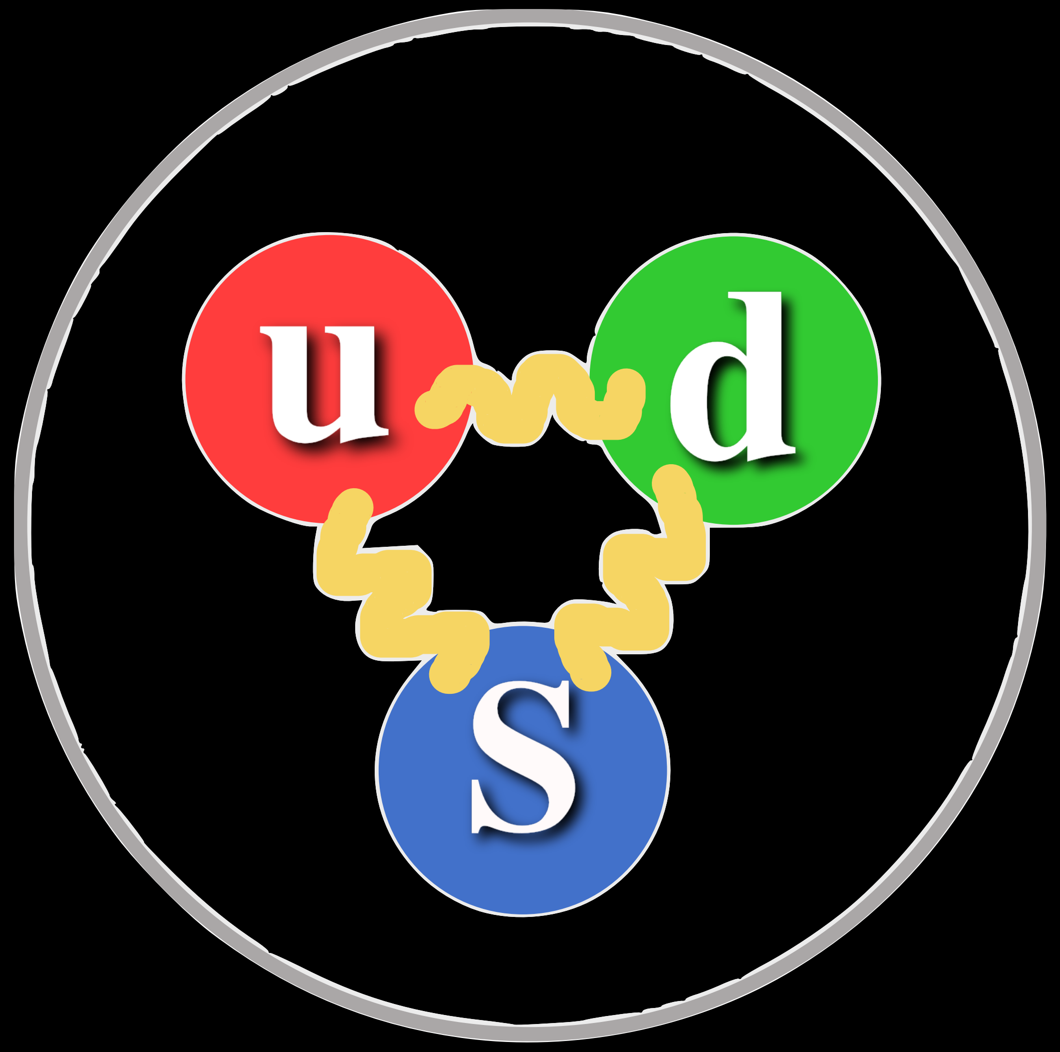
Physical Riemann surfaces of the Λ baryon form factors ratio

Francesco Rosini
Simone Pacetti

06/06/2024



Baryon - photon vertex

Given a baryon $\mathscr{B}$, the electromagnetic current is

$$\langle P_i | J^\mu_{\text{EM}}(0) | P_f \rangle = e\bar{u}(p_f)\left[\gamma^\mu F_1^{\mathscr{B}}(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2M_{\mathscr{B}}}F_2^{\mathscr{B}}(q^2)\right]u(p_i)$$

