Modular Invariance and the Strong CP problem

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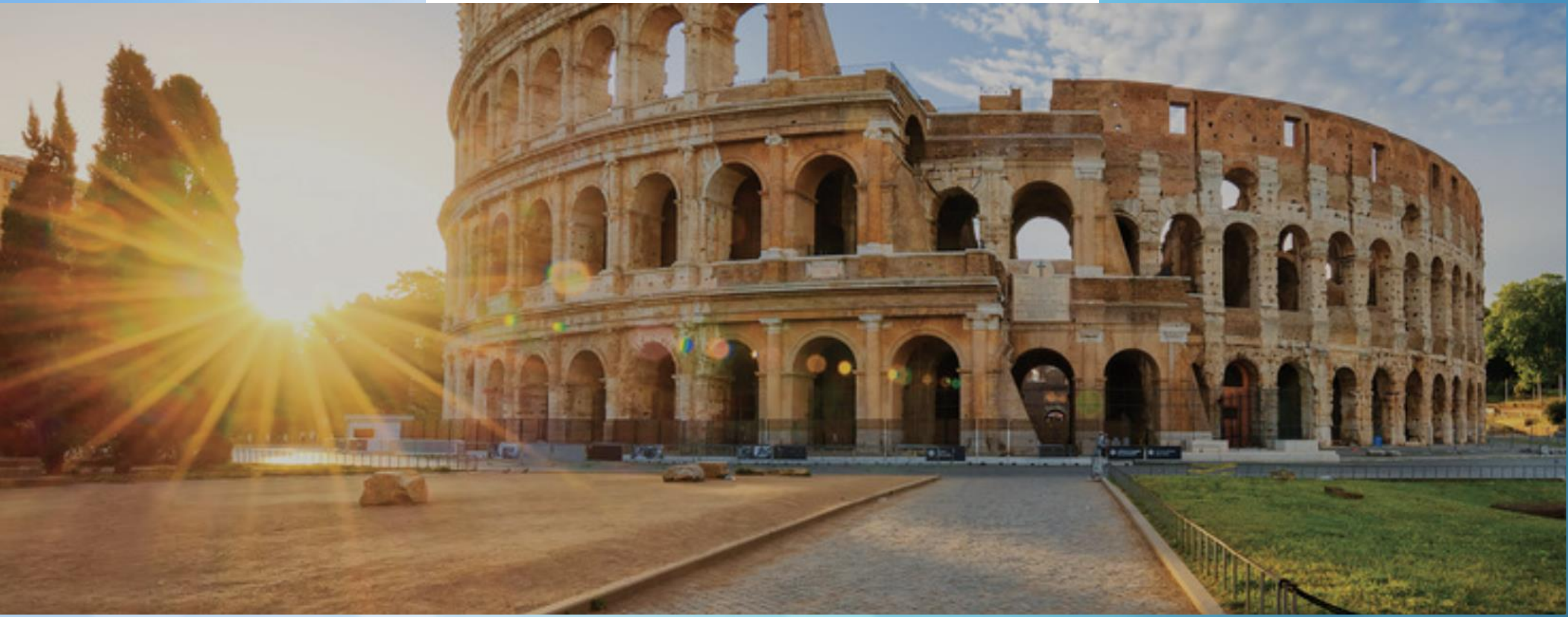

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in collaboration with:

Antonio Marrone, Alessandro Strumia and Arsenii Titov, 2505.20395

[Alessandro Strumia and Arsenii Titov, 2305.08908

Matteo Parricciatu,Alessandro Strumia and Arsenii Titov, 2406.01689

Robert Ziegler, 2411.08101]

the strong CP problem

$$\mathcal{L}_{QCD} = \bar{q}(i\not{D} - m)q - \frac{1}{4g_3^2}\mathcal{G}^a_{\mu\nu}\mathcal{G}^{a\mu\nu} + \frac{\theta_{QCD}}{32\pi^2}\mathcal{G}^a_{\mu\nu}\tilde{\mathcal{G}}^{a\mu\nu}$$

$$\bar{\theta} = \theta_{QCD} + \arg\det m \qquad d_n \approx 1.2 \times 10^{-16}\,\bar{\theta}\; e \cdot cm$$

$$|\bar{\theta}| \lesssim 10^{-10} \quad \& \quad \delta_{CKM} \approx \mathcal{O}(1)$$

Axion solution

$\bar{\theta}$ promoted to a field, the axion, pseudoGB of a global, anomalous $U(1)_{PQ}$ symmetry
VEV dynamically relaxed to zero by QCD dynamics

here: a superstring-inspired model

1. supersymmetry in the UV

2. modular invariance

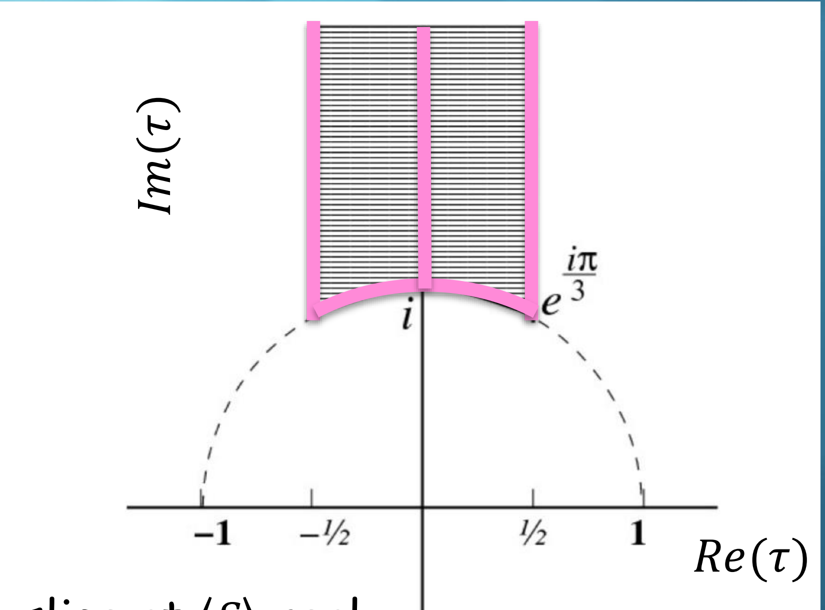
inequivalent vacua parametrized by (τ, S)

