Multiparticle celestial states and their pairwise quantum numbers

Francesco Alessio

2403.03760 with M. Arzano

NORDITA
The Nordic Institute for Theoretical Physics

UPPSALA UNIVERSITET

18-19/04/24 Scuola Superiore Meridionale, Napoli QUAGRAP meeting

Outline

1) Motivations:

- G-W physics and black-hole coalescence
- Infrared effects in gravity (asymptotic symmetries, soft theorems etc.)

2) Pairwise helicity VS gravitational/electromagnetic scoot

3) Conformal primary basis

4) Celestial pairwise little group and quantum numbers

5) Conclusions

Outline

1) Motivations:

- G-W physics and black-hole coalescence
- Infrared effects in gravity (asymptotic symmetries, soft theorems etc.)

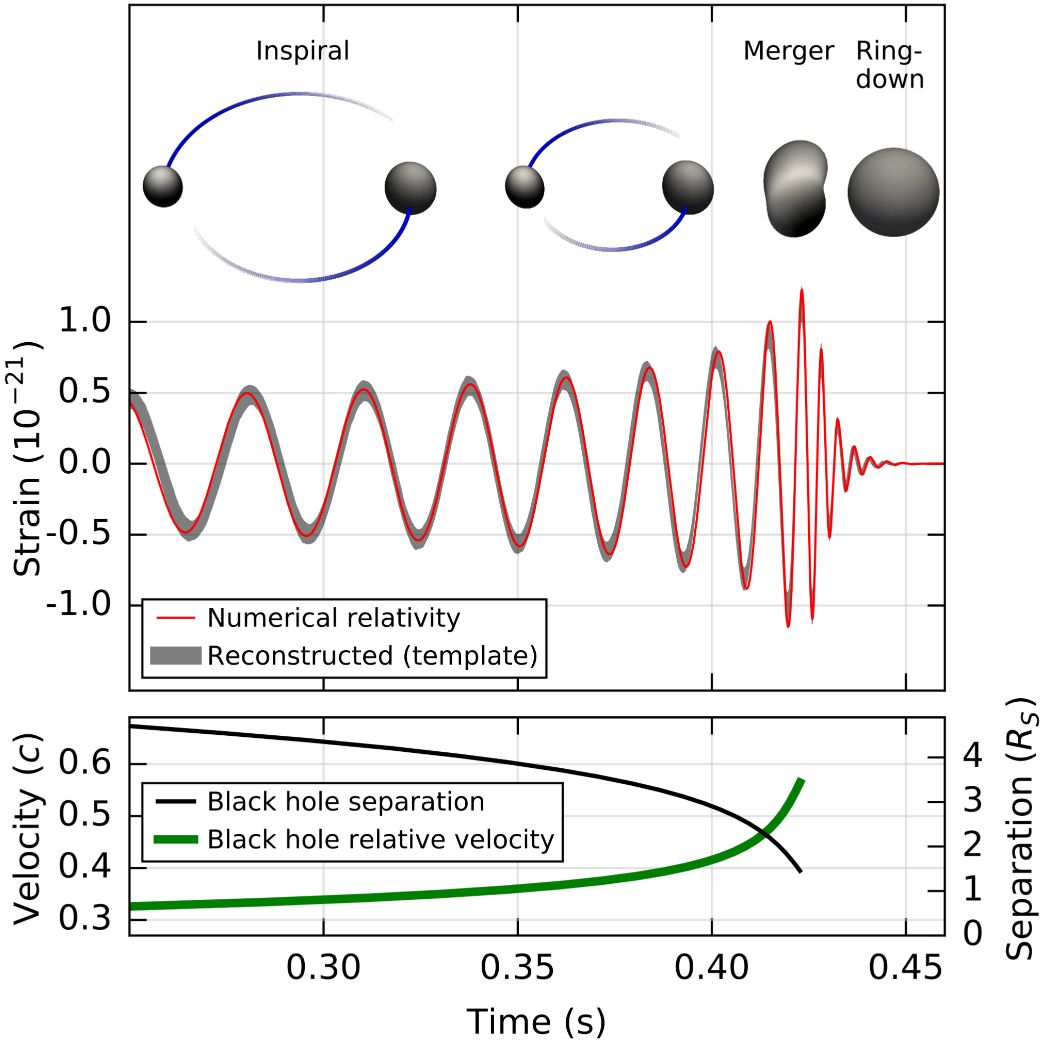
2) Pairwise helicity VS gravitational/electromagnetic scatt

3) Conformal primary basis

4) Celestial pairwise little group and quantum numbers

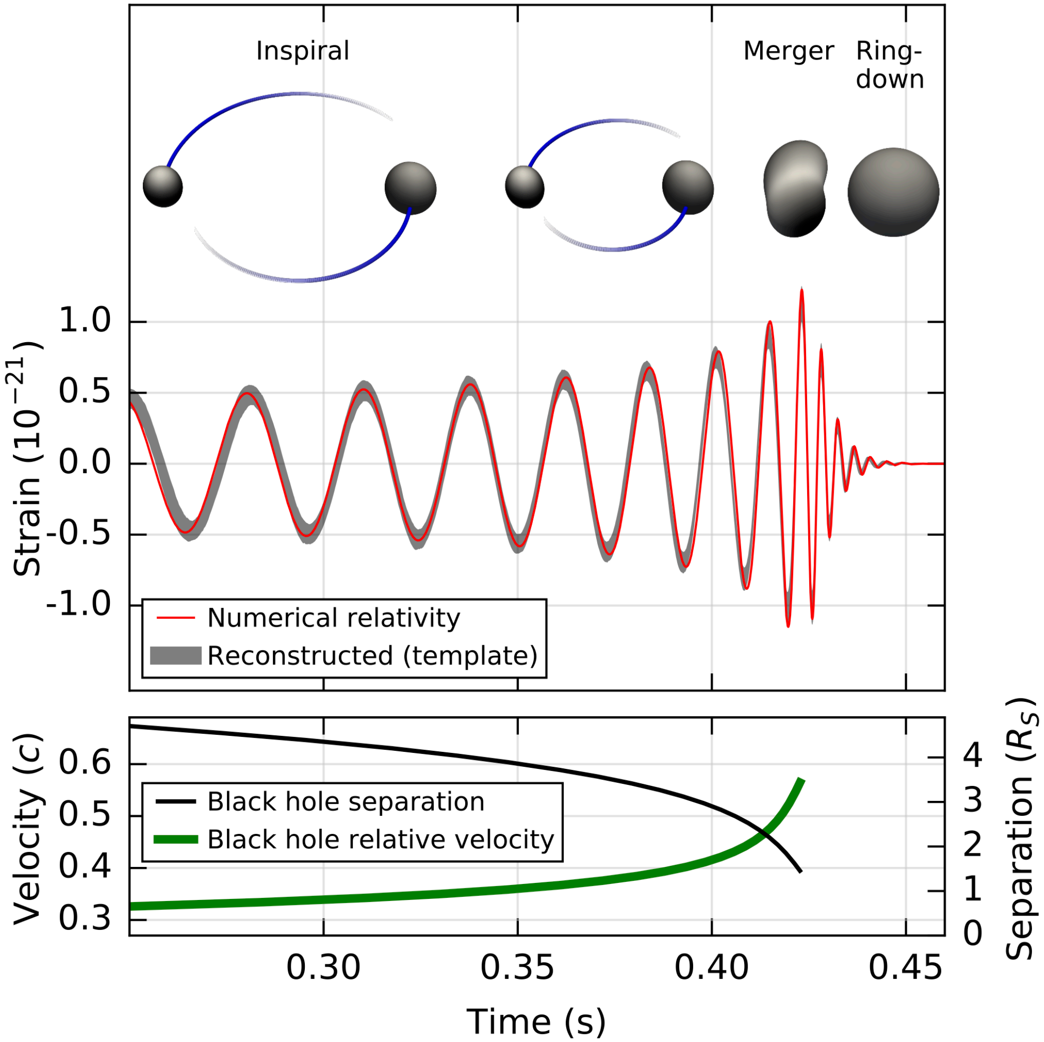
5) Conclusions

Black-hole mergers & gravitational waves



[LIGO Scientific collaboration]

Black-hole mergers & gravitational waves



Full dynamics encoded in:

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

- hard to solve because of non-linearities, gauge invariance ...

