Classification of Modular Symmetries in Type IIB Flux Landscape

Takafumi Kai (Kyushu University)

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In collaboration with

K. Ishiguro (KEK), T. Kobayashi (Hokkaido U.), Y. Koga (Kyushu U.) and H. Otsuka (Kyushu U.)

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Flavor structure and modular invariance

Feruglio, 1706.08749

Flavor puzzle

What is **the origin of the parameters** in the flavor sector?

One possibility : **Modular symmetry**

Modular invariance can provide **rich structures** for the flavor sector

Top-down approach

- String theory
- **Extra-dimensional space**

Modular group

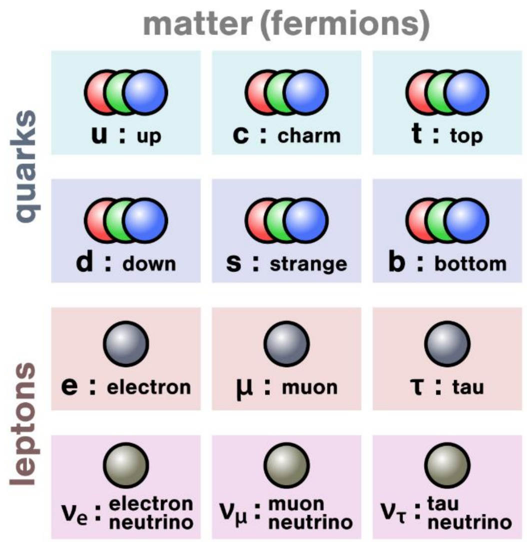
$$SL(2,\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a,b,c,d \in \mathbb{Z}, ad-bc=1 \right\} .$$

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad S^2 = \mathbb{1} \quad , \quad (ST)^3 = \mathbb{1} \quad .$$

Extra-dimensional space

Cremades, Ibáñez, Marchesano, hep-th/0404229

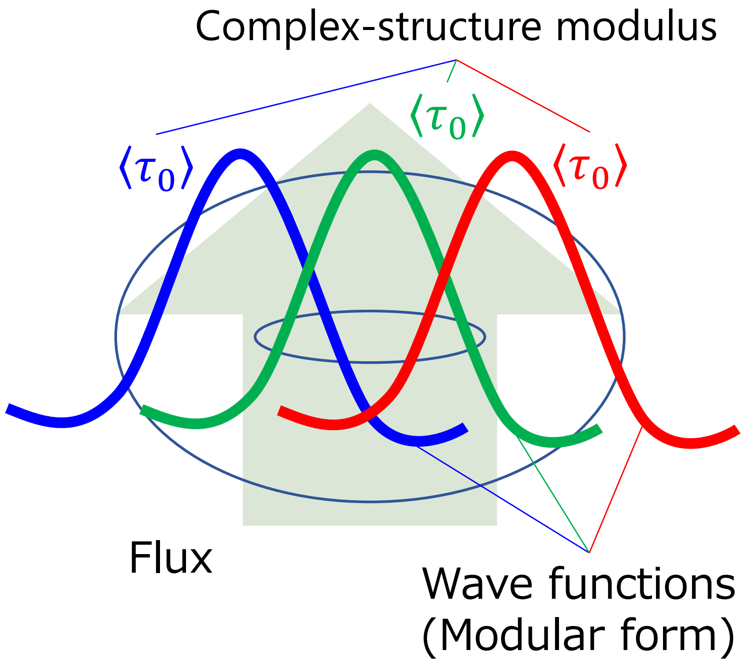
The flavor structure appearing
in the Standard Model



higgstan.com

Yukawa coupling

$Y_{ijk}(\tau)$



VEVs of complex-structure modulus

→ **The input parameters** can be determined through the Yukawa coupling which is described as modular form

Flux landscape

Ishiguro, Kobayashi, Otsuka, 2011.09154

Ishiguro, TK, Otsuka, Kobayashi, 2311.12425

Type IIB flux compactification on $T^6/(\mathbb{Z}_2 \times \mathbb{Z}'_2)$ orientifold

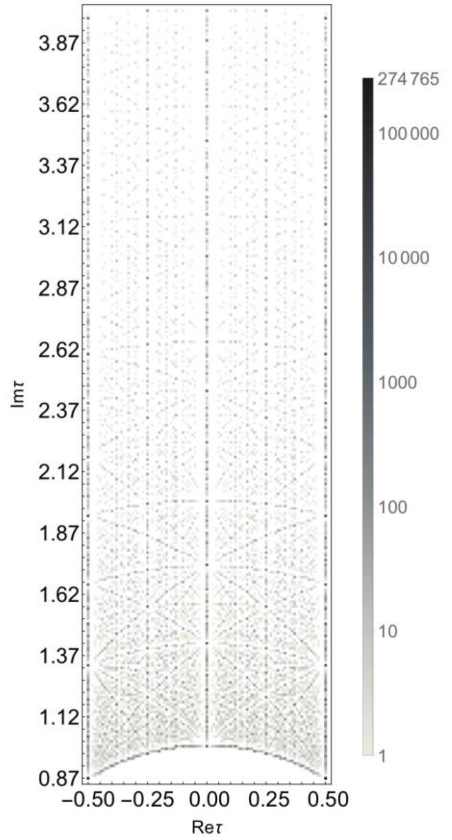
Background 3-form flux: $G_3 \to \{a^0, a^i, b_i, b_0\}$

$\Rightarrow$ Various VEVs of complex-structure moduli

The distribution of VEVs regarding **complex structure moduli** is known to have a peak **at the fixed points of $SL(2,\mathbb{Z})$**.