$$\tau \to \gamma\tau \equiv \frac{a\tau + b}{c\tau + d}$$

$$S \to S$$

3. CP spontaneously broken

CP violation depends on the specific vacuum chosen by the theory



slice at $\langle S \rangle$ real

supersymmetry

- rigid SUSY

gauge kinetic function

$$\mathcal{L} = \int d^2\theta d^2\bar{\theta}\, K + \left[\int d^2\theta\, w + \frac{1}{16}\int d^2\theta\, f\, WW + h.c\right]$$

kinetic terms

Yukawa couplings $\mathcal{Y}$

$$f_3 = \frac{1}{g_3^2} - i\,\frac{\theta_{QCD}}{8\pi^2}$$

not zero, field-dependent

real VEVs for $H_{u,d}$

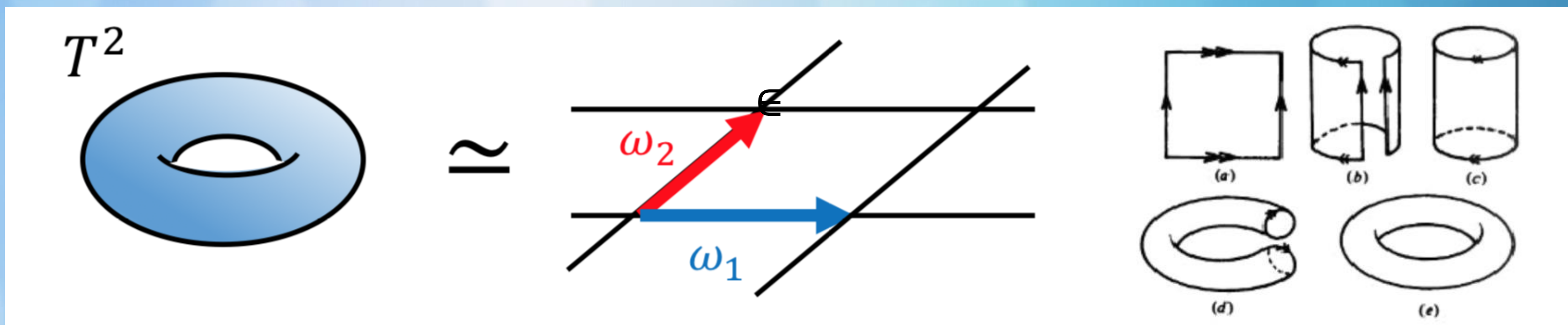
no dependence on K

- $$\bar{\theta} = \arg\left[e^{-8\pi^2 f_3} \det Y_q\right]$$

G. Hiller, M. Schmaltz, 'Solving the Strong CP Problem with Supersymmetry', Phys.Lett.B 514 (2001) 263 [arXiv:hep-ph/0105254].

modular invariance

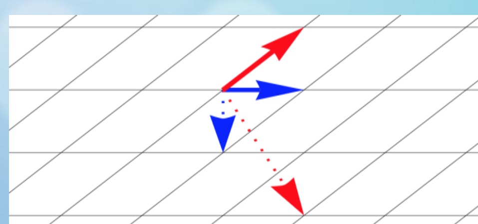
- a discrete gauge symmetry removing redundancy in parametrization of a torus



tori parametrized by

$$\mathcal{M} = \left\{ \tau = \frac{\omega_2}{\omega_1} \quad Im(\tau) > 0 \right\}$$

lattice left invariant by modular transformations:



