Horizon temperature and entanglement entropy in conformal quantum mechanics

Michele Arzano

Università di Napoli Federico II
and
INFN Napoli

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In quantum field theory the notion of **vacuum state** has no universal meaning

"Before the 70s nobody thought very much about "for whom" the vacuum state appears devoid of "stuff"..."

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- **vacuum state** ambiguity = <u>different possible choices of **time-like Killing vectors**</u>

in particular: the vacuum of **"horizonless"** Killing vector
=
thermal state for a **non-globally time-like** Killing vector

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This talk:

MA, JHEP **05**, 072 (2020) [arXiv:2002.01836 [gr-qc]], JHEP **07**, 003 (2021) [arXiv:2103.07228 [hep-th]]

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This talk:

- provide a unified, group-theoretical description of (the conformal quantum mechanics counterparts of) **Milne and diamond temperature**
- **simplest (toy) model** where analytic calculation of **entanglement entropy** associated to a **partition** induced by the modular Hamiltonian is possible

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Radial conformal motions in Minkowski space-time

Minkowski metric in spherical coordinates

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$$\xi = \left(a(t^2 + r^2) + bt + c\right)\partial_t + r(2at + b)\,\partial_r$$

with a, b, c <u>real constants</u> **(Herrero and Morales, J. Math. Phys. 40, 3499 (1999))**

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Key observation: this conformal Killing vector can be written as

$$\xi = aK_0 + bD_0 + cP_0\,,$$

where K_0, D_0 and P_0 generate, respectively,
special conformal transformations, **dilations** and **time translations**

Families of radial conformal Killing vectors

The generators K_0, D_0 and P_0

$$P_0 = \partial_t \,, \qquad D_0 = r\,\partial_r + t\,\partial_t \,, \qquad K_0 = 2tr\,\partial_r + (t^2 + r^2)\,\partial_t$$

close the $\mathfrak{sl}(2,\mathbb{R})$ Lie algebra

$$[P_0, D_0] = P_0 \,, \qquad [K_0, D_0] = -K_0 \,, \qquad [P_0, K_0] = 2D_0$$

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- $\Delta < 0$: **elliptic transformations** ($\mathfrak{sl}(2,\mathbb{R}) \simeq \mathfrak{so}(2,1) \to$ rotations)

$$R_0 = \frac{1}{2}\left(\alpha P_0 + \frac{K_0}{\alpha}\right)$$

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α **is a constant with dimensions of length**, crucial in what follows...

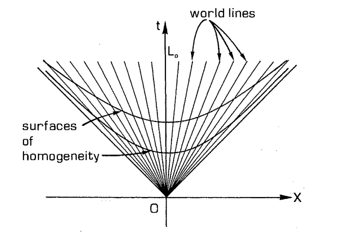
Milne time evolution

$$D_0 = r\,\partial_r + t\,\partial_t$$

generates **conformal time evolution** in a **Milne universe** (**Minkowski future cone**)

$$ds^2 = -d\bar{t}^2 + \bar{t}^2\left(d\chi^2 + \sinh\chi^2 d\Omega^2\right)$$

where $t = \bar{t}\cosh\chi$ and $r = \bar{t}\sinh\chi$ (notice similarity with Rindler coordinates...)

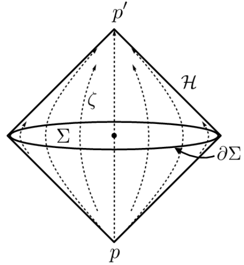


(from Ellis and Williams, "Flat and Curved Space-Times")

Diamond time

$$S_0 = \frac{1}{2}\left(\alpha P_0 - \frac{K_0}{\alpha}\right) = \frac{1}{2}\left(\alpha \partial_t - \frac{1}{\alpha}\left(2t\, r\, \partial_r - \left(t^2 + r^2\right)\partial_t\right)\right)$$

maps a **causal diamond of radius** α into itself **(Jacobson, PRL 116(2016)20)**



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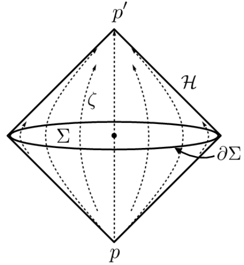

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Worldline conformal time evolution

Along $r = \mathrm{const}$ worldlines and on the light cones $u = t - r = \mathrm{const}$, $v = t + r = \mathrm{const}$

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- $S_0 = \frac{1}{2\alpha} \left(\alpha^2 - \tau^2 \right) \partial_\tau$ generates translation in **"diamond time"** σ: $S_0/\alpha = \partial_\sigma$

  $$\tau = \alpha \tanh \sigma/2\alpha$$

  covering the region $|\tau| < \alpha$: the "diamond"

Conformal quantum mechanics

As it turns out

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Starting from the Lagrangian

$$\mathcal{L} = \frac{1}{2}\left(\dot{q}(t)^2 + \frac{g}{q(t)^2}\right), \qquad g > 0$$

the $\mathfrak{sl}(2,\mathbb{R})$ algebra can be canonically realized

$$\begin{aligned} H = iP_0 &= \frac{1}{2}\left(p^2 + \frac{g}{q^2}\right) \\ D = iD_0 &= t\,H - \frac{1}{4}\left(pq + qp\right) \\ K = iK_0 &= -t^2\,H + 2t\,D + \frac{1}{2}q^2 \end{aligned}$$

Conformal quantum mechanics as a CFT_1

Tha dAFF model can be interpreted as CFT_1

(Chamon, Jackiw, Pi and Santos, Phys. Lett. B 701, 503 (2011); Jackiw and Pi, Phys. Rev. D 86, 045017 (2012))

