Initial state and thermalization

Hard Probes, Cagliari, May 2012

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Outline

1. **Factorization and Universality**
2. **Thermalization and Isotropization**
3. **Summary**

In collaboration with :

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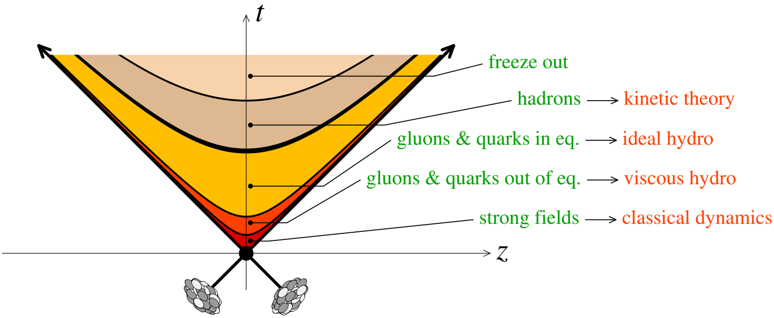
Stages of a collision



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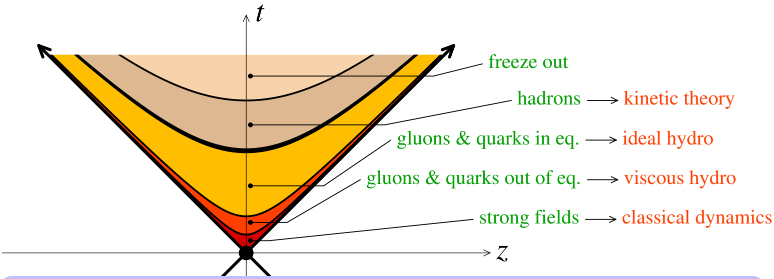


This talk : evolution up to times ~ 1 fm/c

- **i.** Partonic content of high energy nuclei
- **ii.** Gluon production in the collision
- **iii.** Evolution shortly after the collision, Thermalization

Color Glass Condensate = effective theory of small x gluons

[McLerran, Venugopalan (1994), Jalilian-Marian, Kovner, Leonidov, Weigert (1997), Iancu, Leonidov, McLerran (2001)]

- The fast partons ($k^+ > \Lambda^+$) are frozen by time dilation
  ▷ described as static color sources on the light-cone :

$$J^\mu = \delta^{\mu +}\rho(x^-, \vec{x}_\perp) \qquad (0 < x^- < 1/\Lambda^+)$$

- The color sources ρ are random, and described by a probability distribution $W_{\Lambda^+}[\rho]$
- Slow partons ($k^+ < \Lambda^+$) may evolve during the collision
  ▷ treated as standard gauge fields
  ▷ eikonal coupling to the current J^μ : $J_\mu A^\mu$

$$\mathcal{S} = \underbrace{-\frac{1}{4}\int F_{\mu\nu}F^{\mu\nu}}_{\mathcal{S}_{YM}} + \int \underbrace{J^\mu A_\mu}_{\text{fast partons}}$$

Factorization and Universality

[FG, Lappi, Venugopalan (2008)]

Leading Order

- All observables can be expressed in terms of classical solutions of Yang-Mills equations :

$$[\mathcal{D}_\mu, \mathcal{F}^{\mu\nu}] = J^\nu$$

- Boundary conditions for inclusive observables : retarded, with $\mathcal{A} \to 0$ at $x_0 = -\infty$

Inclusive spectra at LO

$$\left.\frac{dN_1}{d^3\vec{p}}\right|_{LO} \sim \int d^4x d^4y \; e^{ip\cdot(x-y)} \; \square_x \square_y \; \mathcal{A}(x) \, \mathcal{A}(y)$$

$$\left.\frac{dN_n}{d^3\vec{p}_1 \cdots d^3\vec{p}_n}\right|_{LO} = \left.\frac{dN_1}{d^3\vec{p}_1}\right|_{LO} \cdots \left.\frac{dN_1}{d^3\vec{p}_n}\right|_{LO}$$

Next to Leading Order

Master relation between LO and NLO

$$\mathcal{O}_{\rm NLO} = \left[\frac{1}{2}\iint_{u,v}\int_k \left[a_k\,\mathbb{T}\right]_u \left[a_k^*\,\mathbb{T}\right]_v + \int_u \left[\alpha\,\mathbb{T}\right]_u\right]\mathcal{O}_{\rm LO}$$

$$\mathbb{T}_u \sim \frac{\delta}{\delta\mathcal{A}(0,u)} \quad , \quad a_k, \alpha \ \text{known analytically}$$