$F_1^{\mathscr{B}}(q^2)$ and $F_2^{\mathscr{B}}(q^2)$ are the Dirac and Pauli form factors

$$F_1^{\mathscr{B}}(0) = Q_{\mathscr{B}}$$

$Q_{\mathscr{B}}$ is the electric charge

$$F_2^{\mathscr{B}}(0) = \kappa_{\mathscr{B}}$$

$Q_{\mathscr{B}}$ is the anomalous magnetic moment

Breit frame

$$(p_f - p_i)^\mu = q^\mu = (0, \vec{q})$$

Sachs form factors

$$G_E^{\mathscr{B}}(q^2) = F_1^{\mathscr{B}}(q^2) + \frac{q^2}{4M_{\mathscr{B}}^2}F_2^{\mathscr{B}}$$

$$G_M^{\mathscr{B}}(q^2) = F_1^{\mathscr{B}}(q^2) + F_2^{\mathscr{B}}$$

$$G_E^{\mathscr{B}}(0) = Q_{\mathscr{B}} \qquad G_M^{\mathscr{B}}(0) = Q_{\mathscr{B}} + \kappa_{\mathscr{B}} = \mu_{\mathscr{B}}$$

$\mu_{\mathscr{B}}$ is the total magnetic moment

Cross section

Scattering cross section

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2 E'_e \cos^2(\theta/2)}{4E_e^3 \sin^4(\theta/2)} \left[\left(G_E^{\mathscr{B}}\right)^2 - \tau \left(1 + 2(1-\tau)\tan^2(\theta/2)\right) \left(G_M^{\mathscr{B}}\right)^2 \right] \frac{1}{1-\tau}$$

Annihilation cross section

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2 \beta \mathscr{C}}{16E^2} \left[\left(1 + cos^2(\theta)\right) \left|G_E^{\mathscr{B}}\right|^2 + \frac{1}{\tau} \sin^2(\theta) \left|G_E^{\mathscr{B}}\right|^2 \right]$$

Coulomb correction

$$\mathscr{C} = \frac{\pi\alpha\beta}{1 - e^{-\pi\alpha\beta}}$$

$\mathscr{C}$ is a final state interaction effect

Asymptotic behaviour

The asymptotic form factors behaviour is given in pQCD by counting rules as $q^2 \to -\infty$

Helicity conservation

- $J^{\lambda,\lambda}(q^2) \propto G_M^{\mathscr{B}}(q^2)$
- 2 gluon propagators distributing the momentum transfer of the virtual photon
- $G_M^{\mathscr{B}}(q^2) \sim (q^2)^{-2}$

Helicity flip

- $J^{\lambda,-\lambda}(q^2) \propto G_E^{\mathscr{B}}(q^2)/\sqrt{-q^2}$
- [2 gluon propagators] / $\sqrt{-q^2}$
- $G_E^{\mathscr{B}}(q^2) \sim (q^2)^{-2}$

Dirac and Pauli Form Factors

$$F_1^{\mathscr{B}} \underset{q^2\to-\infty}{\sim} (q^2)^{-2}$$

$$F_2^{\mathscr{B}} \underset{q^2\to-\infty}{\sim} (q^2)^{-3}$$

Sachs Form Factor Ratio

$$\frac{G_E^{\mathscr{B}}(q^2)}{G_M^{\mathscr{B}}(q^2)} \underset{q^2\to-\infty}{\sim} \text{constant}$$

Form factors in the time-like region

In the time-like region, $G_E^{\mathscr{B}}(q^2)$ and $G_M^{\mathscr{B}}(q^2)$ are complex functions

Crossing symmetry: $\left\langle P(p') \middle| J^\mu \middle| P(p) \right\rangle \rightarrow \left\langle \bar{P}(p')P(p) \middle| J^\mu \middle| 0 \right\rangle$

Optical theorem

$$\text{Im}\left(\left\langle \bar{P}(p')P(p) \middle| J^\mu \middle| 0 \right\rangle\right) \approx \sum_n \left\langle \bar{P}(p')P(p) \middle| J^\mu \middle| n \right\rangle \left\langle n \middle| J^\mu \middle| 0 \right\rangle \Rightarrow \begin{cases} \text{Im}\left(F_{1,2}^{\mathscr{B}}\right) \neq 0 \\ \text{for } q^2 > 4M_\pi^2 \end{cases}$$

Where $|n\rangle$ are intermediate states, i.e. $|n\rangle = 2\pi, 3\pi, \ldots$

Phragmén Lindelöf theorem

If $f(z) \rightarrow f_1$ as $z \rightarrow \infty$ along the straight line L_1
and $f(z) \rightarrow f_2$ as $z \rightarrow \infty$ along the straight line L_2,
and $f(z)$ is regular and bounded in the angle
between the lines, then $f_1 \equiv f_2 = f_{12}$ and
$f(z) \rightarrow f_{12}$ in the region between L_1 and L_2