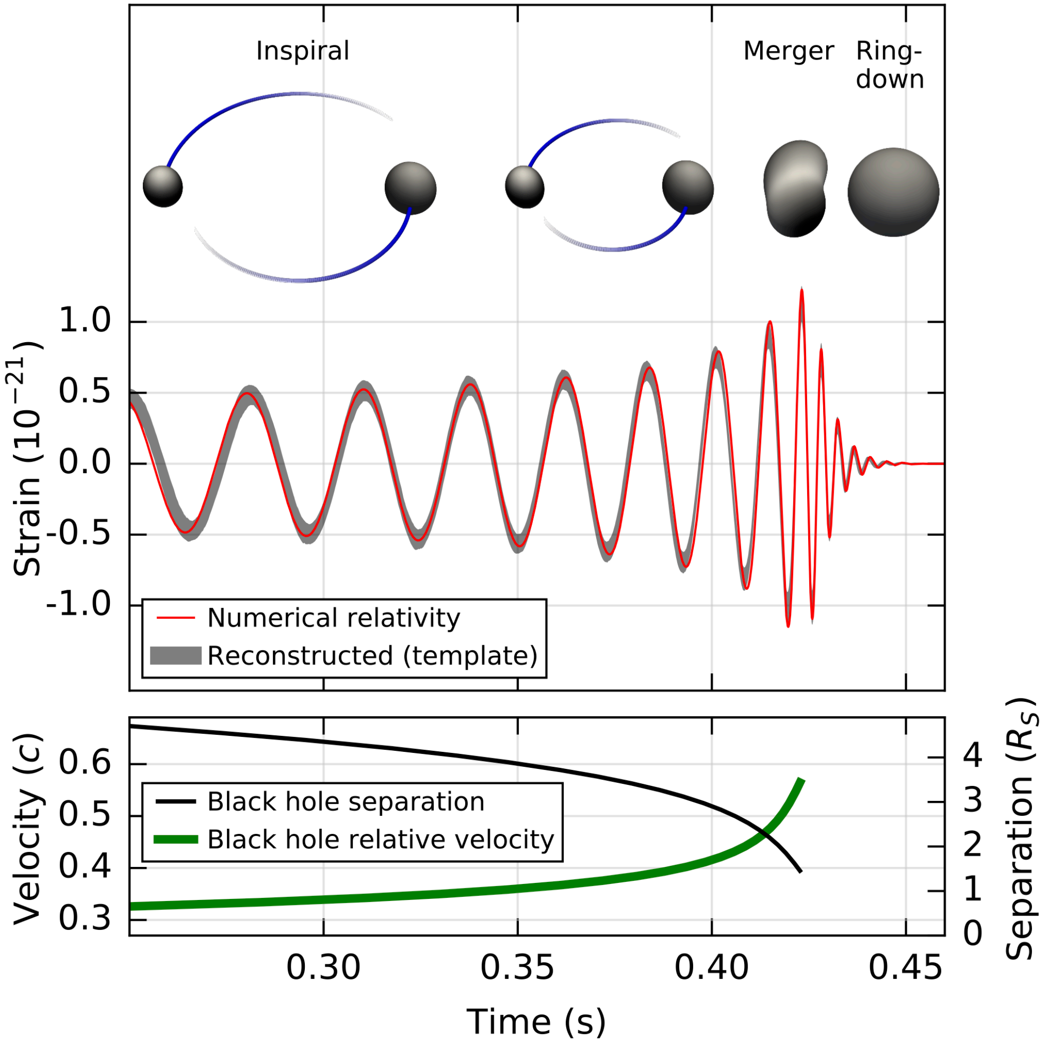
[LIGO Scientific collaboration]

Black-hole mergers & gravitational waves

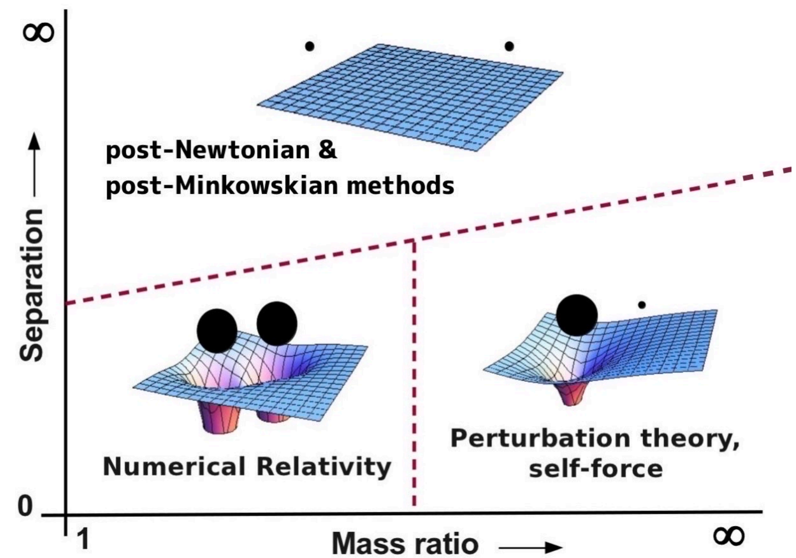
Full dynamics encoded in:

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

- hard to solve because of non-linearities, gauge invariance ...

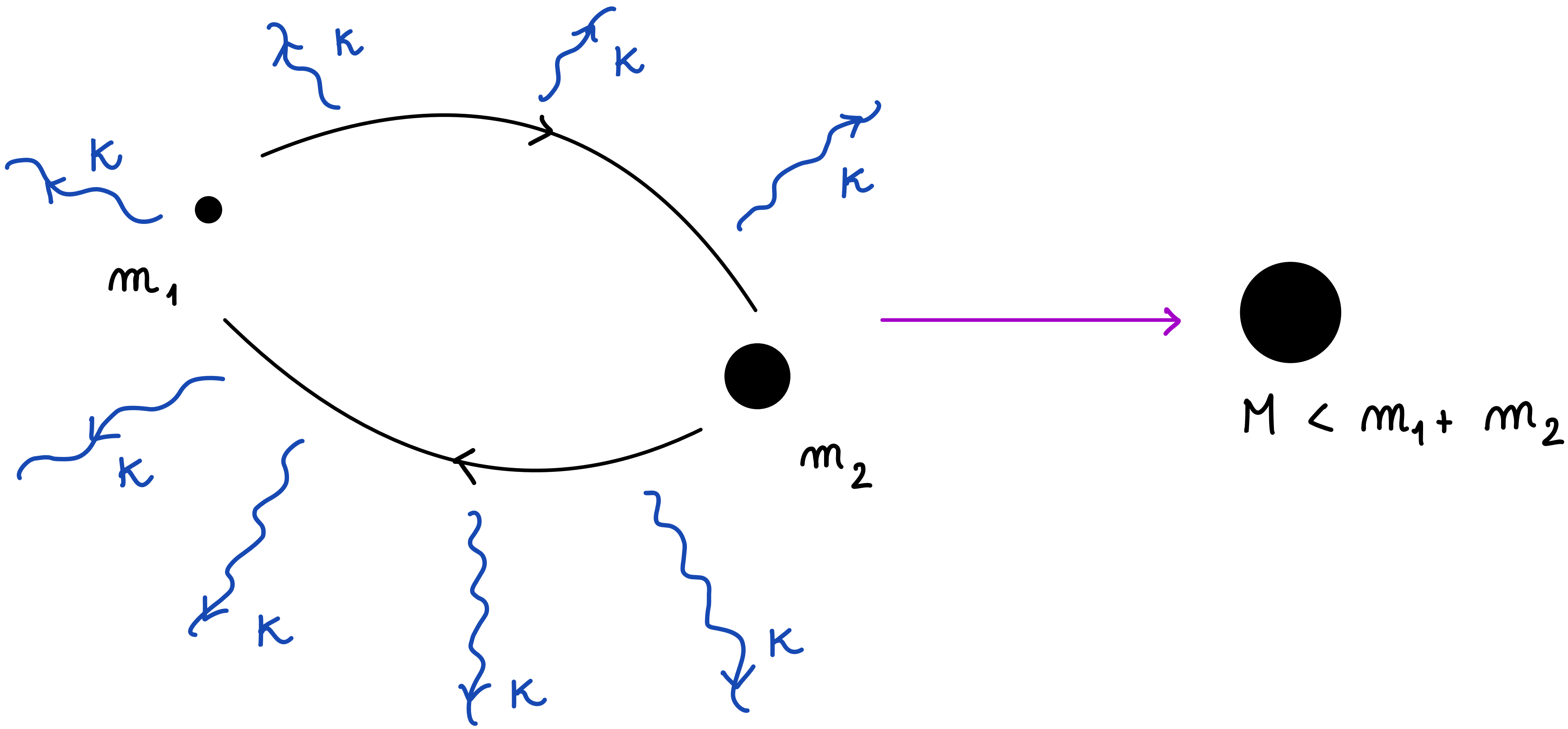


[LIGO Scientific collaboration]