- Valid for all inclusive observables,
  e.g. the energy-momentum tensor
- In the CGC, upper cutoff on the loop momentum : $k^\pm < \Lambda$,
  to avoid double counting with the sources $J_{1,2}$
  ▷ large logarithms of the cutoff

Initial state logarithms

Central result

$$\frac{1}{2}\iint\limits_{u,v}\int_{k}\left[a_{k}\,\mathbb{T}\right]_{u}\left[a_{k}^{*}\,\mathbb{T}\right]_{v}+\int\limits_{u}\left[\alpha\,\mathbb{T}\right]_{u}=$$

$$=\log\left(\Lambda^{+}\right)\,\mathcal{H}_{1}+\log\left(\Lambda^{-}\right)\,\mathcal{H}_{2}+\text{terms w/o logs}$$

$\mathcal{H}_{1,2}$ = JIMWLK Hamiltonians of the two nuclei

- No mixing between the logs of the two nuclei
- Since the LO↔NLO relationship is the same for all inclusive observables, these logs have a universal structure

Factorization of the logarithms

- The JIMWLK Hamiltonian $\mathcal{H}$ is self-adjoint :

$$\int [D\rho]\; W\;(\mathcal{H}\,\mathcal{O}) = \int [D\rho]\;(\mathcal{H}\,W)\;\mathcal{O}$$

Inclusive observables at Leading Log accuracy

$$\langle\mathcal{O}\rangle_{\text{Leading Log}} = \int \big[D\rho_1\; D\rho_2\big]\; W_1\big[\rho_1\big]\; W_2\big[\rho_2\big]\; \underbrace{\mathcal{O}_{\text{LO}}[\rho_1,\rho_2]}_{\text{fixed }\rho_{1,2}}$$

- Logs absorbed into the scale evolution of $W_{1,2}$

$$\Lambda\frac{\partial W}{\partial\Lambda} = \mathcal{H}\,W \qquad \text{(JIMWLK equation)}$$

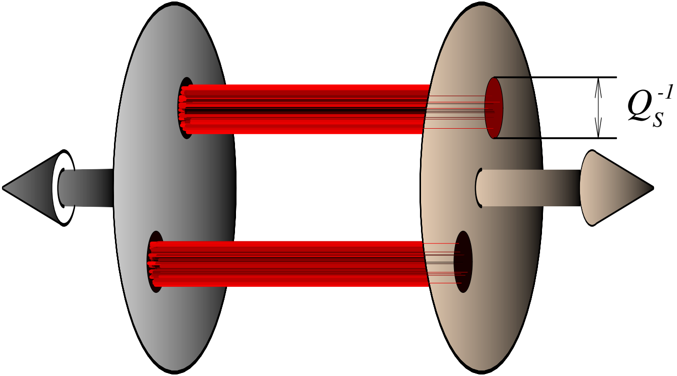
Thermalization and Isotropization

[Dusling, Epelbaum, FG, Venugopalan (2010-12)]

[Dusling, FG, Venugopalan (2011)]

[Epelbaum, FG (2011)]

Energy momentum tensor at LO



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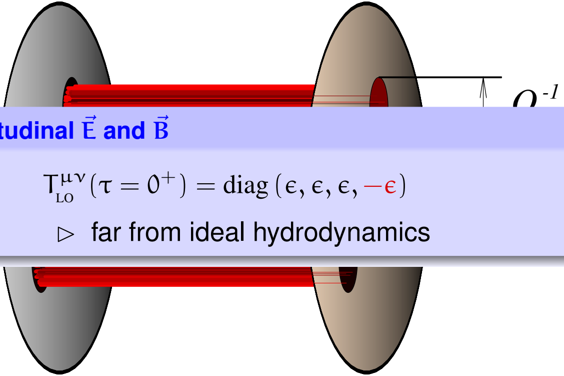