Asymptotic behaviour in the time-like region

$$\lim_{q^2 \to +\infty} G_M^{\mathscr{B}}(q^2) = \lim_{q^2 \to -\infty} G_M^{\mathscr{B}}(q^2)$$

Analyticity of form factors

Spacelike region	Unphysical region	Timelike region
$q^2 < 0$	$q^2_{th} < q^2 \leq q^2_{phys}$	$q^2 > q^2_{phys}$
$e\mathscr{B} \to e\mathscr{B}$	$\mathscr{B}\bar{\mathscr{B}} \to e^+e^-\mathscr{M}_0$	$e^+e^- \leftrightarrow \mathscr{B}\bar{\mathscr{B}}$
$G_E^{\mathscr{B}}(q^2), G_M^{\mathscr{B}}(q^2)$	$\left\lvert G_E^{\mathscr{B}}(q^2) \right\rvert, \left\lvert G_M^{\mathscr{B}}(q^2) \right\rvert$	$\begin{cases} \left\lvert G_E^{\mathscr{B}}(q^2) \right\rvert, \left\lvert G_M^{\mathscr{B}}(q^2) \right\rvert \\ \arg\left(G_E^{\mathscr{B}}/G_M^{\mathscr{B}}\right)^*, \end{cases}$

* Sine of the argument measurable in polarized cross section only

Λ Form Factors

Theoretical threshold

$$q_{th}^2 = \left(2M_\pi + M_{\pi^0}\right)^2$$

$I(\Lambda\bar{\Lambda}) = 0$, and the lightest isoscalar hadronic state is $\pi^+\pi^-\pi^0$

Physical threshold

$$q_{phys}^2 = \left(2M_\Lambda\right)^2$$

Lowest center of mass energy to produce a $\Lambda\bar{\Lambda}$ couple

- Unphysical and space-like regions have no data
- The relative phase is measured through the weak decay $\Lambda \to p\pi^-$, $\bar{\Lambda} \to \bar{p}\pi^+$

- Form factors have nonzero imaginary parts for $q^2 \geq q_{\text{th}}^2$
- $G_E^\Lambda(q^2)$ vanishes for $q^2 = 0$

Dispersion relations

The form factors $G_{E,M}^{\Lambda}$ are analytic functions on the q^2-complex plane with a cut $\left(q_{\text{th}}^2, \infty\right)$ on the real axis.

Dispersion relations are based only on unitarity and analyticity ⇒ **model independent approach**

Dispersion relation for the imaginary part ($q^2 < 0$):

$$G(q^2) = \frac{1}{\pi}\int_{q_{\text{th}}^2}^{\infty} \frac{\text{Im}(G(s))}{s - q^2} ds$$

Dispersion relation for the logarithm ($q^2 < 0$):

$$\ln\left(G(q^2)\right) = \frac{\sqrt{q_{\text{th}}^2 - q^2}}{\pi}\int_{q_{\text{th}}^2}^{\infty} \frac{\ln\left|G(s)\right|}{\left(s - q^2\right)\sqrt{s - q_{\text{th}}^2}} ds$$