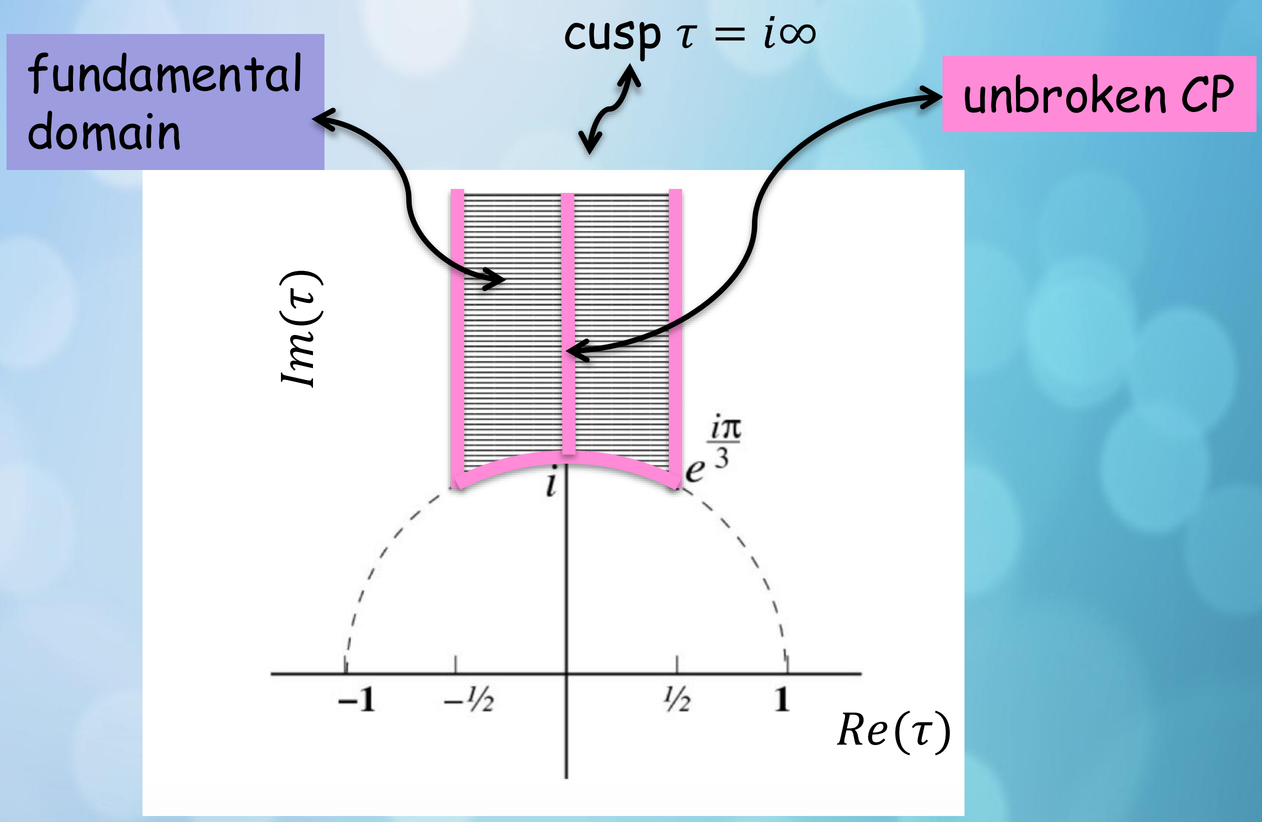
$$\tau \to \frac{a\tau + b}{c\tau + d} \in SL(2, Z)$$

a, b, c, d integers

$ad - bc = 1$

CP spontaneously broken

- τ promoted to a field. Through a gauge choice we can restrict τ to the fundamental domain



- CP

$$\tau \to -\tau^*$$

[up to modular transformations]

[Novichkov, Penedo, Petcov and Titov 1905.11970
Baur, Nilles, Trautner and Vaudrevange, 1901.03251]

Field content

$$\tau \to \gamma\tau \equiv \frac{a\tau + b}{c\tau + d}$$

$$S \to S$$

$$\varphi \to (c\,\tau + d)^{-k_\varphi}\ \varphi$$ matter multiplets

$$V \to V$$

$\bar{\theta}$ becomes field-dependent

$$\bar{\theta} = \arg A(S,\tau)$$

$$A(S,\tau) \equiv e^{-8\pi^2 f_3(S,\tau)} \det Y_q(\tau)$$

holomorphic

main idea:

holomorphic functions with too much symmetry are constants

toy-example

$A(\lambda z) = A(z) \qquad \lambda > 0$ ⇒ $A(z) =$ constant

if constant > 0, then $\bar{\theta} = 0$ (at least in the UV where SUSY is unbroken)

$A(S,\tau)$ in modular invariant theories

1. Yukawa couplings are τ-dependent modular functions

modular function of weight k_{det}

$$\det Y_q(\gamma\tau) = (c\tau + d)^{k_{\det}} \quad \det Y_q(\tau)$$

$$k_{\det} \quad = \sum_{i=1}^{3}\left(2k_{Q_i} + k_{U_i^c} + k_{D_i^c}\right) + 3k_{H_u} + 3k_{H_d}$$

2 anomaly-free theory if

$$e^{-8\pi^2 f_a(S,\gamma\tau)} = (c\tau + d)^{k_{f_a}} e^{-8\pi^2 f_a(S,\tau)}$$

$$k_{f_a} = -\sum_M 2T_a(M)k_M$$

modular function of weight k_{f_a}

$$k_{f_3} = -\sum_{i=1}^{3}\left(2k_{Q_i} + k_{U_i^c} + k_{D_i^c}\right)$$

general result

$$A(S,\tau) \equiv e^{-8\pi^2 f_3(S,\tau)} \det Y_q(\tau)$$

$$A(S,\gamma\tau) = (c\tau + d)^{k_A} A(S,\tau)$$

$$k_A = 3(k_{H_u} + k_{H_d})$$

conditions for $\bar{\theta} = 0$

1. the sum of the weights in the Higgs sector vanishes,

$$k_{H_u} + k_{H_d} = 0$$

→ $A(S, \gamma\tau) = A(S, \tau)$

2. $A(S, \tau)$ has no singularities in the closure $\bar{D}$ of the fundamental domain of $SL(2, Z)$, which includes the cusp $\tau = i\infty$.