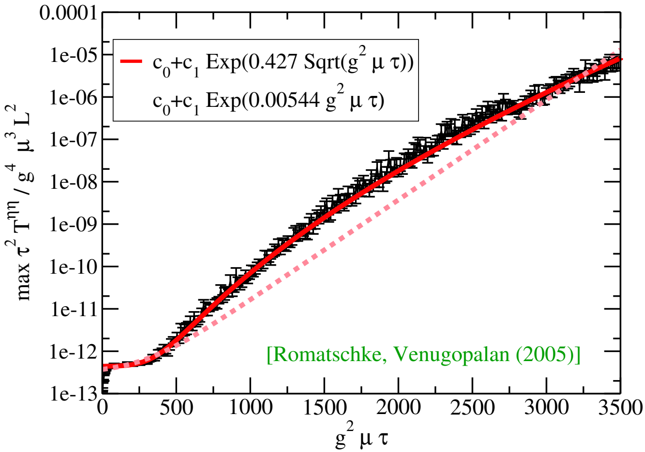
$T^{\mu\nu}$ for longitudinal $\vec{E}$ and $\vec{B}$

$$T^{\mu\nu}_{\mathrm{LO}}(\tau = 0^+) = \mathrm{diag}\,(\epsilon, \epsilon, \epsilon, -\epsilon)$$

▷ far from ideal hydrodynamics

Weibel instabilities for small perturbations

[Mrowczynski (1988), Romatschke, Strickland (2003), Arnold, Lenaghan, Moore (2003), Rebhan, Romatschke, Strickland (2005), Arnold, Lenaghan, Moore, Yaffe (2005), Romatschke, Rebhan (2006), Bodeker, Rummukainen (2007),...]



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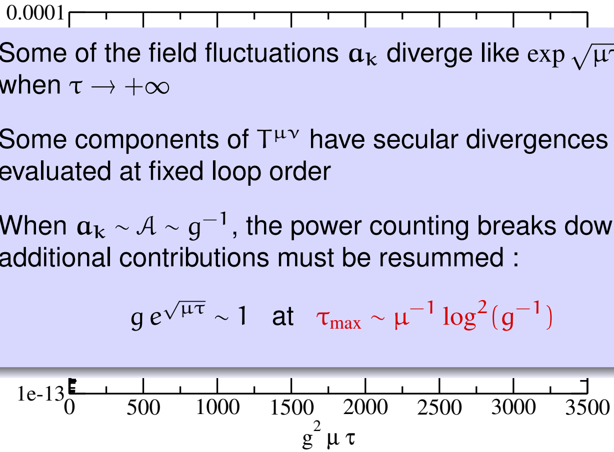
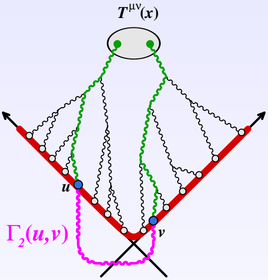


- Some of the field fluctuations a_k diverge like $\exp\sqrt{\mu\tau}$ when $\tau \to +\infty$
- Some components of $T^{\mu\nu}$ have secular divergences when evaluated at fixed loop order
- When $a_k \sim \mathcal{A} \sim g^{-1}$, the power counting breaks down and additional contributions must be resummed :

$$g\, e^{\sqrt{\mu\tau}} \sim 1 \quad \text{at} \quad \tau_{max} \sim \mu^{-1} \log^2(g^{-1})$$

Improved power counting

$$\text{Loop} \sim g^2 \qquad , \qquad \mathbb{T}_{\boldsymbol{u}} \sim e^{\sqrt{\mu\tau}}$$



- 1 loop : $(g e^{\sqrt{\mu\tau}})^2$

Improved power counting

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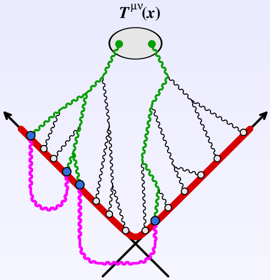

- 1 loop : $(ge^{\sqrt{\mu\tau}})^2$
- 2 disconnected loops : $(ge^{\sqrt{\mu\tau}})^4$

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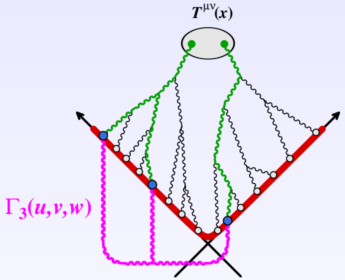

- 1 loop : $(ge^{\sqrt{\mu\tau}})^2$
- 2 disconnected loops : $(ge^{\sqrt{\mu\tau}})^4$
- 2 nested loops : $g(ge^{\sqrt{\mu\tau}})^3 \quad \triangleright$ subleading

