decoupled.

In this case $|n_+ - n_-| = 4 - 1 = 3$ massless states (light ν_i), $\min(n_+, n_-) = 1$ massive Dirac νs (ν_{1R} and $\nu^c_{2L} \equiv C\,(\overline{\nu_{2R}})^T$ form a Dirac N).

C.N. Leung, S.T.P., PL B125 (1983) 461; S.M. Bilenky, S.T.P., RMP 59 (1987) 671

If $L' = L_e + L_\mu + L_\tau + L_1 - L_2$ is conserved ($U(1)_{L'}$ symmetry):
ν_{lL}, $l = e, \mu, \tau$, **can couple only to ν_{1R} (the couplings to ν_{2R} are forbidden), of the couplings between ν_{1R} and ν_{2R} only the term $M_{12}\overline{\nu_{1R}}\,\nu^c_{2L} \equiv M_{12}\overline{\nu_{1R}}\,C(\overline{\nu_{2R}})^T$ is allowed.**

The neutrino mass matrix $\mathcal{M}$ takes the form:

$$\mathcal{M} \rightarrow \left(\begin{array}{ccc|ccc} 0 & 0 & 0 & y_{11}\,v_u & 0 & 0 \\ 0 & 0 & 0 & y_{21}\,v_u & 0 & 0 \\ 0 & 0 & 0 & y_{31}\,v_u & 0 & 0 \\ \hline y_{11}\,v_u & y_{21}\,v_u & y_{31}\,v_u & 0 & M_{12} & 0 \\ 0 & 0 & 0 & M_{12} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & M_3 \end{array}\right).$$

M_R **- “Pauli-like” structure.**

The L' conservation can be broken in different ways. Suppose it is broken by a Majorana mass term of ν_{2R}: $M_{22}\,\overline{\nu_{2R}}\,\nu^c_{2L}$.

In this case ν_{1R} and ν_{2R} form a pseudo-Dirac pair with mass splitting $\Delta M = (M_2 - M_1) \propto M_{22}$.

L. Wolfenstein, NP B186 (1981) 147; S.T.P., PL B110 (1982) 245

$$m_\nu \cong -\,v_u^2\,Y_D\,M_R^{-1}\,Y_D^T \to -\,v_u^2\,Y_D\,M_R^{-1}\,M_{22}\,M_R^{-1}\,Y_D^T\,.$$

If $M_{22}\,M_R^{-1}$ is sufficiently small we can have $|Y_D| \gg 10^{-7}$ for $M_j \sim 10$ GeV, and thus much larger $R \sim v_u|Y_D|/M_R$ and possibly observablle effects of the NHL N_i in low-energy experiments, and testable version of the seesaw mechanism.

L' **conservation can be broken by modifications of m_D or/and M_R. Suppose the breaking is characterised by a small parameter $\epsilon \ll 1$ ($\epsilon \sim 0.01$):**

$$m_D = m_0 + \epsilon^d\, m_1\,,\, M_R = M_0 + \epsilon^r\, M_1\,,\, M^T_{0,1} = M_{0,1}\,,$$

d, r **- the smallest exponents for the powers of ϵ in the perturbation matrices m_1 and M_1 ($d, r = 1$ or 2).**

In the symmetric limit $m_0 M_0^{-1} m_0^T = 0$.

In the case of L'-nonconservation due to $\epsilon^d\, m_1 \neq 0$ and $\epsilon^r\, M_1 \neq 0$ of interest, the mass matrix in the light flavour neutrino Majorana mass term $\mathcal{L} \supset -\frac{1}{2}(m_\nu)_{\alpha\beta}\overline{\nu^c_{\alpha R}}\nu_{\beta L} +$ h.c. is given by:

$$\begin{aligned} m_\nu &\simeq -m_D\, M_R^{-1}\, m_D^T \\ &= \epsilon^r\, m_0 M_0^{-1} M_1 M_0^{-1} m_0^T - (\epsilon^d\, m_1 M_0^{-1} m_0^T + \textbf{transpose}) \\ &+ \mathcal{O}\big(|\epsilon|^{2r}, |\epsilon|^{2d}, |\epsilon|^{r+d}\big) \neq 0\,. \end{aligned}$$

The light flavour neutrino mass scale,

$$m_i^\nu \sim \max(|\epsilon|^r, |\epsilon|^d) \times \frac{y^2 v_u^2}{M}\,, M \sim M_R\,, y\text{−combination of } (Y_D)_{\alpha j}$$

Thus, for given m_i and $M \sim M_R$ ($M \sim M_j$), the neutrino Yukawa coupling y^2 is enhanced with respect to the one in the standard seesaw scenario by the factor $(\max(|\epsilon|^r, |\epsilon|^d))^{-1}$:

$$y^2 \propto (\max(|\epsilon|^r, |\epsilon|^d))^{-1} \quad (\sim (10^2 - 10^4))\,.$$

This opens up the possibility of testing the low-scale seesaw scenario by observing the associated heavy Majorana neutrinos in low-energy experiments.

The diagonalization of the full neutrino mass matrix $\mathcal{M}$ gives also 3 heavy Majorana neutrinos N_j, M_j. Their mass matrix M_N is given by M_R plus corrections:

$$
\begin{aligned}
M_N \simeq\; & M_R + \frac{1}{2}\Big((M_R^*)^{-1} m_D^\dagger m_D + \textbf{transpose}\Big) \\
& = M_0 + \epsilon^r\, M_1 + \frac{1}{2}\Big(\delta M_0 + \delta M_0^T\Big) \\
& \quad + \frac{1}{2}\Big(\epsilon^{d*}\,\delta M_a + \epsilon^d\,\delta M_b - \epsilon^{r*}\,\delta M_c + \textbf{transpose}\Big) \\
& + \mathcal{O}\Big(|\epsilon|^{2r}, |\epsilon|^{2d}, |\epsilon|^{r+d}\Big)\,,
\end{aligned}
$$

$$
\delta M_a \equiv (M_0^*)^{-1} m_1^\dagger m_0\,, \quad \delta M_b \equiv (M_0^*)^{-1} m_0^\dagger m_1\,,
$$

$$
\delta M_c \equiv (M_0^*)^{-1} M_1^* (M_0^*)^{-1} m_0^\dagger m_0\,. \tag{1}
$$

$\delta M_0 \equiv (M_0^*)^{-1} m_0^\dagger m_0$ **does not provide the mass splitting ΔM of ν_{1R} and ν_{2R}.**

If M_1 is non-zero, it is expected to provide the leading contribution to the splitting ΔM,

$$\Delta M \sim |\epsilon|^r \times M_1 \quad (M_1 \sim M \sim M_j)\,.$$

If, instead, $M_1 = 0$ but $\epsilon^d m_1 \neq 0$, to leading order

$$\Delta M \sim |\epsilon|^d \times \frac{y^2 v_u^2}{M} \sim m_i^\nu\,.$$

Thus, the splitting is proportional to the size of the light neutrino mass scale m_i^ν.

The aim of our study: by using the modular symmetry to mimic the conservation and breaking of L' in the low-scale type I seesaw scenario, construct minimal phenomenologically viable models (without flavons), which in addition of explaining the smallness of neutrinos masses provide:

i) non-fine-tuned description of the hierarchies of the charged lepton masses (following P.P. Novichkov et al., arxiv:2102.07488),

ii) prediction for $\Delta M_{jk} = (M_j - M_k)$ of N_j,

iii) enhanced neutrino Yukawa couplings and thus enhanced N_j CC and NC couplings and thus testable low-scale seesaw scenarios,

iv) investigate the phenomenological predictions of the so constructed low-scale seesaw models.

In our study the role of L' - the residual symmetry of the homogeneous modular group $\Gamma \simeq SL(2,\mathbb{Z})$ and a chosen finite modular group Γ'_N, $N = 2,3,4,5$, at one of their two fixed (symmetric) points in the fundamental domain D of $\Gamma \simeq SL(2,\mathbb{Z})$:

- $\tau_{\text{sym}} = i\infty$, invariant under T, preserving $\mathbb{Z}_N^T$;
- $\tau_{\text{sym}} = \omega \equiv \exp(2\pi i/3)$, invariant under ST, preserving $\mathbb{Z}_3^{ST}$.

P.P. Novichkov et al., arXiv:1811.04933 and arXiv:2006.03058

τ_{sym} - the VEV of the modulus τ, τ_{vev}, at the two fixed points; τ - a complex scalar field acquiring a VEV; τ_{vev} - the only source of breaking of $SL(2,\mathbb{Z})$ and Γ'_N; $\mathbb{Z}_N^T$ or $\mathbb{Z}_3^{ST}$ - residual symmetries.

The breaking of L' corresponds to deviation of the VEV of τ from τ_{sym}, $\tau_{\text{vev}} \neq \tau_{\text{sym}}$,
and thus to the breaking of $\mathbb{Z}_N^T$ or $\mathbb{Z}_3^{ST}$;
the L' breaking small parameter introduced earlier $|\epsilon|$ is related to a sufficiently small deviation of τ_{vev} from τ_{sym}.

In the modular invariance approach to the flavour problem the same small parameter $|\epsilon|$ originating from $\tau_{\rm vev}$ having a value in the "vicinity" of $\tau_{\rm sym}$ can be used for a no-fine-tuning description of the charged lepton and quark mass hierarchies

P.P. Novichkov et al., arxiv:2102.07488

The Modular Invariance Approach to Flavour

The Modular-symmetry protected seesaw - many possibilities. One has to be selective.

0. **The group Γ'_N must provide a natural non-fine-tuned realisation of m_e/m_τ, m_μ/m_τ hierarchies in the vicinity of τ_{sym}. The hierarchies reliable only if dictated by the relative sizes of the different components within the same $Y_r^{(k_Y)}$ multiplet. This excludes Γ'_N that do not have triplet irreps.**

1. **Γ'_N must allow for a non-fine-tuned realisation of the Pauli-like structure of the ν_R Majorana mass matrix M_0 at the chosen τ_{sym} (i.e., in the exact L' symmetry limit). Should be achieved without fine-tuning of constant parameters, i.e. should follow from the irreps furnished by N_i^c (ν_R related) superfields and their weights. Implies N_i^c cannot be singlets.**

2. **Γ'_N must allow the realisation of the structure of the neutrino Dirac mass matrix m_0 at the chosen τ_{sym} (i.e., in the exact L' symmetry limit). This fixes the possible irreps of the LH lepton doublets L_i.**

3. **After ensuring that 0,1,2 are fulfilled, one has to verify that the chosen irreps and modular weights allow the non-fine-tuned construction of the m_e/m_τ, m_μ/m_τ hierarchies via the residual symmetry approach of arXiv:2102.07488. One can have, e.g., in the vicinity of the chosen τ_{sym} for M_e (or M_e^T):**

$$M_e \sim \begin{pmatrix} 1 & 1 & 1 \\ \epsilon & \epsilon & \epsilon \\ \epsilon^2 & \epsilon^2 & \epsilon^2 \end{pmatrix}, \quad m_\tau(1:\epsilon:\epsilon^2),$$

with $\epsilon \sim 10^{-2}$ in order to reproduce the observed hierarchies; each column in M_e is a triplet of modular forms (or a combination thereof).

4. **Since $n(Y_r^{(k_Y)})$ grows linearly with k_Y, in order to retain only a limited number of free parameters in the superpotential and to increase the predictive power of the models we choose to limit the weights k_Y by the smallest possible value which still yields viable models: $k_Y \leq 5$ or 6.**

We have explored the first three Γ'_N groups that furnish triplet representations, i.e. $\Gamma'_3 \simeq T'$, $\Gamma'_4 \simeq S'_4$ and $\Gamma'_5 \simeq A'_5$.
In the next two tables we list all potential models passing our criteria, as well as the expected magnitude of the HNL splitting ΔM within each model.
We exclude from our list those models where $m_e = 0$ and/or $M_1 = 0$.
The surviving models are either
i) based on $\Gamma'_3 \simeq T'$, with $\tau \simeq \{\omega, i\infty\}$ and leading to $\Delta M \sim \epsilon\, M$ or $\Delta M \sim m_\nu$ (shown in the 1st table),
or ii) on $\Gamma'_4 \simeq S'_4$, with $\tau \simeq i\infty$ and $\Delta M \sim \epsilon^2\, M$ (shown in the 2nd table).
No A'_5-based models are permitted, irrespective of the upper bound on k_Y.
Note that, even if a model is included in these tables, a fit to lepton data is not guaranteed.

T'-based symmetry-protected seesaw models passing our criteria at $\tau \simeq \tau_{\text{sym}}$, $k_Y \leq 6$.