VEVs with enhanced symmetry are favored in the Flux Landscape.

$\mathbb{Z}_3$: 40.3%



The distribution of VEVs (CS moduli) on the fundamental region

Previous research

Considering Type IIB flux compactification on ***$T^6/\mathbb{Z}_{6-\mathrm{II}}$* and *$T^6/(\mathbb{Z}_2 \times \mathbb{Z}_4)$* orientifolds**, we analyze the structure of vacua in the Flux Landscape of complex structure moduli.

Research purpose

Classification of Modular Symmetries in Type IIB Flux Landscape

$\mathbb{Z}_N$, $\mathbb{Z}_N \times \mathbb{Z}_M$	Lattice	Duality symmetries of U
$\mathbb{Z}_4$	$SU(4)^2$	$PSL(2,\mathbb{Z})$
$\mathbb{Z}_4$	$SU(2) \times SU(4) \times SO(5)$	$\bar{\Gamma}_0(2)$
$\mathbb{Z}_4$	$SU(2)^2 \times SO(5)^2$	$PSL(2,\mathbb{Z})$
$\mathbb{Z}_{6-\mathrm{II}}$	$SU(2) \times SU(6)$	$\bar{\Gamma}_0(3)$
$\mathbb{Z}_{6-\mathrm{II}}$	$SU(3) \times SO(8)$	$\bar{\Gamma}^0(3)$
$\mathbb{Z}_{6-\mathrm{II}}$	$SU(2)^2 \times SU(3)^2$	$PSL(2,\mathbb{Z})$
$\mathbb{Z}_{6-\mathrm{II}}$	$SU(2)^2 \times SU(3) \times G_2$	$PSL(2,\mathbb{Z})$
$\mathbb{Z}_{8-\mathrm{II}}$	$SU(2) \times SO(10)$	$\bar{\Gamma}_0(2)$
$\mathbb{Z}_{8-\mathrm{II}}$	$SO(4) \times SO(9)$	$PSL(2,\mathbb{Z})$
$\mathbb{Z}_{12-\mathrm{II}}$	$SO(4) \times F_4$	$PSL(2,\mathbb{Z})$
$\mathbb{Z}_2 \times \mathbb{Z}_4$	$SU(2)^2 \times SO(5)^2$	$PSL(2,\mathbb{Z})$
$\mathbb{Z}_2 \times \mathbb{Z}_6$	$SU(2)^2 \times SU(3) \times G_2$	$PSL(2,\mathbb{Z})$

Table 1: The duality symmetries regarding the complex-structure modulus on $T^6/\mathbb{Z}_N$ and $T^6/(\mathbb{Z}_N \times \mathbb{Z}_M)$ orbifold. In the case of $SU(3) \times SO(8)$ loot lattice, the complex-structure modulus is defined as $U' \equiv U + 2$.

Motivation

Modular symmetry plays **an important role** in revealing the structure of landscape

Generalization and assumptions

- Being consistent with the structure of type IIB string theory
- Normalization by intersection number
- Scaling duality

Outline

1. **Introduction**
2. **Previous research**
   - Period vector
   - Distribution of VEVs
3. **Conditions for period vectors**
   - Mass spectra of Type IIB closed string
   - Normalization by intersection numbers
   - Scaling duality
4. **Summary**

Moduli stabilization

Scalar potential

$$V = e^K\left(K^{i\bar{j}}(D_iW)(D_{\bar{j}}\overline{W}) - 3|W|^2\right)$$

$i, \bar{j} =$ axio-dilaton S, complex structure moduli τ^α, $K_{i\bar{j}} \equiv \partial_i\partial_{\bar{j}}K$, $D_i \equiv \partial_i W + W\partial_i K$, $M_{\rm Pl} = 1$

Kähler potential

$$K = -2\log\mathcal{V}_W - \log\left(-i(S-\bar{S})\right) - \log\left(i\int \Omega\wedge\bar{\Omega}\right),$$ $\mathcal{V}_W$: volume modulus

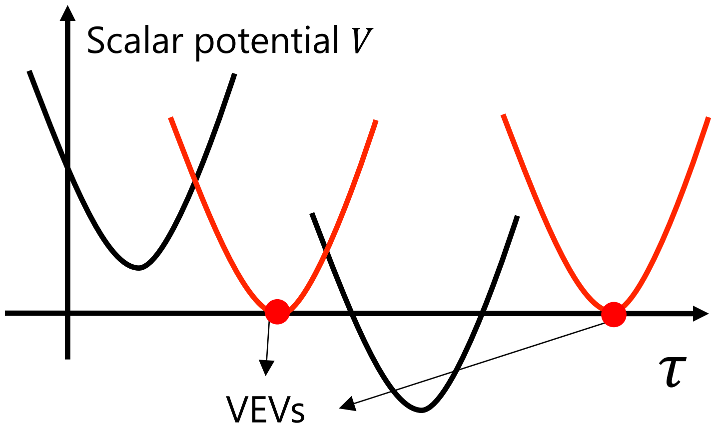
Superpotential

$$W = \int G_3\wedge\Omega,$$

Holomorphic 3-form : $\Omega = X^I\alpha_I - F_I\beta^I$, $\begin{pmatrix} X^\alpha/X^0 = \tau^\alpha, & \alpha = 1, \dots h^{2,1} \\ F_I \equiv \partial_I F, & \text{prepotential}: F \end{pmatrix}$