→ $A(S, \tau)$ is τ-independent

3. τ is the only source of CP-breaking.

$$\langle Im\ S \rangle = 0$$

→ $A(S, \tau)$ is a real constant

we further assume it is positive

→ $\bar{\theta} = \arg A(S, \tau) = 0$

independently from the particular vacuum selected by the modulus τ

singularities

$$A(S,\tau) \equiv e^{-8\pi^2 f_3(S,\tau)} \det Y_q(\tau)$$

cannot be both holomorphic everywhere [have opposite weight]

- $e^{-8\pi^2 f_3(S,\tau)}$ expected to be singular at $\tau = i\infty$ [Gonzalo, Ibanez, Uranga, 1812.06520]

  Distance Conjecture: [Ooguri, Vafa 0605264]

  $\tau = i\infty$ is infinitely far away from any point in D

  explicit computation in string theory compactifications

  $$f_3(S,\tau) = k_3 S - \frac{k_{f_3}}{8\pi^2}\log\eta(\tau) + \cdots$$

- $\det Y_q(\tau)$ exhibits a zero at $\tau = i\infty$ if

$$k_{\det} = 12\,m > 0$$

$$\Rightarrow \quad \det Y_q(\tau) \propto \Delta(\tau)^m$$

discriminant form

$$q \equiv e^{i\,2\pi\tau}$$

$$\Delta(\tau) = q\prod_{n=1}^{\infty}(1-q^n)^{24}$$

Example of $Y_q(\tau)$

Milne, S.C. (2001). Hankel Determinants of Eisenstein Series. In: Garvan, F.G., Ismail, M.E.H. (eds) Symbolic Computation, Number Theory, Special Functions, Physics and Combinatorics. Developments in Mathematics, vol 4. Springer, Boston, MA. https://doi.org/10.1007/978-1-4613-0257-5_10

- $k_{Q_i} = k_{U_i^c} = k_{D_i^c} = (2,4,6)$

$$Y_{u,d}(\tau) = \begin{pmatrix} E_4 & E_6 & E_8 \\ E_6 & E_8 & E_{10} \\ E_8 & E_{10} & E_{12} \end{pmatrix} \Rightarrow \det Y_{u,d}(\tau) \propto \Delta(\tau)^2$$

$$E_{2k} \equiv \sum_{m\neq 0, n\neq 0} \frac{1}{(m+\tau n)^{2k}} \qquad (k>1)$$

- in the basis where kinetic terms are canonical

$$Y^q_{\rm can} = \begin{pmatrix} c_{Q_1^c} c_{Q_1} y^2 E_4 & c_{Q_1^c} c_{Q_2} y^3 E_6 & c_{Q_1^c} c_{Q_3} y^4 E_8 \\ c_{Q_2^c} c_{Q_1} y^3 E_6 & c_{Q_2^c} c_{Q_2} y^4 E_8 & c_{Q_2^c} c_{Q_3} y^5 E_{10} \\ c_{Q_3^c} c_{Q_1} y^4 E_8 & c_{Q_3^c} c_{Q_2} y^5 E_{10} & c_{Q_3^c} c_{Q_3} y^6 E_{12} \end{pmatrix}$$

$$K = \sum_{i=1}^{3} \left[c_{Q_i}^{-2} y^{-k_{Q_i}} |Q_i|^2 + c_{U_i^c}^{-2} y^{-k_{U_i^c}} {U_i^c}^2 + c_{D_i^c}^{-2} y^{-k_{D_i^c}} {D_i^c}^2 \right] \qquad y \equiv 2\, Im\, \tau$$

Observable	Central value $\pm 1\sigma$
m_u/m_c	$(1.93 \pm 0.60) \times 10^{-3}$
m_c/m_t	$(2.82 \pm 0.12) \times 10^{-3}$
m_d/m_s	$(5.05 \pm 0.62) \times 10^{-2}$
m_s/m_b	$(1.82 \pm 0.10) \times 10^{-2}$
m_t/GeV	87.5 ± 2.1
m_b/GeV	0.97 ± 0.01

Observable	Central value $\pm 1\sigma$
$\sin^2\theta_{12}$	$(5.08 \pm 0.03) \times 10^{-2}$
$\sin^2\theta_{13}$	$(1.22 \pm 0.09) \times 10^{-5}$
$\sin^2\theta_{23}$	$(1.61 \pm 0.05) \times 10^{-3}$
$\delta_{\mathrm{CKM}}/\pi$	0.385 ± 0.017

Table 1: *Values of quark masses and mixings renormalized around* $2\ 10^{16}$ GeV, *assuming the supersymmetry breaking scale* $M_{\mathrm{SUSY}} = 10\,\mathrm{TeV}$ *and* $\tan\beta = 10$ *[7].*

