Gymnastics in Phase Space

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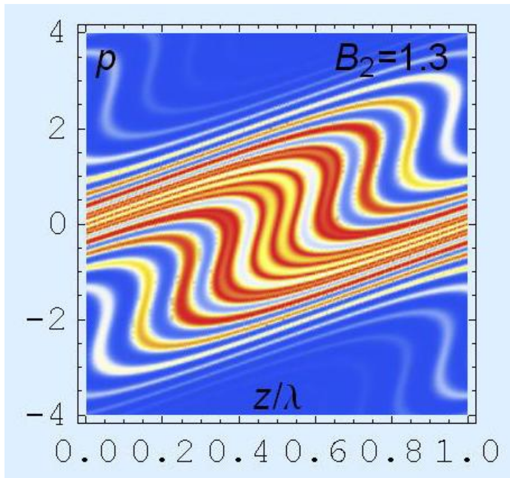
Abstract

As accelerator technology advances, the requirements on accelerator beam quality become increasingly demanding. Facing these new demands, the topic of phase space gymnastics is becoming a newly-focused and critical area of R&D. In a phase space gymnastic, the beam's phase space distribution is manipulated and precision tailored to meet the required beam quality. On the other hand, all realization of such gymnastics will have to obey accelerator physics principles as well as technological limitations. Recent examples of phase space gymnastics include Emittance exchanges, Phase space exchanges, Emittance partitioning, Seeded FELs and Microbunched radiators. In this talk, we will review the emittance related topics. The accelerator physics basis, the lattice designs that provide these phase space manipulations, and the possible applications of these gymnastics, will be discussed. This fascinating field promises to be a focused attention towards the future.

As we demand more and more from accelerators, beam technology gets more advanced, phase space gymnastics becomes very acrobatic and a topic both critical and beautiful.



Ability to manipulate 6D phase space offers many precision-oriented operations of the beam, and opens up many applications in beam manipulations and diagnostics, e.g. for linear colliders and FELs. This is a new and fertile R&D field.

Examples of phase space gymnastics:

- Various RF gymnastics in the past
- Adapters
- Emittance exchanges
- Phase space exchanges
- Emittance partitioning
- Various seeded FEL microbunching schemes

We will discuss only the few emittance related topics (underlined).

Phase space gymnastics permit precision manipulations because phase space is conserved. Liouville theorem is the root cause of these phase space effects. Once something is done to the phase space, no matter how minute, it will be remembered.

This report follows the footstep of many pioneers:

A. Bogacz	R. Brinkmann
A. Burov	Y. Cai
B. Carlsten	M. Cornacchia
V. Danilov	Y. Derbenev
A. Dragt	D. Edwards
P. Emma	K. Floettmann
Y. Jiao	K.J. Kim
V. Lebedev	S. Nagaitsev
J. Peterson	G. Stupakov
R. Talman	L. Teng
D. Xiang	N. Yampolsky
A. Zholents	M. Zolotorev
etc.	

I. Flat-to-round and round-to-flat adapters

Consider the 4D canonical phase space $X_{can}=(x, p_x, y, p_y)$. We have two representations to describe particle motion in phase space:

(1) For an uncoupled lattice, use the Courant-Snyder basis of <u>planar modes</u> (x and y modes): X = Va, where

$$\mathbf{V}=\begin{pmatrix} \sqrt{\beta_x}\cos(\phi_x) & \sqrt{\beta_x}\sin(\phi_x) & 0 & 0 \\ \dfrac{-\alpha_x\cos(\phi_x)-\sin(\phi_x)}{\sqrt{\beta_x}} & \dfrac{-\alpha_x\sin(\phi_x)+\cos(\phi_x)}{\sqrt{\beta_x}} & 0 & 0 \\ 0 & 0 & \sqrt{\beta_y}\cos(\phi_y) & \sqrt{\beta_y}\sin(\phi_y) \\ 0 & 0 & \dfrac{-\alpha_y\cos(\phi_y)-\sin(\phi_y)}{\sqrt{\beta_y}} & \dfrac{-\alpha_y\sin(\phi_y)+\cos(\phi_y)}{\sqrt{\beta_y}} \end{pmatrix}$$

and

$$\mathbf{a}=(\sqrt{2J_x}\sin\chi_x\,,\sqrt{2J_x}\cos\chi_x\,,\;\sqrt{2J_y}\sin\chi_y\,,\sqrt{2J_y}\cos\chi_y)$$

for a planar beam with x- and y-emittances = J_x, J_y.

Lattice parameters = α_x, β_x, ϕ_x, α_y, β_y, ϕ_y ← We are familiar with this representation.

(2) For a fully coupled, rotational symmetric lattice, one uses the basis of <u>circular modes</u> (left-handed and right handed modes): X = Ua where U =

$$\frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{\beta}\cos\phi_+ & \sqrt{\beta}\sin\phi_+ & -\sqrt{\beta}\cos\phi_- & -\sqrt{\beta}\sin\phi_- \\ \frac{-\sin\phi_+ - \alpha\cos\phi_+}{\sqrt{\beta}} & \frac{\cos\phi_+ - \alpha\sin\phi_+}{\sqrt{\beta}} & \frac{\sin\phi_- + \alpha\cos\phi_-}{\sqrt{\beta}} & \frac{-\cos\phi_- + \alpha\sin\phi_-}{\sqrt{\beta}} \\ \sqrt{\beta}\sin\phi_+ & -\sqrt{\beta}\cos\phi_+ & \sqrt{\beta}\sin\phi_- & -\sqrt{\beta}\cos\phi_- \\ \frac{\cos\phi_+ - \alpha\sin\phi_+}{\sqrt{\beta}} & \frac{\sin\phi_+ + \alpha\cos\phi_+}{\sqrt{\beta}} & \frac{\cos\phi_- - \alpha\sin\phi_-}{\sqrt{\beta}} & \frac{\sin\phi_- + \alpha\cos\phi_-}{\sqrt{\beta}} \end{pmatrix}$$

and

$$\mathbf{a} = (\sqrt{2J_+}\sin\chi_+ , \sqrt{2J_+}\cos\chi_+ , \sqrt{2J_-}\sin\chi_- , \sqrt{2J_-}\cos\chi_-)$$

This beam has right-handed and left-handed emittances J_+, J_-.

[Lebedev, Bogacz, 1999]

Lattice parameters = α, β, ϕ_+, ϕ_- ← There is only one β-function but two phases.

Once we have the planar basis V (of a decoupled lattice) and the circular basis U (of a rotational symmetric lattice) --- both are symplectic --- one can consider "adapters" to go from one to the other.

<u>Flat-to-flat adapter</u> from s_1 to s_2 is well known. The job is to design a transport lattice that provides the map from $V(s_1)$ to $V(s_2)$, i.e. the lattice matching one set of lattice parameters into another is given by the map $V(s_2)V(s_1)^{-1}$ =

$$\begin{pmatrix} \sqrt{\frac{bx2}{bx}}\left(\mathrm{Cos}[mux] + ax\,\mathrm{Sin}[mux]\right) & \sqrt{bx\,bx2}\,\mathrm{Sin}[mux] & 0 & 0 \\ \frac{(ax - ax2)\,\mathrm{Cos}[mux]}{\sqrt{bx\,bx2}} + \frac{(-1 - ax\,ax2)\,\mathrm{Sin}[mux]}{\sqrt{bx\,bx2}} & \sqrt{\frac{bx}{bx2}}\left(\mathrm{Cos}[mux] - ax2\,\mathrm{Sin}[mux]\right) & 0 & 0 \\ 0 & 0 & \sqrt{\frac{by2}{by}}\left(\mathrm{Cos}[muy] + ay\,\mathrm{Sin}[muy]\right) & \sqrt{by\,by2}\,\mathrm{Sin}[muy] \\ 0 & 0 & \frac{(ay - ay2)\,\mathrm{Cos}[muy]}{\sqrt{by\,by2}} + \frac{(-1 - ay\,ay2)\,\mathrm{Sin}[muy]}{\sqrt{by\,by2}} & \sqrt{\frac{by}{by2}}\left(\mathrm{Cos}[muy] - ay2\,\mathrm{Sin}[muy]\right) \end{pmatrix}$$

which is a well known result.