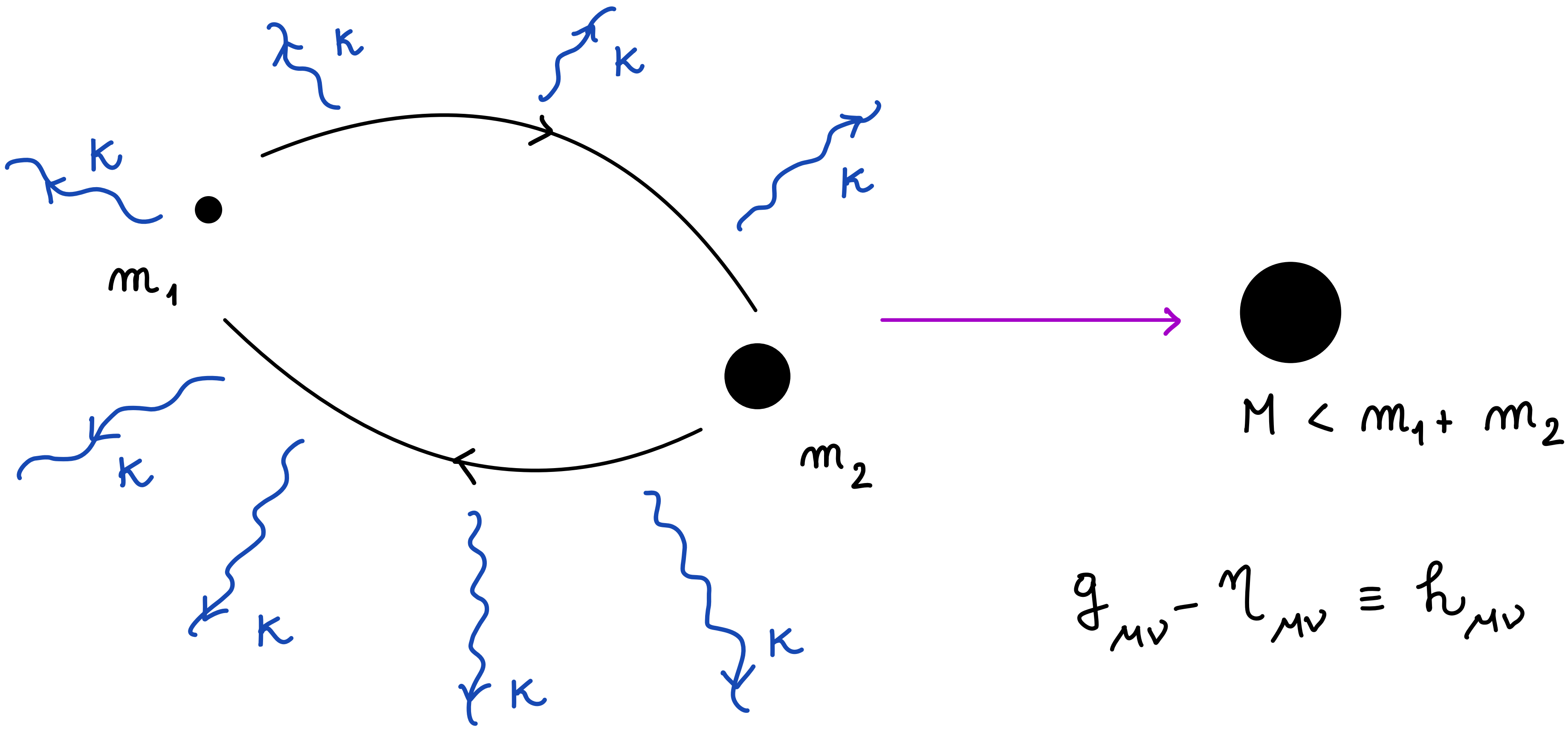


[Barack & Pound]

Black-hole mergers & gravitational waves



Black-hole mergers & gravitational waves



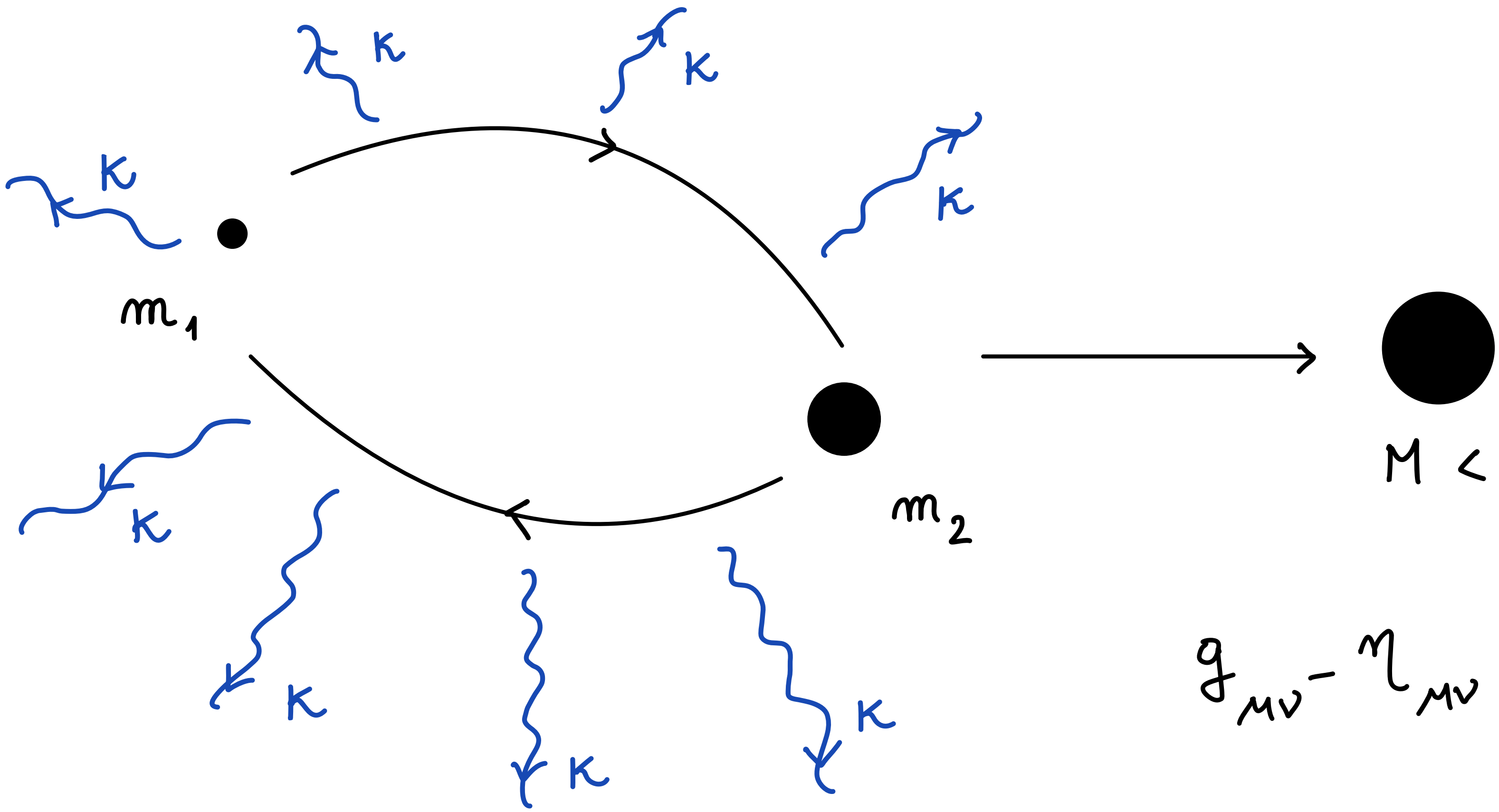
$$g_{\mu\nu} - \eta_{\mu\nu} \equiv h_{\mu\nu}$$

$$P^{\mu}_{rad} = \frac{1}{32\pi G} \int_K \left(\dot{h}_{ij} \right)^2 K^{\mu}$$

$$P^{0}_{rad} = m_1 + m_2 - M$$

$$M^{ij}_{rad} = \frac{1}{8\pi G} \int_K \left(h_k^{[i} \dot{h}^{j]}_k - \frac{1}{2} x^{[i} \partial^{j]} h_{kl} \dot{h}_{kl} \right)$$

Black-hole mergers & gravitational waves



$$M < m_1 + m_2$$

$$g_{\mu\nu} - \eta_{\mu\nu} \equiv h_{\mu\nu}$$

$$P^{\mu}_{rad} = \frac{1}{32\pi G} \int_K \left(\dot{h}_{ij} \right)^2 K^{\mu}$$

$$P^{0}_{rad} = m_1 + m_2 - M$$

$$\boxed{M^{ij}_{rad} = \frac{1}{8\pi G} \int_K \left(h_k^{\ [i} \dot{h}^{j]}_{\ k} - \frac{1}{2} x^{[i} \partial^{j]} h_{kl} \dot{h}_{kl} \right)}$$

Infrared effects in gravity

- Infrared triangle [Strominger et al. ~ '14]

[Weinberg '65, Low '58
Cachazo & Strominger '14
Sen, Laddha & Sahoo '17]

Soft theorems

Memory effects ⟷ Asymptotic symmetries

[Zel'dovich & Polnorev '74]

[Bondi, Metzner, Sachs '62
Barnich, Troessaert ~ '10]

Infrared effects in gravity

- Infrared triangle [Strominger et al. ~ '14]

[Weinberg '65, Low '58
Cachazo & Strominger '14
Sen, Laddha & Sahoo '17]

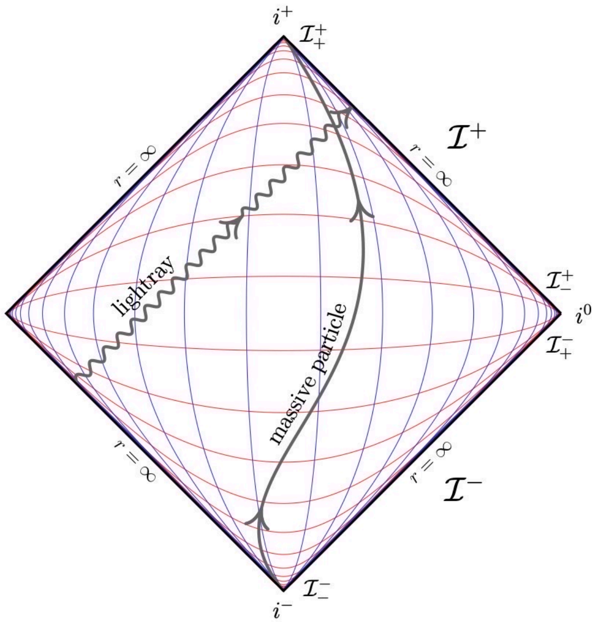
Soft theorems $\longrightarrow$ Soft radiation

[F.A. & Di Vecchia '22, '24]

Memory effects $\longleftrightarrow$ Asymptotic symmetries

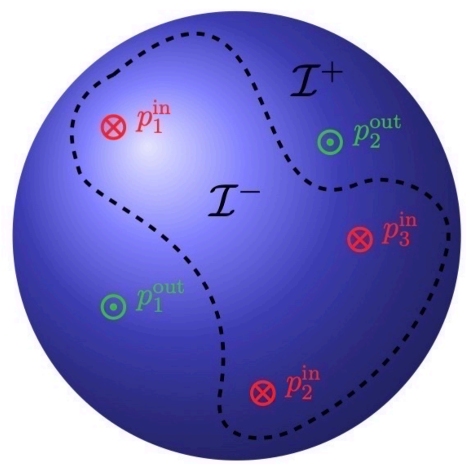
[Zel'dovich & Polnarev '74]