best fit values

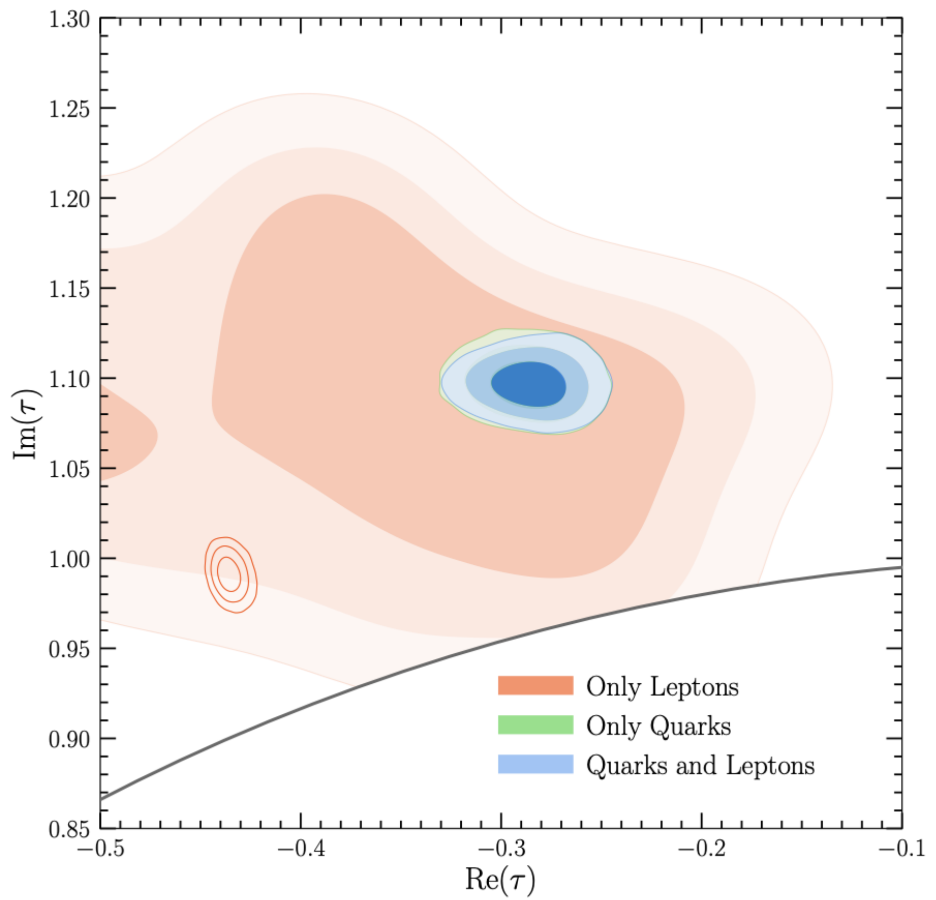
8 parameters + τ

$$\tau = -0.286 + 1.096i$$

$$q_{13} = 0.037\,,\ q_{23} = 0.075\,,\ u_{13} = 0.035\,,\ u_{23} = 19.98\,,\ d_{13} = 3.44\,,\ d_{23} = 0.203\,.$$

$$c_{U_3^c} c_{Q_3} = 2.40\ 10^{-6} \qquad \text{and} \qquad c_{D_3^c} c_{Q_3} = 2.75\ 10^{-5}$$

$$q_{i3} \equiv C_{Q_i}/C_{Q_3} \qquad u_{i3} \equiv C_{U_i^c}/C_{U_3^c} \qquad d_{i3} \equiv C_{D_i^c}/C_{D_3^c} \qquad i = 1,2$$



leptons require 6+2=8 more parameters

normal ordering

$$m_1 \approx 10\ \text{meV}\,, \qquad m_{\beta\beta} \approx 6\ \text{meV}\,, \qquad \sum_{i=1}^{3} m_i \approx 74\ \text{meV}\,,$$

$$\delta_{\text{PMNS}} \approx 0.94\,\pi\,, \qquad \alpha_{21} \approx 1.29\,\pi\,, \qquad \alpha_{31} \approx 0.14\,\pi\,.$$

deviations from $\bar{\theta} = 0$

SUSY unbroken

no corrections from K
no corrections from nonrenormalizable operators: $SL(2,\mathbb{Z})$

SUSY breaking corrections

potentially big if soft terms violate flavour in a generic way

minimized if $\Lambda_{CP} \gg \Lambda_{SUSY}$ (as e.g. in gauge mediation)

and soft breaking terms respect the flavour structure of the SM

$$\bar{\theta} \lesssim \frac{M_t^4 M_b^4 M_c^2 M_s^2}{v^{12}} J_{\mathrm{CP}} \tan^6\beta \sim 10^{-28} \tan^6\beta.$$

J.R. Ellis, M.K. Gaillard, 'Strong and Weak CP Violation', Nucl.Phys.B 150 (1979) 141.

SM corrections

negligible: $\bar{\theta} \leq 10^{-18}$ at four loops

I.B. Khriplovich, 'Quark Electric Dipole Moment and Induced θ Term in the Kobayashi-Maskawa Model', Phys.Lett.B 173 (1986) 193.

to recap

- in a SUSY & CP & modular-invariant theory:
  τ can generate a large CKM phase without contributing to $\bar{\theta}$

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- in a complete theory, the VEVs of S and τ should be determined dynamically

  here $\langle S \rangle$ is real by assumption and $\langle \tau \rangle$ is the result of a fit

to recap

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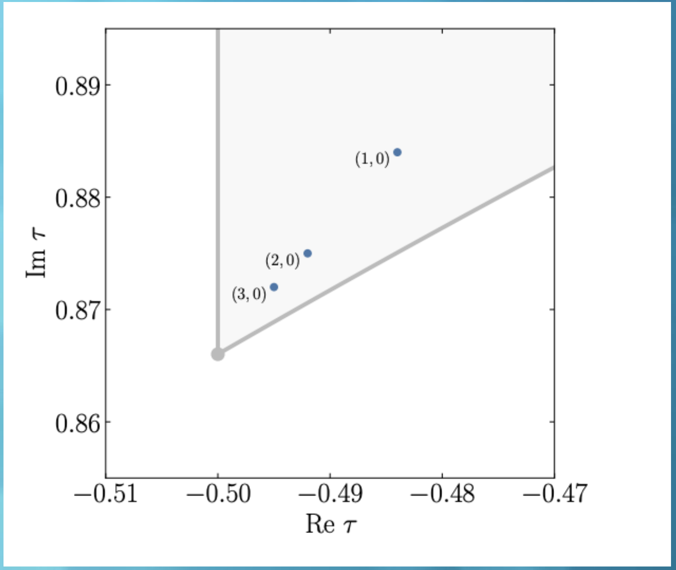
  here $\langle S \rangle$ is real by assumption and $\langle \tau \rangle$ is the result of a fit

- evidence for modular-invariant
  potentials where τ spontaneously breaks CP

  [Novichkov, Penedo, Petcov 2201.02020
  Leedom, Righi, Westphal 2212.03876]