Round-to-round adapter from s_1 to s_2, i.e. one set of circular lattice parameters to another. The map is $U(s_2)U(s_1)^{-1}$, and it can be shown that it has the general form

$$\begin{pmatrix} \mathrm{Cos}[\theta] & 0 & \mathrm{Sin}[\theta] & 0 \\ 0 & \mathrm{Cos}[\theta] & 0 & \mathrm{Sin}[\theta] \\ -\mathrm{Sin}[\theta] & 0 & \mathrm{Cos}[\theta] & 0 \\ 0 & -\mathrm{Sin}[\theta] & 0 & \mathrm{Cos}[\theta] \end{pmatrix} \begin{pmatrix} \sqrt{\frac{b2}{b}}\,(\mathrm{Cos}[mu] + a\,\mathrm{Sin}[mu]) & \sqrt{b\,b2}\,\mathrm{Sin}[mu] & 0 & 0 \\ \frac{(a-a2)\,\mathrm{Cos}[mu]}{\sqrt{b\,b2}} + \frac{(-1-a\,a2)\,\mathrm{Sin}[mu]}{\sqrt{b\,b2}} & \sqrt{\frac{b}{b2}}\,(\mathrm{Cos}[mu] - a2\,\mathrm{Sin}[mu]) & 0 & 0 \\ 0 & 0 & \sqrt{\frac{b2}{b}}\,(\mathrm{Cos}[mu] + a\,\mathrm{Sin}[mu]) & \sqrt{b\,b2}\,\mathrm{Sin}[mu] \\ 0 & 0 & \frac{(a-a2)\,\mathrm{Cos}[mu]}{\sqrt{b\,b2}} + \frac{(-1-a\,a2)\,\mathrm{Sin}[mu]}{\sqrt{b\,b2}} & \sqrt{\frac{b}{b2}}\,(\mathrm{Cos}[mu] - a2\,\mathrm{Sin}[mu]) \end{pmatrix}$$

There are two ways to realize this:

(a) a quadrupole channel, followed by rotating the entire subsequent beamline elements by $-\theta$.

(b) A uniform solenoid (including its two ends) with strength k_s and length z will produce this map with $\theta = \mu = k_s z/2$, $\beta = \beta_2 = 2/k_s$, $\alpha = \alpha_2 = 0$.

Round-to-flat adapter

[Derbenev 1998, Edwards 2001, Kim 2003]

This adapter is given by $U(s_2)V(s_1)^{-1}$, which can be shown to have a general form (flat-to-flat) x (UV^{-1}) x (round-to-round), where

$$UV^{-1} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{\sqrt{b}\,\mathrm{Cos}[mu]}{\sqrt{by}} & \sqrt{b}\,\sqrt{by}\,\mathrm{Sin}[mu] & 0 & 0 \\ -\frac{\mathrm{Sin}[mu]}{\sqrt{b}\,\sqrt{by}} & \frac{\sqrt{by}\,\mathrm{Cos}[mu]}{\sqrt{b}} & 0 & 0 \\ 0 & 0 & \frac{\sqrt{b}\,\mathrm{Sin}[mu]}{\sqrt{by}} & -\sqrt{b}\,\sqrt{by}\,\mathrm{Cos}[mu] \\ 0 & 0 & \frac{\mathrm{Cos}[mu]}{\sqrt{b}\,\sqrt{by}} & \frac{\sqrt{by}\,\mathrm{Sin}[mu]}{\sqrt{b}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$$

which is a regular quadrupole channel (3 quadrupoles minimum) rotated 45°. Design of the adapter therefore reduces to a regular lattice matching problem.

Inserting an adapter from a round optics with (α, β, ϕ_+, ϕ_-) to a flat optics with (α_x, β_x, ϕ_x, α_y, β_y, ϕ_y)

➔

A round beam with left-handed- and right-handed-emittances = (J_+, J_-) is transformed to a planar beam with x- and y-emittances = (J_+, J_-).

Flat-to-round adapter

Reversing the round-to-flat adapter

➔

A flat beam with x- and y-emittances = (J_x, J_y) is transformed to a round beam with left-handed- and right-handed-emittances = (J_x, J_y).

Applications of flat-to-round and round-to-flat adapters

The idea of adapters was first introduced by Derbenev 1993 to control the beam-beam effect in storage ring colliders. But it has been much extended for other applications:

(1) Storage ring colliders:

planar beam in regular arc cells
(flat-to-round adapter)--->
circular beam in collision region with solenoid
(round-to-flat adapter)--->
planar beam in arc cells.

This possibly reduces the beam-beam effect due to much fewer number of nonlinear resonances.

(2) Linear colliders:

[Brinkmann, Derbenev, Floettmann, 1999, Edwards et al, 2000]

With round beam produced at magnetized cathode, a round-to-flat adapter avoids the need of damping ring.

(3) Relativistic electron beam cooling:

Apply a flat-to-round adapter to a very flat beam ($J_x >> J_y$)
➔ A round beam is produced with $J_+ >> J_-$.
Immersing the beam in a matched solenoid with appropriate B_s
➔ Particles all move straight ahead with no Larmor precession and therefore no temperature!

II. Emittance and Phase space exchangers

There are more adapter types. For example, one may stay with flat-to-flat, but wish to exchange the x and y coordinates. This means we want an adapter that transforms the base vectors from

$$V = \begin{pmatrix} \sqrt{bx}\ \mathrm{Cos}[phx] & \sqrt{bx}\ \mathrm{Sin}[phx] & 0 & 0 \\ \frac{-ax\,\mathrm{Cos}[phx]-\mathrm{Sin}[phx]}{\sqrt{bx}} & \frac{\mathrm{Cos}[phx]-ax\,\mathrm{Sin}[phx]}{\sqrt{bx}} & 0 & 0 \\ 0 & 0 & \sqrt{by}\ \mathrm{Cos}[phy] & \sqrt{by}\ \mathrm{Sin}[phy] \\ 0 & 0 & \frac{-ay\,\mathrm{Cos}[phy]-\mathrm{Sin}[phy]}{\sqrt{by}} & \frac{\mathrm{Cos}[phy]-ay\,\mathrm{Sin}[phy]}{\sqrt{by}} \end{pmatrix}$$

to

$$V' = \begin{pmatrix} 0 & 0 & \sqrt{bx2}\ \mathrm{Cos}[phx2] & \sqrt{bx2}\ \mathrm{Sin}[phx2] \\ 0 & 0 & \frac{-ax2\,\mathrm{Cos}[phx2]-\mathrm{Sin}[phx2]}{\sqrt{bx2}} & \frac{\mathrm{Cos}[phx2]-ax2\,\mathrm{Sin}[phx2]}{\sqrt{bx2}} \\ \sqrt{by2}\ \mathrm{Cos}[phy2] & \sqrt{by2}\ \mathrm{Sin}[phy2] & 0 & 0 \\ \frac{-ay2\,\mathrm{Cos}[phy2]-\mathrm{Sin}[phy2]}{\sqrt{by2}} & \frac{\mathrm{Cos}[phy2]-ay2\,\mathrm{Sin}[phy2]}{\sqrt{by2}} & 0 & 0 \end{pmatrix}$$

The adapter needs to provide the map $V'V^{-1}$. Obviously, when $\alpha_{x2}= \alpha_y$, $\alpha_{y2}= \alpha_x$, $\beta_{x2}= \beta_y$, $\beta_{y2}=\beta_y$, $\phi_{x2}=\phi_y$ and $\phi_{y2}=\phi_x$, we have $V'V^{-1} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$. One way to produce the map $\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ is a solenoid with $k_s z = \pi$. When inserted, this adapter will cause x- and y-phase spaces and emittances to be exchanged.