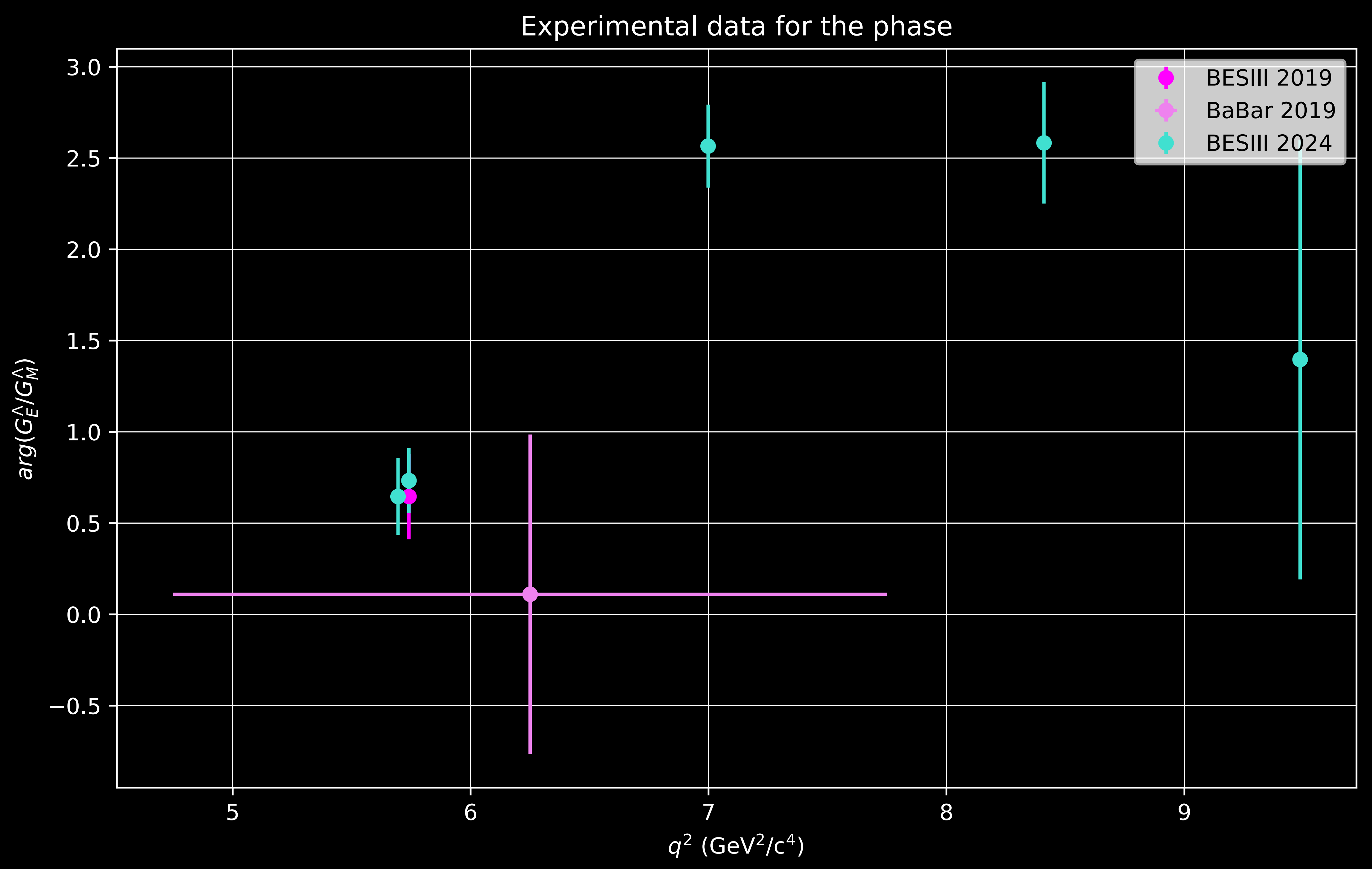
Experimental Inputs

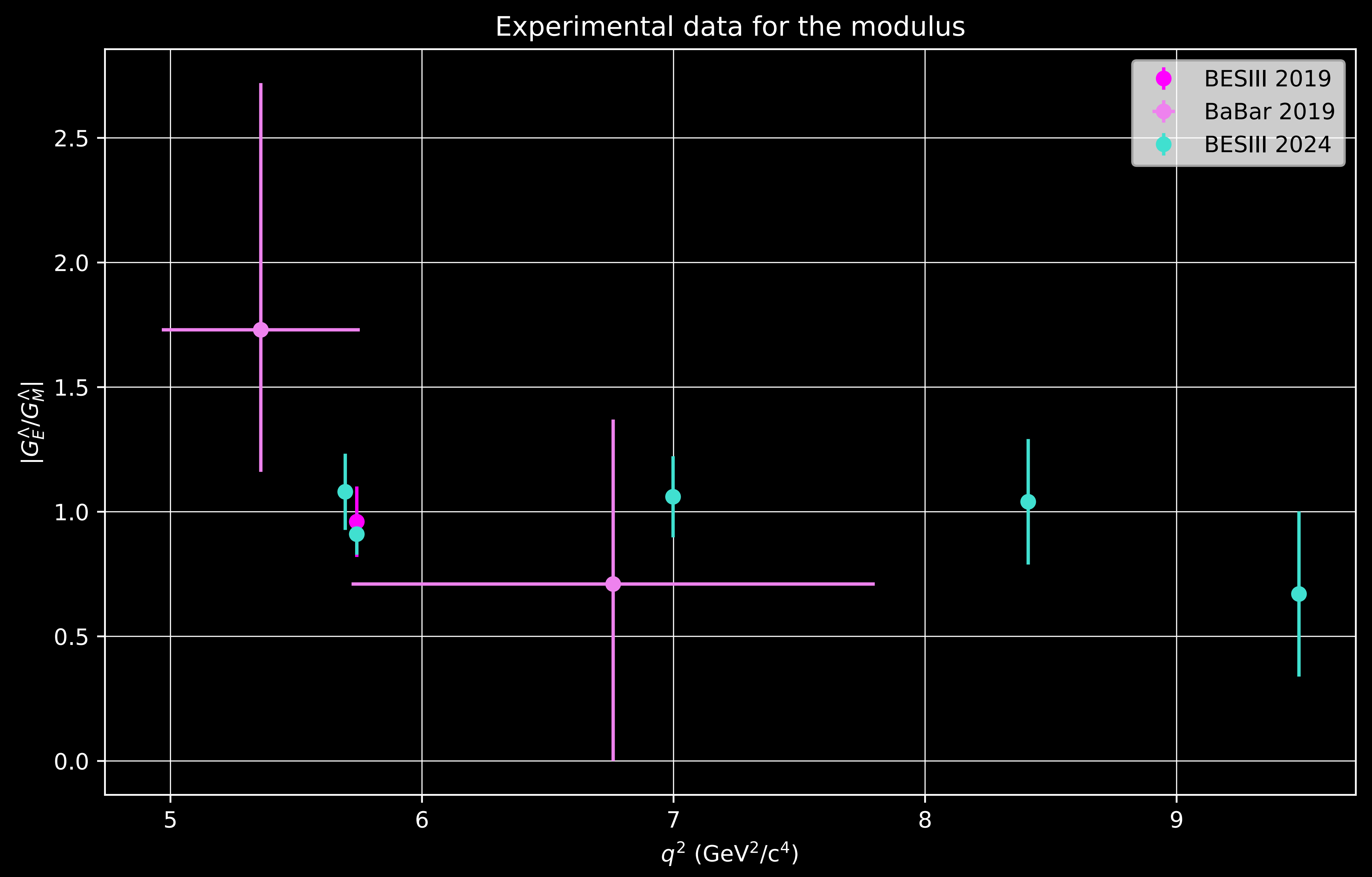
- Time-like data for form factor's moduli from $e^+e^- \leftrightarrow \mathcal{B}\bar{\mathcal{B}}$
- Time-like data for the relative phase from $e^+e^- \leftrightarrow \mathcal{B}^{\uparrow}\bar{\mathcal{B}}$