τ_{sym}	L	E^c	N^c	ΔM
ω	$(\mathbf{1}'', 0) \oplus (\mathbf{1}, +2) \oplus (\mathbf{1}', +4)$	$(\mathbf{3}, +2)$	$(\hat{\mathbf{2}}, +1)$	$\sim \epsilon\, M$
	$(\mathbf{1}, -1) \oplus (\mathbf{1}', +1) \oplus (\mathbf{1}'', +3)$	$(\mathbf{3}, +3)$	$(\hat{\mathbf{2}}'', +2)$	
	$(\mathbf{1}', -2) \oplus (\mathbf{1}'', 0) \oplus (\mathbf{1}, +2)$	$(\mathbf{3}, +4)$	$(\hat{\mathbf{2}}', +3)$	
	$(\mathbf{1}'', +2) \oplus (\mathbf{1}, +4) \oplus (\mathbf{1}', +6)$ $(\mathbf{1}, +2) \oplus (\mathbf{1}', +4) \oplus (\mathbf{1}'', +6)$	$(\mathbf{3}, 0)$	$(\mathbf{3}, 0)$	$\sim m_\nu$
	$(\mathbf{1}, +1) \oplus (\mathbf{1}', +3) \oplus (\mathbf{1}'', +5)$ $(\mathbf{1}', +1) \oplus (\mathbf{1}'', +3) \oplus (\mathbf{1}, +5)$	$(\mathbf{3}, +1)$	$(\mathbf{3}, +1)$	$\sim \epsilon\, M$
	$(\mathbf{1}', 0) \oplus (\mathbf{1}'', +2) \oplus (\mathbf{1}, +4)$ $(\mathbf{1}'', 0) \oplus (\mathbf{1}, +2) \oplus (\mathbf{1}', +4)$	$(\mathbf{3}, +2)$	$(\mathbf{3}, +2)$	
	$(\mathbf{1}'', -1) \oplus (\mathbf{1}, +1) \oplus (\mathbf{1}', +3)$ $(\mathbf{1}, -1) \oplus (\mathbf{1}', +1) \oplus (\mathbf{1}'', +3)$	$(\mathbf{3}, +3)$	$(\mathbf{3}, +3)$	
$i\infty$	$(\mathbf{1}', 0) \oplus (\mathbf{1}', +2) \oplus (\mathbf{1}', +4)$	$(\mathbf{3}, +2)$	$(\hat{\mathbf{2}}', +1)$	$\sim \epsilon\, M$
	$(\mathbf{1}', -1) \oplus (\mathbf{1}', +1) \oplus (\mathbf{1}', +3)$	$(\mathbf{3}, +3)$	$(\hat{\mathbf{2}}', +2)$	
	$(\mathbf{1}', -2) \oplus (\mathbf{1}', 0) \oplus (\mathbf{1}', +2)$	$(\mathbf{3}, +4)$	$(\hat{\mathbf{2}}', +3)$	
	$(\mathbf{1}', +2) \oplus (\mathbf{1}', +4) \oplus (\mathbf{1}', +6)$ $(\mathbf{1}'', +2) \oplus (\mathbf{1}'', +4) \oplus (\mathbf{1}'', +6)$	$(\mathbf{3}, 0)$	$(\mathbf{3}, 0)$	$\sim m_\nu$
	$(\mathbf{1}', +1) \oplus (\mathbf{1}', +3) \oplus (\mathbf{1}', +5)$ $(\mathbf{1}'', +1) \oplus (\mathbf{1}'', +3) \oplus (\mathbf{1}'', +5)$	$(\mathbf{3}, +1)$	$(\mathbf{3}, +1)$	$\sim \epsilon\, M$
	$(\mathbf{1}', 0) \oplus (\mathbf{1}', +2) \oplus (\mathbf{1}', +4)$ $(\mathbf{1}'', 0) \oplus (\mathbf{1}'', +2) \oplus (\mathbf{1}'', +4)$	$(\mathbf{3}, +2)$	$(\mathbf{3}, +2)$	
	$(\mathbf{1}', -1) \oplus (\mathbf{1}', +1) \oplus (\mathbf{1}', +3)$ $(\mathbf{1}'', -1) \oplus (\mathbf{1}'', +1) \oplus (\mathbf{1}'', +3)$	$(\mathbf{3}, +3)$	$(\mathbf{3}, +3)$	

S'_4, $\tau \simeq i\infty$, $k_Y \leq 5$: L	E^c	N^c	ΔM
$(\hat{\mathbf{3}},+1)$	$(\hat{\mathbf{1}}',+3) \oplus (\hat{\mathbf{2}},+3)$ $(\mathbf{1},+2) \oplus (\hat{\mathbf{1}}',+3) \oplus (\mathbf{1},+4)$		
$(\hat{\mathbf{3}}',+1)$	$(\hat{\mathbf{1}},+3) \oplus (\hat{\mathbf{2}},+3)$ $(\mathbf{1},0) \oplus (\mathbf{1},+2) \oplus (\mathbf{1},+4)$	$(\hat{\mathbf{2}},+1)$	
$(\hat{\mathbf{3}},+3)$	$(\hat{\mathbf{1}}',+1) \oplus (\hat{\mathbf{2}},+1)$ $(\mathbf{1},0) \oplus (\hat{\mathbf{1}}',+1) \oplus (\mathbf{1},+2)$		
$(\hat{\mathbf{3}}',+3)$	$(\hat{\mathbf{1}},+1) \oplus (\hat{\mathbf{2}},+1)$ $(\mathbf{1},-2) \oplus (\mathbf{1},0) \oplus (\mathbf{1},+2)$		$\sim \epsilon^2\, M$
$(\hat{\mathbf{3}},0)$	$(\hat{\mathbf{1}}',+4) \oplus (\hat{\mathbf{2}},+4)$ $(\mathbf{1},+3) \oplus (\hat{\mathbf{1}}',+4) \oplus (\mathbf{1},+5)$		
$(\hat{\mathbf{3}}',0)$	$(\hat{\mathbf{1}},+4) \oplus (\hat{\mathbf{2}},+4)$ $(\mathbf{1},+1) \oplus (\mathbf{1},+3) \oplus (\mathbf{1},+5)$	$(\hat{\mathbf{2}},+2)$	
$(\hat{\mathbf{3}},+2)$	$(\hat{\mathbf{1}}',+2) \oplus (\hat{\mathbf{2}},+2)$ $(\mathbf{1},+1) \oplus (\hat{\mathbf{1}}',+2) \oplus (\mathbf{1},+3)$		
$(\hat{\mathbf{3}}',+2)$	$(\hat{\mathbf{1}},+2) \oplus (\hat{\mathbf{2}},+2)$ $(\mathbf{1},-1) \oplus (\mathbf{1},+1) \oplus (\mathbf{1},+3)$		
$(\hat{\mathbf{1}},+3) \oplus (\mathbf{1},+4) \oplus (\hat{\mathbf{1}},+5)$ $(\hat{\mathbf{1}}',+3) \oplus (\mathbf{1},+4) \oplus (\hat{\mathbf{1}}',+5)$ $(\hat{\mathbf{1}},+3) \oplus (\mathbf{1}',+4) \oplus (\hat{\mathbf{1}},+5)$	$(\mathbf{3},0)$ $(\mathbf{3}',0)$	$(\mathbf{3},0)$ $(\mathbf{3}',0)$	$\sim m_\nu$
$(\hat{\mathbf{1}},+2) \oplus (\mathbf{1},+3) \oplus (\hat{\mathbf{1}},+4)$ $(\hat{\mathbf{1}}',+2) \oplus (\mathbf{1},+3) \oplus (\hat{\mathbf{1}}',+4)$ $(\hat{\mathbf{1}},+2) \oplus (\mathbf{1}',+3) \oplus (\hat{\mathbf{1}},+4)$ $(\hat{\mathbf{1}},+1) \oplus (\mathbf{1},+2) \oplus (\hat{\mathbf{1}},+3)$ $(\hat{\mathbf{1}}',+1) \oplus (\mathbf{1},+2) \oplus (\hat{\mathbf{1}}',+3)$ $(\hat{\mathbf{1}},+1) \oplus (\mathbf{1}',+2) \oplus (\hat{\mathbf{1}},+3)$	$(\mathbf{3},+1)$ $(\mathbf{3}',+1)$ $(\mathbf{3},+2)$ $(\mathbf{3}',+2)$	$(\mathbf{3},+1)$ $(\mathbf{3}',+1)$ $(\mathbf{3},+2)$ $(\mathbf{3}',+2)$	$\sim \epsilon^2\, M$

In what follows, we focus on four benchmark models that successfully accommodate charged-lepton masses and oscillation data, and explore their phenomenological predictions.

Model	Group	τ_{sym}	L	E^c	N^c	ΔM
A	A_4	ω	$(1,+2)\oplus(1',+4)\oplus(1'',+6)$	$(\mathbf{3},0)$	$(\mathbf{3},0)$	$\sim m_\nu$
B			$(1',+1)\oplus(1'',+3)\oplus(1,+5)$	$(\mathbf{3},+1)$	$(\mathbf{3},+1)$	$\sim \epsilon\, M$
C	S_4'	$i\infty$	$(\hat{1},+2)\oplus(1,+3)\oplus(\hat{1},+4)$	$(\mathbf{3},+1)$	$(\mathbf{3},+1)$	$\sim \epsilon^2\, M$
D			$(\hat{1}',+2)\oplus(1,+3)\oplus(\hat{1}',+4)$			

Summary of benchmark models. For each model, we specify the finite modular group, the value of the symmetric point, with $\tau \simeq \tau_{\text{sym}}$, the modular assignments of lepton superfields, with $\psi \sim (\mathrm{r}, k_\psi)$, and the expected magnitude of the HNL splitting ΔM.

The superpotential in the considered models has the form:

$$W = (Y_e)_{\alpha\beta} L_\alpha E^c_\beta H_d + (Y_D)_{\alpha\kappa} L_\alpha N^c_\kappa H_u + \frac{1}{2}(M_R)_{\kappa\rho} N^c_\kappa N^c_\rho \,.$$

The constant parameters are included in the Yukawa couplings and Majorana mass matrices Y_e, Y_D and M_R.

To increase the predictivity of the models the gCP symmetry is imposed on each of them, so that all constants in W are real (in a suitable basis, $S^T = S$, $T^T = T$).

P.P. Novichkov et al., arXiv:1905.11970

τ_{vev} - the only source of flavour (modular) and CP symmetries breaking.
Under the CP-transformation, $\tau_{\text{vev}} \to -\tau^*_{\text{vev}}$.
Thus, $|\tau_{\text{vev}}| = 1$ and $|Re(\tau_{\text{vev}})| = 0, 1/2$ conserve CP.

We assume the minimal form for the Khler potential:

$$K(\Phi,\bar{\Phi}) = -h\Lambda_\tau^2 \log(-i\tau + i\bar{\tau}) + \sum_I (-i\tau + i\bar{\tau})^{-k_I} |\varphi_I|^2 \,,$$

which gives rise to the kinetic terms. Here, $h > 0$ and Λ_τ has mass-dimension of one, φ_I - all the matter superfields.

More general $K(\tau, \overline{\tau}, \psi, \overline{\psi})$ and the possible consequences they can have for flavour model building are discussed in
Mu-Chun Chen et al., arXiv:1909.06910 and 2108.02240; Y. Almumin et al., arXiv:2102.11286.

After the modulus τ acquires a VEV, φ_I will need to be rescaled as $\varphi_I \to (2\,\mathrm{Im}\,\tau_{\mathbf{vev}})^{k_I/2}\varphi_I$ to yield canonical kinetic terms.
Hence, the original superpotential parameters - hereafter denoted with hats $\widehat{\beta}_i$, are rescaled by the relevant powers of $2\,\mathrm{Im}\,\tau_{\mathbf{vev}}$. Further we drop the hats,
$\widehat{\beta}_i \to \beta_i = \widehat{\beta}_i (2\,\mathrm{Im}\,\tau_{\mathbf{vev}})^{k_i}$, to indicate that this rescaling has already taken place (see the expressions for Y_e and Y_D).

Finally, we note that, in the presence of reducible representations among different families, each matrix column or row may potentially emerge from different modular form multiplets, for whose absolute normalization there is no top-down prescription. In order to safeguard the reliability of the obtained matrix structures, one should employ a definite prescription to control the relative sizes of these columns or rows. To this end, we choose to normalize the modular forms according to S.T.P., arXiv:2311.04185.