One can also exchange x and z instead of x and y. Consider a planar lattice in X = (x,x',z,δ) coordinates. Let the transformation map be

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

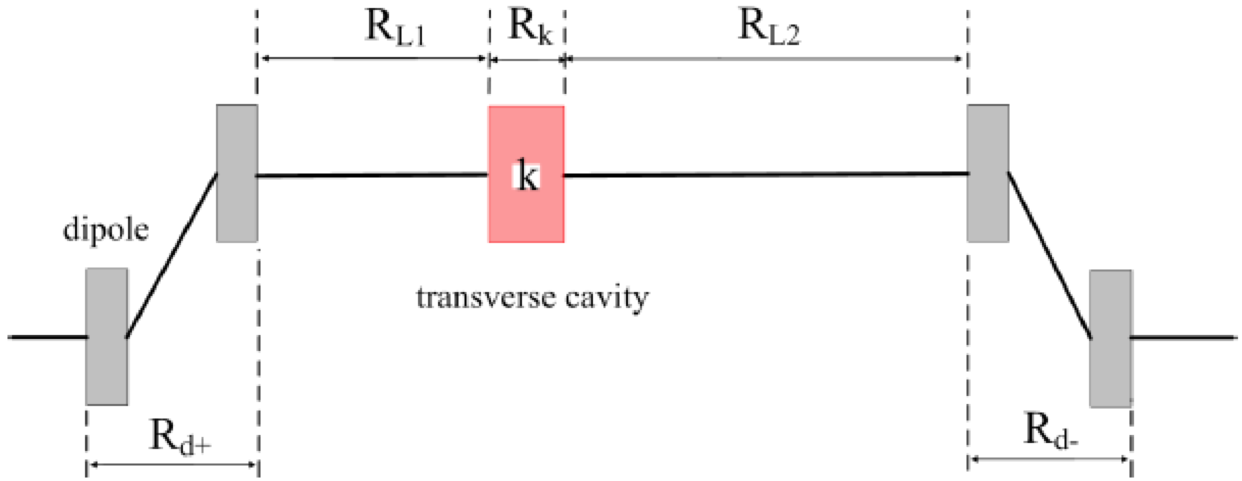
An emittance exchanger (EEX) requires A=D=0:

(x,x') ⇔ (z,δ) switching two blocks

A phase space exchanger requires A=D=0 and B and C = diagonal.

x ⇔ z, x' ⇔ δ switching four coordinates

<u>Cornacchia-Emma EEX</u>

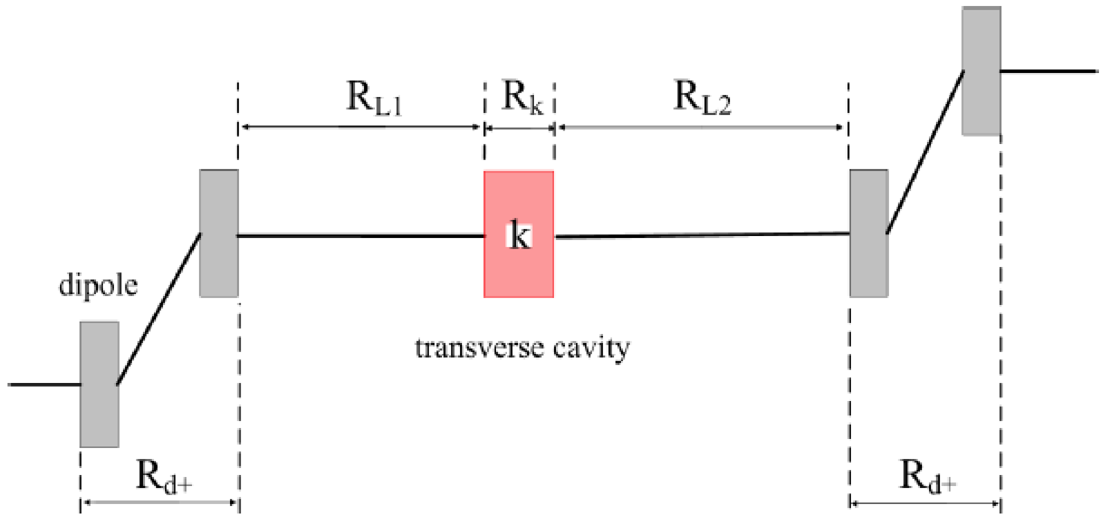


$$\begin{bmatrix} 1 & L & 0 & -\eta \\ 0 & 1 & 0 & 0 \\ 0 & -\eta & 1 & \xi \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} 1 & L2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & k & 0 \\ 0 & 0 & 1 & 0 \\ k & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & L_1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & L & 0 & \eta \\ 0 & 1 & 0 & 0 \\ 0 & \eta & 1 & \xi \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{pmatrix} 0 & 2\,L + 2\,L2 & \frac{L+L2}{\eta} & \frac{L\,Xi + L2\,Xi - \eta^2}{\eta} \\ 0 & 2 & \frac{1}{\eta} & \frac{Xi}{\eta} \\ \frac{Xi}{\eta} & \frac{L\,Xi + L1\,Xi - \eta^2}{\eta} & 0 & 2\,Xi \\ \frac{1}{\eta} & \frac{L+L1}{\eta} & 0 & 2 \end{pmatrix}$$

where $\eta k = 1$ has been chosen.

However, the exchange is incomplete.

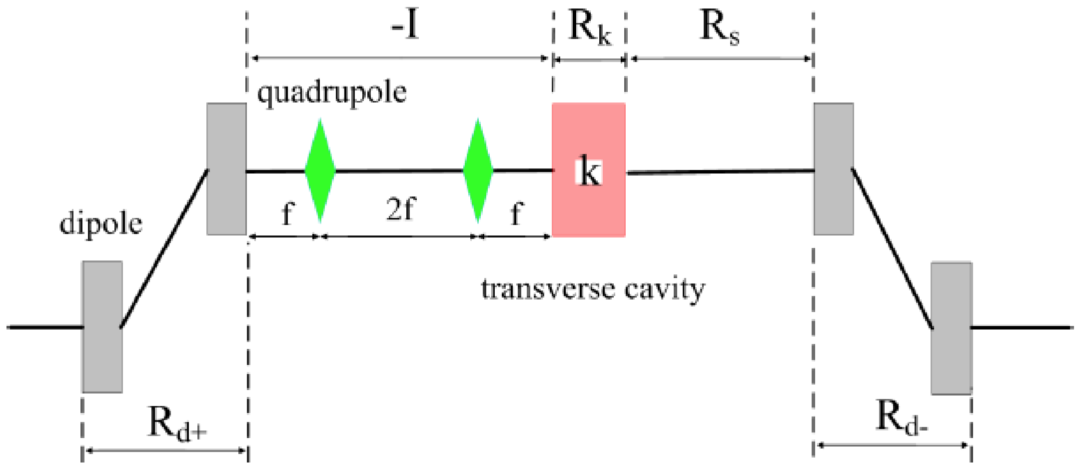
<u>Kim EEX</u>



The transform map is $\begin{pmatrix} 0 & 0 & -\frac{\mathrm{L+L2}}{\eta} & -\frac{(\mathrm{L+L2})\,\mathrm{Xi}}{\eta}+\eta \\ 0 & 0 & -\frac{1}{\eta} & -\frac{\mathrm{Xi}}{\eta} \\ -\frac{\mathrm{Xi}}{\eta} & 2\,\eta-\frac{\mathrm{L\,Xi+L1\,Xi}+\eta^2}{\eta} & 0 & 0 \\ -\frac{1}{\eta} & -\frac{\mathrm{L+L1}}{\eta} & 0 & 0 \end{pmatrix}$ if $\eta k = -1$. The exchange is complete.

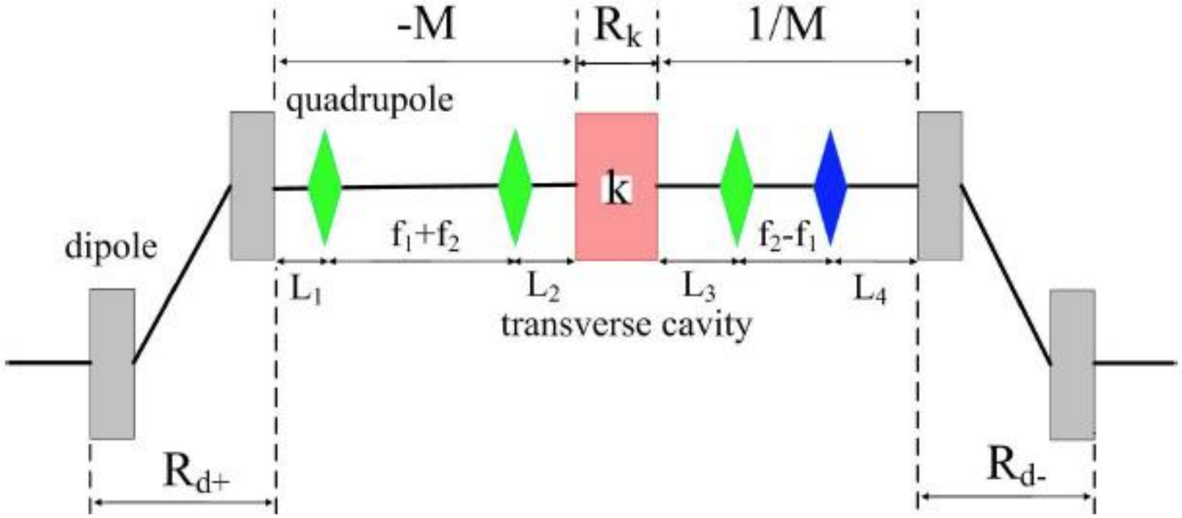
<u>Xiang-Chao EEX</u>

The Kim exchanger requires replacing a chicane by two dog-legs. Chicane can be recovered by inserting a –I map to the (x.x') lattice.