Leading terms at $\tau_{\rm max}$

- All disjoint loops to all orders
  ▷ exponentiation of the 1-loop result

Resummation of the leading secular terms

$$T^{\mu\nu}_{\text{resummed}} = \exp\left[\frac{1}{2}\int_{u,v\in\Sigma}\underbrace{\int_k [a_k\mathbb{T}]_u[a_k^*\mathbb{T}]_v}_{\mathcal{G}(u,v)} + \int_{u\in\Sigma}[\alpha\mathbb{T}]_u\right] T^{\mu\nu}_{\text{LO}}[\mathcal{A}_{\text{init}}]$$

Resummation of the leading secular terms

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$$= \int[D\chi]\,\exp\left[-\frac{1}{2}\int_{u,v\in\Sigma}\chi(u)\mathcal{G}^{-1}(u,v)\chi(v)\right]T^{\mu\nu}_{\text{LO}}[\mathcal{A}_{\text{init}}+\chi+\alpha]$$

- The evolution remains classical, but we must average over a Gaussian ensemble of initial conditions
- The shift α can be absorbed into a redefinition of $\mathcal{A}_{\text{init}}$
- The 2-point correlation $\mathcal{G}(u,v)$ of the fluctuations is known analytically

Analogous scalar toy model

ϕ^4 field theory coupled to a source

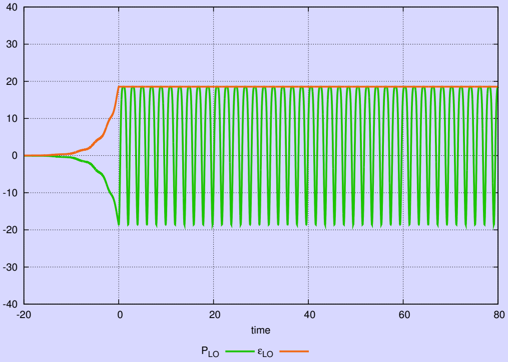
$$\mathcal{L} = \frac{1}{2}(\partial_\alpha \phi)^2 - \frac{g^2}{4!}\phi^4 + J\phi$$

Strong external source: $J \quad \propto \quad \frac{Q^3}{g}$

- In 3+1-dim, g is dimensionless, and the only scale in the problem is Q
- This theory has unstable modes (parametric resonance)
- Two setups have been studied :
  - Fixed volume system
  - Longitudinally expanding system

Pathologies in fixed order calculations

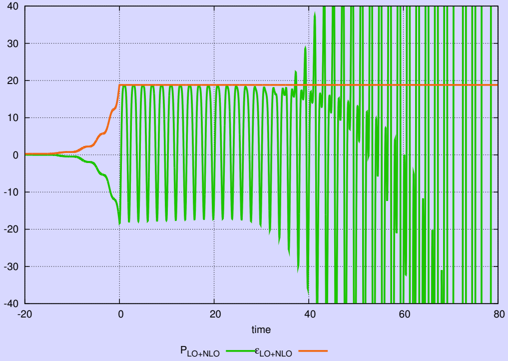
Tree



- Oscillating pressure at LO : no equation of state

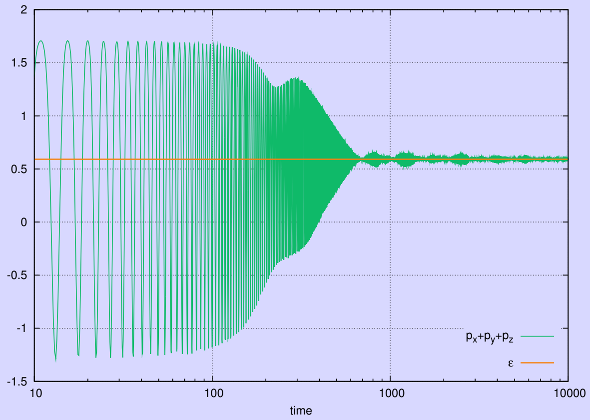
Pathologies in fixed order calculations

Tree + 1-loop



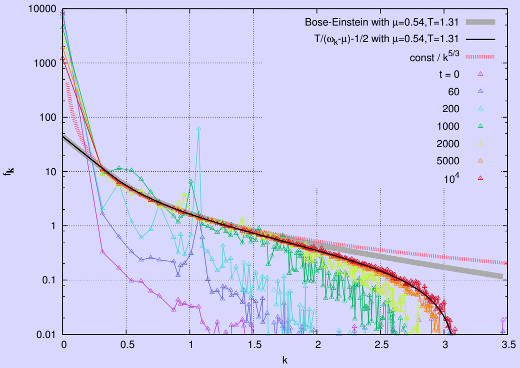
- Oscillating pressure at LO : no equation of state
- Small NLO correction to the energy density (protected by energy conservation)
- Secular divergence in the NLO correction to the pressure

