Measurement Process in GR: the case of Ring Lasers

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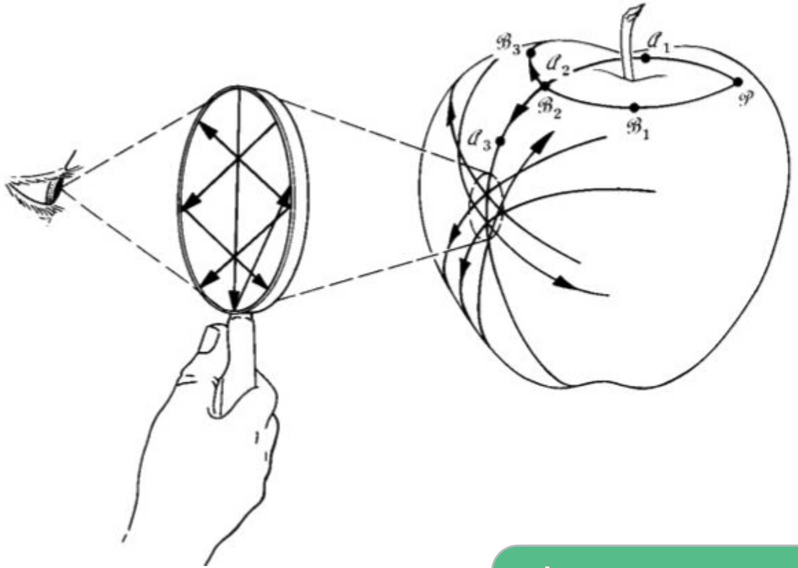
Physics is simple only when analyzed locally



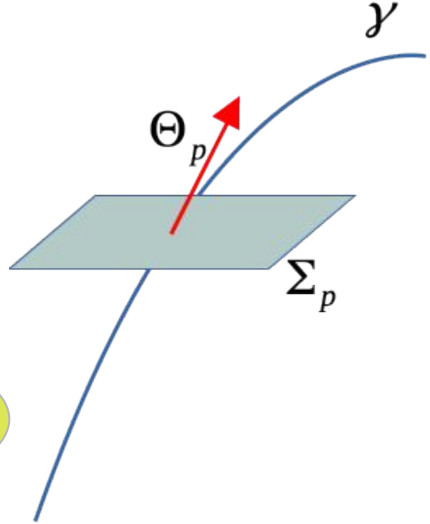
Spacetime is locally Minkowski → Large spacetime volumes are needed to see the effects of GR

Laboratory experiments are difficult to perform

The measurement process

1+3 Splitting Formalism

Splitting spacetime into space plus time is fundamental in GR to get a better understanding on the basis of our 3-dimensional experience of what happens in the 4- dimensional geometry.



PROJECTIONS

4-dimensional quantities → **space-like+time-like quantities**

At each point p in spacetime, the tangent space Tp can be split into the direct sum of two subspaces: the local time direction Θp, and Σp, the local space platform: a 3-dimensional subspace which is orthogonal to Θp

$$T_p = \Theta_p \oplus \Sigma_p.$$

The measurement process

1+3 Splitting Formalism

MOTION OF A FREE TEST PARTICLE

"relative quantities"

$$\begin{cases} \dfrac{\mathrm{D}^{\perp} p}{\mathrm{d}T} = F^G, \\ \dfrac{\mathrm{d}E}{\mathrm{d}T} = v \cdot F^G, \end{cases}$$

$$\left(F^G\right)_\alpha = -m\gamma \left[a(u)_\alpha + \theta_{\alpha\beta} v^\beta + (w \times_u v)_\alpha\right]$$