$$\text{Map} = \begin{pmatrix} 0 & 0 & \frac{L+L2}{\eta} & \frac{(L+L2)\,Xi}{\eta} - \eta \\ 0 & 0 & \frac{1}{\eta} & \frac{Xi}{\eta} \\ -\frac{Xi}{\eta} & -\frac{L\,Xi}{\eta} + \eta & 0 & 0 \\ -\frac{1}{\eta} & -\frac{L}{\eta} & 0 & 0 \end{pmatrix} \text{ when } \eta k = 1.$$

Furthermore, by inserting telescopic sections,



the condition for complete exchange can be relaxed to $\eta k = 1/M$.

x-z Phase Space Exchanger

A clean x-z phase space exchanger can be obtained by adding

$$\begin{pmatrix} -\frac{L}{\eta} & \frac{L\,Xi}{\eta}-\eta & 0 & 0 \\ \frac{1}{\eta} & -\frac{Xi}{\eta} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

upstream and inserting

$$\begin{pmatrix} \frac{Xi}{\eta} & -\frac{(L+L2)\,Xi-\eta^2}{\eta} & 0 & 0 \\ -\frac{1}{\eta} & \frac{L+L2}{\eta} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

downstream the Xiang-Chao EEX.

Then the map becomes

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

, i.e. we then get a clean phase space exchanger.

Note that these added sections are straightforward, each consisting of two quadrupoles, and staying outside of the EEX.

x-y Phase Space Exchanger

A clean phase space exchanger between x and y was discussed before, i.e. a solenoid with $k_s z = \pi$:

$$\begin{pmatrix} 0 & 0 & 0 & \frac{2}{ks} \\ 0 & 0 & -\frac{ks}{2} & 0 \\ 0 & -\frac{2}{ks} & 0 & 0 \\ \frac{ks}{2} & 0 & 0 & 0 \end{pmatrix}$$

i.e. $x \Leftrightarrow y'$, $x' \Leftrightarrow y$

Applications of x-z exchangers

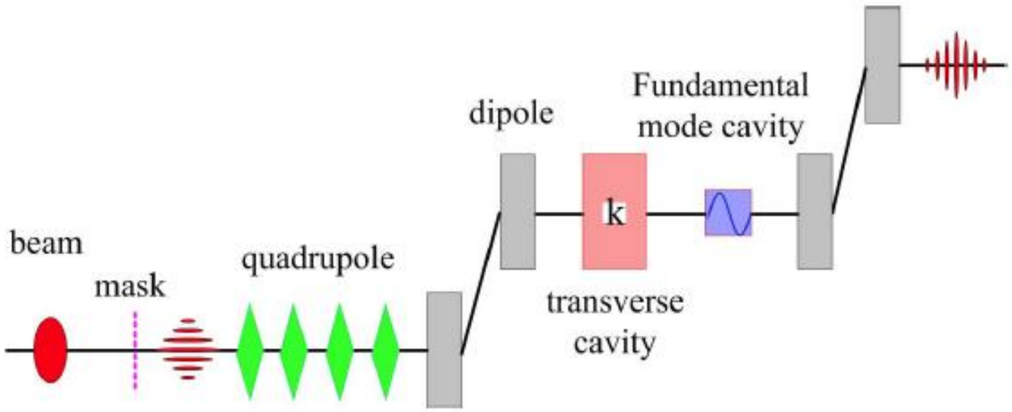
[list from D. Xiang 2010]

(1) When $\varepsilon_z << \varepsilon_x$, EEX allows small ex for FEL
(2) When $\varepsilon_z >> \varepsilon_x$, EEX allows bunch compression
(3) Observing z-distribution by an x-profile monitor
(4) Tailoring z-distribution by an x-scraper
(5) Measuring slice energy spread by an x-profile monitor
(6) Cleaning the z- and δ-tails by x-scraper
(7) Observing beam microbunching in z by an x-profile monitor
(8) Generate z-microbunching by modulating the x-profile of a beam
(9) Generating z-double bunches by x-wire scraper
(10) Longitudinal phase space linearizer by a sextupole
(11) Study CSR effect by converting CSR-induced z-δ correlation to x-x' correlation
(12) Suppressing CSR by $\varepsilon_z > \varepsilon_x$ and exchange afterwards
(13) Observing curvature of z(y) by an x-y profile monitor
(14) Bunch compression without energy chirp
(15) etc.

Example (8)

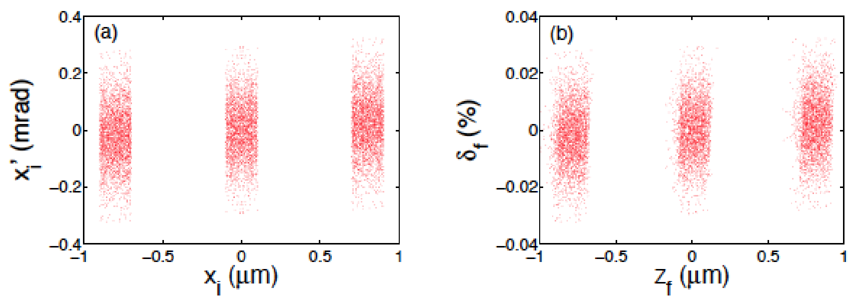
[Experiment Y-E Sun et al 2010, Xiang 2011]

An x-z phase space exchanger with magnification factor of 1:



x-mask with 0.8 μm slits ➔ beam with 0.8 μm microbunches in z.

Simulation (ELEGANT) with thick lenses and nonlinear optics but no CSR:



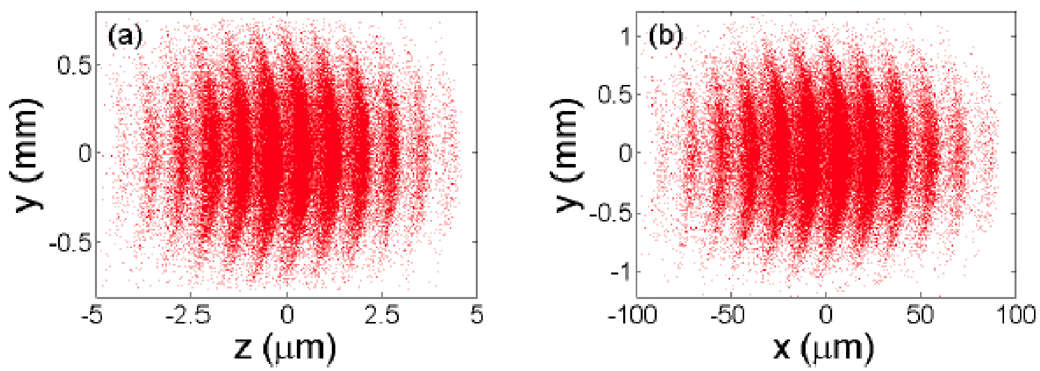
Note: Shorter microbunches can be obtained by a demagnification factor.

Example (13)

A curvature in z(y) in the microbunches hurts the FEL mechanism and needs to be cured by beam conditioning. A x-z phase space exchanger will allow observation of z(y) curvature on an x-y profile.