[Bondi, Metzner, Sachs '62
Barnich, Troessaert ~ '10]



"Celestial holography"

$\Longleftrightarrow$



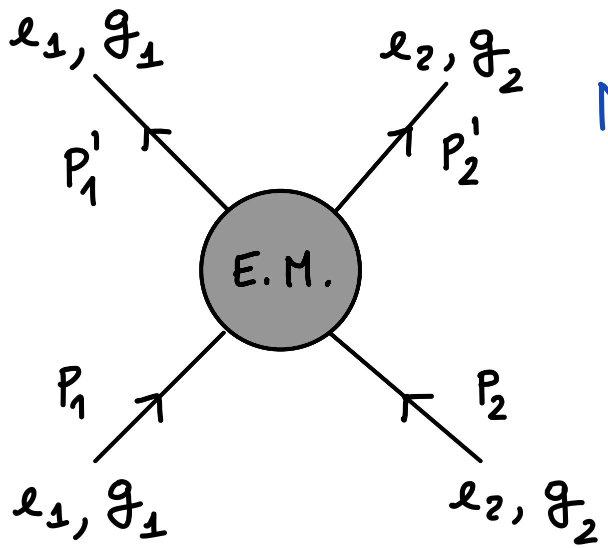
[Credits: Strominger '17]

Outline

1) Motivations:
   - G-W physics and black-hole coalescence
   - Infrared effects in gravity (asymptotic symmetries, soft theorems etc.)
2) Pairwise helicity VS gravitational/electromagnetic scoot
3) Conformal primary basis
4) Celestial pairwise little group and quantum numbers
5) Conclusions

Zwanziger Pairwise Little group

- Scattering of dyons and radiated angular momentum



$$M_{\pm}^{\mu\nu}(p_1, p_2) = \pm \underbrace{(e_1 g_2 - e_2 g_1)}_{\mu_{12}} M^{\mu\nu}(p_1, p_2)$$

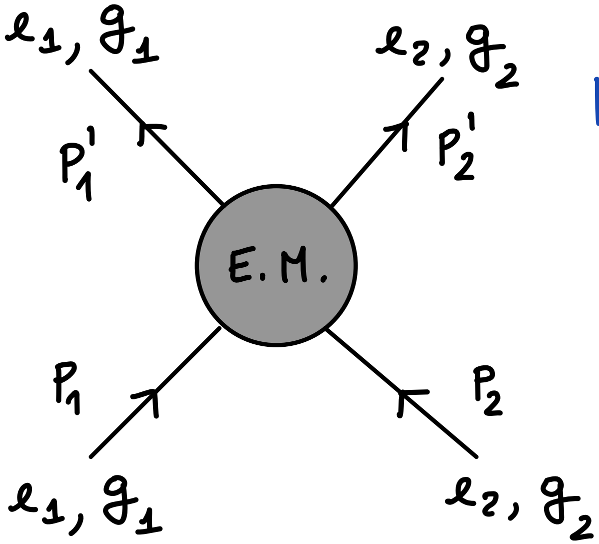
[Zwanziger '72]

$$M^{\mu\nu}(p_1, p_2) = \frac{\epsilon^{\mu\nu}{}_{\rho\sigma}\, p_1^{\rho} p_2^{\sigma}}{\left[(p_1 \cdot p_2)^2 - p_1^2 p_2^2 \right]^{1/2}}$$

- Pairs of dyons carry, asymptotically, a non-vanishing rotation-like angular momentum

Zwanziger Pairwise Little group

- Scattering of dyons and radiated angular momentum



$$M_{\pm}^{\mu\nu}(p_1, p_2) = \pm \underbrace{(e_1 g_2 - e_2 g_1)}_{\mu_{12}} M^{\mu\nu}(p_1, p_2)$$

[Zwanziger '72]

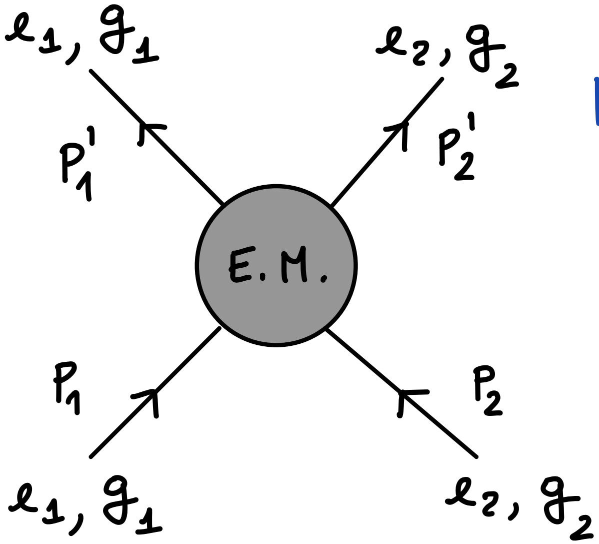
$$M^{\mu\nu}(p_1, p_2) = \frac{\epsilon^{\mu\nu}{}_{\rho\sigma}\, p_1^{\rho} p_2^{\sigma}}{\left[(p_1 \cdot p_2)^2 - p_1^2 p_2^2\right]^{1/2}}$$

- Pairs of dyons carry, asymptotically, a non-vanishing rotation-like angular momentum

- Asymptotic two-particle states must be labelled by a new quantum number $\mu_{12} \equiv$ "pairwise helicity" [Zwanziger '72]

Zwanziger Pairwise Little group

- Scattering of dyons and radiated angular momentum



$$M_{\pm}^{\mu\nu}(p_1, p_2) = \pm \underbrace{(e_1 g_2 - e_2 g_1)}_{\mu_{12}} M^{\mu\nu}(p_1, p_2)$$

[Zwanziger '72]

$$M^{\mu\nu}(p_1, p_2) = \frac{\epsilon^{\mu\nu}{}_{\rho\sigma}\, p_1^\rho p_2^\sigma}{\left[(p_1 \cdot p_2)^2 - p_1^2 p_2^2\right]^{1/2}}$$

- Pairs of dyons carry, asymptotically, a non-vanishing rotation-like angular momentum

- Asymptotic two-particle states must be labelled by a new quantum number $\mu_{12} \equiv$ "pairwise helicity" [Zwanziger '72]

Q: How do we understand μ_{12} from group theory?

Zwanziger Pairwise Little group

Induced representations a la Wigner : single particle states

- Reference state : $k = (m, \vec{0})$
- General state : $p = L(p,k) \cdot k$

$$
\begin{aligned}
\mathcal{U}(\Lambda)|p\rangle &= \mathcal{U}(\Lambda)\,\mathcal{U}(L(p,k))|k\rangle = \mathbb{1}\,\mathcal{U}(\Lambda)\,\mathcal{U}(L(p,k))|k\rangle \\
&= \mathcal{U}(L(\Lambda \cdot p, k))\,\underbrace{\mathcal{U}(L(k, \Lambda \cdot p))\,\mathcal{U}(\Lambda)\,\mathcal{U}(L(p,k))}_{k \to k}|k\rangle \\
&= |\Lambda \cdot p\rangle
\end{aligned}
$$

Zwanziger Pairwise Little group

Induced representations a la Wigner : single particle states

- Reference state : $k = (m, \vec{0})$
- General state : $p = L(p,k)\cdot k$

$$\begin{aligned} \mathcal{U}(\Lambda)|p\rangle &= \mathcal{U}(\Lambda)\,\mathcal{U}(L(p,k))|k\rangle = \mathbb{1}\,\mathcal{U}(\Lambda)\,\mathcal{U}(L(p,k))|k\rangle \\ &= \mathcal{U}(L(\Lambda\cdot p,k))\underbrace{\mathcal{U}(L(k,\Lambda\cdot p))\,\mathcal{U}(\Lambda)\,\mathcal{U}(L(p,k))}_{k\to k}|k\rangle \\ &= |\Lambda\cdot p\rangle \end{aligned}$$

If there is spin:

$$\mathcal{U}(\Lambda)|p,\sigma\rangle = D_{\sigma'\sigma}\,|\Lambda\cdot p,\sigma'\rangle \quad \text{with}$$

$$D_{\sigma'\sigma} = \begin{cases} D^{(s)}_{s_z' s_z} & \text{massive (Wigner matrices)} \\ e^{ih\phi} & \text{massless} \end{cases}$$

Zwanziger Pairwise Little group

Induced representations a la Wigner : multi-particle states

- Reference state : $k_1 = (E_1, 0, 0, p)$ $\quad k_2 = (E_2, 0, 0, -p)$
- General state : $p_1 = L(p_i, k_i) \cdot k_1$ $\quad p_2 = L(p_i, k_i) \cdot k_2$

Zwanziger Pairwise Little group

Induced representations a la Wigner : multi-particle states

- Reference state : $k_1 = (E_1, 0, 0, p)$ $\quad k_2 = (E_2, 0, 0, -p)$
- General state : $p_1 = L(p_i, k_i) \cdot k_1$ $\quad p_2 = L(p_i, k_i) \cdot k_2$

$$M^{12}_{\pm}(k_1, k_2) = \pm \mu_{12}$$

[Zwanziger '72]