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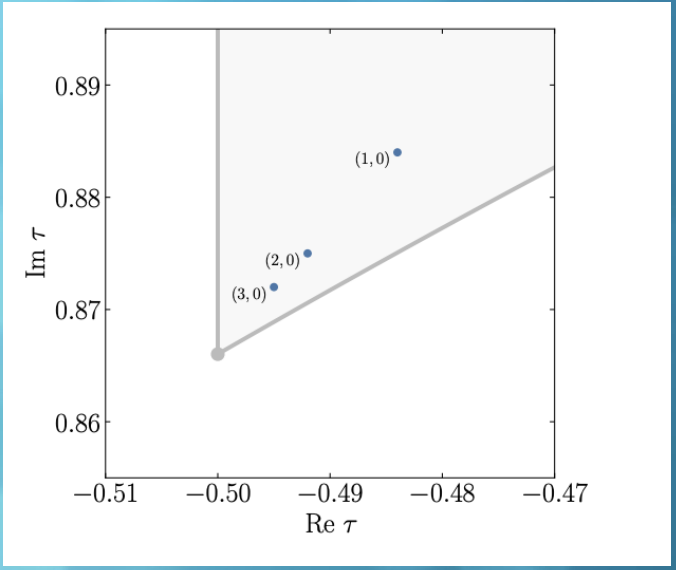


- options for $\langle Im\ S \rangle = 0$

  - dynamics dominated by QCD → axion

  - S stabilized at a CP-conserving minimum by Planck-scale dynamics
    if so, $\bar{\theta} = 0$ not altered by flavour physics.

    [Higaki, Kobayashi, Nasu and Otsuka 2405.18813]

THANK YOU!

back-up slides

transformation of $f_a(S,\tau)$

$$f_a(S,\tau) \to f_a(S,\gamma\tau) - \frac{1}{8\pi^2}\sum_M 2T_a(M)k_M \log(c\tau + d)$$

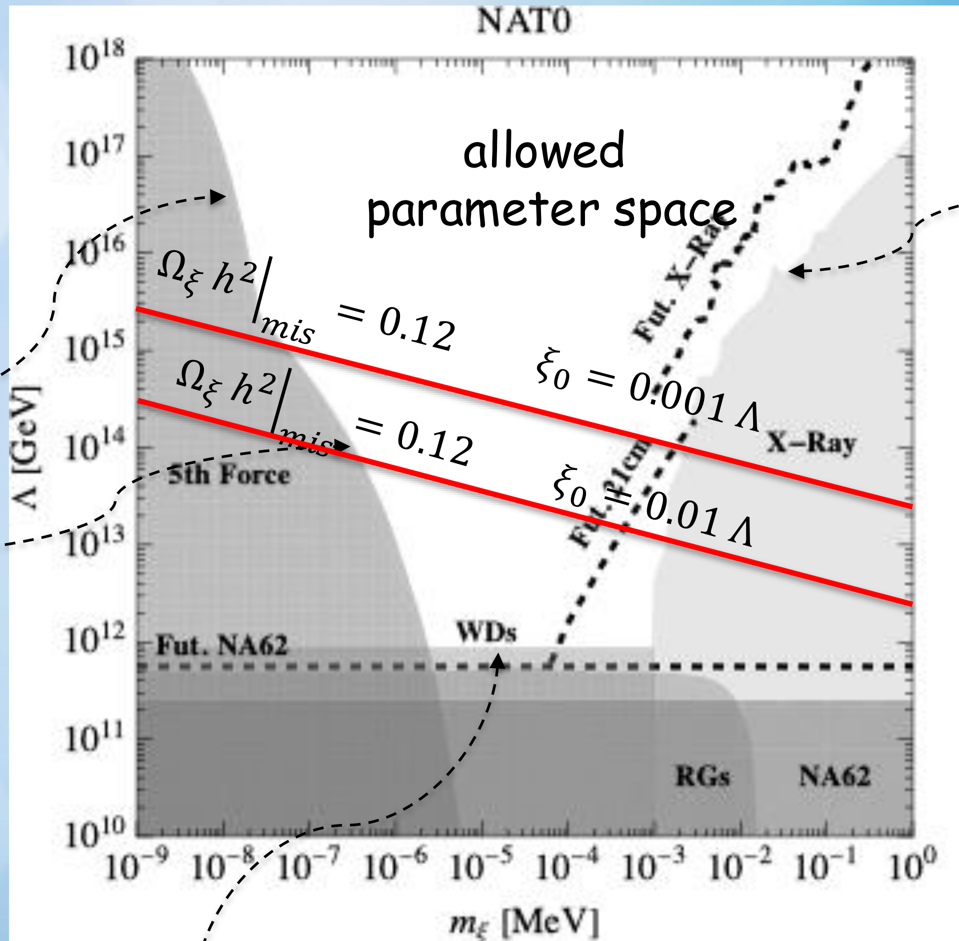
intrinsic
τ-dependence

anomaly

[Konishi, Shizuya 1985;
Ferrara, Kounnas, Lust, Zwirner, 1991;
Dixon, Kaplunovsky, Louis, 1991;
N. Arkani-Hamed, H. Murayama 9707133]

a light spin-0 component in τ ?