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CFT_1 two-point function

The $|\tau\rangle$ states can be characterized by their overlap with $|n\rangle$ states

$$\langle\tau|n\rangle = (-1)^n \left[\frac{\Gamma(2r_0+n)}{n!}\right]^{\frac{1}{2}} \left(\frac{\alpha - i\tau}{\alpha + i\tau}\right)^{r_n} \left(1+\frac{\tau^2}{\alpha^2}\right)^{-r_0}$$

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from which one obtains the inner product

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For $r_0 = 1$: two-point function of a **massless scalar field in Minkowski space-time**, evaluated along the **worldline of an inertial observer** sitting at the origin

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$$\langle\tau_1|\tau_2\rangle = \frac{\Gamma\left(2r_0\right)\alpha^{2r_0}}{\left[2i\left(\tau_1-\tau_2\right)\right]^{2r_0}}$$

which Jackiw, Pi et al. interpret as **the two-point function** of the CFT_1

For $r_0 = 1$: two-point function of a **massless scalar field in Minkowski space-time**, evaluated along the **worldline of an inertial observer** sitting at the origin

ASIDE: this is reminiscent of the $SL(2,\mathbb{R})$-invariant **wordline quantum mechanics** for **static patch observers** in de Sitter space-time

(Anninos, Hartnoll and Hofman, Class. Quant. Grav. 29, 075002 (2012))

A bi-partite vacuum state

As shown by Jackiw, Pi et al. we can re-write the CFT_1 two-point function as

$$G_2(\tau_1, \tau_2) \equiv \langle \tau_1 | \tau_2 \rangle = \langle \tau = 0 | e^{-i(\tau_1 - \tau_2)H} | \tau = 0 \rangle$$

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so that

$$|n = 0\rangle = |0\rangle_L \otimes |0\rangle_R$$

the ground state $|n = 0\rangle$ has a **bi-partite structure**!

CFT_1 vacua

Notice now that the Lie algebra

$$[L_-, L_+] = 2L_0\,, \quad [L_0, L_\pm] = \pm L_\pm$$

can be realized via another combination of H, D and K, namely

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as in the "real world" the Hartle-Hawking vacuum is a **thermofield double state** built on the bi-partite Boulware vacuum

The thermofield double of CFT_1

With simple manipulations

$$
\begin{aligned}
|\tau = 0\rangle &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(a_L^\dagger a_R^\dagger \right)^n |0\rangle_L \otimes |0\rangle_R = \sum_{n=0}^{\infty} (-1)^n |n\rangle_L \otimes |n\rangle_R \\
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mini-detour: given a set of eigenstates $H|k\rangle = E_k|k\rangle$ for a quantum system, the **thermofield double state** is built by "doubling" the system

$$
|TFD\rangle = \frac{1}{Z(\beta)} \sum_{k=0}^{\infty} e^{-\beta E_k/2} |k\rangle_L \otimes |k\rangle_R
$$

tracing over the degrees of freedom of one copy $\Rightarrow$ thermal density matrix at $T = 1/\beta$

$$
Tr_L(|TFD\rangle\langle TFD|) = e^{-\beta H}
$$

Diamond temperature

The inertial vacuum

$$|\tau = 0\rangle = -\sum_{n=0}^{\infty} e^{-\pi S}\, |n\rangle_L \otimes |n\rangle_R$$

has the structure of a thermofield double state with temperature

$$T_S = \frac{1}{2\pi\alpha}$$

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This is just the **diamond temperature** for diamond observers at the origin

(Su and Ralph, Phys. Rev. D 93, no.4, 044023 (2016))

Indeed the **two-point function** for a **diamond observer** sitting at the origin coincides with the CFT_1 two-point function in **diamond time** and both are **periodic in imaginary time** i.e. thermal at temperature $T_S = \frac{1}{2\pi\alpha}$

(MA, JHEP 05, 072 (2020) [arXiv:2002.01836 [gr-qc]])

From the diamond to Milne

S and D belong to the same class of generators of **hyperbolic time evolution**

$$\downarrow$$

one can find a $SL(2,\mathbb{R})$ transformation **mapping one into another**

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note: this is the map from the **causal diamond to the Rindler wedge** used to derive the **diamond modular Hamiltonian** from the Rindler one (in light-cone coordinates)

(Casini, Huerta and Myers, JHEP 05, 036 (2011))

The Milne temperature

The conformal map

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the "inertial" vacuum $|\tau = 0\rangle$ is the thermofield double for the Hamiltonian D/α at the **Milne temperature** **(Olson and Ralph, PRL 106, 110404 (2011), arXiv:1003.0720)**

$$T_D = \frac{1}{2\pi\alpha}$$

Getting hot without accelerating

Observers whose worldlines are **integral curves of time-like RCKV**

$$\xi = aK_0 + bD_0 + cP_0$$

are **accelerated** **(Herrero and Morales, J. Math. Phys. 40, 3499 (1999))**

$$|\mathbf{a}| = \frac{2|a|}{\sqrt{\omega - \Delta}}$$

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You can get hot without accelerating!
(if you enjoy conformal symmetry...)

Entanglement entropy

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$$S = \log\left(\frac{\alpha}{\epsilon}\right) + \mathrm{const} + \mathcal{O}(\epsilon^2)$$

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group-theoretic evidence for Milne and diamond temperatures

the inertial vacuum is a thermal state for observers whose **time evolution is not eternal**

thermodynamic properties of the **Milne "patch"** and of **causal diamonds** are deeply connected...

$\Longrightarrow$ new tools for studying **entanglement and "modular fluctuations"** (Verlinde and Zurek Phys.Lett.B **822** (2021)) in Minkowski space-time?

$\Longrightarrow$ run same arguments using **affine transformations of the real line**

(MA and J.Kowalski-Glikman, Phys. Lett. B **788**, 82-86 (2019) [arXiv:1804.10550 [hep-th]].)