Resummed energy momentum tensor



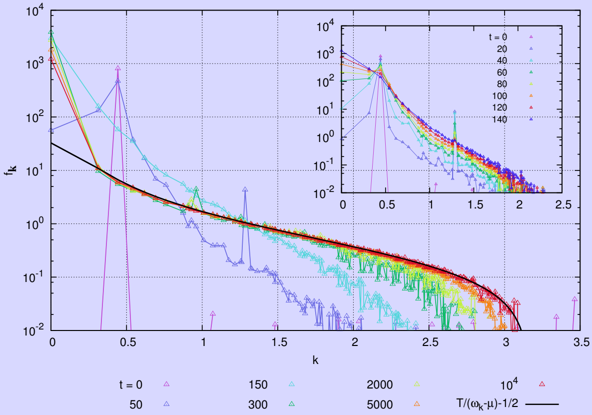
- No secular divergence in the resummed pressure
- The pressure relaxes to the equilibrium equation of state

Time evolution of the occupation number



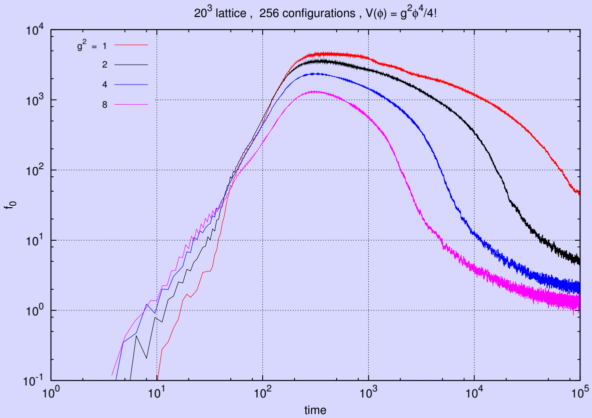
- Resonant peak at early times
- Turbulent Kolmogorov spectrum in the intermediate k-range?
- Late times : classical equilibrium with a chemical potential
- $\mu \approx m$ + excess at $k = 0$: Bose-Einstein condensation?

Bose-Einstein condensation



- Start with the same energy density, but an empty zero mode
- Very quickly, the zero mode becomes highly occupied
- Same distribution as before at late times

Evolution of the condensate



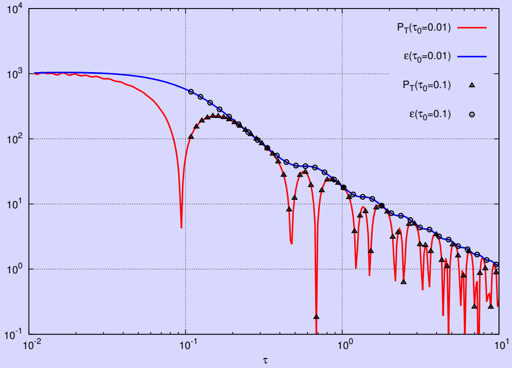
- Formation time almost independent of the coupling
- Condensate lifetime much longer than its formation time
- Smaller amplitude and faster decay at large coupling

Longitudinal expansion

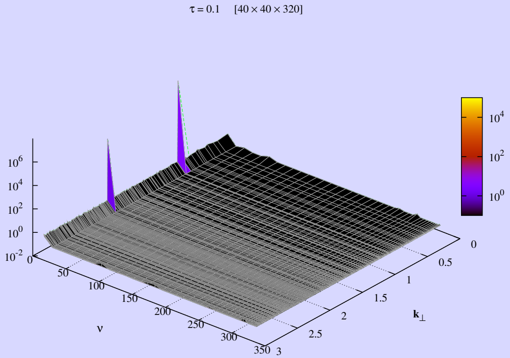
- The EoM is singular when $\tau \to 0$: one must start at $\tau_0 \neq 0$

$$\partial_\tau^2 \phi + \frac{1}{\tau}\partial_\tau \phi - \frac{1}{\tau^2}\partial_\eta^2 \phi - \nabla_\perp^2 \phi + \frac{g^2}{6}\phi^3 = 0$$