[Xiang 2010]

ELEGANT simulation:

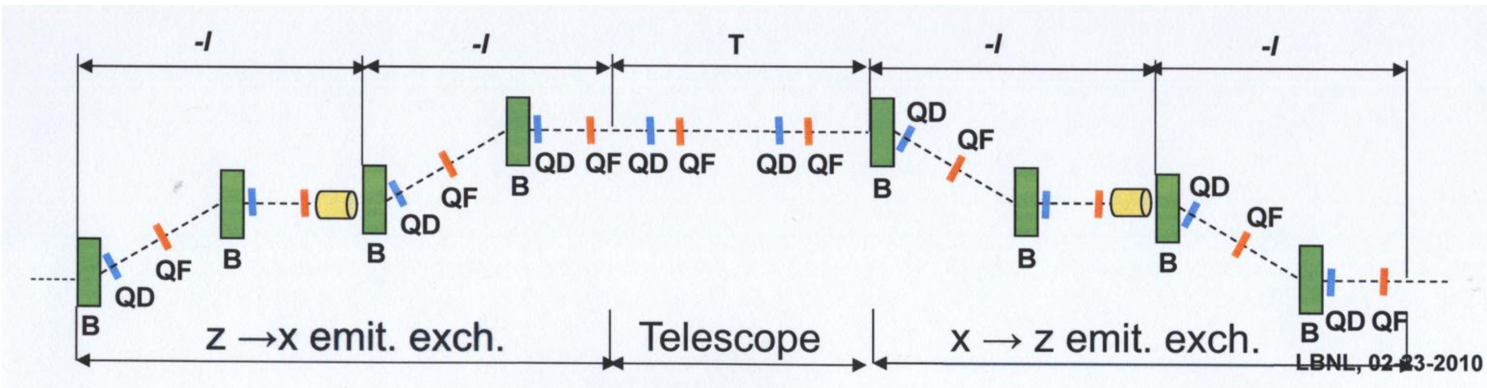


A magnification factor of 20 has been applied.

Example (14)

[Zholents/Zolotorev 2010]

With two back-to-back x-z phase space exchangers:



➔ bunch compressor is replaced by a telescope in between the two exhchangers. No large energy chirp needed, and avoiding rf nonlinearities. No net emittance exchange.

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} -\mathrm{mm} & 0 & 0 & 0 \\ 0 & -\frac{1}{\mathrm{mm}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\mathrm{mm} & 0 \\ 0 & 0 & 0 & -\frac{1}{\mathrm{mm}} \end{pmatrix}$$

III. Emittance partitioning

So far, we have been talking about exchanging the emittances. How about changing the emittances? The EEX and phase space exchangers are adapters. Adapters are symplectic. They diagonalize the phase spaces, but do not alter eigen-emittances. They can exchange emittances, but not change them.

Theorem: Eigenemittance are invariant under symplectic transformations.

Since all beamline elements are symplectic, the beam's eigen-emittances cannot be changed once the beam was born at the cathode. Once the beam was born at the cathode, Dragt: "the game is over".

- Transformations are symplectic only if we use $X_{can} = (x, p_x, y, p_y)$.
- $X_{can} = X$ if (and only if) in field-free regions!
- To affect the eigen-emittances, there are only two ways:
    a) Try to affect the way the beam is born at the cathode
    b) If it has to be done after the beam is born, then try to implement non-symplectic beamline elements.

Magnetized cathode

Consider a photocathode immersed in solenoid B_s. Consider the case when incident laser is normal to the cathode and is rotational symmetric. The 4 x 4 beam second-moment matrix at the cathode is round, with

$$\Sigma_0 = \begin{pmatrix} \sigma_{x0}^2 & 0 & 0 & 0 \\ 0 & \sigma_{x0}'^2 & 0 & 0 \\ 0 & 0 & \sigma_{x0}^2 & 0 \\ 0 & 0 & 0 & \sigma_{x0}'^2 \end{pmatrix}$$

where we have used X = (x,x',y,y') coordinates because that is how the beam gets produced at the cathode, even if magnetized.

As soon as the beam is born at the cathode, its eigen-emittances are determined. The eigen-emittances however are not $\sigma_{x0}\sigma_{x0}'$. To find the eigen-emittances, one must not use the X coordinates but has to use

$$X_{can} = (x,\ p_x = x' - k_s y/2,\ y,\ p_y = y' + k_s x/2)$$

or

$$X_{can} = MX$$

where M = $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\frac{ks}{2} & 0 \\ 0 & 0 & 1 & 0 \\ \frac{ks}{2} & 0 & 0 & 1 \end{pmatrix}$

In terms of X_{can}, the beam matrix is

$$\Sigma = M\Sigma_0 M^T$$

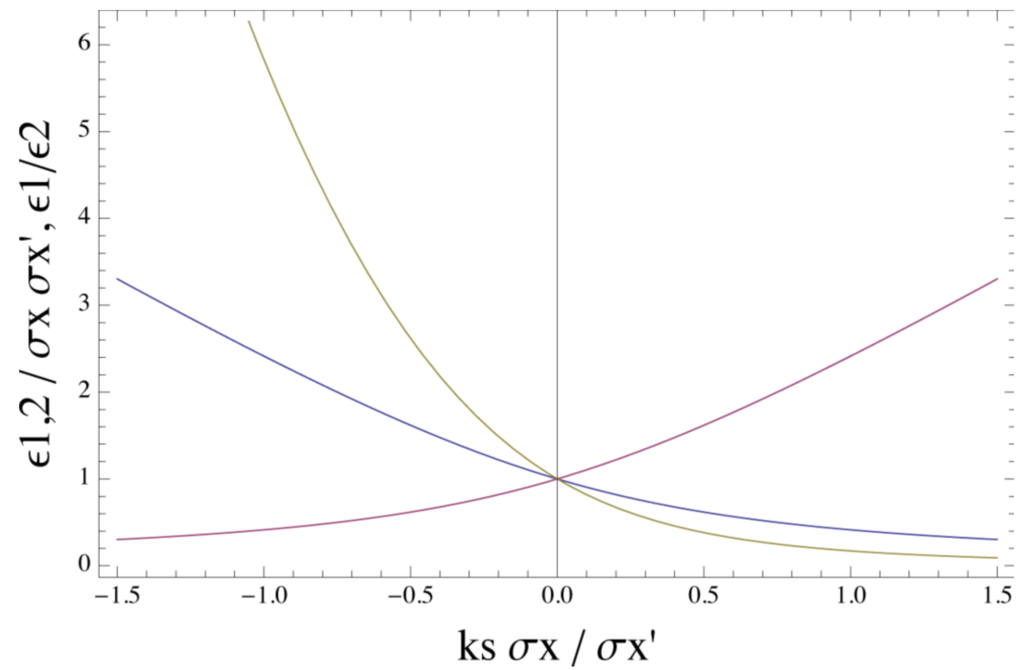
The new-born beam always has the same Σ_0 distribution in the X coordinates regardless of B_s. However, when projected to X_{can}, it changes according to B_s. Magnetizing the cathode is therefore one way to control the eigen-emittances.

How to compute the eigen-emittances? Answer: They are given by the eigen-values of $iJ\Sigma$, where J = $\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$.

Explicit calculation gives

$$\varepsilon_{1,2} = \sigma_{x0}\sqrt{\frac{k_s^2}{4}\sigma_{x0}^2 + \sigma_{x0}'^2} \pm \frac{k_s}{2}\sigma_{x0}^2$$

The parameter $\xi = k_s \sigma_{x0} / \sigma_{x0}'$ controls the eigen-emittances:



To produce a very flat beam (from a round laser cathode!), one takes a large value of ξ. The aspect ratio $\varepsilon_1/\varepsilon_2$=1/250 if ξ=8.
(If σ_{x0}/σ_{x0}'=1 m, B_s=0.3 T, E= 100 keV, then ξ=8)

By choosing ξ, we control the eigen-emittances after exiting the solenoid. However, at the solenoid exit, phase space is entangled. We insert a round-to-flat adapter so that the eigen-planes align with x and y.