FF, Robert Ziegler, 2411.0810]1



limits from Inverse Square Law of gravity

X-rays diffuse emissions from DM decay in galaxy clusters

stellar energy loss in Red Giants

What can solve the Strong CP problem? #1

David E. Kaplan (Johns Hopkins U.), Tom Melia (Tokyo U., IPMU), Surjeet Rajendran (Johns Hopkins U.) (May 13, 2025)

e-Print: 2505.08358 [hep-ph]

- θ_{QCD} is not a Lagrangian parameter as a mass, a coupling, …
  it is a variable that labels a vacuum

  strong CP problem = a problem of vacuum selection: why do experience $\bar{\theta} = 0$, if the universe started with a generic θ_{QCD}?

- cannot be solved by

$$\bar{\theta} = \theta_{QCD} + \arg\det m$$

setting this to zero by CP

the vacuum can violate CP, even in a CP-invariant theory

- the only viable solution to the strong CP problem is the axion
  where $\bar{\theta} = 0$ from dynamics

What can solve the Strong CP problem?

#1

David E. Kaplan (Johns Hopkins U.), Tom Melia (Tokyo U., IPMU), Surjeet Rajendran (Johns Hopkins U.) (May 13, 2025)

e-Print: 2505.08358 [hep-ph]

- in our framework CP is conserved but $\bar{\theta}$ is a dynamical variable
  i.e. we do not set $\theta_{QCD} = 0$ by CP invariance

$$\bar{\theta} = \arg[e^{-8\pi^2 f_3(S,\tau)} \det Y_q(\tau)]$$

CP invariance ➡ $$f_3(S^*,\tau^*) = f_3^*(S,\tau)$$

θ_{QCD} is inside S ➡ $$f_3(S,\tau) = S + \cdots$$

axion solution

- $\bar{\theta}$ dynamically relaxed to zero by the axion, would-be GB of a global, anomalous $U(1)_{PQ}$ symmetry

- provides a candidate for DM
- many axion candidates in e.g. superstring theories

- axion quality problem

minimum of $V(a)$ should be at $a = 0$

$$V(a) = V_{QCD}(a) - M^4 e^{-S} \cos(\frac{a}{f_a} + \delta)$$

$M = M_P$
$\delta = \mathcal{O}(1)$

$\Rightarrow$ $S \geq 200$

- axion undetected, so far

Nelson-Barr solution | our solution

CP ia a symmetry of the UV,
SB to get $\bar{\theta} = 0$ & $\delta_{CKM} = \mathcal{O}(1)$

CP $\Rightarrow$ $\theta_{QCD} = 0$

Nelson-Barr solution

- heavy vector-like quark sector

$$\begin{array}{cc} Q & q \end{array}$$

$$m = \begin{pmatrix} \mu & \lambda_a \eta_a \\ 0 & y\, v \end{pmatrix}$$

- CP spontaneously broken by $\langle \eta_a \rangle$ complex
  [one is not enough]

- $\mu \approx \lambda_a \eta_a$ [tuning]

our solution

- no extra matter
- CP spontaneously broken by τ alone
- no tuning

$\mathcal{N} = 1$ supergravity

- K and w no more independent

$$\mathcal{G} = \frac{K}{M_{Pl}^2} + \log\left|\frac{w}{M_{Pl}^3}\right|^2$$

$$K = -k_W M_{Pl}^2 \log(-i\tau + i\tau^+) + \cdots$$

$$w(\tau) \to (c\tau + d)^{-k_W}\, w(\tau)$$

- $$\bar{\theta} = \arg A \qquad \text{where now} \qquad A = e^{-8\pi^2 f_3} W^{-C_3} \det Y_u \det Y_d\,.$$

$[\arg M_3 = -\arg W]$

- $$k_{f_a} + \sum_M 2T_a(M) k_M + k_W \left[C_a - \sum_M T_a(M)\right] = 0.$$

$$A(S, \gamma\tau) = (c\tau + d)^{k_A}\, A(S, \tau)$$

$$k_A = 3(k_{H_u} + k_{H_d})$$

gauge coupling unification

$$f_a = \frac{1}{g_a^2} - i\,\frac{\theta_{QCD}}{8\pi^2}$$

holomorphic coupling ≠ physical

$$K_S = -\bar{M}_{\rm Pl}^2 \ln(S+\bar{S})\,, \qquad f_a = \kappa_a S - \frac{k_{f_a}}{8\pi^2}\ln\eta^2(\tau)$$

$$\frac{16\pi^2}{g_a^{c2}(\mu)} = 16\pi^2\kappa_a {\rm Re}\,S + 2C_a \ln\frac{\kappa_a}{2} + b_a \ln\frac{\Lambda^2}{2{\rm Re}\,S\,\mu^2} + \Delta_a(\tau,\bar{\tau})$$

unification condition

1-loop running

threshold corrections

$$\Delta_a(\tau,\bar{\tau}) \equiv -k_{f_a}\ln y|\eta(\tau)|^4$$

dependence on: SUSY-breaking scale, sparticle spectrum, k_a levels, …

modular forms

$$f(\gamma\tau) = (c\tau + d)^k\, f(\tau)$$

& $f(\tau)$ holomorphic everywhere included at $\tau = i\,\infty$

$k < 0$: no modular forms

$k = 0$: modular forms are constants

$k > 0$: modular forms polynomials in $E_4(\tau), E_6(\tau)$

Modular weight k	0	2	4	6	8	10	12	14
Number of forms	1	0	1	1	1	1	2	1
Modular forms	1	–	E_4	E_6	$E_8 = E_4^2$	$E_{10} = E_4 E_6$	E_4^3, E_6^2	$E_{14} = E_4^2 E_6$

variants

Solving the strong CP problem without axions #1

Ferruccio Feruglio (INFN, Padua), Matteo Parriciatu (INFN, Rome and Rome III U.), Alessandro Strumia (Pisa U.), Arsenii Titov (Pisa U.) (Jun 3, 2024)

e-Print: 2406.01689 [hep-ph]