Model A: A_4, $\tau_{\text{vev}} \simeq \tau_{\text{sym}} = \omega$, $\mathbb{Z}_3^{ST}$

$\epsilon(\equiv u) = (\tau - \omega)/(\tau - \omega^2)$ **(P.P. Novichkov et al.,arXiv:2102.07488)**

$$Y_D = \begin{pmatrix} g_1 \left(Y_{\mathbf{3}}^{(2)}\right)_1 & g_1 \left(Y_{\mathbf{3}}^{(2)}\right)_3 & g_1 \left(Y_{\mathbf{3}}^{(2)}\right)_2 \\ g_2 \left(Y_{\mathbf{3}}^{(4)}\right)_3 & g_2 \left(Y_{\mathbf{3}}^{(4)}\right)_2 & g_2 \left(Y_{\mathbf{3}}^{(4)}\right)_1 \\ g_{3,1} \left(Y_{\mathbf{3},1}^{(6)}\right)_2 + g_{3,2} \left(Y_{\mathbf{3},2}^{(6)}\right)_2 & g_{3,1} \left(Y_{\mathbf{3},1}^{(6)}\right)_1 + g_{3,2} \left(Y_{\mathbf{3},2}^{(6)}\right)_1 & g_{3,1} \left(Y_{\mathbf{3},1}^{(6)}\right)_3 + g_{3,2} \left(Y_{\mathbf{3},2}^{(6)}\right)_3 \end{pmatrix},$$

$$Y_e = \begin{pmatrix} \alpha_1 \left(Y_{\mathbf{3}}^{(2)}\right)_1 & \alpha_1 \left(Y_{\mathbf{3}}^{(2)}\right)_3 & \alpha_1 \left(Y_{\mathbf{3}}^{(2)}\right)_2 \\ \alpha_2 \left(Y_{\mathbf{3}}^{(4)}\right)_3 & \alpha_2 \left(Y_{\mathbf{3}}^{(4)}\right)_2 & \alpha_2 \left(Y_{\mathbf{3}}^{(4)}\right)_1 \\ \alpha_{3,1} \left(Y_{\mathbf{3},1}^{(6)}\right)_2 + \alpha_{3,2} \left(Y_{\mathbf{3},2}^{(6)}\right)_2 & \alpha_{3,1} \left(Y_{\mathbf{3},1}^{(6)}\right)_1 + \alpha_{3,2} \left(Y_{\mathbf{3},2}^{(6)}\right)_1 & \alpha_{3,1} \left(Y_{\mathbf{3},1}^{(6)}\right)_3 + \alpha_{3,2} \left(Y_{\mathbf{3},2}^{(6)}\right)_3 \end{pmatrix},$$

g_i and α_i are constant superpotential parameters (including the $(2\,\mathrm{Im}\,\tau_{\text{vev}})^{k_i}$ factors).
In the ST-diagonal basis,

$$M_R = \Lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad Y'_D \sim \begin{pmatrix} \epsilon^2 & 1 & \epsilon \\ \epsilon^2 & 1 & \epsilon \\ \epsilon^2 & 1 & \epsilon \end{pmatrix}, \quad Y'_e \sim \begin{pmatrix} 1 & 1 & 1 \\ \epsilon & \epsilon & \epsilon \\ \epsilon^2 & \epsilon^2 & \epsilon^2 \end{pmatrix}, \quad m_\tau(1 : \epsilon : \epsilon^2),$$

Λ has the dimension of a mass and corresponds to the HNL mass scale.
The relevant flavour-fit parameters are:

$$\Lambda > 0, \quad \mathrm{Re}\,\tau \in [-1/2, 1/2], \quad \mathrm{Im}\,\tau > \sqrt{3}/2,$$
$$g_1 \geq 0, \quad g_2 \geq 0, \quad g_{3,1} \geq 0, \quad g_{3,2} \in \mathbb{R},$$
$$\alpha_1 \geq 0, \quad \alpha_2 \in \mathbb{R}, \quad \alpha_{3,1} \in \mathbb{R}, \quad \alpha_{3,2} \in \mathbb{R}.$$

There are, in total, 11 real parameters (including τ) for 15 observables (the 12 lepton observables + 3 NHL masses M_j).

Model C: S_4', $\tau_{\rm vev} \simeq \tau_{\rm sym} = i\infty$, $\mathbb{Z}_4^{ST}$; T-diagonal basis.

$\epsilon = q^{1/4} \equiv q_4 = \exp(i\pi\tau_{\rm vev}/2)$ **(P.P. Novichkov et al.,arXiv:2102.07488)**

$$M_R = \Lambda \begin{pmatrix} \frac{2}{\sqrt{3}}\left(Y_{\mathbf{2}}^{(2)}\right)_1 & 0 & 0 \\ 0 & \left(Y_{\mathbf{2}}^{(2)}\right)_2 & -\frac{1}{\sqrt{3}}\left(Y_{\mathbf{2}}^{(2)}\right)_1 \\ 0 & -\frac{1}{\sqrt{3}}\left(Y_{\mathbf{2}}^{(2)}\right)_1 & \left(Y_{\mathbf{2}}^{(2)}\right)_2 \end{pmatrix} \sim \Lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & \epsilon^2 & 1 \\ 0 & 1 & \epsilon^2 \end{pmatrix} \text{ at } \tau_{\rm vev} \simeq \tau_{\rm sym}\,.$$

$$Y_D = \begin{pmatrix} g_1\left(Y_{\hat{\mathbf{3}}'}^{(3)}\right)_1 & g_1\left(Y_{\hat{\mathbf{3}}'}^{(3)}\right)_3 & g_1\left(Y_{\hat{\mathbf{3}}'}^{(3)}\right)_2 \\ g_2\left(Y_{\mathbf{3}}^{(4)}\right)_1 & g_2\left(Y_{\mathbf{3}}^{(4)}\right)_3 & g_2\left(Y_{\mathbf{3}}^{(4)}\right)_2 \\ g_3\left(Y_{\hat{\mathbf{3}}'}^{(5)}\right)_1 & g_3\left(Y_{\hat{\mathbf{3}}'}^{(5)}\right)_3 & g_3\left(Y_{\hat{\mathbf{3}}'}^{(5)}\right)_2 \end{pmatrix} \sim \begin{pmatrix} \epsilon^3 & \epsilon^2 & 1 \\ \epsilon^2 & \epsilon & \epsilon^3 \\ \epsilon^3 & \epsilon^2 & 1 \end{pmatrix}\;;\; m_0 - \text{symmetry protected}\,.$$

$$Y_e = \begin{pmatrix} \alpha_1\left(Y_{\hat{\mathbf{3}}'}^{(3)}\right)_1 & \alpha_1\left(Y_{\hat{\mathbf{3}}'}^{(3)}\right)_3 & \alpha_1\left(Y_{\hat{\mathbf{3}}'}^{(3)}\right)_2 \\ \alpha_2\left(Y_{\mathbf{3}}^{(4)}\right)_1 & \alpha_2\left(Y_{\mathbf{3}}^{(4)}\right)_3 & \alpha_2\left(Y_{\mathbf{3}}^{(4)}\right)_2 \\ \alpha_3\left(Y_{\hat{\mathbf{3}}'}^{(5)}\right)_1 & \alpha_3\left(Y_{\hat{\mathbf{3}}'}^{(5)}\right)_3 & \alpha_3\left(Y_{\hat{\mathbf{3}}'}^{(5)}\right)_2 \end{pmatrix} \sim \begin{pmatrix} \epsilon^3 & \epsilon^2 & 1 \\ \epsilon^2 & \epsilon & \epsilon^3 \\ \epsilon^3 & \epsilon^2 & 1 \end{pmatrix}\,,\quad m_\tau(1:\epsilon:\epsilon^3)\,.$$

g_i **and** α_i **are 6 real constant superpotential parameters (including the** $(2\,{\rm Im}\,\tau_{\rm vev})^{k_i}$ **factors),**
Λ **has the dimension of a mass and corresponds to the HNL mass scale.**
Thus, the relevant parameters are:

$$\Lambda > 0\,,\quad {\rm Re}\,\tau \in [-1/2, 1/2]\,,\quad {\rm Im}\,\tau > \sqrt{3}/2\,,$$
$$g_1 \geq 0\,,\quad g_2 \geq 0\,,\quad g_3 \geq 0\,,\quad \alpha_1 \geq 0\,,\quad \alpha_2 \in \mathbb{R}\,,\quad \alpha_3 \in \mathbb{R}\,.$$

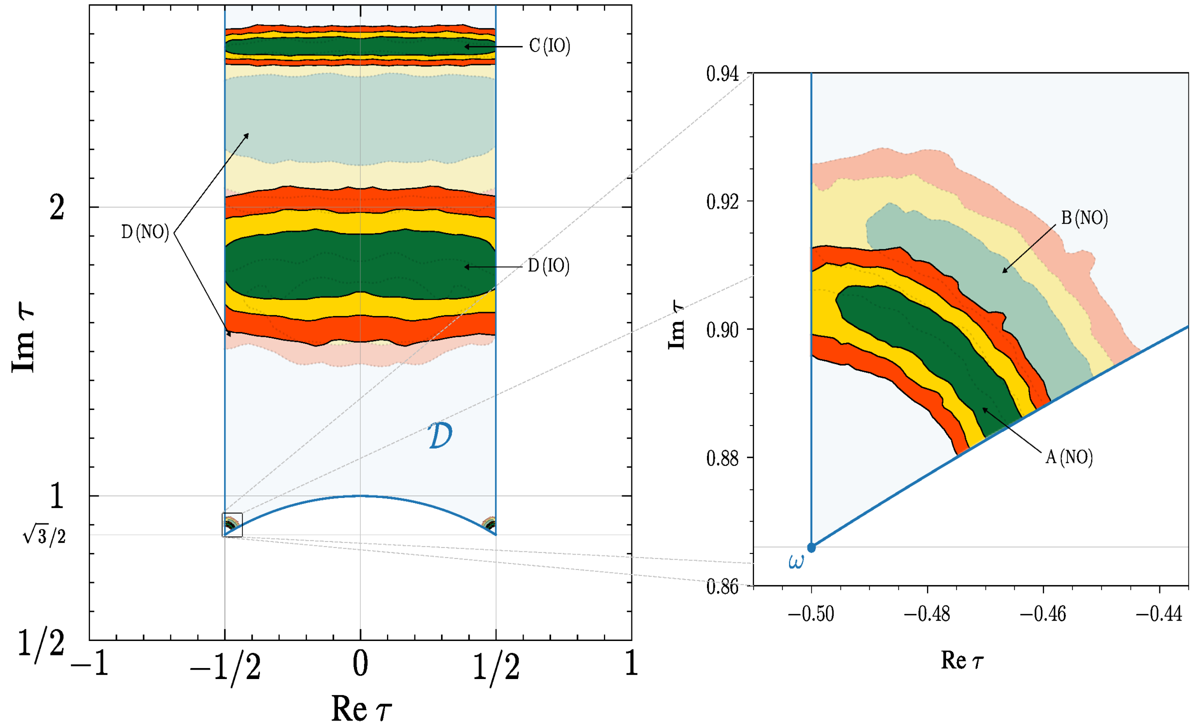
There are, in total, 9 real parameters (including τ**) for 15 observables.**

Input Data in the Statistical analysis

Observable	Best-fit value and 1σ range	
m_e/m_μ	0.0048 ± 0.0002	
m_μ/m_τ	0.0565 ± 0.0045	
	NO	IO
$\Delta m^2_{21}/(10^{-5}\ \text{eV}^2)$	7.49 ± 0.19	
$\lvert\Delta m^2_{31(32)}\rvert/(10^{-3}\ \text{eV}^2)$	2.513 ± 0.020	2.484 ± 0.020
$r \equiv \Delta m^2_{21}/\lvert\Delta m^2_{31(32)}\rvert$	0.0298 ± 0.0008	0.0301 ± 0.0008
$\sin^2\theta_{12}$	0.308 ± 0.012	0.308 ± 0.012
$\sin^2\theta_{13}$	0.02215 ± 0.00057	0.02231 ± 0.00056
$\sin^2\theta_{23}$	0.510 ± 0.025	0.512 ± 0.024

Best-fit values and 1σ ranges for neutrino oscillation parameters obtained from the `NuFit 6.0` global analysis, I. Esteban et al., arXiv:2410.05380, and for charged-lepton mass ratios given at the scale 2×10^{16} GeV with $\tan\beta \equiv v_u/v_d \cong 5$. For $\sin^2\theta_{23}$, in place of the non-Gaussian one-dimensional projections, we have considered a Gaussian approximation based on the 3σ ranges given in arXiv:2410.05380.

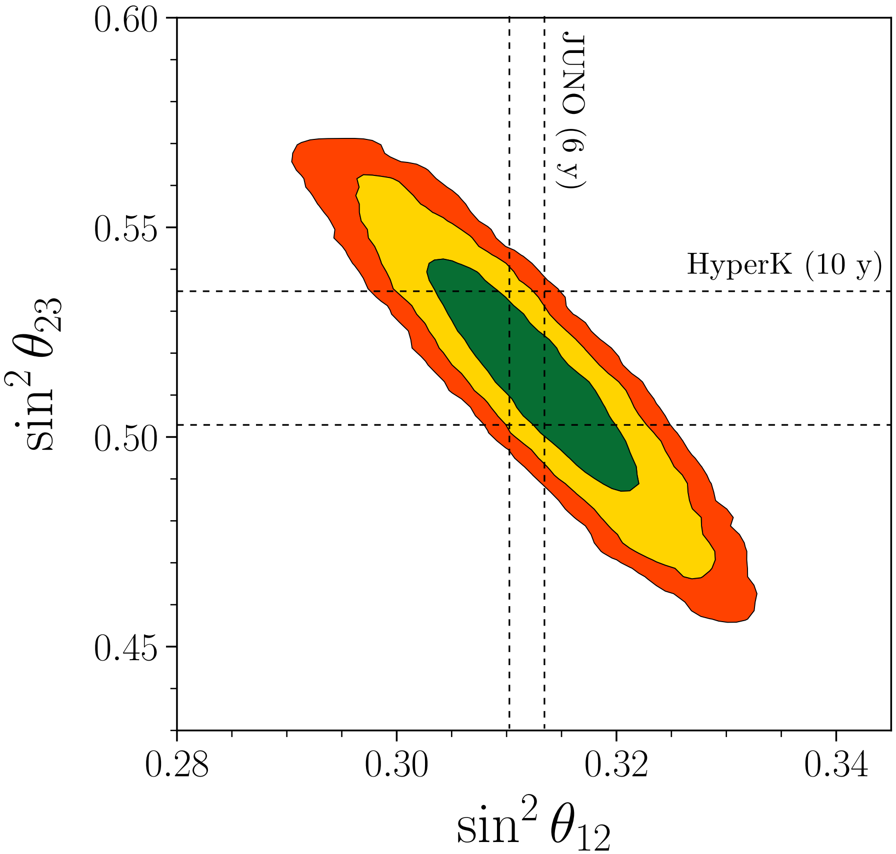
We use the standard parametrisation of the PMNS neutrino mixing matrix.