- The spectrum of fluctuations of the initial field also depends on τ_0, precisely in such a way that the end result is independent of τ_0



Longitudinal expansion : occupation number



- Note : ν is the Fourier conjugate of the rapidity η
- Initially, only the $\nu = 0$ modes are occupied

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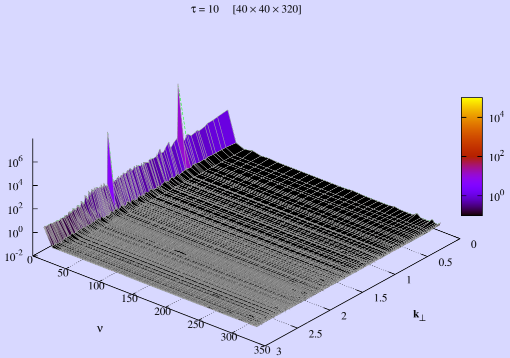

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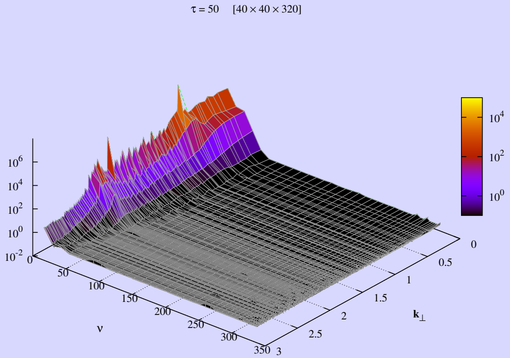

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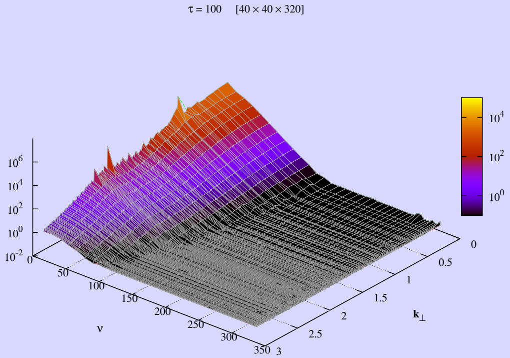

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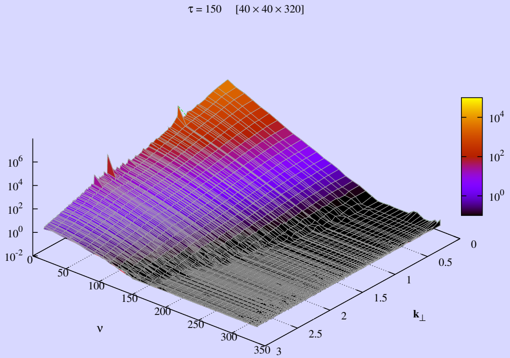

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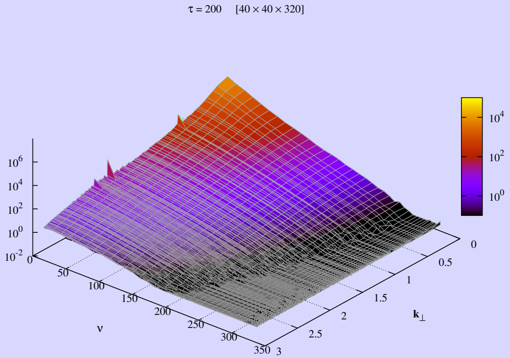

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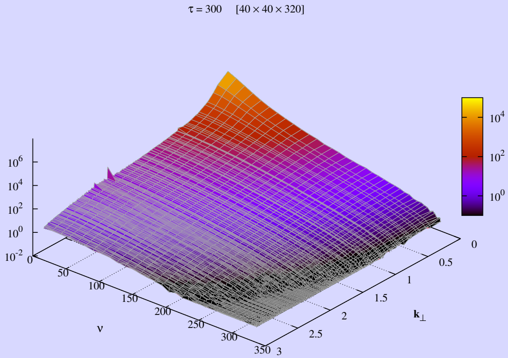