inertial effects

$$a(u) \to \frac{M}{r^2}$$

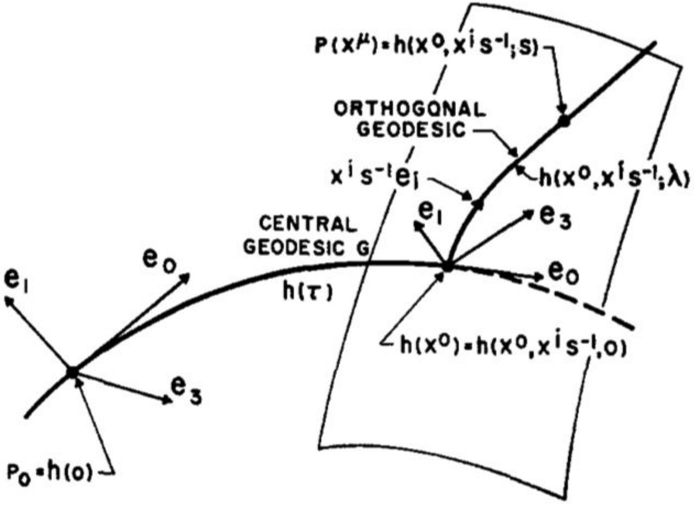
Static observers in Schwarzschild spacetime

The laboratory frame

Near the observer’s worldline

In the tangent space along the observer's world-line we consider an orthonormal tetrad. The time coordinate is the proper time, while the space coordinates are defined using space-like geodesics.

Fermi Coordinates



from Misner, Manasse (1963)

The expression of the spacetime element in Fermi coordinates depends both on the properties of the reference frame (the acceleration and rotation of the congruence) and on the spacetime curvature, through the Riemann curvature tensor.

The laboratory frame

Near the observer's worldline

$R_{\alpha\beta\gamma\delta}(T)$ the projection of the Riemann curvature tensor on the orthonormal tetrad of the reference observer, parameterized by the proper time

Inertial effects

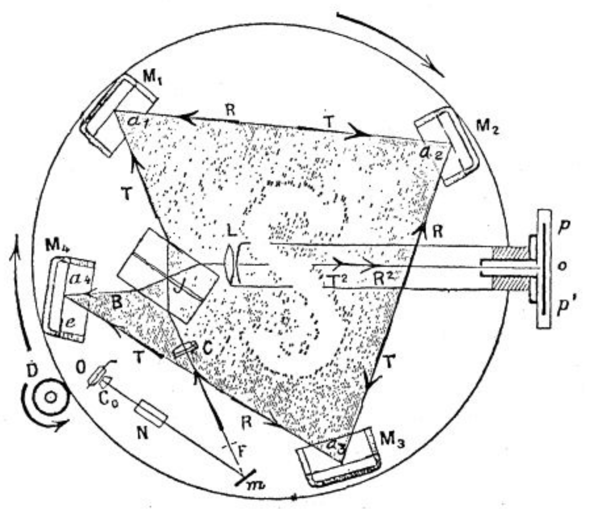
$$ds^2 = -\left[\left(1 + \frac{\mathbf{a}\cdot\mathbf{X}}{c^2}\right)^2 - \frac{1}{c^2}(\boldsymbol{\Omega}\wedge\mathbf{X})^2 + R_{0i0j}X^iX^j\right]c^2dT^2 + \left[\frac{1}{c}(\boldsymbol{\Omega}\wedge\mathbf{X})_i - \frac{4}{3}R_{0jik}X^jX^k\right]cdTdX^i +$$
$$+\left(\delta_{ij} - \frac{1}{3}R_{ikjl}X^kX^l\right)dX^idX^j.$$

curvature effects

AT LINEAR ORDER $$g_{0i} = \frac{1}{c}(\boldsymbol{\Omega}\times\mathbf{X})$$

$\boldsymbol{\Omega}$ is the rotation rate of the laboratory frame w.r.t. the Fermi-Walker non rotating one

The Sagnac Effect



A ring laser converts a time difference into a frequency difference

Proper time difference:

$$\delta\tau = -2\sqrt{g_{00}(x_0^i)} \oint_S \frac{g_{0i}}{g_{00}} ds^i$$

$$\delta\tau = -4\int_A \mathbf{\Omega}\cdot d\mathbf{A} = -4\mathbf{\Omega}\cdot\mathbf{A}$$

$$\delta\tau = 4\mathbf{\Omega}_\oplus\cdot\mathbf{A} + 4\mathbf{\Omega}'\cdot\mathbf{A}$$