Viable regions for the modulus VEV τ within the fundamental domain $\mathcal{D}$, for the benchmark modular-symmetry-protected models A, B, C and D. Green, yellow and red fills correspond to the 1σ, 2σ and 3σ credible regions. These regions are symmetric under the gCP transformation that flips the sign of $\mathrm{Re}\,\tau$. The panel on the right shows a zoomed-in view near $\tau_{\mathbf{sym}} = \omega$.

In the next Table we show the central values and limits of the 3σ credible regions for the parameters and observables in each of the benchmark models. Also shown are the $N\sigma$, the root of a Gaussian χ^2, at the point of maximum likelihood.
$\epsilon = (\tau - \omega)/(\tau - \omega^2)$ for cases A and B and $\epsilon = e^{\pi i \tau_{vev}/2}$ for cases C and D. In cases with a nearly massless neutrino, we report the single relevant Majorana phase: $\alpha_{23} \equiv \alpha_{21} - \alpha_{31}$ for NO and α_{21} for IO.

Model (ordering)	A (NO)	B (NO)	C (IO)	D (NO)	D (IO)
$\mathrm{Re}\,\tau$	$-0.472^{+0.017}_{-0.028}$	$-0.475^{+0.036}_{-0.024}$	$[-1/2,+1/2]$	$[-1/2,+1/2]$	$[-1/2,+1/2]$
$\mathrm{Im}\,\tau$	$0.892^{+0.019}_{-0.042}$	$0.912^{+0.015}_{-0.062}$	$2.55^{+0.07}_{-0.04}$	$2.31^{+0.25}_{-0.82}$	$1.68^{+0.37}_{-0.24}$
$\hat{\alpha}_2/\hat{\alpha}_1$	$-0.286^{+0.674}_{-0.106}$	$0.590^{+0.125}_{-0.083}$	$-0.593^{+0.102}_{-0.078}$	$0.31^{+1.06}_{-1.05}$	$-0.058^{+0.348}_{-0.084}$
$\hat{\alpha}_{3(,1)}/\hat{\alpha}_1$	$-0.160^{+0.145}_{-0.481}$	$-0.0502^{+0.0501}_{-0.1540}$	$0.0339^{+0.0026}_{-0.0024}$	$0.33^{+2.08}_{-0.12}$	$-1.38^{+0.70}_{-0.85}$
$\hat{\alpha}_{3,2}/\hat{\alpha}_1$	$0.0202^{+0.0094}_{-0.0514}$	$-0.0185^{+0.0037}_{-0.0062}$	—	$0.20^{+4.25}_{-0.79}$	$-0.71^{+2.30}_{-2.39}$
$\hat{g}_2/\hat{g}_1$	$0.712^{+0.052}_{-0.076}$	$0.219^{+0.011}_{-0.008}$	$0.0799^{+0.0137}_{-0.0125}$	$1.56^{+1.30}_{-1.19}$	$2.55^{+1.29}_{-0.61}$
$\hat{g}_{3(,1)}/\hat{g}_1$	$0.491^{+0.196}_{-0.461}$	$0.336^{+0.062}_{-0.161}$	$13.2^{+1.1}_{-1.2}$	$0.129^{+0.011}_{-0.073}$	$0.0651^{+0.0172}_{-0.0163}$
$\hat{g}_{3,2}/\hat{g}_1$	$0.214^{+0.119}_{-0.551}$	$0.289^{+0.018}_{-0.020}$	—	$0.853^{+0.077}_{-0.734}$	$0.798^{+0.224}_{-0.231}$
$v_d\,\hat{\alpha}_1$, GeV	$0.594^{+0.070}_{-0.121}$	$0.363^{+0.047}_{-0.055}$	$0.176^{+0.005}_{-0.007}$	$0.0689^{+0.0132}_{-0.0427}$	$0.0384^{+0.0126}_{-0.0061}$
$v_u^2\,\hat{g}_1^2/\Lambda$, eV	$0.0328^{+0.0145}_{-0.0070}$	$0.0568^{+0.0192}_{-0.0120}$	$0.00032^{+0.00005}_{-0.00003}$	$0.0184^{+0.0069}_{-0.0066}$	$0.0120^{+0.0108}_{-0.0035}$
m_e/m_μ	$0.0048^{+0.0006}_{-0.0005}$	$0.0047^{+0.0006}_{-0.0005}$	$0.0048^{+0.0005}_{-0.0006}$	$0.0047^{+0.0005}_{-0.0006}$	$0.0048^{+0.0005}_{-0.0006}$
m_μ/m_τ	$0.0560^{+0.0115}_{-0.0114}$	$0.0571^{+0.0114}_{-0.0121}$	$0.0577^{+0.0109}_{-0.0134}$	$0.0550^{+0.0129}_{-0.0105}$	$0.0572^{+0.0099}_{-0.0117}$
$\sin^2\theta_{12}$	$0.307^{+0.034}_{-0.030}$	$0.308^{+0.032}_{-0.030}$	$0.312^{+0.019}_{-0.018}$	$0.310^{+0.030}_{-0.034}$	$0.308^{+0.032}_{-0.034}$
$\sin^2\theta_{13}$	$0.0220^{+0.0017}_{-0.0014}$	$0.0221^{+0.0014}_{-0.0014}$	$0.0222^{+0.0016}_{-0.0014}$	$0.0222^{+0.0016}_{-0.0016}$	$0.0224^{+0.0013}_{-0.0015}$
$\sin^2\theta_{23}$	$0.506^{+0.067}_{-0.065}$	$0.507^{+0.065}_{-0.062}$	$0.519^{+0.049}_{-0.058}$	$0.512^{+0.070}_{-0.065}$	$0.507^{+0.070}_{-0.030}$
m_1, eV	$< 10^{-4}$	$< 10^{-4}$	$0.0491^{+0.0002}_{-0.0002}$	$0.00267^{+0.00048}_{-0.00132}$	$0.0620^{+0.0025}_{-0.0032}$
m_3, eV	$0.0501^{+0.0001}_{-0.0002}$	$0.0501^{+0.0002}_{-0.0002}$	$< 10^{-4}$	$0.0502^{+0.0002}_{-0.0002}$	$0.0379^{+0.0038}_{-0.0054}$
$\Sigma_i m_i$, eV	$0.0588^{+0.0001}_{-0.0001}$	$0.0588^{+0.0001}_{-0.0001}$	$0.0990^{+0.0003}_{-0.0004}$	$0.0619^{+0.0006}_{-0.0017}$	$0.163^{+0.009}_{-0.012}$
$m_{\beta\beta}$, meV	$1.49^{+0.31}_{-0.28}$	$1.49^{+0.27}_{-0.28}$	$17.8^{+1.7}_{-1.8}$	$2.06^{+0.98}_{-0.22}$	$60.0^{+2.4}_{-3.1}$
δ	$\simeq 0,\, \pi$	$\simeq 0$	$\simeq 0$	$\simeq \pi$	$\simeq 0,\, \pi$
$\alpha_{21(23)}$	$\simeq \pi$	$\simeq \pi$	$\simeq 0$	$\simeq \pi$	$\simeq 0$
α_{31}	—	—	—	$\simeq \pi$	$\simeq \pi$
$\lvert\epsilon(\tau)\rvert$	$0.0218^{+0.0047}_{-0.0046}$	$0.0292^{+0.0076}_{-0.0067}$	$0.0182^{+0.0013}_{-0.0018}$	$0.0267^{+0.0700}_{-0.0087}$	$0.0713^{+0.0320}_{-0.0312}$
$\min N\sigma$	0.412	0.411	0.548	0.488	0.362



Correlation between the solar and atmospheric mixing angles in model C. Green, yellow and red fills correspond to the 1σ, 2σ and 3σ credible regions. For comparison, we show the prospective 1σ sensitivities of future long-baseline and reactor experiments HyperKamiokande and JUNO taking into account the found central values for $\sin^2\theta_{23}$ and $\sin^2\theta_{12}$, respectively.

Heavy Majorana Neutrino Phenomenology

The heavy Majorana neutrinos that result from the diagonalization of the neutrino mass terms (m_D and M_R) mix with the active flavour ones. In the basis for which the charged-lepton Yukawa and the RHν mass matrices are diagonal, the mixing relation reads:

$$\nu_{lL} \simeq U_{li}\,\nu_{iL} + \frac{v_u \widehat{Y}_{lj}}{M_j}\,N_{jL}\,, \;\; l = e, \mu, \tau\,,$$

$\widehat{Y}_{\alpha j}$ are the neutrino Yukawa couplings in the considered basis; ν_{iL} and N_{jL} - the LH components of the massive light and heavy Majorana neutrinos with masses m_i, $i = 1, 2, 3$, and M_j, $j = 1, 2, 3$.

Due to this mixing N_j appear in the CC and NC weak interaction Lagrangian:

$$\mathcal{L}^N_{\mathbf{CC}} \simeq -\frac{g}{\sqrt{2}}\,\Theta_{lj}\,\overline{l_L}\,W\,N_{jL} + \text{h.c.}\,,$$

$$\mathcal{L}^N_{\mathbf{NC}} \simeq -\frac{g}{2c_w}\,\Theta_{lj}\,\overline{\nu_{lL}}\,Z\,N_{jL} + \text{h.c.}\,, \qquad l = e, \mu, \tau\,,$$

$\Theta_{lj} \equiv v_u \widehat{Y}_{lj}/M_j$, **$\widehat{Y}_{\alpha j} = U_e^{\dagger}(Y_D^* V^*)$, Y_D^* is the neutrino Yukawa coupling in the "non-diagonal" basis, V is a unitary matrix diagonalising the RH neutrino Majorana mass matrix M_R, $U_e^{\dagger}$ is one of the two unitary matrices diagonalising the charged lepton mass matrix with $U_{PMNS} = U_e^{\dagger} U_{\nu}$.**

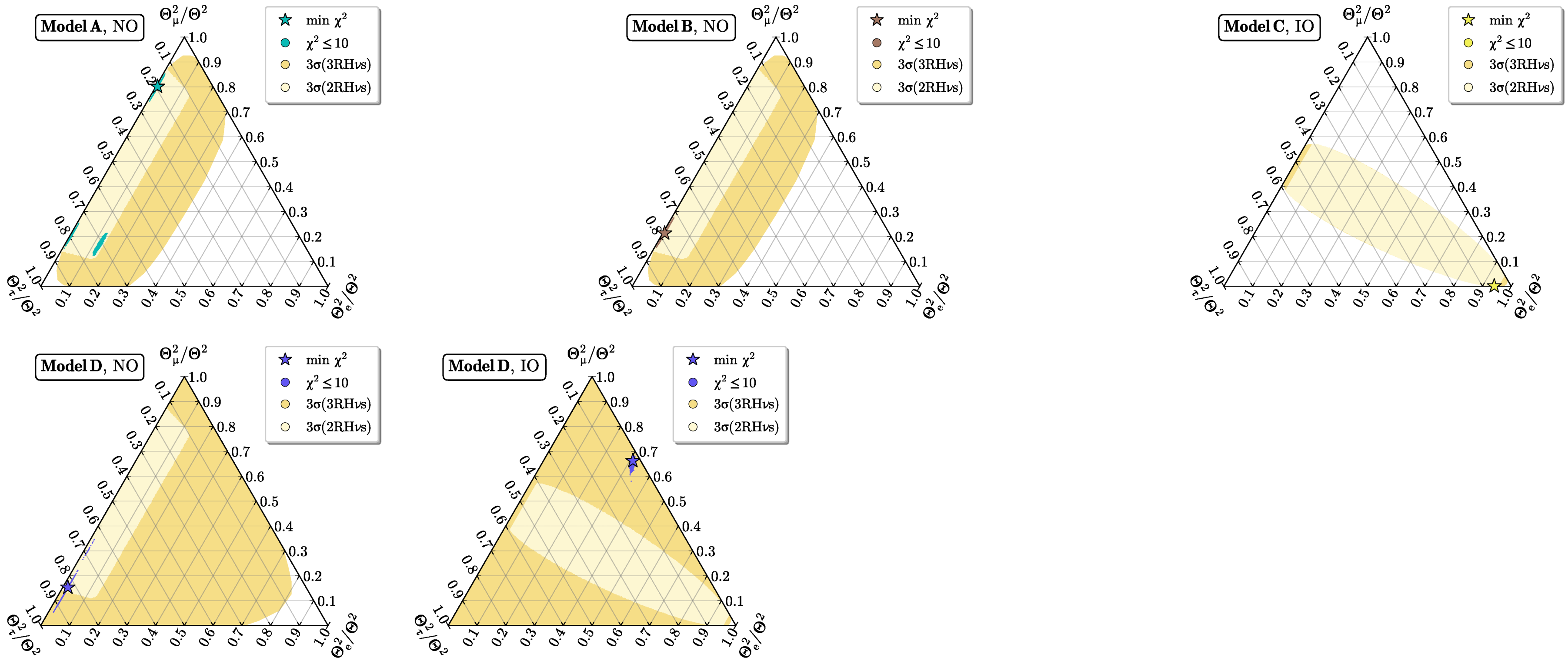
The fact that the heavy Majorana neutrinos N_j (NHL) couple to the weak bosons implies testable phenomenology, which has been extensively studied.