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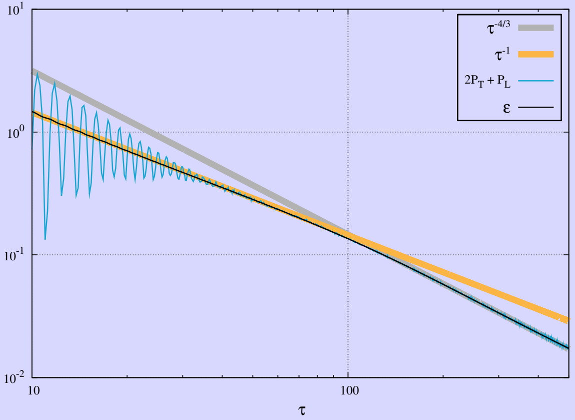
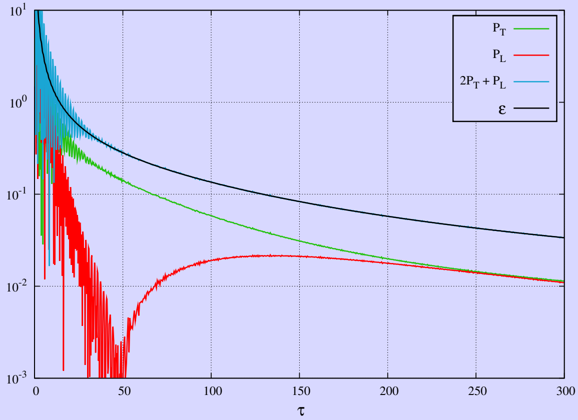
Longitudinal expansion : equation of state



- After a short time, one has $2P_T + P_L \approx \epsilon$
- Change of behavior of the energy density: $\tau^{-1} \to \tau^{-4/3}$. Since one has $\partial_\tau \epsilon + (\epsilon + P_L)/\tau = 0$, this suggests that P_L gets close to $\epsilon/3$

Longitudinal expansion : isotropization



- At early times, P_L drops much faster than P_T (redshifting of the longitudinal momenta due to the expansion)
- Drastic change of behavior when the expansion rate becomes smaller than the growth rate of the unstability
- Eventually, isotropic pressure tensor : $P_L \approx P_T$

Summary and Outlook

Summary

- Factorization of high energy logarithms in AA collisions
  - universal distributions, also applicable in pA or DIS
  - controls the rapidity dependence of correlations
  - limited to inclusive observables

- Resummation of secular terms in the final state evolution
  - stabilizes the NLO calculation
  - leads to the equilibrium equation of state
  - isotropization even with longitudinal expansion
  - full thermalization on longer time-scales
  - Bose-Einstein condensation if overoccupied initial state

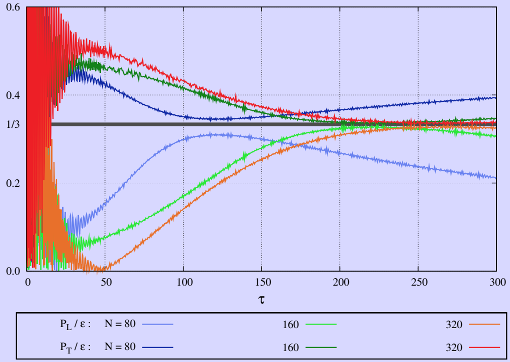
(so far, all numerical studies done for a toy scalar model)

What's next? QCD

- approach straightforwardly generalizable to QCD
- seamless integration with the CGC description of AA collisions
- gauge invariant
- computationally expensive ($\sim$ [scalar case] $\times 3(N_c^2 - 1)$)