As mentioned before, by inserting after the solenoid a 45°-rotated channel of normal quadrupoles that produces the map

$$\begin{pmatrix} \frac{\sqrt{b}\,\text{Cos}[\text{mu}]}{\sqrt{\text{by}}} & \sqrt{b}\,\sqrt{\text{by}}\,\text{Sin}[\text{mu}] & 0 & 0 \\ -\frac{\text{Sin}[\text{mu}]}{\sqrt{b}\,\sqrt{\text{by}}} & \frac{\sqrt{\text{by}}\,\text{Cos}[\text{mu}]}{\sqrt{b}} & 0 & 0 \\ 0 & 0 & \frac{\sqrt{b}\,\text{Sin}[\text{mu}]}{\sqrt{\text{by}}} & -\sqrt{b}\,\sqrt{\text{by}}\,\text{Cos}[\text{mu}] \\ 0 & 0 & \frac{\text{Cos}[\text{mu}]}{\sqrt{b}\,\sqrt{\text{by}}} & \frac{\sqrt{\text{by}}\,\text{Sin}[\text{mu}]}{\sqrt{b}} \end{pmatrix}$$

with

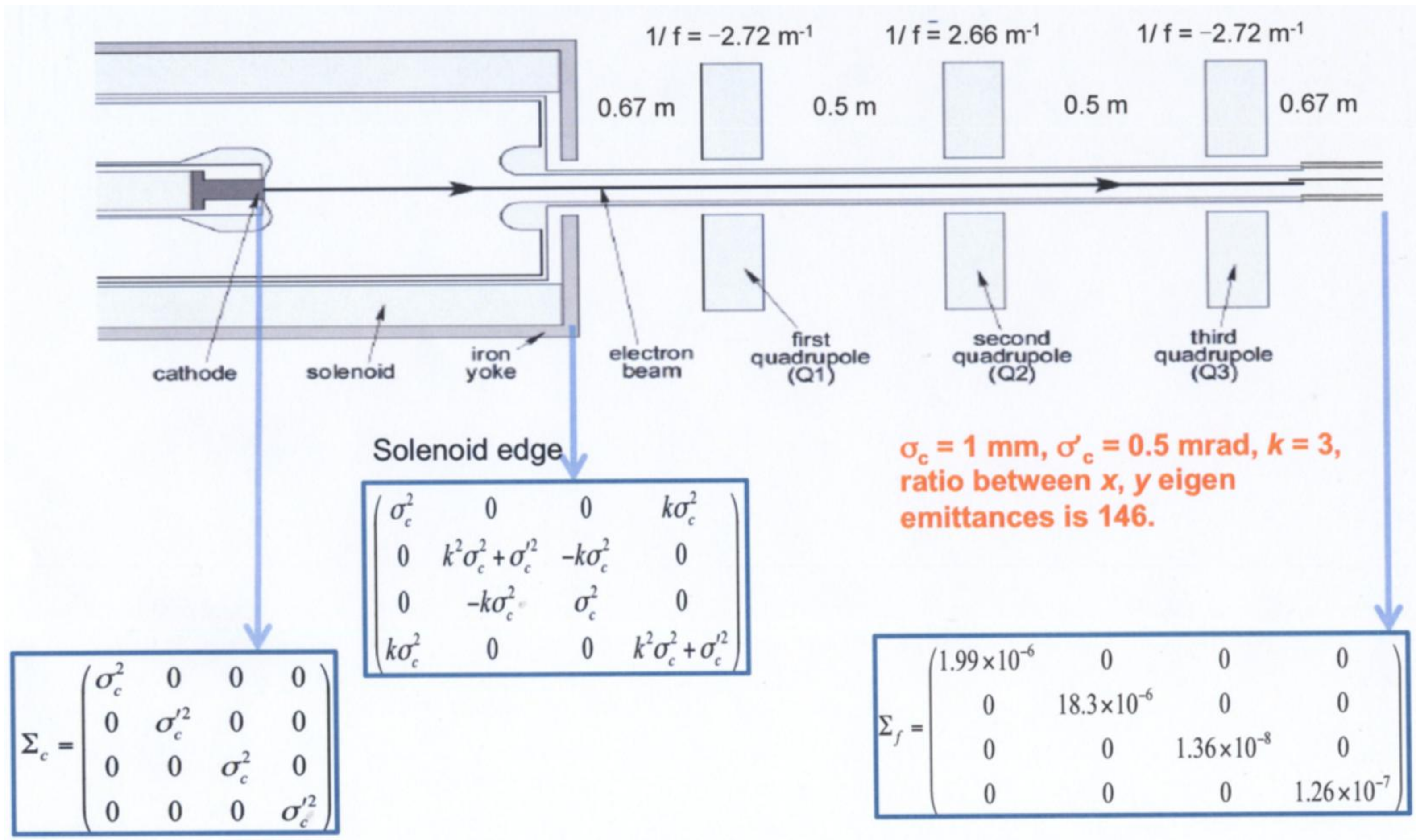
$$1/\beta^2 = 1/\beta_y^2 = k_s^2/4 + \sigma_{x0}'^2/\sigma_{x0}^2$$

➔ the beam distribution matrix becomes diagonal,

$$\Sigma_1 = \begin{pmatrix} \sigma_{x1}^2 & 0 & 0 & 0 \\ 0 & \sigma_{x1}'^2 & 0 & 0 \\ 0 & 0 & \sigma_{x2}^2 & 0 \\ 0 & 0 & 0 & \sigma_{x2}'^2 \end{pmatrix}$$

and one of the dimensions will have a very small emittance!

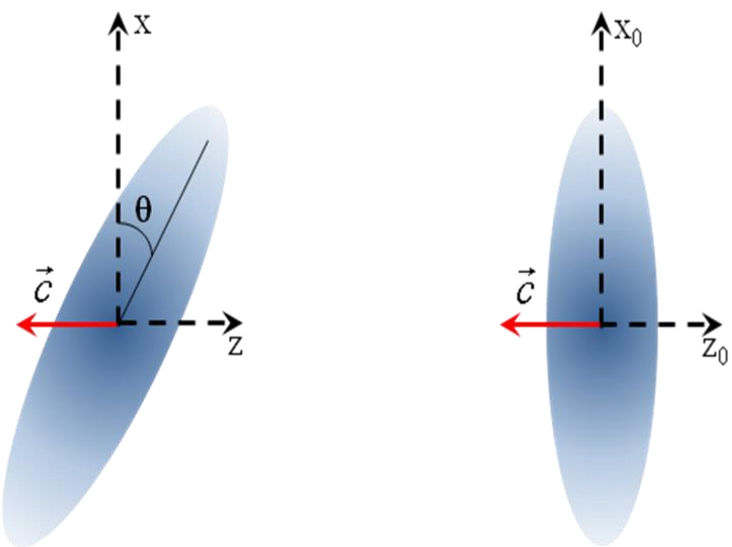
Example design [Jiao 2011]:



Partitioning by tilt laser

Magnetized cathode is a way to control x and y emittances, leaving z emittance intact. But how to control x and z emittances (leaving y emittance intact)? One way is to tilt the laser pulse-front.

[Carlsten 2010, Yampolsky 2010, Jiao 2011]



Tilt pulse   Original pulse

With tilt laser,

$$x = x_0$$

$$z = z_0 - x_0 \tan\theta$$

This coordinate correlation is non-symplectic! (The tilt is applied to the laser, not the electrons.)

The laser tilt modifies the eigen-emittances of the electron beam at its birth. Assume the beam distribution (in x-z plane) produced by the laser without tilt is

$$\Sigma_0 = \begin{pmatrix} \sigma_{x0}^2 & 0 & 0 & 0 \\ 0 & \sigma_{x0}'^2 & 0 & 0 \\ 0 & 0 & \sigma_{z0}^2 & 0 \\ 0 & 0 & 0 & \sigma_{\delta 0}^2 \end{pmatrix}$$

With tilt, the beam matrix becomes

$$\Sigma_1 = \begin{pmatrix} \sigma_{x0}^2 & 0 & \alpha\sigma_{x0}^2 & 0 \\ 0 & \sigma_{x0}'^2 & 0 & 0 \\ \alpha\sigma_{x0}^2 & 0 & \sigma_{z0}^2 + \alpha^2\sigma_{x0}^2 & 0 \\ 0 & 0 & 0 & \sigma_{\delta 0}^2 \end{pmatrix}$$

where $\alpha = -\tan\theta$.