3-form flux : $G_3 = F_3 - SH_3$, $(F_3 = a^I\alpha_I + b_I\beta^I, \quad H_3 = c^I\alpha_I + d_I\beta^I,)$

Cohomology basis : $\int \alpha_I\wedge\beta^J = \delta_I^J$, $(I, J = 0, \dots, h^{2,1})$



SUSY Minkowski solutions

$$\partial_{\tau^\alpha}W = 0, \qquad \partial_S W = 0, \qquad W = 0$$

By taking arbitrary integer values for the 3-form flux, the moduli fields are given a wide variety of VEVs.

$Sp(4,\mathbb{Z})$ transformation on $T^6/\mathbb{Z}_{6-\mathrm{II}}$ orbifold

Ishiguro, TK, Kobayashi, Otsuka, 2311.12425

Period vector

$$\Pi = \begin{pmatrix} \int_{A^0}\Omega \\ \int_{A^1}\Omega \\ \int_{B^0}\Omega \\ \int_{B^1}\Omega \end{pmatrix} = \begin{pmatrix} \int_{T^6/\mathbb{Z}_{6-\mathrm{II}}}\Omega\wedge\mathbf{1}_4 \\ \int_{T^6/\mathbb{Z}_{6-\mathrm{II}}}\Omega\wedge(-\mathbf{1}_3) \\ \int_{T^6/\mathbb{Z}_{6-\mathrm{II}}}\Omega\wedge\mathbf{1}_1 \\ \int_{T^6/\mathbb{Z}_{6-\mathrm{II}}}\Omega\wedge\mathbf{1}_2 \end{pmatrix} = -\frac{1}{6}\begin{pmatrix} i \\ \sqrt{3} \\ 3i\tau \\ \sqrt{3}\tau \end{pmatrix}.$$

holomorphic 3-form : Ω

By Kähler transformation and $Sp(4,\mathbb{Z})$ transformation, $\Pi \to \Pi' = \left(1,\ -\sqrt{3}i\tau,\ 3\tau,\ \sqrt{3}i\right)^T$

$SL(2,\mathbb{Z})$ transformation for period vector

$c \equiv 0 \mod 3$

$$\Pi' \to (c\tau+d)^{-1}\begin{pmatrix} d & 0 & \frac{c}{3} & 0 \\ 0 & a & 0 & -b \\ 3b & 0 & a & 0 \\ 0 & -c & 0 & d \end{pmatrix}\Pi' \equiv (c\tau+d)^{-1}M\Pi' \qquad \left(\tau \to \frac{a\tau+b}{c\tau+d}\right)$$

Since we consider **a basis transformation of $H_3(T^6/\mathbb{Z}_{6-\mathrm{II}},\mathbb{Z})$**,
the modular transformation must be $\Gamma_0(3)$ (Hecke congruence subgroup).

“Scaling duality” on $T^6/\mathbb{Z}_{6-\mathrm{II}}$ orbifold

Ishiguro, TK, Kobayashi, Otsuka, 2311.12425

Another symmetry for period vector

“Scaling duality transformation”: $S_{(3)} \equiv \tau \to -\frac{1}{3\tau}$

Taking account of $\Pi' = \left(1,\ -\sqrt{3}i\tau,\ 3\tau,\ \sqrt{3}i\right)^T$, the scaling transformation is

$$\Pi' \to \tau^{-1}\frac{i}{\sqrt{3}}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}\Pi' \equiv \tau^{-1}\frac{i}{\sqrt{3}}M\Pi'.$$