If N_j have masses $M_j <$ EW scale and Θ_{lj} is sufficiently large - can be copiously produced at colliders or beam-line facilities and identified via the subsequent decay (either prompt or displaced) into charged particles (see, e.g., A.M. Abdullahi et al., arXiv:2203.08039, C. Antel et al., arXiv:2305.01715 and references therein).

In these N_j (HNL) searches, the phenomenologically-relevant parameters are combinations of $|\Theta_{lj}|^2$; we consider the following two key combinations:

$$\Theta_l^2 \equiv \sum_{j=1}^{3} |\Theta_{lj}|^2 \qquad \textbf{and} \qquad \Theta^2 \equiv \sum_{\alpha=e,\mu,\tau} \Theta_l^2 \,,$$

Θ_l^2 quantifies the overall contribution of the N_j (NHL) fixed flavour couplings to the N_j (HNL) signals, Θ^2 quantifies the total mixing irrespective of the flavour and the specific heavy neutrino state.



The ratios $\Theta^2_e/\Theta^2 - \Theta^2_\mu/\Theta^2 - \Theta^2_\tau/\Theta^2$ associated to the considered models. The coloured points – top panel in turquoise for model A, middle left in brown for model B, middle right in yellow for model C, and bottom panels in blue for model D (left for NO, right for IO) – are those for which $\chi^2 \leq 10$, with the stars marking the point of maximum posterior probability (minimum Gaussian χ^2). The orange regions correspond to the full parameter space of the type-I seesaw, in the cases with either two (lighter colour) and three (darker colour) RHνs, with the oscillation data varied within the 3σ regions obtained in the `NuFit 6.0` global analysis, I. Esteban et al., arXiv:2410.05380. To obtain the region associated to generic type-I scenario with 3 RHνs, the lightest neutrino mass is varied randomly in the range allowed by the corresponding model when compared against oscillation data.

Model A: $\Theta^2_\mu \cong 4.0 \times 10^{-10}$, $\Theta^2_\tau \cong 0.97 \times 10^{-10}$, $\Theta^2_e \cong 0.03 \times 10^{-10}$ **at** $M_{\rm av} = 1.0$ **GeV.**
There is a second "solution" with dominating Θ^2_τ**.**

Model B: $\Theta^2_\tau \cong 2.5 \times 10^{-10}$, $\Theta^2_\mu \cong 0.7 \times 10^{-10}$, $\Theta^2_e \cong 0.02 \times 10^{-10}$ **at** $M_{\rm av} = 1.0$ **GeV.**

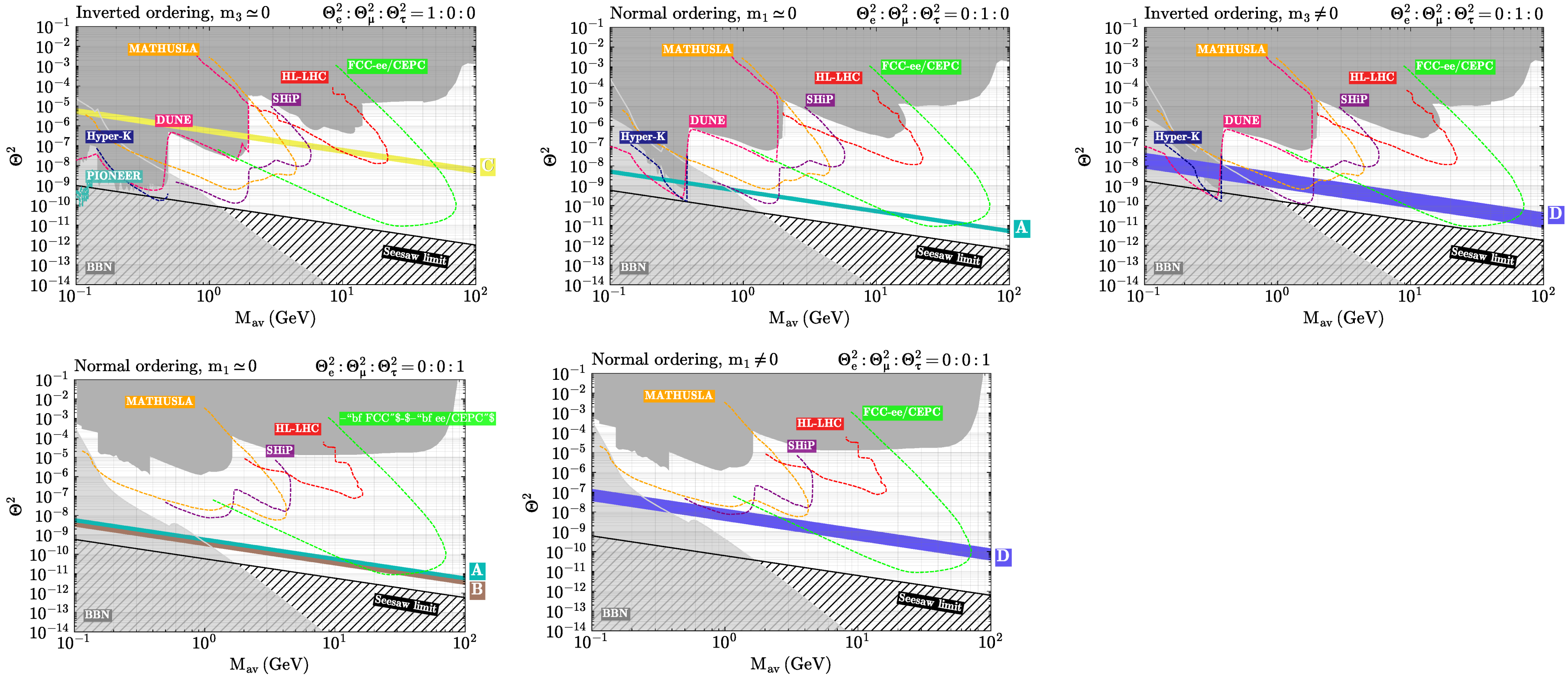
Model C: $\Theta^2_e \cong 1.2 \times 10^{-6}$, $\Theta^2_\tau \cong 0.08 \times 10^{-10}$, $\Theta^2_\mu \cong 0.001 \times 10^{-10}$ **at** $M_{\rm av} = 1.0$ **GeV.**

Model D(NO): $\Theta^2_\tau \cong 1.4 \times 10^{-8}$, $\Theta^2_\mu \cong 0.3 \times 10^{-8}$, $\Theta^2_e \cong 0.03 \times 10^{-8}$ **at** $M_{\rm av} = 1.0$ **GeV.**

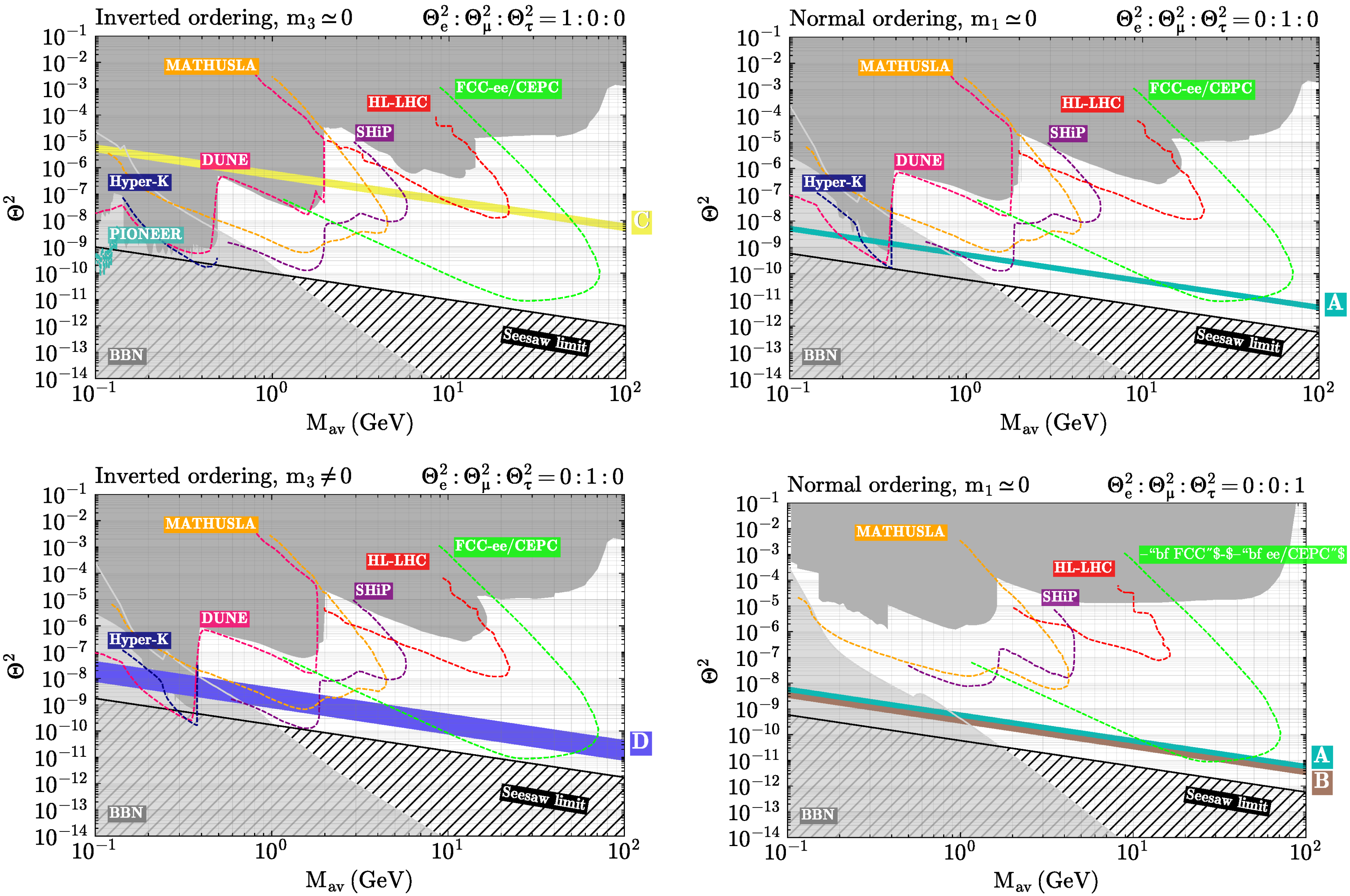
Model D(IO): $\Theta^2_\mu \cong 3.4 \times 10^{-9}$, $\Theta^2_e \cong 1.6 \times 10^{-9}$, $\Theta^2_\tau \cong 1.4 \times 10^{-9}$ **at** $M_{\rm av} = 1.0$ **GeV.**

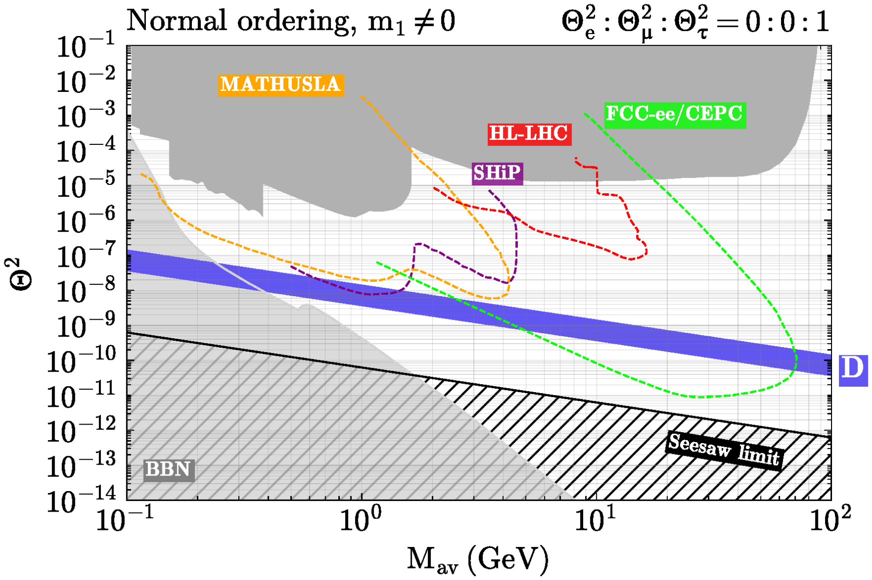
$$M_{\rm av} = \frac{1}{3}(M_1 + M_2 + M_3)$$

Parts of the parameter space in the $\Theta_l^2 - M_{\rm av}$ plane of each of the models can be probed in upcoming, planned and proposed experiments (Hyper-Kamiokande, DUNE, SHiP, HL-LHC, FCC-ee), as is illustrated by the following figures.