The eigen-emittances are readily obtained,

$$\varepsilon_{1,2} = \sqrt{\frac{\sigma_{z0}^2\sigma_{\delta 0}^2 + \sigma_{x0}^2\sigma_{x0}'^2 + \alpha^2\sigma_{x0}^2\sigma_{\delta 0}^2 \pm \sqrt{(\sigma_{z0}^2\sigma_{\delta 0}^2 + \sigma_{x0}^2\sigma_{x0}'^2 + \alpha^2\sigma_{x0}^2\sigma_{\delta 0}^2)^2 - 4\sigma_{z0}^2\sigma_{\delta 0}^2\sigma_{x0}^2\sigma_{x0}'^2}}{2}}$$

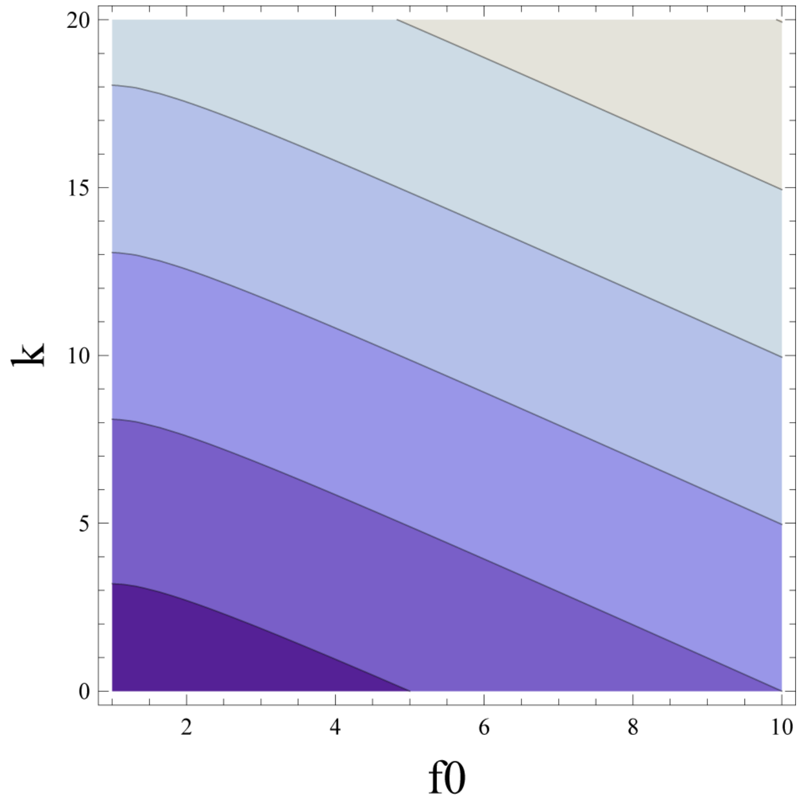
The emittance aspect ratio is given by

$$f = \frac{\varepsilon_1}{\varepsilon_2} = \sqrt{\frac{1}{2}t^2 - 1 + t\sqrt{\frac{1}{4}t^2 - 1}}$$

with

$$t = \frac{\varepsilon_{x0}}{\varepsilon_{z0}} + \frac{\varepsilon_{z0}}{\varepsilon_{x0}} + \frac{\alpha^2\sigma_{x0}\sigma_{\delta 0}}{\sigma_{x0}'\sigma_{z0}}$$

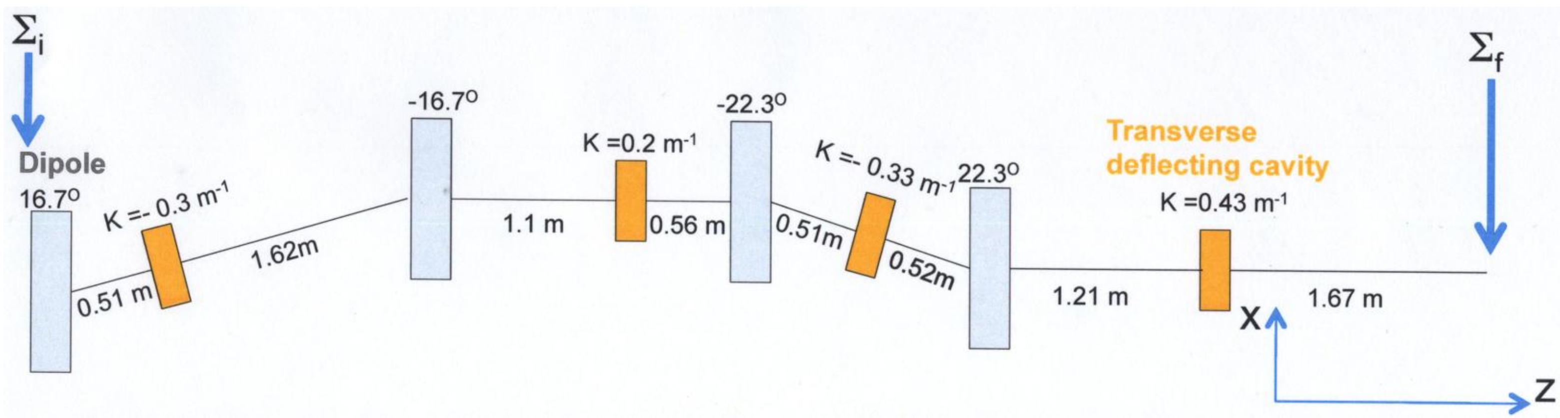
f as a function of $f_0 = \varepsilon_{z0}/\varepsilon_{x0}$ and $k = \alpha^2\sigma_{x0}\sigma_{\delta 0}/\sigma'_{x0}\sigma_{z0}$:



Take normalized σ_{x0} = 1.3 mm, σ_{x0}' = 0.3 mrad, ε_{x0} = 0.4 μm, σ_{z0} = 0.8 mm, $\sigma_{\delta 0} = 5\text{x}10^{-3}$, ε_{z0} = 4 μm ➔ f_0 =10, k=27 α^2 ➔ f ranges up to 30.

We still need to diagonalize the coordinates after the tilt laser gun. That can be done by an appropriate adapter.

Example design [Jiao 2011]:



Combining magnetized cathode with tilt laser

Consider the beam emittances at the LCLS photo cathode gun:

$$(\varepsilon_{x0}, \varepsilon_{y0}, \varepsilon_{z0}) = (0.4\ \mu\text{m}, 0.4\ \mu\text{m}, 4\ \mu\text{m})$$

We want to change to (0.1 μm, 0.1 μm, 64 μm) for the FEL. This involves both x-y and x-z exchanges, so how about applying tilt laser to a magnetized photocathode?

We need 3D analysis. The 3D beam distribution in X_{can} coordinates is found to be

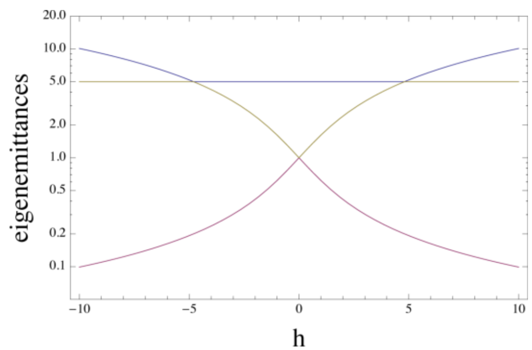
$$\Sigma_2 = \begin{pmatrix} \sigma_{x0}^2 & 0 & 0 & \frac{k_s}{2}\sigma_{x0}^2 & \alpha\sigma_{x0}^2 & 0 \\ 0 & \sigma_{x0}'^2 + \frac{k_s^2}{4}\sigma_{x0}^2 & -\frac{k_s}{2}\sigma_{x0}^2 & 0 & 0 & 0 \\ 0 & -\frac{k_s}{2}\sigma_{x0}^2 & \sigma_{x0}^2 & 0 & 0 & 0 \\ \frac{k_s}{2}\sigma_{x0}^2 & 0 & 0 & \sigma_{x0}'^2 + \frac{k_s^2}{4}\sigma_{x0}^2 & \frac{k_s\alpha}{2}\sigma_{x0}^2 & 0 \\ \alpha\sigma_{x0}^2 & 0 & 0 & \frac{k_s\alpha}{2}\sigma_{x0}^2 & \sigma_{z0}^2 + \alpha^2\sigma_{x0}^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sigma_{\delta 0}^2 \end{pmatrix}$$