"Bare" tensor product construction is too naive : pairwise states

[Csaba Csáki et al '20,'21,'22]

$$|p_1, p_2\rangle = |p_1\rangle \otimes |p_2\rangle \longrightarrow |p_1, p_2, \mu_{12}\rangle \equiv |p_1, p_2\rangle \otimes |k_1, k_2, \mu_{12}\rangle$$

Zwanziger Pairwise Little group

Induced representations a la Wigner : multi-particle states

- Reference state : $k_1 = (E_1, 0, 0, P)$ $\quad k_2 = (E_2, 0, 0, -P)$
- General state : $p_1 = L(p_i, k_i) \cdot k_1$ $\quad p_2 = L(p_i, k_i) \cdot k_2$

$$M^{12}_{\pm}(k_1, k_2) = \pm \mu_{12}$$

[Zwanziger '72]

"Bare" tensor product construction is too naive : pairwise states

[Csaba Csáki et al '20,'21,'22]

$$|p_1, p_2\rangle = |p_1\rangle \otimes |p_2\rangle \longrightarrow |p_1, p_2, \mu_{12}\rangle \equiv |p_1, p_2\rangle \otimes |k_1, k_2, \mu_{12}\rangle$$

Induced representations:

$$\mathcal{U}(\Lambda)|p_1, p_2, \mu_{12}\rangle = \exp\left\{\pm i \mu_{12}\, \phi(\Lambda; p_1, p_2)\right\} |\Lambda \cdot p_1, \Lambda \cdot p_2, \mu_{12}\rangle$$

$$\cos\phi(\Lambda; p_1, p_2) = \hat{\epsilon}_\mu(p_1, p_2, \Lambda^{-1} n)\, \hat{\epsilon}^\mu(p_1, p_2, n)$$

arbitrary 4-vector

[Zwanziger '72]

$$\left[\hat{\epsilon}^\mu \equiv \epsilon^\mu / \sqrt{\epsilon \cdot \epsilon} \quad , \quad \epsilon^\mu(a, b, c) \equiv \epsilon^\mu{}_{\nu\rho\sigma}\, a^\nu b^\rho c^\sigma\right]$$

Zwanziger Pairwise Little group

$$U(\Lambda)|p_1, p_2, \mu_{12}\rangle = e^{\pm i \mu_{12} \phi(\Lambda; p_1, p_2)} |\Lambda\cdot p_1, \Lambda\cdot p_2, \mu_{12}\rangle$$

Pairwise little group $\supseteq$ pairwise helicity μ_{12}

- Lorentz generators for multiparticle states have an additional internal contribution

$$M^{\mu\nu} = p_1^{[\mu} \frac{\partial}{\partial x_{1\,\nu]}} + p_2^{[\mu} \frac{\partial}{\partial x_{2\,\nu]}} + \mu_{12} \frac{\left[(p_1\cdot p_2)^2 - m_1^2 m_2^2\right]^{1/2}}{\epsilon^2(p_1, p_2, n)} n^{[\mu} \epsilon^{\nu]}(p_1, p_2, n)$$

Zwanziger Pairwise Little group

$$\mathcal{U}(\Lambda)|p_1, p_2, \mu_{12}\rangle = e^{\pm i \mu_{12} \phi(\Lambda; p_1, p_2)} |\Lambda \cdot p_1, \Lambda \cdot p_2, \mu_{12}\rangle$$

Pairwise little group $\supseteq$ pairwise helicity μ_{12}

- Lorentz generators for multiparticle states have an additional internal contribution

$$M^{\mu\nu} = p_1^{[\mu} \frac{\partial}{\partial x_{1\,\nu]}} + p_2^{[\mu} \frac{\partial}{\partial x_{2\,\nu]}} + \mu_{12} \frac{\left[(p_1 \cdot p_2)^2 - m_1^2 m_2^2\right]^{1/2}}{\epsilon^2(p_1, p_2, n)} n^{[\mu} \epsilon^{\nu]}(p_1, p_2, n)$$

- Hilbert space : $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_{12}$

Pairwise helicity vs Scoot

- There is an additional Coulombic contribution:

$$M^{\mu\nu}_{\pm}(p_1,p_2) = \pm\, \mu_{12}\, M^{\mu\nu}(p_1,p_2) + J^{\mu\nu}_{\pm}(p_1,p_2) \propto e_1 e_2$$

At present we note only that the Coulombic contribution to $M^{\mu\nu}$, Eq. (2.5), must vanish because for each pair of particles i and j the total contribution to $M^{\mu\nu}$ is symmetric in i and j, whereas the only antisymmetric covariant tensor that can be formed out of u_i and u_j is antisymmetric in i and j [i.e., $u_i \wedge u_j$ or $(u_i\wedge u_j)^d$], and in the Coulombic term this is multiplied by the symmetric coefficient $e_i e_j + g_i g_j$. (Actually, the Coulombic contribution to the integral (2.5) is ambiguous and depends on the order of integration.)

[Zwanziger '72]

Pairwise helicity vs Scoot

- There is an additional Coulombic contribution:

$$M^{\mu\nu}_{\pm}(p_1,p_2) = \pm \mu_{12} M^{\mu\nu}(p_1,p_2) + J^{\mu\nu}_{\pm}(p_1,p_2) \propto e_1 e_2$$

At present we note only that the Coulombic contribution to $M^{\mu\nu}$, Eq. (2.5), must vanish because for each pair of particles i and j the total contribution to $M^{\mu\nu}$ is symmetric in i and j, whereas the only antisymmetric covariant tensor that can be formed out of u_i and u_j is antisymmetric in i and j [i.e., $u_i \wedge u_j$ or $(u_i \wedge u_j)^d$], and in the Coulombic term this is multiplied by the symmetric coefficient $e_i e_j + g_i g_j$. (Actually, the Coulombic contribution to the integral (2.5) is ambiguous and depends on the order of integration.)

$$J^{\mu\nu}_{\pm}(p_1,p_2) \sim \pm e_1 e_2 \frac{p_1^{[\mu} p_2^{\nu]}}{\left[(p_1\cdot p_2)^2 - p_1^2 p_2^2\right]^{3/2}} \log\left|\frac{I_1}{I_2}\right|$$

[Zwanziger '72]

[Bhardwaj, Lippstreu '22]

Pairwise helicity VS Scoot

- There is an additional Coulombic contribution:

$$M^{\mu\nu}_{\pm}(p_1,p_2) = \pm\, \mu_{12}\, M^{\mu\nu}(p_1,p_2) + J^{\mu\nu}_{\pm}(p_1,p_2) \propto e_1 e_2$$

At present we note only that the Coulombic contribution to $M^{\mu\nu}$, Eq. (2.5), must vanish because for each pair of particles i and j the total contribution to $M^{\mu\nu}$ is symmetric in i and j, whereas the only antisymmetric covariant tensor that can be formed out of u_i and u_j is antisymmetric in i and j [i.e., $u_i \wedge u_j$ or $(u_i\wedge u_j)^d$], and in the Coulombic term this is multiplied by the symmetric coefficient $e_i e_j + g_i g_j$. (Actually, the Coulombic contribution to the integral (2.5) is ambiguous and depends on the order of integration.)

$$J^{\mu\nu}_{\pm}(p_1,p_2) \sim \pm\, e_1 e_2 \frac{p_1^{[\mu}\, p_2^{\nu]}}{\left[(p_1\cdot p_2)^2 - p_1^2 p_2^2\right]^{3/2}} \log\left|\frac{\tau_1}{\tau_2}\right|$$

[Zwanziger '72]

[Bhardwaj, Lippstreu '22]

in gravity:

$$J^{\mu\nu}_{\pm}(p_1,p_2) \sim \pm\, G m_1 m_2 \frac{p_1^{[\mu}\, p_2^{\nu]}\; p_1\cdot p_2 \left(2\, p_1\cdot p_2^{\,2} - 3\, p_1^2 p_2^2\right)}{\left[(p_1\cdot p_2)^2 - p_1^2 p_2^2\right]^{3/2}} \log\left|\frac{\tau_1}{\tau_2}\right|$$

[F.A., Arzano '24]

- Pairs of particles carry, asymptotically, a non-vanishing boost-like angular momentum

Pairwise helicity vs Scoot

- Electromagnetic and gravitational scoot: net shift of the boost charge during the scattering

$$\Delta J^{03} = J_{+}^{03}(k_1,k_2) - J_{-}^{03}(k_1,k_2)$$ [Gralla & Lobo '21]

$$\Delta J^{03}_{E.M.} = 2\,\frac{e_1 e_2}{\gamma^2-1}\log\left|\frac{\tau_1}{\tau_2}\right| \qquad \Delta J^{03}_{G} = \frac{2G\,m_1 m_2\,\gamma(3-2\gamma)}{\gamma^2-1}\log\left|\frac{\tau_1}{\tau_2}\right|$$

Pairwise helicity vs Scoot

- Electromagnetic and gravitational scoot : net shift of the boost charge during the scattering

$$\Delta J^{03} = J_+^{03}(k_1,k_2) - J_-^{03}(k_1,k_2)$$ [Gralla & Lobo '21]

$$\Delta J^{03}_{E.M.} = 2\,\frac{e_1 e_2}{\gamma^2 - 1}\log\left|\frac{\tau_1}{\tau_2}\right| \qquad \Delta J^{03}_{G} = \frac{2G\,m_1 m_2\,\gamma(3-2\gamma)}{\gamma^2-1}\log\left|\frac{\tau_1}{\tau_2}\right|$$

It's an infrared effect : memory? soft theorems?