Earth Rotation Rate

Spacetime effects

Frequency difference:

$$\delta f = \frac{4A}{\lambda P}\mathbf{u}_n\cdot\mathbf{\Omega},$$

The Sagnac Effect

Spacetime effects: $$\mathbf{\Omega}' = \mathbf{\Omega}_G + \mathbf{\Omega}_B + \mathbf{\Omega}_W + \mathbf{\Omega}_T$$ **+...**

de Sitter:

$$\mathbf{\Omega}_G = -(1+\gamma)\nabla U(R) \wedge \boldsymbol{V},$$

Lense-Thirring:

$$\mathbf{\Omega}_B = -\frac{1+\gamma+\alpha_1/4}{2}\left(\frac{\boldsymbol{J}_\oplus}{R^3} - \frac{3\boldsymbol{J}_\oplus \cdot \boldsymbol{R}}{R^5}\boldsymbol{R}\right)$$

Preferred frames:

$$\mathbf{\Omega}_W = \alpha_1 \tfrac{1}{4} \nabla U(R) \wedge \boldsymbol{W},$$

Thomas:

$$\mathbf{\Omega}_T = -\frac{1}{2}\boldsymbol{V} \wedge \frac{d\boldsymbol{V}}{dT}.$$

The Sagnac Effect

$$ds^2 = -\left[\left(1+\frac{\mathbf{a}\cdot\mathbf{X}}{c^2}\right)^2 - \frac{1}{c^2}\left(\mathbf{\Omega}\wedge\mathbf{X}\right)^2 + R_{0i0j}X^iX^j\right]c^2dT^2 + \left[\frac{1}{c}\left(\mathbf{\Omega}\wedge\mathbf{X}\right)_i - \frac{4}{3}R_{0jik}X^jX^k\right]cdTdX^i +$$
$$+\left(\delta_{ij} - \frac{1}{3}R_{ikjl}X^kX^l\right)dX^idX^j.$$

AT QUADRATIC ORDER

$$g_{0i} = -\frac{2}{3}R_{0jik}X^jX^k \quad \Rightarrow \quad \delta\tau \simeq \frac{1}{c}\mathbf{B}\cdot\mathbf{u}_n A$$

Curvature effects:

$$\mathrm{d}s^2 = -c^2\left(1-2\frac{\Phi}{c^2}\right)\mathrm{d}T^2 - \frac{4}{c}A_i\mathrm{d}X^i\mathrm{d}T + \left(\delta_{ij}+2\frac{\Psi_{ij}}{c^2}\right)\mathrm{d}X^i\mathrm{d}X^j$$

$$\Phi(T,X^i) = -\frac{c^2}{2}R_{0i0j}(T)X^iX^j, \qquad A_i(T,X^i) = \frac{c^2}{3}R_{0jik}(T)X^jX^k, \qquad \Psi_{ij}(T,X^i) = -\frac{c^2}{6}R_{ikjl}(T)X^kX^l,$$

with the definition of the fields:

$$\mathbf{E} = -\nabla\Phi - \frac{2}{c}\frac{\partial\mathbf{A}}{\partial T}, \qquad \mathbf{B} = \nabla\wedge\mathbf{A},$$

A plane gravitational wave

For a gravitational wave propagating along the x direction, in TT gauge: $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$

$$h_{xx} = 1, \quad h_{yy} = 1 - A^{+} \sin\left(\omega t - kx\right), \quad h_{zz} = 1 + A^{+} \sin\left(\omega t - kx\right), \quad h_{zy} = -A^{\times} \cos\left(\omega t - kx\right)$$

$$B_X = 0, \quad B_Y = -\frac{\omega^2}{2}\left[-A^{\times}\cos\left(\omega T\right) Y + A^{+}\sin\left(\omega T\right) Z\right], \quad B_Z = -\frac{\omega^2}{2}\left[A^{+}\sin\left(\omega T\right) Y + A^{\times}\cos\left(\omega T\right) Z\right].$$

Additional Sagnac time delay $$\delta\tau \simeq \frac{\omega^2 L^2}{c^2}\frac{L}{c}h$$

$h = A^{+}, A^{\times}$ *L* is the scale length of the measuring device

Summary and Perspectives

- Definition of the measurement process in the laboratory frame
- Spacetime metric in the laboratory (linear effects, quadratic effects…)

- Theories other than GR can be tested (extended, torsion…) in this framework