Theoretical Inputs

- Analyticity
- Threshold values
- Asymptotic behaviour

Data for modulus and phase of $G_E^\Lambda / G_M^\Lambda$

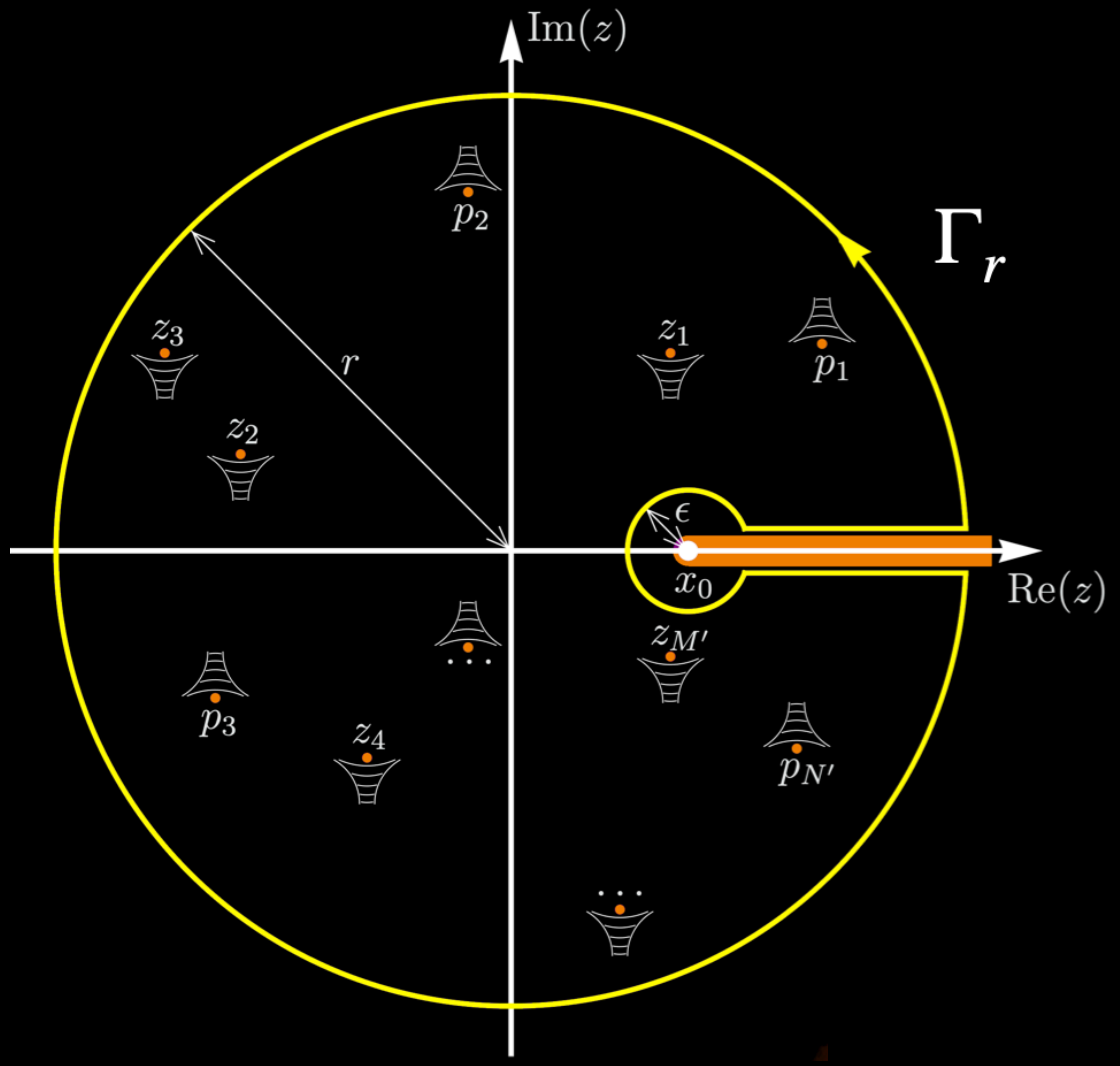




- Sine of the relative phase accessible through polarization
- No hints on the determination of the relative phase

$$\mathscr{P}_y = -\frac{2M_\Lambda\sqrt{q^2}\,\sin(2\theta)\left|G_E^\Lambda/G_M^\Lambda\right|\,\sin\left(\arg(G_E^\Lambda/G_M^\Lambda)\right)}{q^2\left(1+\cos^2(\theta)\right)+4M_\Lambda^2\left|G_E^\Lambda/G_M^\Lambda\right|\sin^2(\theta)}$$

The meaning of the phase



- Consider the complex function $R(z)$ with N poles $\{p_j\}_{j=1}^{N}$ and M zeroes $\{z_k\}_{k=1}^{M}$ and a branch cut $\left(x_0, \infty\right)$

- Taking the integral over the contour Γ_r gives the Cauchy's argument principle

$$\lim_{r\to\infty} \frac{1}{2i\pi} \oint_{\Gamma_r} \frac{d \ln\left(R(z)\right)}{dz} dz = M - N$$

- By taking each contribution into account

$$\lim_{r\to\infty} \frac{1}{2i\pi} \oint_{\Gamma_r} \frac{d \ln\left(R(z)\right)}{dz} dz = \frac{1}{\pi} \left(\arg(R(\infty)) - \arg(R(x_0))\right)$$

$$\left(\arg(R(\infty)) - \arg(R(x_0))\right) = \pi\,(M - N)$$

Levinson's Theorem

Dispersive Procedure

We define the ratio $R(q^2) = \dfrac{G_E^\Lambda(q^2)}{G_M^\Lambda(q^2)} \Rightarrow \begin{cases} G_E^\Lambda(0) = 0 \\ G_E^\Lambda(q^2_{\text{phy}}) = G_E^\Lambda(q^2_{\text{phy}}) \end{cases} \Rightarrow \begin{cases} R(0) = 0 \\ R(q^2_{\text{phy}}) = 1 \end{cases}$

The asymptotic behaviour

$$\lim_{q^2 \to \pm\infty} R(q^2) = \frac{G_E^\Lambda(q^2)}{G_M^\Lambda(q^2)} = \mathcal{O}(1)$$