Pairwise helicity VS Scoot

- Electromagnetic and gravitational scoot: net shift of the boost charge during the scattering

$$\Delta J^{03} = J_{+}^{03}(k_1,k_2) - J_{-}^{03}(k_1,k_2)$$ [Gralla & Lobo '21]

$$\Delta J^{03}_{E.M.} = 2\,\frac{e_1 e_2}{\gamma^2-1}\log\left|\frac{\tau_1}{\tau_2}\right| \qquad \Delta J^{03}_{G} = \frac{2G\,m_1 m_2\,\gamma(3-2\gamma)}{\gamma^2-1}\log\left|\frac{\tau_1}{\tau_2}\right|$$

It's an infrared effect: memory? soft theorems?

Q: Can we repeat Zwanziger's pairwise little group argument for this boost-like angular momentum?

Q: How the deformed boost generator look like?

Pairwise helicity vs Scoot

- Electromagnetic and gravitational scoot: net shift of the boost charge during the scattering

$$\Delta J^{03} = J_+^{03}(\kappa_1, \kappa_2) - J_-^{03}(\kappa_1, \kappa_2)$$ [Gralla & Lobo '21]

$$\Delta J^{03}_{E.M.} = 2\,\frac{e_1 e_2}{\gamma^2 - 1} \log\left|\frac{\tau_1}{\tau_2}\right| \qquad \Delta J^{03}_{G} = \frac{2G\, m_1 m_2\, \gamma(3-2\gamma)}{\gamma^2 - 1} \log\left|\frac{\tau_1}{\tau_2}\right|$$

It's an infrared effect: memory? soft theorems?

Q: Can we repeat Zwanziger's pairwise little group argument for this boost-like angular momentum?

Q: How the deformed boost generator look like?

A: Need to use celestial holography conformal primaries.

Outline

1) Motivations:

- G-W physics and black-hole coalescence
- Infrared effects in gravity (asymptotic symmetries, soft theorems etc.)

2) Pairwise helicity VS gravitational/electromagnetic scoot

3) Conformal primary basis

4) Celestial pairwise little group and quantum numbers

5) Conclusions

Conformal primary basis

On-shell massive momenta p^μ: $p^2 = -m^2$

- $$p^\mu = \frac{m}{2y}\left(1+y^2+|z|^2,\; z+\bar{z},\; i(\bar{z}-z),\; 1-y^2-|z|^2\right) = m\,\hat{p}^\mu, \qquad \hat{p}^2 = -1$$

[Pasterski, Shao, Strominger '16...]

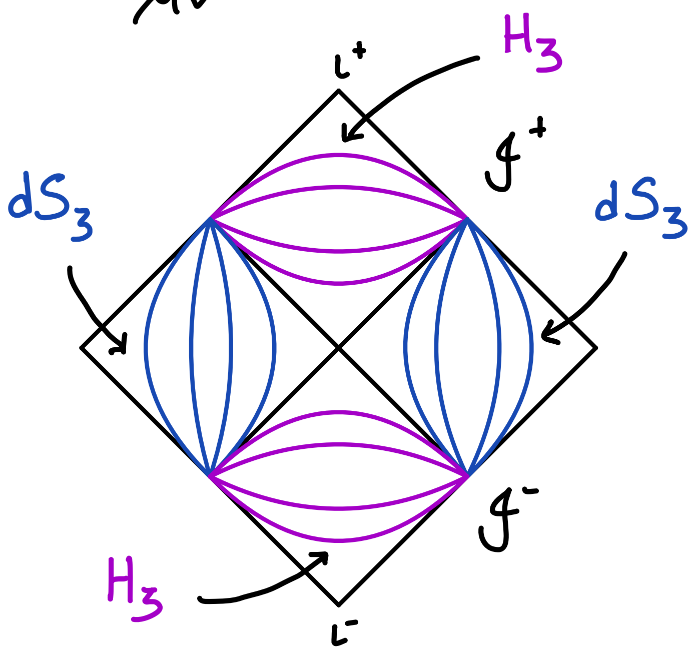
Conformal primary basis

On-shell massive momenta p^μ: $p^2 = -m^2$

- $p^\mu = \frac{m}{2y}\left(1+y^2+|z|^2,\ z+\bar{z},\ i(\bar{z}-z),\ 1-y^2-|z|^2\right) = m\,\hat{p}^\mu, \qquad \hat{p}^2 = -1$

[Pasterski, Shao, Strominger '16...]

- $ds^2 = \eta_{\mu\nu}\, dp^\mu dp^\nu \overset{\text{on-shell}}{=} m^2 \frac{1}{y^2}\left(dy^2 + dz\,d\bar{z}\right) = m^2\, ds^2_{H_3}$



- $y = 0$ conformal boundary
- $y > 0$ bulk

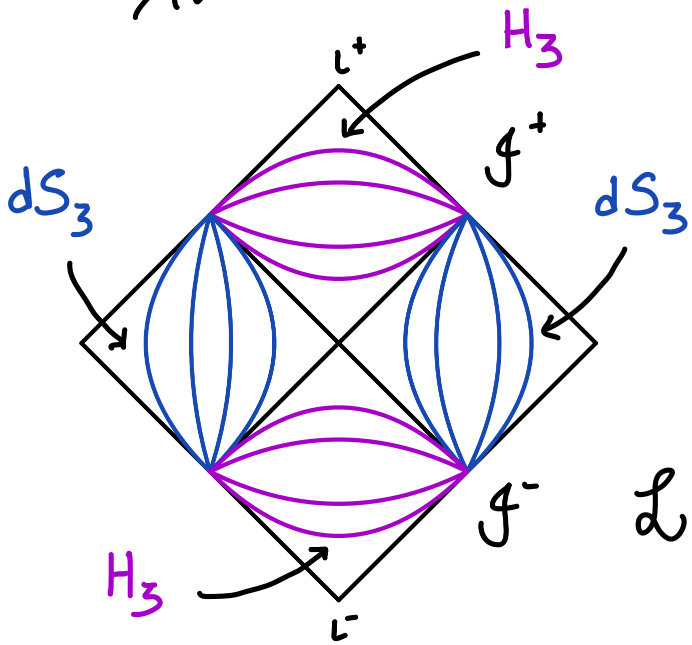
Conformal primary basis

On-shell massive momenta p^μ: $p^2 = -m^2$

- $p^\mu = \dfrac{m}{2y}\left(1+y^2+|z|^2,\ z+\bar z,\ i(\bar z - z),\ 1-y^2-|z|^2\right) = m\,\hat p^\mu, \qquad \hat p^2 = -1$

[Pasterski, Shao, Strominger '16...]

- $ds^2 = \eta_{\mu\nu}\, dp^\mu dp^\nu \overset{\text{on-shell}}{=} m^2 \dfrac{1}{y^2}\left(dy^2 + dz\, d\bar z\right) = m^2\, ds^2_{H_3}$



- $y = 0$ conformal boundary
- $y > 0$ bulk

$\mathcal{L} \simeq SL(2,\mathbb{C})/\mathbb{Z}_2 \supset \Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad ad - bc = 1$

$$y' = \frac{y}{|cz+d|^2 + |c|^2 y^2} \equiv \Lambda y \qquad z' = \frac{(az+b)(\bar c \bar z + \bar d) + a \bar c\, y^2}{|cz+d|^2 + |c|^2 y^2} \equiv \Lambda z$$

Conformal primary basis

- On-shell momentum basis $|P(y,z,\bar{z})\rangle \equiv |y,z\rangle$

Completeness

$$|y,z\rangle = \int_0^\infty \frac{dy'}{y'^3}\int dz'd\bar{z}'\langle y',z'|y,z\rangle|y',z'\rangle$$

Orthonormality

$$\langle y',z'|y,z\rangle = y^3\delta(y-y')\delta^{(2)}(z-z')$$

Plane waves $\quad \psi(\hat{p};x) = \langle x|y,z\rangle \sim e^{\pm im\hat{p}\cdot x}$

Conformal primary basis

- On-shell momentum basis $|P(y,z,\bar{z})\rangle \equiv |y,z\rangle$

Completeness

$$|y,z\rangle = \int_0^\infty \frac{dy'}{y'^3}\int dz' d\bar{z}' \langle y',z'|y,z\rangle |y',z'\rangle$$

Orthonormality

$$\langle y',z'|y,z\rangle = y^3\,\delta(y-y')\,\delta^{(2)}(z-z')$$

Plane waves $\quad \psi(\hat{p};x) = \langle x|y,z\rangle \sim e^{\pm i m \hat{p}\cdot x}$

- Conformal primary basis $|\Delta,w,\bar{w}\rangle \equiv |\Delta,w\rangle$ $\quad \Delta = 1+i\lambda \quad \lambda>0$

"conformal dimension"

$$|\Delta,w\rangle \equiv \int_0^\infty \frac{dy}{y^3}\int dz\,d\bar{z}\, G_\Delta(\hat{p};w)\,|y,z\rangle \qquad G_\Delta(\hat{p};w) \sim \left(\frac{y}{y^2+|w-z|^2}\right)^\Delta$$

[Witten '98]

Conformal primary basis

- On-shell momentum basis $|p(y,z,\bar{z})\rangle \equiv |y,z\rangle$

Completeness

$$|y,z\rangle = \int_0^\infty \frac{dy'}{y'^3}\int dz' d\bar{z}' \langle y',z'|y,z\rangle |y',z'\rangle$$

Orthonormality

$$\langle y',z'|y,z\rangle = y^3\,\delta(y-y')\,\delta^{(2)}(z-z')$$

Plane waves $\quad \psi(\hat{p};x) = \langle x|y,z\rangle \sim e^{\pm i m \hat{p}\cdot x}$

"conformal dimension"

- Conformal primary basis $|\Delta,w,\bar{w}\rangle \equiv |\Delta,w\rangle \qquad \Delta = 1+i\lambda \quad \lambda>0$

$$|\Delta,w\rangle \equiv \int_0^\infty \frac{dy}{y^3}\int dz\,d\bar{z}\, G_\Delta(\hat{p};w)\,|y,z\rangle \qquad G_\Delta(\hat{p};w) \sim \left(\frac{y}{y^2+|w-z|^2}\right)^\Delta$$

[Witten '98]

Completeness

$$|\Delta,w\rangle = \int_0^\infty d\lambda' \int dw' d\bar{w}' \langle \Delta',w'|\Delta,w\rangle |\Delta',w'\rangle$$

Orthonormality

$$\langle \Delta',w'|\Delta,w\rangle = \delta(\lambda-\lambda')\,\delta^{(2)}(w-w')$$

Conformal primary wavefunctions

$$\phi_\Delta(x;w) = \langle x|\Delta,w\rangle \sim \frac{(-x^2)^{(\Delta-1)/2}}{(-q\cdot x \mp i\epsilon)^\Delta} K_{\Delta-1}\left[im(-x^2)^{1/2}\right]$$

[Pasterski, Shao, Strominger '16...]