$\notin \Gamma_0(3)$

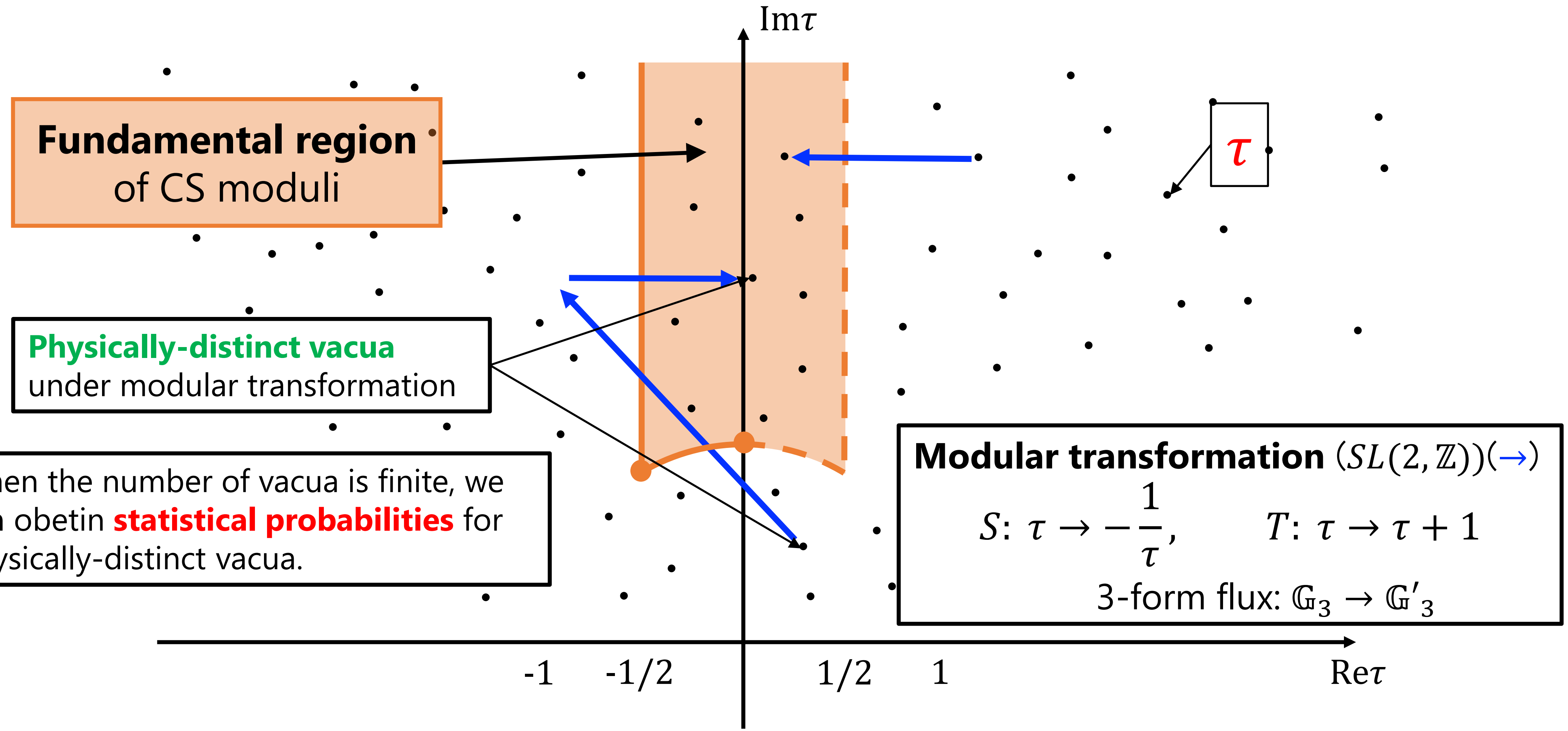
Congruence subgroup : $\Gamma_0(3)$

$$\Pi' \to (c\tau + d)^{-1}\begin{pmatrix} d & 0 & \frac{c}{3} & 0 \\ 0 & a & 0 & -b \\ 3b & 0 & a & 0 \\ 0 & -c & 0 & d \end{pmatrix}\Pi', \quad \left(\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})\right)$$

The outer semidirect product group

$$\Gamma_0(3) \rtimes_{\varphi(S_{(3)})} \mathbb{Z}_2$$

The distribution of VEVs



When the number of vacua is finite, we can obetin **statistical probabilities** for physically-distinct vacua.

Modular transformation $(SL(2,\mathbb{Z}))(\rightarrow)$

$$S:\ \tau \rightarrow -\frac{1}{\tau},\qquad T:\ \tau \rightarrow \tau + 1$$

3-form flux: $\mathbb{G}_3 \rightarrow \mathbb{G}'_3$

The distribution of VEVs on $T^6/\mathbb{Z}_{6-\mathrm{II}}$

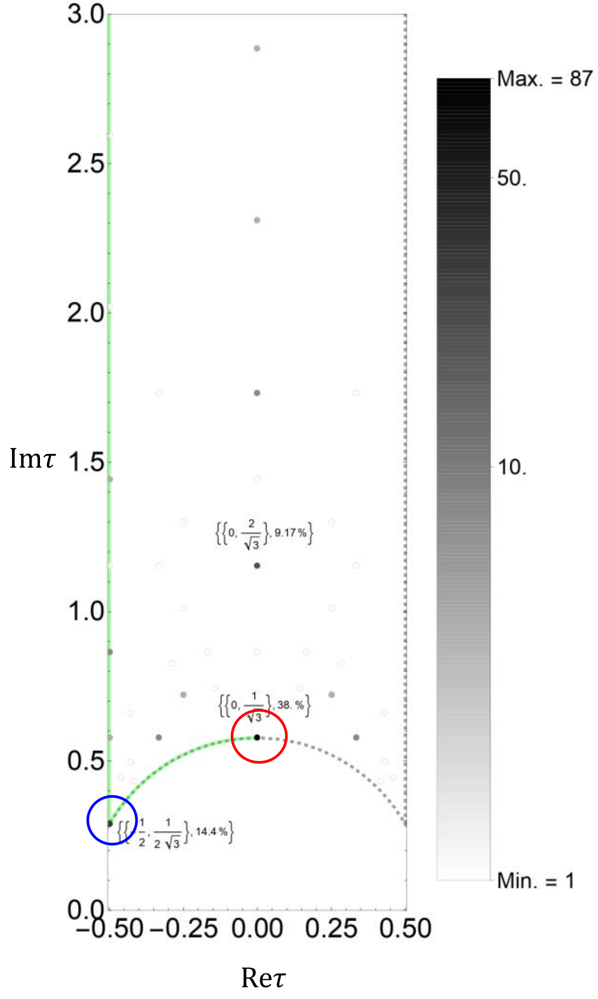
Ishiguro, TK, Kobayashi, Otsuka, 2311.12425

- **The largest number of VEVs** are clustered at the fixed point (elliptic point) associated with **Scaling duality**

- **The fixed point related to $\Gamma_0(3)$** has **the second highest concentration of VEVs**

In the fundamental region, the fixed points $(\mathbb{Z}_2, \mathbb{Z}_3)$ are very strong candidates for Landscape.