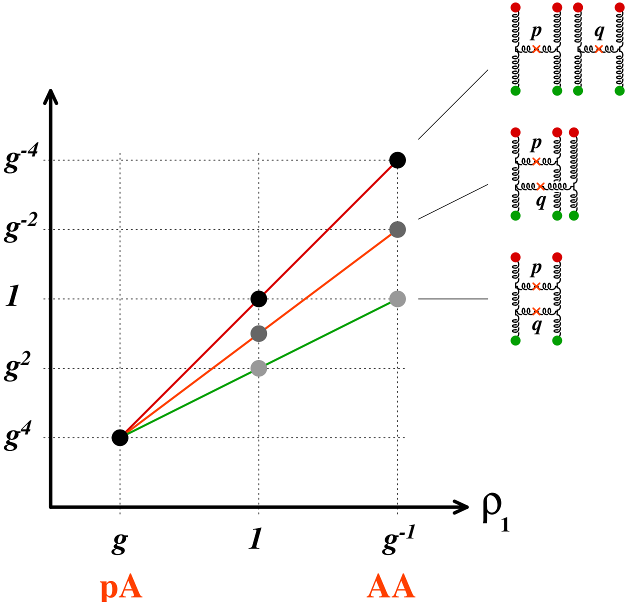
Extra bits

Lattice spacing dependence



- When $\nu_{max} \sim \tau^{2/3} \gtrsim 2N$, the longitudinal modes are artificially cut-off, and the longitudinal pressure decreases
- If N increases, the field acquires a mass $m^2 \sim g^2 \log(N)$
  Taking the limit $N \to \infty$ requires a proper renormalization of the bare parameters of the theory

Dense-dilute collisions



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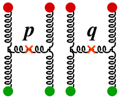
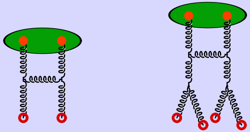


Expected complications

- More diagrams to consider even at Leading Order
- More terms in the evolution Hamiltonian if $\rho \sim g$:

$$g^2 \rho^2 \left(\frac{\partial}{\partial \rho}\right)^2 \sim g^4 \rho^2 \left(\frac{\partial}{\partial \rho}\right)^4$$

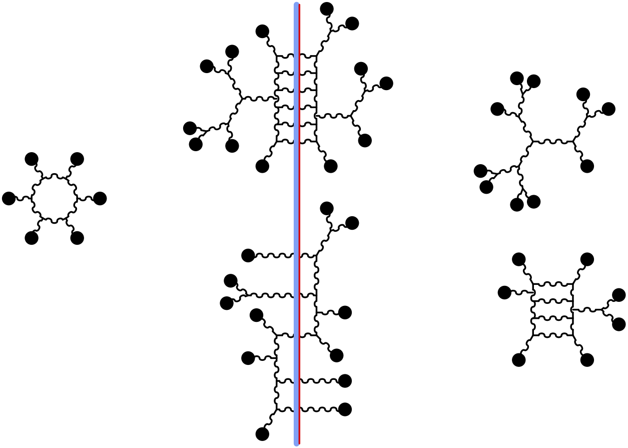


ρ_1

g 1 g^{-1}

pA AA

Exclusive processes

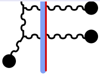


Exclusive processes

Example : differential probability to produce 1 particle at LO

$$\left.\frac{dP_1}{d^3\vec{p}}\right|_{\text{LO}} = F[0] \times \int d^4x d^4y \; e^{ip\cdot(x-y)} \, \square_x \square_y \, \mathcal{A}_+(x)\mathcal{A}_-(y)\Big|_{z=0}$$

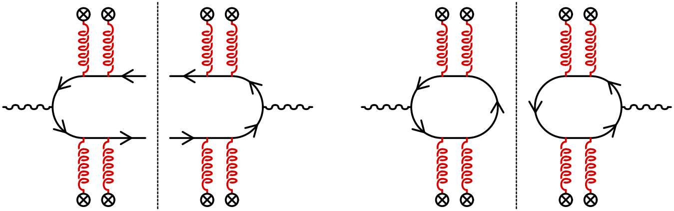
- The vacuum-vacuum graphs do not cancel in exclusive quantities : $F[0] \neq 1$ (in fact, $F[0] = \exp(-c/g^2) \ll 1$)
- $\mathcal{A}_+$ and $\mathcal{A}_-$ are classical solutions of the Yang-Mills equations, but with non-retarded boundary conditions



Thermalization in Yang-Mills theory

- Recent analytical work : Kurkela, Moore (2011)
- Going from scalars to gauge fields :
  - More fields per site (3 Lorentz components $\times$ 8 colors)
  - More complicated spectrum of initial conditions

BEC and dilepton production



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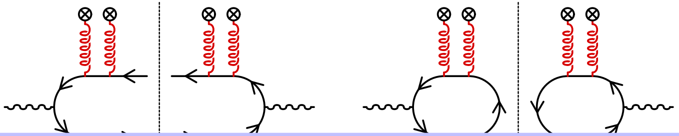

Two topologies for virtual photons at LO

Connected	$\omega \sim M_{\rm inv} \sim Q_s$	$k_\perp \sim Q_s$
Disconnected	$\omega \sim M_{\rm inv} \sim Q_s$	$k_\perp \ll Q_s$

▷ excess of dileptons with $k_\perp \ll M_{\rm inv}$