The parameter space in the Θ^2- plane of the type-I seesaw scenario for the discussed benchmark models. The black solid curve represents the seesaw limit. The darker gray regions are excluded by several experiments on HNL production via meson decays (PS191, BEBC, PIENU, E949, NA62, T2K, NuTeV, MicroBooNE, CHARM, searches at KEK, tau lepton decays (BELLE) and at colliders (DELPHI, CMS, ATLAS. The lighter gray region is excluded by BBN. The dashed curves represent the sensitivities of the upcoming, planned and proposed experiments PIONEER(cyan), Hyper-K(blue), DUNE(pink), MATHUSLA(orange), SHiP(purple), and searches at HL-LHC(red), FCC-ee and CEPC(green). Current constraints and future sensitivities are only indicative, as they are given for $\Theta^2_e : \Theta^2_\mu : \Theta^2_\tau = 1:0:0$ (upper panel), $0:1:0$ (middle panels) and $0:0:1$ (lower panels) – such ratios hold only approximately in the scenarios considered here – and in the case of a single HNL.


Inverted ordering, $m_3 \simeq 0$
$\Theta_e^2 : \Theta_\mu^2 : \Theta_\tau^2 = 1:0:0$
Normal ordering, $m_1 \simeq 0$
$\Theta_e^2 : \Theta_\mu^2 : \Theta_\tau^2 = 0:1:0$
Inverted ordering, $m_3 \neq 0$
$\Theta_e^2 : \Theta_\mu^2 : \Theta_\tau^2 = 0:1:0$
Normal ordering, $m_1 \simeq 0$
$\Theta_e^2 : \Theta_\mu^2 : \Theta_\tau^2 = 0:0:1$
MATHUSLA
FCC-ee/CEPC
HL-LHC
SHiP
DUNE
Hyper-K
PIONEER
BBN
Seesaw limit
C
A
D
B
–“bf FCC”$-$–“bf ee/CEPC”$
Θ^2
M_{av} (GeV)


Normal ordering, $m_1 \neq 0$
$\Theta_e^2 : \Theta_\mu^2 : \Theta_\tau^2 = 0 : 0 : 1$
MATHUSLA
FCC-ee/CEPC
HL-LHC
SHiP
D
Seesaw limit
BBN
Θ^2
M_{av} (GeV)

Summary

The proposed link between modular symmetry and L' symmetry, in the context of the considered low-scale type-I seesaw mechanism, which connects charged-lepton masses, neutrino masses, neutrino mixing, leptonic CP violation and heavy Majorana neutrino (HNL) phenomenology, while also avoiding fine-tuning and retaining minimality and predictivity, opens up the possibility to probe the modular paradigm in new ways.

Supporting Slides

Matter Fields and Modular Forms

The matter(super)fields (charged lepton, neutrino, quark) transform under $\overline{\Gamma} \simeq PSL(2,\mathbb{Z}) = SL(2,\mathbb{Z})/\mathbb{Z}_2$**,** $\mathbb{Z}_2 = \{I, -I\}$ **(**$\Gamma \simeq SL(2,\mathbb{Z})$**) as "weighted" multiplets:**

$$\psi_i \xrightarrow{\gamma} (c\tau + d)^{-k_\psi}\, \rho_{ij}(\tilde{\gamma})\psi_j\,, \gamma \in \overline{\Gamma}\ (\gamma \in \Gamma)\,,$$

$$\left(\gamma\tau = \frac{a\tau+b}{c\tau+d}\,, \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\,, a,b,c,d \in \mathbb{Z}\,, ad - bc = 1\,, \mathrm{Im}\tau > 0\right)$$

k_ψ **is the weight of** ψ**;** $k_\psi \in \mathbb{Z}$ **(or rational number).**
$\Gamma(N)$ **- principal congruence (normal) subgroup of** $SL(2,\mathbb{Z})$**.**
$\rho(\tilde{\gamma})$ **is a unitary representation of the** *inhomogeneous (homogeneous)* **finite modular group** $\Gamma_N = \overline{\Gamma}/\overline{\Gamma}(N)$ **(**$\Gamma'_N = \Gamma/\Gamma(N)$**),** $\tilde{\gamma}$ **− representation of** γ **in** Γ_N **(**Γ'_N**)**

F. Feruglio, arXiv:1706.08749; S. Ferrara et al., Phys.Lett. B233 (1989) 147, B225 (1989) 363

As we have indicated in brackets, one can consider also the case of Γ **and** $\gamma \in \Gamma(N)$**. Then** $\rho(\gamma)$ **will be a unitary representation of the** *homogeneous* **finite modular group** Γ'_N**.**

The group $\Gamma \equiv SL(2,\mathbb{Z})$ is generated by the matrices

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad R = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$

obeying $S^2 = R$, $(ST)^3 = R^2 = \mathbf{I}$, and $RT = TR$.

Thus,

$$S\,\tau = -\frac{1}{\tau} \qquad T\,\tau = \tau + 1\,.$$

Remarkably, for $N \leq 5$, the inhomogeneous finite modular groups Γ_N are isomorphic to non-Abelian discrete groups widely used in flavour model building:
$\Gamma_2 \simeq S_3$, $\Gamma_3 \simeq A_4$, $\Gamma_4 \simeq S_4$ **and** $\Gamma_5 \simeq A_5$.
Γ_N **is presented by two generators** S **and** T **satisfying:**

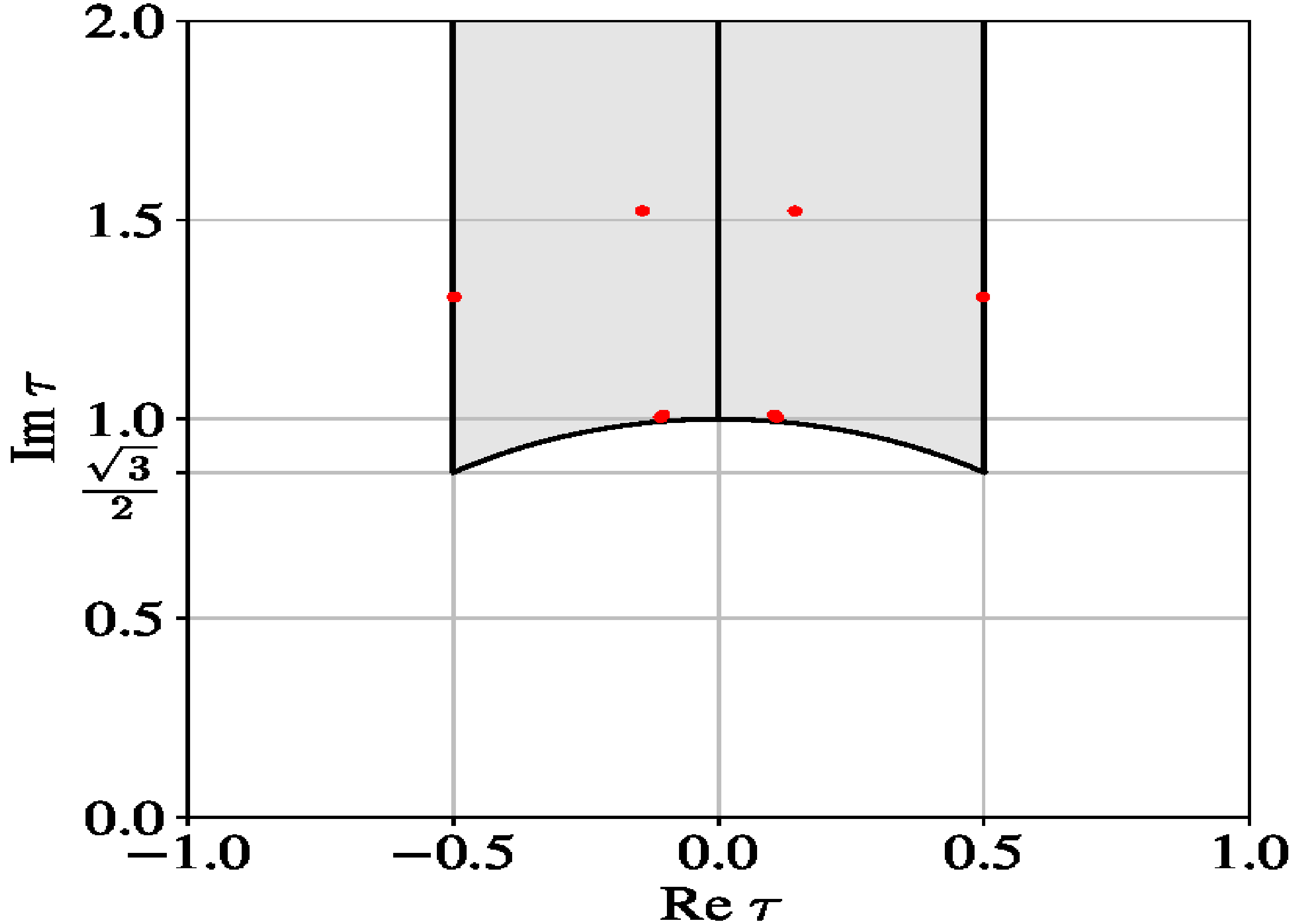
$$S^2 = (ST)^3 = T^N = I\,.$$

The group theory of $\Gamma_2 \simeq S_3$, $\Gamma_3 \simeq A_4$, $\Gamma_4 \simeq S_4$ **and** $\Gamma_5 \simeq A_5$ **is summarized, e.g., in P.P. Novichkov** *et al.*, **JHEP 07 (2019) 165, arXiv:1905.11970.**

$\Gamma \simeq SL(2,\mathbb{Z})$ – *homogeneous* **modular group,** $\Gamma(N)$ **and the quotient groups** $\Gamma'_N \equiv \Gamma/\Gamma(N)$ **– homogeneous finite modular groups. For** $N = 3,4,5$, Γ'_N **are isomorphic to the double covers of the corresponding non-Abelian discrete groups:**
$\Gamma'_3 \simeq A'_4 \equiv T'$, $\Gamma'_4 \simeq S'_4$ **and** $\Gamma'_5 \simeq A'_5$.
Γ'_N **is presented by two generators** S **and** T **satisfying:**

$$S^4 = (ST)^3 = T^N = I\,,\;\; S^2\,T = T\,S^2 \quad (S^2 = R)\,.$$

The group theory of $\Gamma'_3 \simeq A'_4$, $\Gamma'_4 \simeq S'_4$ **and** $\Gamma'_5 \simeq A'_5$ **for flavour model building was developed in X.-G. Liu, G.-J. Ding, arXiv:1907.01488 (**A'_4**);**
P.P. Novichkov et al., arXiv:2006.03058 (S'_4**); C.-Y. Yao et al., arXiv:2011.03501 (**A'_5**).**



The Fundamental Domain of $\overline{\Gamma}$ shown for $\mathrm{Im}\tau \leq 2$ (the red dots correspond to solutions of the lepton flavour problem, see further).

P.P. Novichkov, J.T. Penedo, STP, A.V. Titov, arXiv:1811.04933.

Relevant sub-groups of Γ_N and Γ'_N: :

$\mathbb{Z}_3^{ST} = \{I, ST, (ST)^2\}$

$\mathbb{Z}_N^{T} = \{I, T, (T)^2, ..., T^{N-1}\}$

Γ_N: $\mathbb{Z}_2^{S} = \{I, S\}$

Γ'_N: $\mathbb{Z}_4^{S} = \{I, S, S^2, S^3\}$ ($R^2 = I$, $\mathbb{Z}_2^{R} = \{I, R\}$, $R\tau = \tau$)

Modular Forms

Within the considered framework the elements of the Yukawa coupling and fermion mass matrices in the Lagrangian of the theory are expressed in terms of modular forms of a certain level N and weight k_f.

The modular forms are functions of a single complex scalar field – the modulus τ – and have specific transformation properties under the action of the modular group.

Both the Yukawa couplings and the matter fields (supermultiplets) are assumed to transform in representations of an inhomogeneous (homogeneous) finite modular group $\Gamma_N^{(\prime)}$.

Once τ acquires a VEV, the modular forms and thus the Yukawa couplings and the form of the mass matrices get fixed, and a certain flavour structure arises.