Ratio	38.0%	14.4%	9.17%	3.49%	1.75%
τ	$\frac{1}{\sqrt{3}}i$	$-\frac{1}{2}+\frac{1}{2\sqrt{3}}i$	$\frac{2}{\sqrt{3}}i$	$\frac{1}{3}+\frac{1}{\sqrt{3}}i$ $\sqrt{3}i$ $-\frac{1}{3}+\frac{1}{\sqrt{3}}i$ $-\frac{1}{2}+\frac{\sqrt{3}}{2}i$	$-\frac{1}{2}+\frac{1}{\sqrt{3}}i$ $-\frac{1}{4}+\frac{5}{4\sqrt{3}}i$ $\frac{5}{\sqrt{3}}i$ $\frac{4}{\sqrt{3}}i$ $\frac{1}{4}+\frac{5}{4\sqrt{3}}i$ $-\frac{1}{2}+\frac{5}{2\sqrt{3}}i$



Outline

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2. **Previous research**
   - Period vector
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3. **Conditions for period vectors**
   - Mass spectra of Type IIB closed string
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4. **Summary**

Problem and motivation

Period vector

$$\Pi = \begin{pmatrix} \int_{A^0} \Omega \\ \int_{A^1} \Omega \\ \int_{B^0} \Omega \\ \int_{B^1} \Omega \end{pmatrix} = \begin{pmatrix} \int_{T^6/\mathbb{Z}_{6-\mathrm{II}}} \Omega \wedge \mathbf{1}_4 \\ \int_{T^6/\mathbb{Z}_{6-\mathrm{II}}} \Omega \wedge (-\mathbf{1}_3) \\ \int_{T^6/\mathbb{Z}_{6-\mathrm{II}}} \Omega \wedge \mathbf{1}_1 \\ \int_{T^6/\mathbb{Z}_{6-\mathrm{II}}} \Omega \wedge \mathbf{1}_2 \end{pmatrix}$$

There is **a variety of the choice concerning the cycles** on the toroidal orbifolds

$$\Pi' \to (c\tau + d)^{-1} \begin{pmatrix} d & 0 & \frac{c}{3} & 0 \\ 0 & a & 0 & -b \\ 3b & 0 & a & 0 \\ 0 & -c & 0 & d \end{pmatrix} \Pi' \equiv (c\tau + d)^{-1} M \Pi'$$

The different choice of the cycles → **The different transformation matrix**

To identify the modular symmetries for the classification, we discuss **a generalization** and **assumptions**

Consistency concerning string theory

Ishiguro, TK, Koga, Kobayashi, Otsuka, 2502.20743

The moduli-dependent part in the untwisted sectors
(the one-loop partition function for the orbifolds)

$$Z_{(1,\theta^2)}(\tau,\bar{\tau},T,U)=\sum_{m_1,m_2,n_1,n_2\in\mathbb{Z}} e^{2\pi i\tau_1(2m_1n_1+m_2n_2)}e^{-\frac{\pi\tau_2}{T_2U_2}|TUn_2+Tn_1-2Um_1+m_2|^2},$$

We can read off **the mass spectrum of strings** for the two-dimensional **sub-torus**

$$m_\perp^2=\sum_{m_1,m_2,n_1,n_2\in\mathbb{Z}}\frac{1}{T_2U_2}|TUn_2+Tn_1-2Um_1+m_2|^2,$$

This is invariant under the following transformations

$$T\to T+2,\quad T\to\frac{T}{T+1},$$
$$U\to U+1,\quad U\to-\frac{U}{2U-1}.$$

The duality group for $T^6/\mathbb{Z}_4$ orbifold with $SU(2)\times SU(4)\times SO(5)$ root lattice

$$\Gamma^0(2)_T\times\Gamma_0(2)_U.$$

Normalization by intersection number

Ishiguro, TK, Koga, Kobayashi, Otsuka, 2502.20743

Intersection number

$$\int_{A^J} \mathbf{1}_{A^I} = \int_{T^6/\mathbb{Z}_4} \mathbf{1}_{A^I} \wedge \mathbf{1}_{B_J} = c_I \delta^I_J, \qquad \int_{B_J} \mathbf{1}_{B_I} = \int_{T^6/\mathbb{Z}_4} \mathbf{1}_{B_I} \wedge \mathbf{1}_{A^J} = -c_J \delta^J_I.$$

Coordinates and functions of prepotential

$$X^I = \frac{1}{\sqrt{c_I}} \int_{A^I} \Omega, \qquad F_I = \frac{1}{\sqrt{c_I}} \int_{B_I} \Omega, \qquad (I = 0, \ldots, h^{2,1}_{\mathrm{untw.}}).$$

Considering **the above modification**, we can choose the bases including **the modular symmetry** which is consistent with **the structure of the 10d bosonic string sector** of the type IIB closed string