Subtracted dispersion relations for real and imaginary part

$$R(q^2) = R(0) + \frac{q^2}{\pi}\int_{q^2_{\text{th}}}^{\infty} \frac{\text{Im}(R(s))}{s(s-q^2)}ds, \quad \forall q^2 \in \left[q^2_{\text{th}}, \infty\right)$$

$$\text{Re}\left(R(q^2)\right) = \frac{q^2}{\pi}\text{Pr}\int_{q^2_{\text{th}}}^{\infty} \frac{\text{Im}(R(s))}{s(s-q^2)}ds, \quad \forall q^2 \in \left[q^2_{\text{th}}, \infty\right)$$

The subtracted dispersion relations ensure the normalization at $q^2 = 0$

Parametrization for the form factors ratio

Parametrization through the set of **Chebyshev polynomials** $\left\{T_j(x)\right\}_{j=0}^{N}$.

$$\mathrm{Im}(\mathrm{R}(\mathrm{q}^2)) \equiv Y\left(q^2; \vec{C}, q^2_{\mathrm{asy}}\right) = \begin{cases} \sum_{j=0}^{N} C_j T_j(x(q^2)), & q^2_{\mathrm{th}} < q^2 < q^2_{\mathrm{asy}} \\ 0, & q^2 \geq q^2_{\mathrm{asy}} \end{cases}$$

$$x(q^2) = 2\frac{q^2 - q^2_{\mathrm{th}}}{q^2_{\mathrm{asy}} - q^2_{\mathrm{th}}} - 1$$

$$q^2 \in [q^2_{\mathrm{th}}, q^2_{\mathrm{asy}}] \Rightarrow x(q^2) \in [-1,1]$$

Theoretical constraints on $Y(q^2; \vec{C}, q^2_{\mathrm{asy}})$

$R(q^2_{\mathrm{th}})$ is real $\Rightarrow Y(q^2_{\mathrm{th}}; \vec{C}, q^2_{\mathrm{asy}}) = 0$

$R(q^2_{\mathrm{phy}})$ is real $\Rightarrow Y(q^2_{\mathrm{phy}}; \vec{C}, q^2_{\mathrm{asy}}) = 0$

$R(q^2 \geq q^2_{\mathrm{asy}})$ is real $\Rightarrow Y(q^2 \geq q^2_{\mathrm{asy}}; \vec{C}, q^2_{\mathrm{asy}}) = 0$

Theoretical constraints on $\mathrm{Re}(R(q^2))$

$$\mathrm{Re}\left(R(q^2_{\mathrm{th}})\right) = \frac{q^2_{\mathrm{th}}}{\pi}\mathrm{Pr}\int_{q^2_{\mathrm{th}}}^{q^2_{\mathrm{asy}}} \frac{Y(s; \vec{C}, q^2_{\mathrm{asy}})}{s(s - q^2_{\mathrm{th}})} ds = 1$$

$$\left|\mathrm{Re}\left(R(q^2_{\mathrm{asy}})\right)\right| = \frac{q^2_{\mathrm{asy}}}{\pi}\left|\mathrm{Pr}\int_{q^2_{\mathrm{th}}}^{q^2_{\mathrm{asy}}} \frac{Y(s; \vec{C}, q^2_{\mathrm{asy}})}{s(s - q^2_{\mathrm{asy}})} ds\right| = 1$$

Experimental constraints for the time-like region $(q^2 > q^2_{\mathrm{phy}})$

8 experimental points for the modulus and 7 for the phase from Babar (2019), BESIII (2019) and BESIII (2024)

The χ^2 definition

$$\chi^2\left(\vec{C}, q^2_{\rm asy}\right) = \chi^2_{|R|} + \chi^2_\phi + \tau_{\rm phy}\chi^2_{\rm phys} + \tau_{\rm asy}\chi^2_{\rm asy} + \tau_{\rm curv}\chi^2_{\rm curv}$$

$$\chi^2_{|R|} = \sum_{j=1}^{8}\left(\frac{\sqrt{X^2(q_j^2) + Y^2(q_j^2)} - \left|R_j\right|}{\delta\left|R_j\right|}\right)^2$$

$$X(q^2) \equiv \mathrm{Re}(R(q^2))$$

$$\chi^2_\phi = \sum_{j=1}^{7}\left(\frac{\sin\left(\arctan(Y(q_k^2)/X(q_k^2)\right) - \sin(\phi_k)}{\delta\sin(\phi_k)}\right)^2$$