Quantitatively and barring fine-tuning, the magnitude of the values of the non-zero elements of the fermion mass matrices and therefore the fermion mass ratios are determined by the modular form values (which in turn are functions of the τ's VEV).

Modular Forms (contd.)

The key elements of the considered framework are modular forms $f(\tau)$ of *weight* k_f *and level* N **– holomorphic functions of τ, which transform under $\overline{\Gamma}$ (Γ) as follows:**

$$F(\gamma\tau) = (c\tau + d)^{k_F}\, \rho_{\mathbf{r}}(\tilde{\gamma}) F(\tau)\,, \quad \gamma \in \overline{\Gamma} \quad (\gamma \in \Gamma)\,,$$

F. Feruglio, arXiv:1706.08749

$\rho_{\mathbf{r}}$ is a unitary representation of the finite modular group Γ_N (Γ'_N).
In the case of $\overline{\Gamma}$ (Γ) *non-trivial* **modular forms exist only for positive even integer (positive integer) weight k_F.**
For given k, N (N is a natural number), the modular forms span a linear space of finite dimension:
of weight k and level 3, $\mathcal{M}_k(\Gamma_3^{(\prime)} \simeq A_4^{(\prime)})$, is $k+1$;
of weight k and level 4, $\mathcal{M}_k(\Gamma_4^{(\prime)} \simeq S_4^{(\prime)})$, is $2k+1$;
of weight k and level 5, $\mathcal{M}_k(\Gamma_5^{(\prime)} \simeq A_5^{(\prime)})$, is $5k+1$.
Thus, $\dim\mathcal{M}_1(\Gamma'_3 \simeq A'_4) = 2$, $\dim\mathcal{M}_1(\Gamma'_4 \simeq S'_4) = 3$, $\dim\mathcal{M}_1(\Gamma'_5 \simeq A'_5) = 6$.

Multiplets of Γ_N (Γ'_N) of higher weight modular forms can be constructed from tensor products of the lowest weight 2 (weigh 1) multiplets (they represent homogeneous polynomials of the lowest weight modular forms).

Following arXiv:1706.08749, it was of highest priority and of crucial importance for model building to find the basis of modular forms of the lowest weight 2 (weight 1) transforming in irreps of Γ_N (Γ'_N).

It took about two years to find the requisite bases for Γ_N (Γ'_N), $N = 2, 3, 4, 5$.

F. Feruglio, 1706.08749 ($\Gamma_3 \simeq A_4$, $k_f = 2$: the 3 mod.forms form a 3 of A_4);
T. Kobayashi et al., 1803.10391 ($\Gamma_2 \simeq S_3$, $k_f = 2$: the 2 mod. forms form a **2** of S_3);
J. Penedo, STP, 1806.11040 ($\Gamma_4 \simeq S_4$, $k_f = 2$: the 5 mod. forms form a **2** and $3'$ of S_4);
P.P. Novichkov et al., 1812.02158; G.-J. Ding et al., 1903.12588 (($\Gamma_5 \simeq A_5$), $k_f = 2$: the 11 basis modular forms were shown to form a **3**, a $3'$ and a **5** of A_5).

More elegant constuction: modular forms for A'_4, S'_4, A'_5 (and A_4, S_4, A_5).

The weight 1 modular forms
i) of A'_4 form a 2 of A'_4, ii) of S'_4 form a $\hat{3}$ of S'_4, iii) of A'_5 form a 5 of A'_5,
as was proven respectively in X.-G. Liu, G.-J. Ding, 1907.01488, P.P. Novichkov et al., 2006.03058 and C.-Y. Yao et al., 2011.03501.

In each of the cases of A'_4, S'_4 and A'_5 the lowest weight 1 modular forms, and thus all higher weight modular forms, icluding those (of even weight) associated with A_4, S_4 and A_5, constructed from tensor products of the weight 1 multiplets, were shown in the three quoted articles to be expressed in terms of only two independent functions of τ.

These pairs of functions are different for the three different groups; but they all are related (in different ways) to the Dedekind η-function (in the case of A'_5 (A_5) - to two Jacobi theta constants also) and have similar (fastly converging) $q-$expansions, i.e., power series expansions in $q = e^{2\pi i\tau}$.

Thus, in the case of a flavour symmetry described by a finite modular group $\Gamma^{(\prime)}_{\rm N}$, $N = 2, 3, 4, 5$, the elements of the matices of the Yukawa couplings in the considered approach represent homogeneous polynomials of various degree of only two (holomorphic) functions of τ. They include also a limited (relatively small) number of constant parameters.

The modular forms of level $N = 2, 3, 4, 5$ for $\Gamma^{(\prime)}_{2,3,4,5} \simeq S_3, A^{(\prime)}_4, S^{(\prime)}_4, A^{(\prime)}_5$ have been constructed by use of the Dedekind eta function, $\eta(\tau)$:

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1-q^n) = q^{\frac{1}{24}} \sum_{n=-\infty}^{\infty} (-1)^n \, q^{\frac{n(3n-1)}{2}} \, , q = e^{i2\pi\tau} \, .$$

In the cases of $\Gamma^{(\prime)}_5 \simeq A^{(\prime)}_5$ two "Jacobi theta constants" are also used.

Modular forms of level $N = 4$ for $\Gamma'_4 \simeq S'_4$ ($\Gamma_4 \simeq S_4$) − in terms of $\theta(\tau)$, $\varepsilon(\tau)$:

$$\theta(\tau) \equiv \frac{\eta^5(2\tau)}{\eta^2(\tau)\eta^2(4\tau)} = \Theta_3(2\tau) \, , \; \varepsilon(\tau) \equiv \frac{2\,\eta^2(4\tau)}{\eta(2\tau)} = \Theta_2(2\tau) \, .$$

$\Theta_2(\tau)$ and $\Theta_3(\tau)$ are the Jacobi theta constants, $\eta(a\tau)$, $a = 1, 2, 4$, is the Dedekind eta.

Modular forms of level $N = 3$ for $\Gamma'_3 \simeq A'_4$ ($\Gamma_3 \simeq A_4$) − in terms of $\hat{e}_1$ and $\hat{e}_2$:

$$\hat{e}_1 = \frac{\eta^3(3\tau)}{\eta(\tau)} , \quad \hat{e}_2 = \frac{\eta^3(\tau/3)}{\eta(\tau)} \, .$$

Modular forms of level $N = 5$ for $\Gamma'_5 \simeq A'_5$ ($\Gamma_5 \simeq A_5$) − in terms of $\theta_5(\tau)$ and $\varepsilon_5(\tau)$:
$\theta_5(\tau) = \exp(-\,i\pi/10)\,\Theta_{\frac{1}{10},\frac{1}{2}}(5\tau)\,\eta^{-3/5}(\tau)$, $\varepsilon_5(\tau) = \exp(-\,i\,3\pi/10)\,\Theta_{\frac{3}{10},\frac{1}{2}}(5\tau)\,\eta^{-3/5}(\tau)$.

Example: S_4'

P.P. Novichkov, J.T. Penedo. S.T.P., arXiv:2006.03058

Weight 1 modular forms furnishing a $\hat{3}$ of S_4':

$$Y_{\hat{\mathbf{3}}}^{(1)}(\tau) = \begin{pmatrix} \sqrt{2}\,\varepsilon\,\theta \\ \varepsilon^2 \\ -\theta^2 \end{pmatrix}$$

Modular S_4 lowest-weight 2 multiplets furnish a 2 and a $3'$ irreducible representations of S_4 (S_4') and are given by: :

$$Y_{\mathbf{2}}^{(2)}(\tau) = \begin{pmatrix} \frac{1}{\sqrt{2}}\left(\theta^4+\varepsilon^4\right) \\ -\sqrt{6}\,\varepsilon^2\,\theta^2 \end{pmatrix} = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}, \qquad Y_{\mathbf{3'}}^{(2)}(\tau) = \begin{pmatrix} \frac{1}{\sqrt{2}}\left(\theta^4-\varepsilon^4\right) \\ -2\,\varepsilon\,\theta^3 \\ -2\,\varepsilon^3\,\theta \end{pmatrix} = \begin{pmatrix} Y_3 \\ Y_4 \\ Y_5 \end{pmatrix}.$$

At weight $k=3$, a non-trivial singlet and two triplets exclusive to S_4' arise:

$$Y_{\hat{\mathbf{1}}'}^{(3)}(\tau) = \sqrt{3}\left(\varepsilon\,\theta^5 - \varepsilon^5\,\theta\right),$$

$$Y_{\hat{\mathbf{3}}}^{(3)}(\tau) = \begin{pmatrix} \varepsilon^5\,\theta + \varepsilon\,\theta^5 \\ \frac{1}{2\sqrt{2}}\left(5\,\varepsilon^2\,\theta^4 - \varepsilon^6\right) \\ \frac{1}{2\sqrt{2}}\left(\theta^6 - 5\,\varepsilon^4\,\theta^2\right) \end{pmatrix}, \qquad Y_{\hat{\mathbf{3}}'}^{(3)}(\tau) = \frac{1}{2}\begin{pmatrix} -4\sqrt{2}\,\varepsilon^3\,\theta^3 \\ \theta^6 + 3\,\varepsilon^4\,\theta^2 \\ -3\,\varepsilon^2\,\theta^4 - \varepsilon^6 \end{pmatrix}.$$

At weight $k=4$ one again recovers the S_4 result: the modular forms furnish a 1, 2, 3 and $3'$ irreducible representations of S_4 (S_4').

$$Y_{\mathbf{1}}^{(4)}(\tau) = \frac{1}{2\sqrt{3}}\left(\theta^8 + 14\,\varepsilon^4\,\theta^4 + \varepsilon^8\right), \qquad Y_{\mathbf{2}}^{(4)}(\tau) = \begin{pmatrix} \frac{1}{4}\left(\theta^8 - 10\,\varepsilon^4\,\theta^4 + \varepsilon^8\right) \\ \sqrt{3}\left(\varepsilon^2\,\theta^6 + \varepsilon^6\,\theta^2\right) \end{pmatrix},$$

$$Y_{\mathbf{3}}^{(4)}(\tau) = \frac{3}{2\sqrt{2}}\begin{pmatrix} \sqrt{2}\left(\varepsilon^2\,\theta^6 - \varepsilon^6\,\theta^2\right) \\ \varepsilon^3\,\theta^5 - \varepsilon^7\,\theta \\ -\varepsilon\,\theta^7 + \varepsilon^5\,\theta^3 \end{pmatrix}, \qquad Y_{\mathbf{3'}}^{(4)}(\tau) = \begin{pmatrix} \frac{1}{4}\left(\theta^8 - \varepsilon^8\right) \\ \frac{1}{2\sqrt{2}}\left(\varepsilon\,\theta^7 + 7\,\varepsilon^5\,\theta^3\right) \\ \frac{1}{2\sqrt{2}}\left(7\,\varepsilon^3\,\theta^5 + \varepsilon^7\,\theta\right) \end{pmatrix},$$

The functions $\theta(\tau)$ and $\varepsilon(\tau)$ are given by:

$$\theta(\tau) \equiv \frac{\eta^5(2\tau)}{\eta^2(\tau)\eta^2(4\tau)} = \Theta_3(2\tau)\,, \qquad \varepsilon(\tau) \equiv \frac{2\,\eta^2(4\tau)}{\eta(2\tau)} = \Theta_2(2\tau)\,.$$

$\Theta_2(\tau)$ and $\Theta_3(\tau)$ are the Jacobi theta constants, $\eta(a\tau)$, $a = 1, 2, 4$, is the Dedekind eta function.

The functions $\theta(\tau)$ and $\varepsilon(\tau)$ admit the following q-expansions - power series expansions in $q_4 \equiv \exp(i\pi\tau/2)$ ($\mathrm{Im}(\tau) \geq \sqrt{3}/2$, $|q_4| \lesssim 0.26$) :

$$\theta(\tau) = 1 + 2\sum_{k=1}^{\infty} q_4^{(2k)^2} = 1 + 2\,q_4^4 + 2\,q_4^{16} + \ldots\,,$$

$$\varepsilon(\tau) = 2\sum_{k=1}^{\infty} q_4^{(2k-1)^2} = 2\,q_4 + 2\,q_4^9 + 2\,q_4^{25} + \ldots\,.$$

In the "large volume" limit $\mathrm{Im}\,\tau \to \infty$, $\theta \to 1$, $\varepsilon \to 0$.
In this limit $\varepsilon \sim 2\,q_4$ and ε can be used as an expansion parameter instead of q_4.
Due to quadratic dependence in the exponents of q_4, the $q-$expansion series converge rapidly in the fundamental domain of the modular group, where $\mathrm{Im}(\tau) \geq \sqrt{3}/2$ and $|q_4| \leq \exp(-\pi\sqrt{3}/4) \simeq 0.26$.

Similar conclusions are valid for the pair of functions in terms of which the lowest weight 1 modular forms, and thus all higher weight modular forms of A'_4 and A'_5 are expressed.