$T^6/\mathbb{Z}_4$ orbifold
with $SU(2) \times SU(4) \times SO(5)$ root lattice

$$\Pi \equiv \begin{pmatrix} X^0 \\ X^1 \\ F_0 \\ F_1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \int_{T^6/\mathbb{Z}_4} \Omega \wedge \mathbf{1}_{B_0} \\ \frac{1}{2\sqrt{2}} \int_{T^6/\mathbb{Z}_4} \Omega \wedge \mathbf{1}_{B_1} \\ \frac{1}{2} \int_{T^6/\mathbb{Z}_4} \Omega \wedge \mathbf{1}_{A^0} \\ \frac{1}{2\sqrt{2}} \int_{T^6/\mathbb{Z}_4} \Omega \wedge \mathbf{1}_{A^1} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -iU \\ \frac{U}{\sqrt{2}} \\ \frac{i}{2} \end{pmatrix},$$

Scaling duality

Ishiguro, TK, Koga, Kobayashi, Otsuka, 2502.20743

Two matrices regarding the symplectic basis transformation

$$X_1 = \begin{pmatrix} d' & 0 & 2c' & 0 \\ 0 & a' & 0 & -2b' \\ b' & 0 & a' & 0 \\ 0 & -c' & 0 & d' \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & c'' & 0 & 2d'' \\ -b'' & 0 & -2a'' & 0 \\ 0 & a'' & 0 & -b'' \\ d'' & 0 & c'' & 0 \end{pmatrix}. \qquad \begin{matrix} (a', b', c', d' \in \mathbb{Z}) \\ (a'', b'', c'', d'' \in \mathbb{Z}), \end{matrix}$$

Scaling duality bridging between two matrices

$$S_{\rm SD} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \qquad S_{\rm SD} x_1 = x_2, \qquad (x_1 \in X_1,\ x_2 \in X_2)\,,$$

Scaling duality including Kähler transformation
is denoted as **a generalized *S*-transformation**

$$S_{(2)} \equiv \frac{i}{\sqrt{2}U} S_{\rm SD} \qquad U \to -\frac{1}{2U} \qquad S_{(2)} = \begin{pmatrix} 0 & -\frac{1}{\sqrt{2}} \\ \sqrt{2} & 0 \end{pmatrix}$$

The outer semidirect product group

$$\bar{\Gamma}_0(2) \rtimes \mathbb{Z}_2.$$

Classification of Modular Symmetries

Ishiguro, TK, Koga, Kobayashi, Otsuka, 2502.20743

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$\mathbb{Z}_{6-\mathrm{II}}$	$SU(2)^2 \times SU(3) \times G_2$	$PSL(2,\mathbb{Z})$
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$\mathbb{Z}_{8-\mathrm{II}}$	$SO(4) \times SO(9)$	$PSL(2,\mathbb{Z})$
$\mathbb{Z}_{12-\mathrm{II}}$	$SO(4) \times F_4$	$PSL(2,\mathbb{Z})$
$\mathbb{Z}_2 \times \mathbb{Z}_4$	$SU(2)^2 \times SO(5)^2$	$PSL(2,\mathbb{Z})$
$\mathbb{Z}_2 \times \mathbb{Z}_6$	$SU(2)^2 \times SU(3) \times G_2$	$PSL(2,\mathbb{Z})$

Table 1: The duality symmetries regarding the complex-structure modulus on $T^6/\mathbb{Z}_N$ and $T^6/(\mathbb{Z}_N \times \mathbb{Z}_M)$ orbifold. In the case of $SU(3) \times SO(8)$ loot lattice, the complex-structure modulus is defined as $U' \equiv U + 2$.

Generalization and assumptions

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- Normalization by intersection number
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Summary

Motivation and previous research

The modular symmetry of the modulus on $T^6/\mathbb{Z}_{6-\mathrm{II}}$ orbifold is **different from $\boldsymbol{PSL(2,\mathbb{Z})}$**

The different structure of distribution for the VEVs based on $T^6/\mathbb{Z}_{6-\mathrm{II}}$ orbifold

The modular symmetry plays an important role in **revealing the structure of landscape**

Conclusion (necessary conditions for the classification)

- Being consistent with the structure of type IIB string theory
- Normalization by intersection number
- Scaling duality

To work on …

- Constraints for moduli by considering threshold corrections
- Modular symmetry of Hecke congruence subgroup and Scaling duality

Back up

Moduli

<u>10d metric</u>

$$G_{MN} = \begin{pmatrix} g_{\mu\nu} & g_{\mu n} \\ g_{m\nu} & g_{mn} \end{pmatrix}$$

$(M, N = 0, \dots 9, \quad \mu, \nu = 0, \dots 3, \quad m, n = 4, \dots 9)$

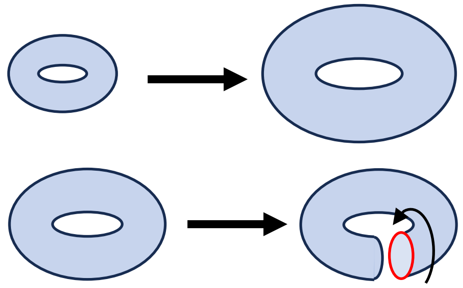
<u>Deformation of extra-dimensional space</u>

$$g_{mn} \to g_{mn} + \delta g_{mn}$$

Kähler moduli (real (1,1)-form) $\delta g_{i\bar{j}} dz^i \wedge \overline{dz^j}$