Boosts and rotations in $(\Delta, w, \bar{w})$

- $(w, \bar{w})$ stereographic coordinates on $\mathcal{I}^\pm$ "$w = z\big|_{y\to 0}$".
- Under Lorentz transformations $\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C})/\mathbb{Z}_2$

$$w' = \frac{aw+b}{cw+d} \equiv \Lambda w \longrightarrow \phi_\Delta(\Lambda\cdot x; \Lambda w) = |cw+d|^{2\Delta}\phi_\Delta(x;w)$$

Boosts and rotations in $(\Delta, w, \bar{w})$

- $(w, \bar{w})$ stereographic coordinates on $\mathcal{I}^{\pm}$ "$w = z\big|_{y \to 0}$".
- Under Lorentz transformations $\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{C})/\mathbb{Z}_2$

$$w' = \frac{a w + b}{c w + d} \equiv \Lambda w \longrightarrow \phi_{\Delta}(\Lambda \cdot x; \Lambda w) = |c w + d|^{2\Delta} \phi_{\Delta}(x; w)$$

- Generators:

$$\phi_{\Delta}(\Lambda \cdot x; \Lambda w) - \phi_{\Delta}(x; w) \overset{\Lambda << 1}{\simeq} \hat{M}(\Lambda; \Delta, w)\, \phi_{\Delta}(x; w)$$

Boost along x^3: $\hat{M}^{03} = \bar{w}\, \partial_{\bar{w}} + w\, \partial_w - \Delta$

orbital part

internal part

[Law, Zlotnikov '20]

The conformal primary basis diagonalizes the action of x^3 boosts!

Boosts and rotations in $(\Delta, w, \bar{w})$

- $(w, \bar{w})$ stereographic coordinates on $\mathcal{C}^{\pm}$ "$w = z\big|_{y\to 0}$".
- Under Lorentz transformations $\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C})/\mathbb{Z}_2$

$$w' = \frac{aw+b}{cw+d} \equiv \Lambda w \longrightarrow \phi_{\Delta}(\Lambda\cdot x; \Lambda w) = |cw+d|^{2\Delta}\phi_{\Delta}(x;w)$$

- Generators:

$$\phi_{\Delta}(\Lambda\cdot x; \Lambda w) - \phi_{\Delta}(x;w) \overset{\Lambda \ll 1}{\simeq} \hat{M}(\Lambda; \Delta, w)\,\phi_{\Delta}(x;w)$$

Boost along x^3: $\hat{M}^{03} = \bar{w}\partial_{\bar{w}} + w\partial_w - \Delta$

orbital part — internal part

[Law, Zlotnikov '20]

The conformal primary basis diagonalizes the action of x^3 boosts!

Spinning conformal primary wavefunctions $\phi_{h,\bar{h}}(x;w) = \langle x | h, \bar{h}; w\rangle$

$\Delta = h + \bar{h}$, $\sigma = h - \bar{h}$

$$\hat{M}^{12} = -i\,\bar{w}\partial_{\bar{w}} + i\,w\partial_w - i\sigma$$

$$\hat{M}^{03} = \bar{w}\partial_{\bar{w}} + w\partial_w - \Delta$$

[Pasterski, Puhm '21]

Outline

1) Motivations:

- G-W physics and black-hole coalescence
- Infrared effects in gravity (asymptotic symmetries, soft theorems etc.)

2) Pairwise helicity vs gravitational/electromagnetic scoot

3) Conformal primary basis

4) Celestial pairwise little group and quantum numbers

5) Conclusions

Little group in celestial variables

stereographic map $(w, \bar{w}) \to (\theta, \phi)$: $w = \tan\frac{\theta}{2} e^{i\phi}$

- $w = 0$ reference direction (north pole) [Banerjee '18]

Little group in celestial variables

stereographic map $(w,\bar{w}) \to (\theta, \phi)$: $w = \tan\frac{\theta}{2} e^{i\phi}$

- $w=0$ reference direction (north pole) [Banerjee '18]

"Dual" little group $D(w=0) \ni \Lambda : w=0 \longmapsto \Lambda w = 0$

$D(w=0) = \{ \hat{M}^{12}, \hat{M}^{03} \}$ rotations and boosts along x^3 preserve $w=0$

$$|h,\bar{h},w\rangle = |\Delta,\sigma,w\rangle \to \begin{array}{l} \mathcal{U}(\hat{M}^{12})|\Delta,\sigma,0\rangle = -i\sigma\,|\Delta,\sigma,0\rangle \\ \mathcal{U}(\hat{M}^{03})|\Delta,\sigma,0\rangle = -\Delta\,|\Delta,\sigma,0\rangle \end{array}$$

Little group in celestial variables

stereographic map $(w,\bar{w}) \to (\theta,\phi)$: $w = \tan\frac{\theta}{2}\, e^{i\phi}$

- $w=0$ reference direction (north pole) [Banerjee '18]

"Dual" little group $D(w=0) \ni \Lambda : w=0 \mapsto \Lambda w = 0$

$D(w=0) = \{\hat{M}^{12}, \hat{M}^{03}\}$ rotations and boosts along x^3 preserve $w=0$

$$|h,\bar{h},w\rangle = |\Delta,\sigma,w\rangle \to \begin{array}{l} \mathcal{U}(\hat{M}^{12})|\Delta,\sigma,0\rangle = -i\sigma\,|\Delta,\sigma,0\rangle \\ \mathcal{U}(\hat{M}^{03})|\Delta,\sigma,0\rangle = -\Delta\,|\Delta,\sigma,0\rangle \end{array}$$

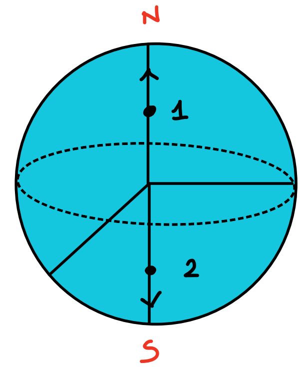
- Induced representations:

$$\mathcal{U}(\Lambda)|h,\bar{h},w\rangle = (cw+d)^{-2h}(\bar{c}\bar{w}+\bar{d})^{-2\bar{h}}\,|h,\bar{h},\Lambda w\rangle \Rightarrow$$

$$\mathcal{U}(\Lambda)|\Delta,\sigma,w\rangle = |cw+d|^{-2\Delta}\left(\frac{cw+d}{\bar{c}\bar{w}+\bar{d}}\right)^{-\sigma}|\Delta,\sigma,\Lambda w\rangle$$

[Banerjee '18]

Pairwise Little group in celestial variables



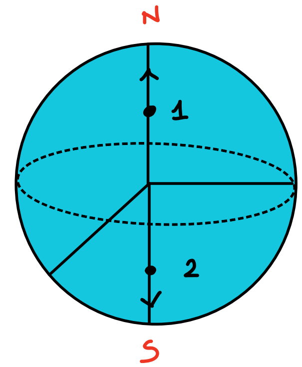
[Lippstreu'21]

- Reference state: $w_1 = 0$, $w_2 = \infty$ (CM)

- Pairwise Dual little group $D(0,\infty) \ni \Lambda$:

$$w_1 = 0 \longmapsto \Lambda w_1 = 0$$
$$w_2 = \infty \longmapsto \Lambda w_2 = \infty$$

Pairwise Little group in celestial variables



[LippsTreu'21]

- Reference state: $w_1 = 0$, $w_2 = \infty$ (CM)

- Pairwise Dual little group $D(0,\infty) \ni \Lambda$: $w_1 = 0 \longmapsto \Lambda w_1 = 0$
  $w_2 = \infty \longmapsto \Lambda w_2 = \infty$

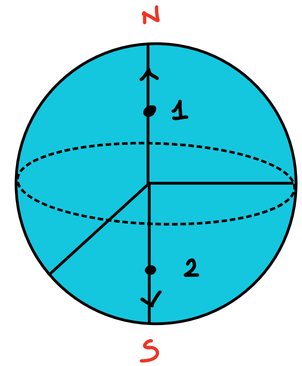
$$D(0,\infty) = \{ \hat{M}^{12}, \hat{M}^{03} \} \rightarrow \begin{array}{l} \mathcal{U}(\hat{M}^{12}) |\Delta_{12}, \sigma_{12}\rangle_{0,\infty} = \mp i \sigma_{12} |\Delta_{12}, \sigma_{12}\rangle_{0,\infty} \\ \mathcal{U}(\hat{M}^{03}) |\Delta_{12}, \sigma_{12}\rangle_{0,\infty} = \mp \Delta_{12} |\Delta_{12}, \sigma_{12}\rangle_{0,\infty} \end{array}$$