The Framework

$\mathcal{N}=1$ **rigid (global) SUSY, the matter action** $\mathcal{S}$ **reads:**

$$\mathcal{S} = \int \mathrm{d}^4x\, \mathrm{d}^2\theta\, \mathrm{d}^2\overline{\theta}\;\; K(\tau, \overline{\tau}, \psi, \overline{\psi}) + \left(\int \mathrm{d}^4x\, \mathrm{d}^2\theta\;\; W(\tau, \psi) + \text{h.c.}\right)\,,$$

K **is the Kähler potential,** W **is the superpotential,** ψ **denotes a set of chiral supermultiplets** ψ_i**,** θ **and** $\overline{\theta}$ **are Grassmann variables;**
τ **is the modulus chiral superfield, whose lowest component is the complex scalar field acquiring a VEV (we use in what follows the same notation** τ **for the lowest complex scalar component of the modulus superfield and call this component also "modulus").**
τ **and** ψ_i **transform under the action of** $\overline{\Gamma}$ **(**Γ**) in a certain way (S. Ferrara et al., PL B225 (1989) 363 and B233 (1989) 147). Assuming that** $\psi_i = \psi_i(x)$ **transform in a certain irrep** $\mathbf{r}_i$ **of** Γ_N **(**Γ'_N**), the transformations read:**

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \overline{\Gamma}\;(\Gamma)\,: \qquad \begin{cases} \tau \to \dfrac{a\tau + b}{c\tau + d}\,, \\ \psi_i \to (c\tau + d)^{-k_i}\, \rho_{\mathbf{r}_i}(\gamma)\, \psi_i\,. \end{cases}$$

ψ_i **is not a modular form multiplet, the integer** $(-k_i)$ **can be** >0**,** <0**, 0.**
Invariance of $\mathcal{S}$ **under these transformations implies (global SUSY):**

$$\begin{cases} W(\tau,\psi) \rightarrow W(\tau,\psi)\,, \\ K(\tau,\overline{\tau},\psi,\overline{\psi}) \rightarrow K(\tau,\overline{\tau},\psi,\overline{\psi}) + f_K(\tau,\psi) + \overline{f_K}(\overline{\tau},\overline{\psi})\,. \end{cases}$$

The second line represents a Kähler transformation.
An example Kähler potential that is widely used in model building reads:

$$K(\tau,\overline{\tau},\psi,\overline{\psi}) = -\Lambda_0^2 \log(-i\tau + i\overline{\tau}) + \sum_i \frac{|\psi_i|^2}{(-i\tau + i\overline{\tau})^{k_i}}\,,$$

$\Lambda_0 > 0$ **having mass dimension one.**
More general $K(\tau,\overline{\tau},\psi,\overline{\psi})$ **and the possible consequences they can have for flavour model building are discussed in**
Mu-Chun Chen et al., arXiv:1909.06910 and 2108.02240; Y. Almumin et al., arXiv:2102.11286.

$$W(\tau,\psi) \to W(\tau,\psi)\,,$$

The superpotential can be expanded in powers of ψ_i:

$$W(\tau,\psi) = \sum_n \sum_{\{i_1,\dots,i_n\}} \sum_s g_{i_1\dots i_n,s}\left(Y_{i_1\dots i_n,s}(\tau)\,\psi_{i_1}\dots\psi_{i_n}\right)_{\mathbf{1},s}\,,$$

1 stands for an invariant singlet of Γ_N (Γ'_N). For each set of n fields $\{\psi_{i_1},\dots,\psi_{i_n}\}$, the index s labels the independent singlets. Each of these is accompanied by a coupling constant $g_{i_1\dots i_n,s}$ and is obtained using a modular multiplet $Y_{i_1\dots i_n,s}$ of the requisite weight. To ensure invariance of W under Γ_N (Γ'_N), $Y_{i_1\dots i_n,s}(\tau)$ must transform as:

$$Y(\tau) \xrightarrow{\gamma} (c\tau+d)^{k_Y}\rho_{\mathbf{r}_Y}(\gamma)\,Y(\tau)\,,$$

r_Y is a representation of Γ_N (Γ'_N), and k_Y and r_Y are such that

$$k_Y = k_{i_1} + \dots + k_{i_n}\,, \tag{2}$$

$$\mathbf{r}_Y \otimes \mathbf{r}_{i_1} \otimes \dots \otimes \mathbf{r}_{i_n} \supset \mathbf{1}\,. \tag{3}$$

Thus, $Y_{i_1\dots i_n,s}(\tau)$ represents a multiplet of weight k_Y and level N modular forms transforming in the representation r_Y of Γ_N (Γ'_N).

Mass Matrices

Consider the bilinear (i.e., mass term)

$$\psi_i^c \, M(\tau)_{ij} \, \psi_j \,,$$

where the superfields ψ and ψ^c transform as

$$\psi \xrightarrow{\gamma} (c\tau + d)^{-k} \rho_r(\gamma) \, \psi \quad (\rho(\gamma)\,,\ \Gamma_N^{(\prime)}\,,\ N = 2,3,4,5)\,,$$
$$\psi^{\mathbf{c}} \xrightarrow{\gamma} (c\tau + d)^{-k^c} \rho^c_{r^c}(\gamma) \, \psi^c\,, \quad (\rho^c(\gamma)\,,\ \Gamma_N^{(\prime)})\,.$$

Modular invariance: $M(\tau)_{ij}$ must be modular form of level N and weight $K \equiv k + k^c$,

$$M(\tau) \xrightarrow{\gamma} M(\gamma\tau) = (c\tau + d)^K \rho^c(\gamma)^* M(\tau) \rho(\gamma)^\dagger \,.$$

CP Symmetry in Modular Invariant Flavour Models

The formalism of combined finite modular and generalised CP (gCP) symmetries for theories of flavour was developed in P.P. Novichkov et al., arXiv:1905.11970.

gCP invariance was shown to imply that the constants g, which accompany each invariant singlet in the superpotential, must be real (in a symmetric basis of S and T and at least for $\Gamma_N^{(\prime)}$, $N \leq 5$). Thus, the number of free parameters in modular-invariant models which also enjoy a gCP symmetry gets reduced, leading to "minimal" models which have higher predictive power.

In these models, the only source of both modular symmetry breaking and CP violation is the VEV of the modulus τ.

The "minimal" phenomenologically viable modular-invariant flavour models with gCP symmetry constructed so far
– of the lepton sector with massive Majorana neutrinos (12 observables) contain $\geq$ 7 (6) real parameters – 5 (4) real couplings + the complex τ (6 (5) real constants + 1 phase);
– of the quark sector contain $\geq$ 9 real parameters – 7 real coulplings + the complex τ;
– while the models of lepton and quark flavours (22 observables) have $\geq$ 14 real parameters - 12 real couplings + the complex τ.

See, e.g., B.-Y. Qu et al., arXiv:2106.11659

Under the CP transformatoion,

$$\tau \xrightarrow{\text{CP}} -\tau^* \,.$$

P.P. Novichkov et al., 1905.11970; A. Baur et al., 1901.03251 and 1908.00805

It was further demonstrated that CP is conserved for

$$\text{Re}\tau = \pm 1/2\,; \;\; \tau = e^{i\theta}\,,\; \theta = [\pi/3, 2\pi/3]\,; \text{Re}\tau = 0\,,\; \text{Im}\tau \geq 1\,.$$

i.e., for the values of τ's VEV at the boundary of the fundamental domain and on the imaginary axis.

Residual Symmetries

The breakdown of modular symmetry is parameterised by the VEV of τ.
There is no value of τ's VEV which preserves the full symmetry $\Gamma^{(\prime)}$ ($\Gamma_N^{(\prime)}$).

At certain "symmetric points" $\tau = \tau_{\rm sym}$, $\Gamma^{(\prime)}$ ($\Gamma_N^{(\prime)}$) is only partially broken, with the unbroken generators giving rise to residual symmetries.
The $R = S^2$ generator ($\Gamma_N^{(\prime)}$) is unbroken for any value of τ, thus a $\mathbb{Z}_2^R$ symmetry is always preserved.

There are only 3 inequivalent symmetric points in $\mathcal{D}$:

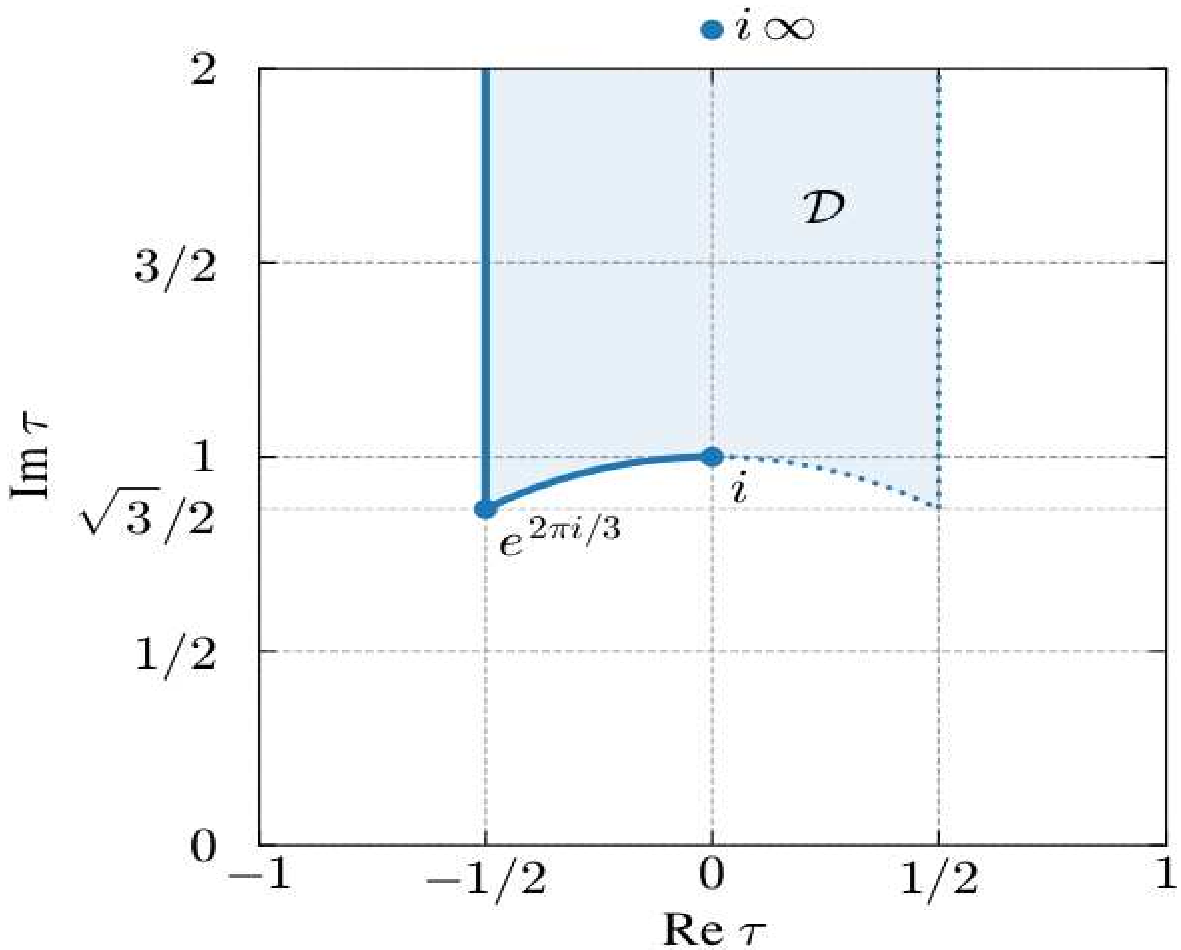
- $\tau_{\rm sym} = i\infty$, invariant under T, preserving $\mathbb{Z}_N^T$;
- $\tau_{\rm sym} = i$, invariant under S, preserving $\mathbb{Z}_2^S$ ($\mathbb{Z}_4^S$, $S^2 = R$);
- $\tau_{\rm sym} = \omega \equiv \exp(2\pi i/3)$, invariant under ST, preserving $\mathbb{Z}_3^{ST}$.

P.P. Novichkov et al., arXiv:1811.04933 and arXiv:2006.03058

These symmetric values of τ preserve the CP ($\mathbb{Z}_2^{CP}$) symmetry of a CP- and modular-invariant theory (e.g. a modular theory where the couplings satisfy a reality condition).

P.P. Novichkov et al., arXiv:1911.04933 and arXiv:2006.03058

The CP ($\mathbb{Z}_2^{\rm CP}$) symmetry is preserved for ${\rm Re}\,\tau = 0$ or for τ lying on the border of the fundamental domain $\mathcal{D}$, but is broken at generic values of τ.



The fundamental domain $\mathcal{D}$ of the modular group Γ and its three symmetric points $\tau_{\rm sym} = i\infty, i, \omega$. At the solid and dotted lines (which include the three points) CP is also preserved. The value of τ can always be restricted to $\mathcal{D}$ by a suitable modular transformation.

Figure from P.P. Novichkov et al., arXiv:2006.03058