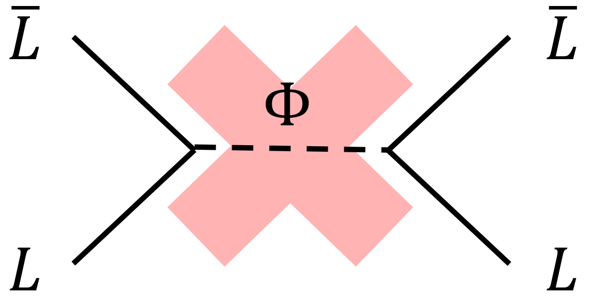
Complex structure moduli
(complex (2,1)-form) $\Omega_{ijk} g^{k\bar{l}} \delta g_{\bar{l}\bar{m}} dz^i \wedge dz^j \wedge \overline{dz^m}$

$i, j, k, l, m = 1, \dots, 3$



They appear as scalar fields in low-energy effective theories

→ **Moduli fields**



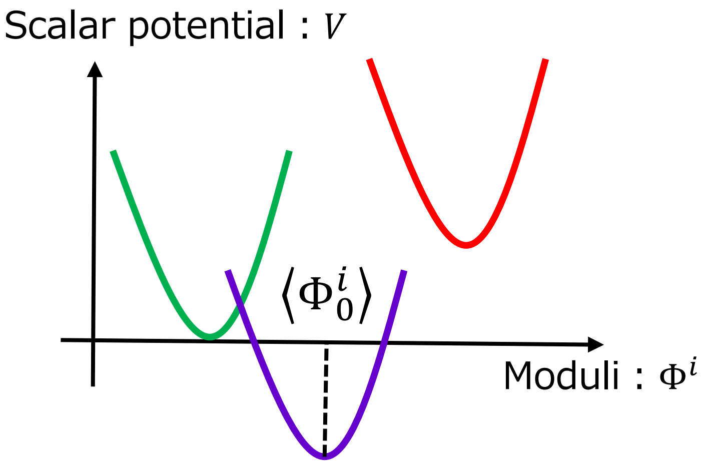
Moduli fields produce **unknown interactions** with elementary particles that appear in the Standard Model

Moduli stabilization

The effective action of massless moduli fields

$$S = \int dx^4 \left[K_{\bar{j}i} \partial_\mu \Phi^{*\bar{j}} \partial^\mu \Phi^{\mathrm{i}} + \underline{V\left(\Phi^{\mathrm{i}}, \Phi^{*\bar{j}}, N_{F_i}, N_{H_i}\right)} \right]$$

→ **Scalar potential**



Moduli stabilization

Fixing the moduli fields at the local minimum value (VEV $\langle \Phi_0^i \rangle$) of the scaler potential

Conservation law of charge in extra-dimensional space (tadpole cancelation condition)

$$N_{\mathrm{flux}} + D3 \text{ brane charge} + O3 \text{ plane charge} = 0$$

Induced **3-form flux** $\{N_{F_i}, N_{H_i}\}$ take various integers ➡ Scalar potential varies and gives **a wide variety of VEVs**

Construction of complex bases on the orbifold

Lüst, Reffert, Schulgin, Steinberger, hep-th/0506090

Coxeter element on the $SU(6) \times SU(2)$ root lattice on $\mathbb{Z}_{6-\mathrm{II}}$ orbifold

$$Q = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}, \qquad Q^6 = 1$$

Bases dz^i which satisfy with $Q^t dz^i = e^{2\pi i v^i} dz^i$, $(v^1, v^2, v^3) = \left(-\frac{1}{6}, -\frac{2}{6}, \frac{3}{6}\right)$

$$dz^1 = dx^1 + e^{-\frac{2\pi i}{6}} dx^2 + e^{-\frac{2\pi i}{3}} dx^3 - dx^4 + e^{\frac{2\pi i}{3}} dx^5,$$
$$dz^2 = dx^1 + e^{-\frac{2\pi i}{3}} dx^2 + e^{\frac{2\pi i}{3}} dx^3 + dx^4 + e^{-\frac{2\pi i}{3}} dx^5,$$
$$dz^3 = \frac{1}{2\sqrt{3}} \left[\frac{1}{3} (dx^1 - dx^2 + dx^3 - dx^4 + dx^5) + \tau\, dx^6 \right]$$

Holomorphic 3-form: $\Omega \equiv dz^1 \wedge dz^2 \wedge dz^3$,

Construction of orbits on the orbifold

Four independent orbits on $T^6/\mathbb{Z}_{6-\mathrm{II}}$ orbifold $\quad (h^{2,1}_{\mathrm{untw.}} = 1)$

$$\mathbf{1}_1 \equiv \sum \Gamma(\alpha_0)\,, \qquad \mathbf{1}_2 \equiv \sum \Gamma(\alpha_1)\,, \qquad \mathbf{1}_3 \equiv -\left(\sum \Gamma(\beta^1) + \sum \Gamma(\beta^0)\right), \qquad \mathbf{1}_1 \equiv \sum \Gamma(\beta^0)\,,$$

e.g. $\alpha_0 = dx^1 \wedge dx^3 \wedge dx^5$

Γ : orbifold twist on the cohomology basis induced by Coxeter element (Q)

$$\sum \Gamma(\alpha_0) = \alpha_0 + Q\alpha_0 + Q^2\alpha_0 + Q^3\alpha_0 + Q^4\alpha_0 + Q^5\alpha_0, \qquad (dx^j = Q^j{}_i dx^i, \qquad Q^6 = 1)$$

e.g. $Q\alpha_0 = 3\alpha_0 - \alpha_1 - \alpha_2 + \beta_3 - \delta_5 + \delta_6 + \gamma_1 - \gamma_2 - \gamma_3 + \gamma_4$