"pairwise state"

Δ_{12} ≡ pairwise boost quantum number

σ_{12} ≡ pairwise helicity quantum number

Pairwise Little group in celestial variables



[Lippstreu'21]

- Reference state: $w_1 = 0$, $w_2 = \infty$ (CM)

- Pairwise Dual little group $D(0,\infty) \ni \Lambda$: $w_1 = 0 \longmapsto \Lambda w_1 = 0$

  $w_2 = \infty \longmapsto \Lambda w_2 = \infty$

$$D(0,\infty) = \{\hat{M}^{12}, \hat{M}^{03}\} \rightarrow \begin{aligned} &\mathcal{U}(\hat{M}^{12})|\Delta_{12}, \sigma_{12}\rangle_{0,\infty} = \mp i\sigma_{12}|\Delta_{12}, \sigma_{12}\rangle_{0,\infty} \\ &\mathcal{U}(\hat{M}^{03})|\Delta_{12}, \sigma_{12}\rangle_{0,\infty} = \mp \Delta_{12}|\Delta_{12}, \sigma_{12}\rangle_{0,\infty} \end{aligned}$$

"pairwise state"

Δ_{12} ≡ pairwise boost quantum number

σ_{12} ≡ pairwise helicity quantum number

- The conformal primary basis "naturally" accounts for pairwise boost and helicity quantum numbers.

Pairwise Little group in celestial variables

- Induced representations? [Csaba Csáki et al. '21]

Two-particle state $|1,2\rangle \equiv |\Delta_1, \sigma_1, w_1\rangle \otimes |\Delta_2, \sigma_2, w_2\rangle \otimes |\Delta_{12}, \sigma_{12}\rangle_{0,\infty}$

$$\mathcal{U}(\Lambda)|1,2\rangle = \prod_{i=1}^{2} |c w_i + d|^{-2(\Delta_i \mp \Delta_{12})} \left(\frac{c w_i + d}{\bar{c}\bar{w}_i + \bar{d}}\right)^{-(\sigma_i \mp \sigma_{12})} |1,2\rangle$$

[Lippstreu '21]

In practice: $\Delta_i \longrightarrow \Delta_i \mp \frac{\Delta_{12}}{2}$ and $\sigma_i \longrightarrow \sigma_i \mp \frac{\sigma_{12}}{2}$

Pairwise Little group in celestial variables

- Induced representations? [Csaba Csáki et al. '21]

Two-particle state $|1,2\rangle \equiv |\Delta_1, \sigma_1, w_1\rangle \otimes |\Delta_2, \sigma_2, w_2\rangle \otimes |\Delta_{12}, \sigma_{12}\rangle_{0,\infty}$

$$\mathcal{U}(\Lambda)|1,2\rangle = \prod_{i=1}^{2} |c w_i + d|^{-2(\Delta_i \mp \Delta_{12})} \left(\frac{c w_i + d}{\bar{c}\bar{w}_i + \bar{d}}\right)^{-(\sigma_i \mp \sigma_{12})} |1,2\rangle$$

[Lippstreu '21]

In practice: $\Delta_i \longrightarrow \Delta_i \mp \frac{\Delta_{12}}{2}$ and $\sigma_i \longrightarrow \sigma_i \mp \frac{\sigma_{12}}{2}$

- Generators:

new pairwise pieces

$$\hat{M}^{12} = -i\,\bar{w}_1 \partial_{\bar{w}_1} + i\, w_1 \partial_{w_1} - i\,\sigma_1 + (1 \longleftrightarrow 2) \pm i\,\sigma_{12}$$

$$\hat{M}^{03} = \bar{w}_1 \partial_{\bar{w}_1} + w_1 \partial_{w_1} - \Delta_1 + (1 \longleftrightarrow 2) \pm \Delta_{12}$$

[F.A., Arzano '24]

Matching the scoot (and the pairwise helicity)

Group theoretically there is space for $(\Delta_{12}, \sigma_{12})$

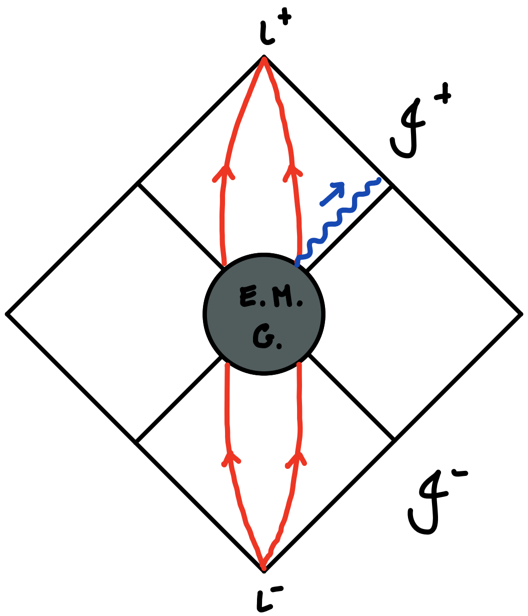
Q: What's their physical interpretation? [F.A., Arzano '24]

Matching the scoot (and the pairwise helicity)

Group theoretically there is space for $(\Delta_{12}, \sigma_{12})$

Q: What's their physical interpretation? [F.A., Arzano '24]

$2 \to 2$ scattering in electromagnetism or gravity:



Asymptotic angular momenta:

$$_{\text{E.M.}}M^{12}_{\pm}(k_1,k_2) = \pm\, \mu_{12}$$

$$_{\text{E.M.}}M^{03}(k_1,k_2) = \pm \frac{e_1 e_2}{\gamma^2-1} \log\left|\frac{\tau_1}{\tau_2}\right|$$

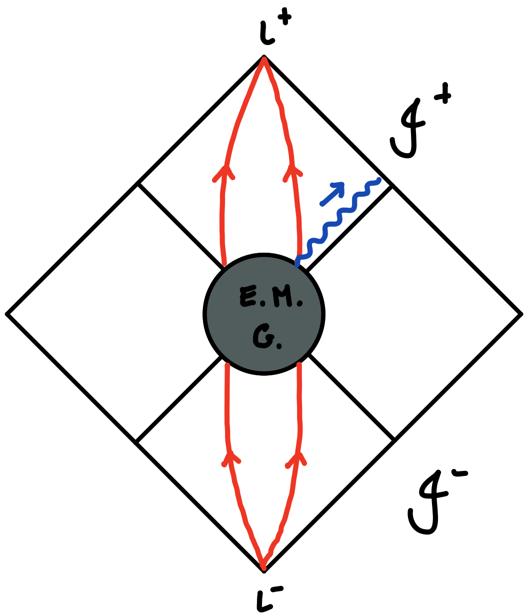
$$_{\text{G.}}M^{03}(k_1,k_2) = \pm \frac{2G\, m_1 m_2\, \gamma\,(3-2\gamma)}{\gamma^2-1} \log\left|\frac{\tau_1}{\tau_2}\right|$$

Matching the scoot (and the pairwise helicity)

Group theoretically there is space for $(\Delta_{12}, \sigma_{12})$

Q: What's their physical interpretation? [F.A., Arzano '24]

$2 \to 2$ scattering in electromagnetism or gravity:



Asymptotic angular momenta:

$$ {}_{\text{E.M.}}M^{12}_{\pm}(k_1,k_2) = \pm \mu_{12} $$

$$ {}_{\text{E.M.}}M^{03}(k_1,k_2) = \pm \frac{e_1 e_2}{\gamma^2 - 1} \log\left|\frac{T_1}{T_2}\right| $$

$$ {}_{\text{G.}}M^{03}(k_1,k_2) = \pm \frac{2 G m_1 m_2 \gamma (3 - 2\gamma)}{\gamma^2 - 1} \log\left|\frac{T_1}{T_2}\right| $$

- Matching with the algebra:

$$ {}_{\text{E.M.}}\Delta_{12} = \frac{e_1 e_2}{\gamma^2 - 1} \log\left|\frac{T_1}{T_2}\right| \qquad {}_{\text{G.}}\Delta_{12} = \frac{2 G m_1 m_2 \gamma (3 - 2\gamma)}{\gamma^2 - 1} \log\left|\frac{T_1}{T_2}\right| $$

$$ {}_{\text{E.M.}}\sigma_{12} = \mu_{12} = e_1 g_2 - e_2 g_1 $$

Outline

1) Motivations:

- G-W physics and black-hole coalescence
- Infrared effects in gravity (asymptotic symmetries, soft theorems etc.)

2) Pairwise helicity VS gravitational/electromagnetic scoot

3) Conformal primary basis

4) Celestial pairwise little group and quantum numbers

5) Conclusions

Conclusions

- Non-vanishing asymptotic values of the angular momentum after electromagnetic or gravitational scattering can be taken into account by an independent group-theoretical analysis, using conformal primary states in celestial holography
- The gravitational scoot has a group-theoretical interpretation
- Infrared effects prevent exact factorization of multi-particle Hilbert space
- When gravity is turned on ($G \neq 0$), $\Delta_{12} \neq 0$ and our standard assumptions about asymptotic scattering must be revisited.