Constraint at $q^2 = q^2_{\rm phy}$ $\longrightarrow$ $\chi^2_{\rm phy} = \left(1 - X(q^2_{\rm phy})\right)^2$

Constraint at $q^2 = q^2_{\rm asy}$ $\longrightarrow$ $\chi^2_{\rm asy} = \left(1 - X^2(q^2_{\rm asy})\right)^2$

The values of $\tau_{\rm phys}$ and $\tau_{\rm asy}$ are chosen so that the theoretical conditions are exactly verified

Oscillation damping $\longrightarrow$ $\chi^2_{\rm curv} = \int_{q^2_{\rm th}}^{q^2_{\rm asy}}\left(\frac{d^2Y(s)}{ds^2}\right)^2 ds$

The dispersion relation solution is an ill-posed problem which has to be regularized

The parametrization

The theoretical constraints $Y(q^2_{\text{th}}; \vec{C}, q^2_{\text{asy}}) = Y(q^2_{\text{phy}}; \vec{C}, q^2_{\text{asy}}) = Y(q^2_{\text{asy}}; \vec{C}, q^2_{\text{asy}}) = 0$ remove three degrees of freedom, allowing to determine three coefficients, i.e. C_0, C_1, C_2.

The asymptotic threshold q^2_{asy} is used a a free parameter

If we consider $(N + 1)$ Chebyshev polynomials, we are left with $(N - 2)$ free coefficients.

We used $N = 5$, so we have four free parameters C_3, C_4, C_5 and q^2_{asy}.

- $\tau_{\text{phy}} = 10^4 \Rightarrow$ The real part of the ratio is forced to the unity at $q^2 = q^2_{\text{phy}}$
- $\tau_{\text{asy}} = 0 \Rightarrow$ No constraint for the real part at $q^2 = q^2_{\text{asy}}$
- $\tau_{\text{curv}} = 0.05 \Rightarrow$ Dumping relevant only for high degree polynomials

  If τ_{curv} is too large physical information are canceled.

  If τ_{curv} is too small the solution have too much noise

Results discussion

At the thresholds q^2_{th} and q^2_{asy} the values of the ratio are real, so the relative phases are integer multiples of of π radians.

$$N_{\text{th,asy}} = \frac{1}{\pi} \arg\left(\frac{G_E^\Lambda(q^2_{\text{th,asy}})}{G_M^\Lambda(q^2_{\text{th,asy}})}\right) \in \mathbb{Z}$$

The χ^2 minimization alongside with the theoretical constraints allows to produce 4 $(N_{\text{th}}, N_{\text{asy}})$ possible pairs compatible with the data points.

A Monte Carlo procedure allows to obtain the probability of occurrence of each pair $(N_{\text{th}}, N_{\text{asy}})$.

N_{th}	N_{asy}	%
-1	1	32%
0	3	36%
0	4	6%
1	3	24%

$$\chi^2_{\text{min}} = 21.81$$

Moduli and relative phases

Moduli and relative phases

Final Considerations

The bands represent the one-sigma-error computed with statistical analysis of the Monte Carlo procedure results.

The dispersive procedure, connecting time-like experimental values and theoretical constraints, allows to assign different determinations to the phase, and hence to the measured values of the phase. This gives informations about the space-like behaviour of the form factors ratio.

Assuming no zeroes for the magnetic form factor, the Levinson's Theorem allows to count the number of zeroes of the electric form factor, aside from the theoretical one at $q^2 = 0$

$$\Delta\phi = \phi(\infty) - \phi(q_{\text{th}}^2) = \pi\left(N_{\text{asy}} - N_{\text{th}}\right) \geq \pi$$

The most probable value for $N_{\text{asy}} - N_{\text{th}}$ is 3, hence there are two additional zeroes for $G_E^{\Lambda}(q^2)$

Final Considerations - Work in progress

In the near future, we would like to increase the statistics of the Monte Carlo procedure, in order to obtain a more precise evaluation of the possible cases for $(N_{\text{th}}, N_{\text{asy}})$

The dispersive relation for the imaginary can be used to obtain an estimation of the charge radius of the Λ baryon

$$\langle r_E \rangle^2 = \frac{1}{6} \frac{dG_E(q^2)}{dq^2} \bigg|_{q^2=0} = 6\mu \frac{dR(q^2)}{dq^2} \bigg|_{q^2=0}$$