All of these are real three-forms

Intersection number of dual cycles

$$\int_{T^6} \mathbf{1}_1 \wedge \mathbf{1}_4 = 6, \qquad \int_{T^6} \mathbf{1}_2 \wedge \mathbf{1}_3 = -6$$

The distribution on VEVs on $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_4)$

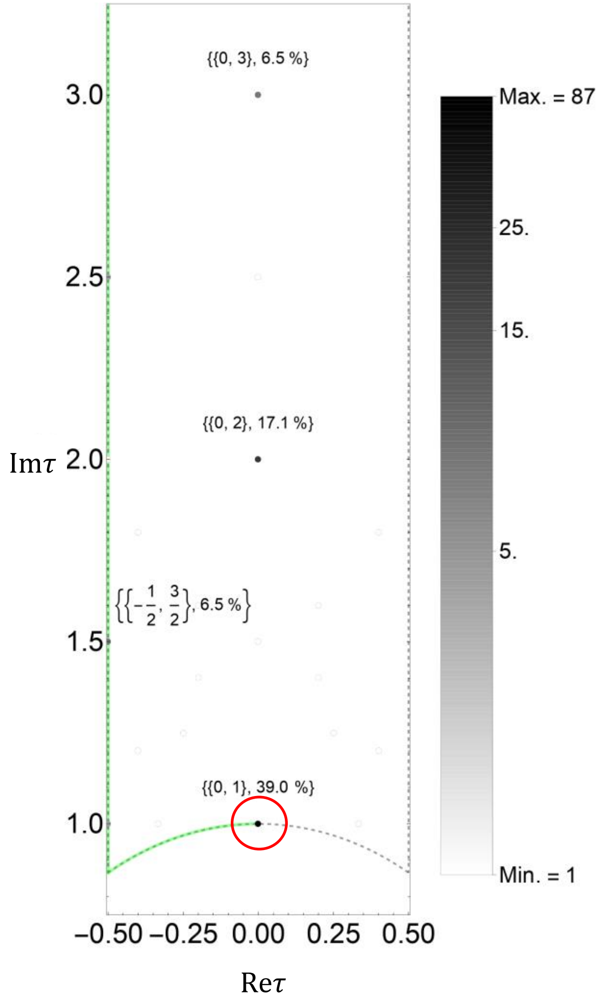
Ishiguro, TK, Kobayashi, Otsuka, 2311.12425

- **The largest number of VEVs** are clustered at the fixed point ($\mathbb{Z}_2$) associated with $\boldsymbol{SL(2,\mathbb{Z})}$

- No vacua are realized at the $\mathbb{Z}_3$ fixed point ($\tau = \omega$)

In the case of $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_4)$, the fixed points ($\mathbb{Z}_2$) is also very strong candidate for Landscape

Ratio	39.0%	17.1%	6.50%	3.25%
τ	i	$2i$	$3i$ $-\frac{1}{2}+\frac{3}{2}i$	$4i$ $-\frac{1}{2}+i$ $5i$ $-\frac{1}{2}+\frac{5}{2}i$



Moduli stabilization on $T^6/\mathbb{Z}_{6-II}$ and $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_4)$

SUSY Minkowski solutions

$$\begin{cases} D_\tau W = \partial_\tau W + K_\tau W = 0 \\ D_S W = \partial_S W + K_S W = 0 \\ W = 0 \end{cases} \Rightarrow \begin{cases} C + DS = 0 \\ B + D\tau = 0 \\ A + C\tau = 0 \end{cases},$$

Superpotential: $W = A + BS + \tau[C + DS]$

The set of (fluxes, moduli VEVs)

$$\langle S \rangle = -\frac{C}{D}, \qquad \langle \tau \rangle = -\frac{B}{D}, \qquad AD - BC = 0$$

$T^6/\mathbb{Z}_{6-\mathrm{II}}$ orientifold

$$A = -ib_0 - \frac{\sqrt{3}}{2} b_1, \qquad C = -ia^0 - \left(\sqrt{3} a^1 + \frac{1}{\sqrt{3}} a^0 \right),$$

$$B = -id_0 - \frac{\sqrt{3}}{2} d_1, \qquad D = -ic^0 - \left(\sqrt{3} c^1 + \frac{1}{\sqrt{3}} c^0 \right),$$

$T^6/(\mathbb{Z}_2 \times \mathbb{Z}_4)$ orientifold

$$A = \frac{-1+i}{2}[a^1 + (-1+i)b_2], \qquad C = \frac{-1+i}{2}[(-1+i)a^2 + (-2i)b_1],$$

$$B = -\frac{-1+i}{2}[c^1 + (-1+i)d_2], \qquad D = -\frac{-1+i}{2}[(-1+i)c^2 + (-2i)d_1],$$