- higher levels, smaller weight

modular forms associated with subgroups of $SL(2,Z)$ → $k_{Q_i} = k_{U_i^c} = k_{D_i^c} = (-1,0+1)\ or\ (-2,0+2)$

perhaps easier to occur in string theory

- With heavy vector-like quarks

anomaly of IR theory canceled by a nontrivial gauge kinetic function

$$f_{IR} = f_{UV} - \frac{1}{8\pi^2}\log\det Y_{Heavy}(\tau)$$

many more viable patterns of quark mass matrices

- can be extended to supergravity

Modular invariance and the QCD angle #3

Ferruccio Feruglio (INFN, Padua), Alessandro Strumia (Pisa U.), Arsenii Titov (Pisa U.) (May 15, 2023)

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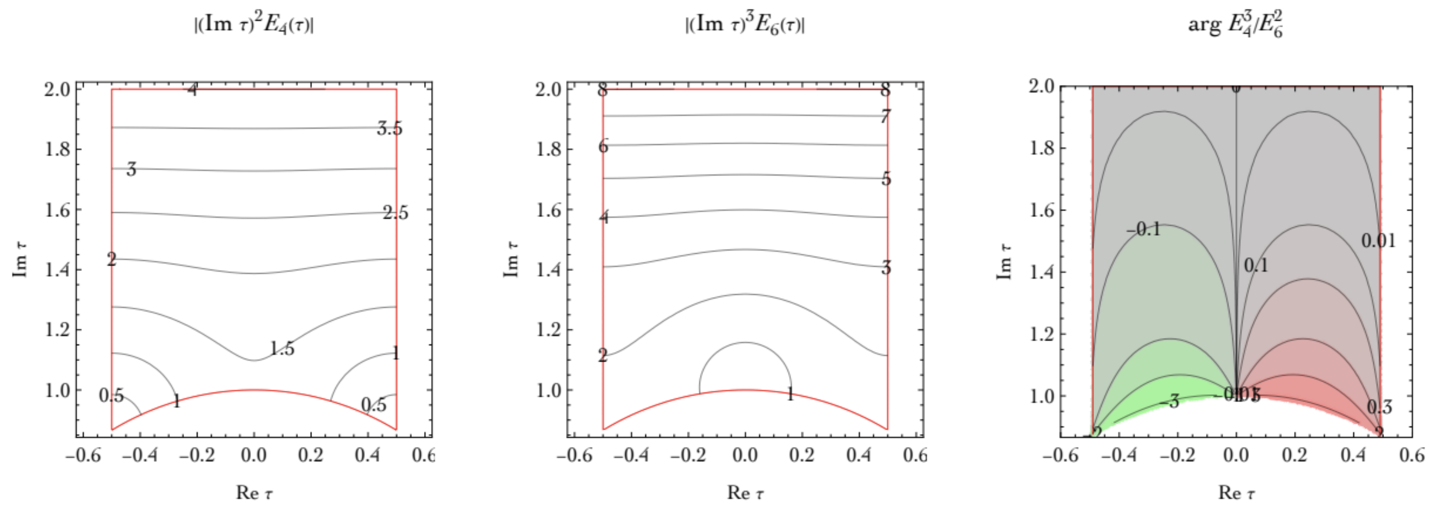

$|(\mathrm{Im}\ \tau)^2 E_4(\tau)|$
$|(\mathrm{Im}\ \tau)^3 E_6(\tau)|$
$\arg E_4^3/E_6^2$
Im τ
Re τ

Ingredients

1. CP in the UV
2. Yukawa couplings are field-dependent quantities
3. the vacuum has a redundant description: vacua related by $SL(2,\mathbb{Z})$ are equivalent
4. CP and $SL(2,\mathbb{Z})$ are unified in a gauge flavour symmetry
5. absence of anomalies
6. singularities in the EFT

Ingredients

String Theory

1. CP in the UV

the four-dimensional CP symmetry is a gauge symmetry in most string theory compactifications.

2. Yukawa couplings are field-dependent quantities

string theory has no free parameters and couplings are set by moduli VEVs

3. the vacuum has a redundant description: vacua related by $SL(2,\mathbb{Z})$ are equivalent

modular invariance is a key ingredient of string theory compactifications

4. CP and $SL(2,\mathbb{Z})$ are unified in a gauge flavour symmetry

Unification of Flavor, CP, and Modular Symmetries

Alexander Baur (Munich, Tech. U.), Hans Peter Nilles (Bonn U. and Bonn U., HISKP and Munic
Trautner (Heidelberg, Max Planck Inst.), Patrick K.S. Vaudrevange (Munich, Tech. U.) (Jan 10

Published in: *Phys.Lett.B* 795 (2019) 7-14 • e-Print: 1901.03251 [hep-th]

5. absence of anomalies

mandatory in string theory

6. singularities in the EFT

emergence of singularities at infinite distances in moduli space.

to recap

- in a SUSY & CP & modular-invariant theory:
  τ can generate a large CKM phase without contributing to $\bar{\theta}$

e.g. $f_3(S,\tau) = k_3 S - \frac{k_{f_3}}{8\pi^2}\log\eta(\tau) + \cdots$

$$\frac{k_3}{16}\int d^2\theta\, S\, W_3 W_3 \underbrace{- \frac{k_{f_3}}{16}\int d^2\theta\, \frac{\log\eta(\tau)}{8\pi^2} W_3 W_3 + \int d^2\theta\, w(\tau) + h.c}_{\delta\bar{\theta} = 0}$$