Eigen-emittances are determined by

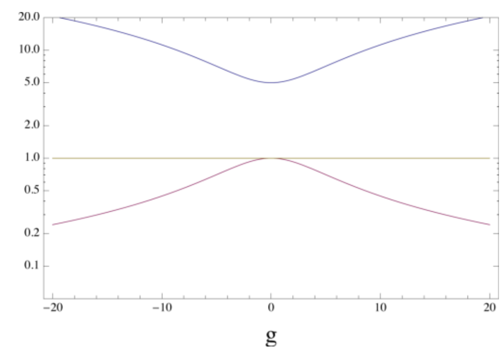
$$E^3 - (2+f_0^2+g^2+h^2)\,E^2 + (1+2f_0^2+g^2+f_0^2h^2)\,E - f_0^2 = 0$$

where

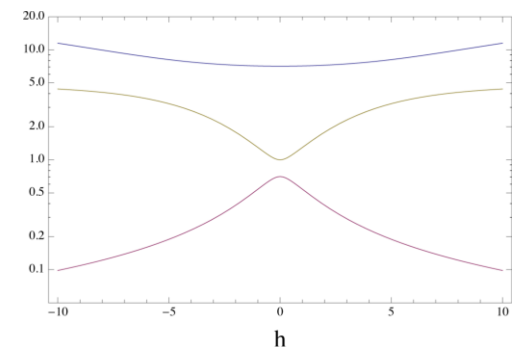
$$E = (\varepsilon/\sigma_{x0}\sigma_{x0}')^2,\ g = \alpha\sigma_{\delta 0}/\sigma_{x0}',\ f_0 = \sigma_{z0}\sigma_{\delta 0}/\sigma_{x0}\sigma_{x0}',\ h = k_s\sigma_{x0}/\sigma_{x0}'.$$



f_0=5, g=0



f_0=5, h=0



f_0=5, g=5

Naively we might think to achieve the desired FEL emittances in two steps:

(1) Tilt laser:

(0.4 μm, 0.4 μm, 4 μm) → (0.025 μm, 0.4 μm, 64 μm)

(2) then with immersed solenoid:

(0.025 μm, 0.4 μm, 64 μm) → (0.1 μm, 0.1 μm, 64 μm)

But this does not work. Both immersed solenoid and the tilt laser are applied at the cathode, not applied in sequence. Combined laser tilt and immersed solenoid can not produce (0.1 μm, 0.1 μm, 64 μm) !

What are available so far:

a) Start with (ε_{x0}, ε_{y0}, ε_{z0}) = (X, Y, 0.1). Apply magnetized cathode to obtain (10*XY, 0.1, 0.1). Then go through x-z emittance exchanger to get (0.1, 0.1, 10*XY).
b) Start with (ε_{x0}, ε_{y0}, ε_{z0}) = (X, 0.1, Z). Apply tilt laser cathode to obtain (0.1, 0.1, 10*XZ).
c) Start with (ε_{x0}, ε_{y0}, ε_{z0}) = (X, Y, 0.1). Apply tilt laser cathode to obtain (0.1, 0.1, 10*XY).

In all cases, at least one of the initial emittances has to be 0.1 μm.

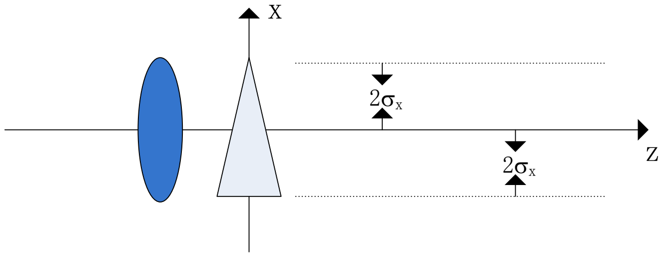
Partitioning by foil

So we cannot combine partitioning steps. Steps must be clearly separated. The second step, necessarily applied after the beam is born, will have to be non-symplectic. One idea, first introduced by Peterson 1983, is to insert a tapered foil.

[J. Peterson 1983, Carlsten 2010, Jiao 2011]

(cathode with tilt laser)
→ (an adapter to diagnalize)
→ (tapered foil)
→ (another adapter to diagonalize again)

Tapered foil:



induces a nonsymplectic map $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ S & 0 & 0 & 1 \end{pmatrix}$ in the coordinates X = (x, x', z, δ).

The beam distribution

$$\Sigma_0 = \begin{pmatrix} \sigma_{x0}^2 & 0 & 0 & 0 \\ 0 & \sigma_{x0}'^2 & 0 & 0 \\ 0 & 0 & \sigma_{z0}^2 & 0 \\ 0 & 0 & 0 & \sigma_{\delta 0}^2 \end{pmatrix}$$

is transformed to

$$\begin{pmatrix} \mathrm{sigx}^2 & 0 & 0 & S\,\mathrm{sigx}^2 \\ 0 & \mathrm{d}\sigma\mathrm{xp}^2 + \mathrm{sigxp}^2 & 0 & 0 \\ 0 & 0 & \mathrm{sigz}^2 & 0 \\ S\,\mathrm{sigx}^2 & 0 & 0 & \mathrm{d}\sigma\mathrm{zp}^2 + S^2\,\mathrm{sigx}^2 + \mathrm{sigzp}^2 \end{pmatrix}$$

where $\Delta\sigma_x'^2$ is added to the $\langle x'^2 \rangle$ and $\Delta\sigma_\delta^2$ added to $\langle \delta^2 \rangle$ to model the effects of Coulomb scattering by the foil. The foil introduces three quantities: S (desired), $\Delta\sigma_x'^2$ (undesired) and $\Delta\sigma_\delta^2$ (undesired).

The x eigen-emittance is found to be

$$\varepsilon_{x,eig} = \frac{1}{\sqrt{2}}\sqrt{\mathcal{A} - \sqrt{\mathcal{A}^2 - \mathcal{B}^2}}$$

where

$$A = (\sigma_x'^2 + \Delta\sigma_x'^2)\sigma_x^2 + (\sigma_\delta^2 + \Delta\sigma_\delta^2)\sigma_z^2 + S^2\sigma_{\square}^2\,\sigma_z^2$$

$$B^2 = 4\sigma_{\square}^2\,\sigma_z^2\,(\sigma_x'^2 + \Delta\sigma_x'^2)\,(\sigma_\delta^2 + \Delta\sigma_\delta^2)$$

Assume carbon foil, let $d_f = L_{foil}/L_{rad}$, and cut 4% of the tail particles, then for $d_f < 10^{-3}$,

$$S\sigma_x = (41.5\, d_f/\gamma)$$

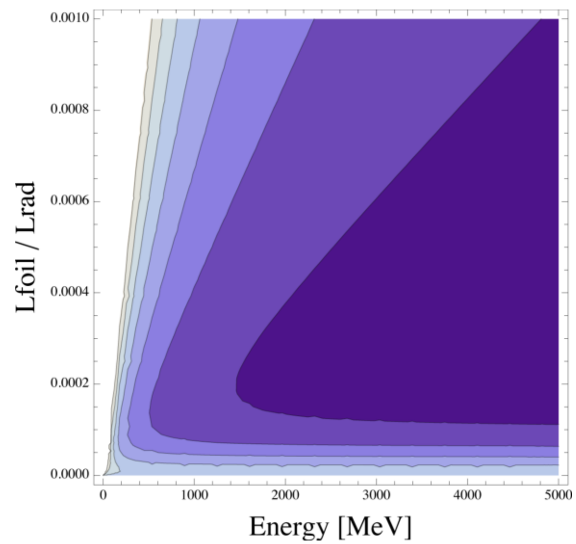
$$\Delta\sigma_x'^2 = 157.4\, d_f/\gamma^2$$

$$\Delta\sigma_\delta^2 = (35.5\, d_f/\gamma)^2$$

Take
σ_z = 0.5 mm,
σ_δ = 2.8 mrad/γ,
σ_x =0.083 mm/$\gamma^{1/2}$,
σ_x' = 8.3 mrad /$\gamma^{1/2}$

x eigen-emittance
contours 0.5 - 0.8 μm

➔



This beam has initial (ε_{x0}, ε_{y0}, ε_{z0}) = (0.7 μm, 0.7 μm, 1.4 μm) and we wish to reduce ε_x as much as possible.

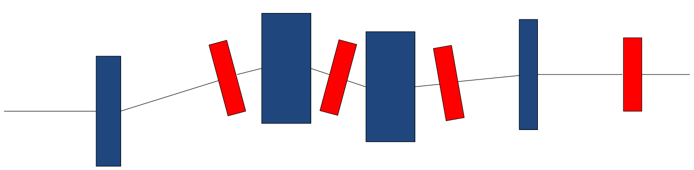
We found that when foil scattering is included, it is difficult to have ε_x <0.5 μm unless much more tail particles are cut. Root cause is that $\Delta\sigma_\delta$ is too large for the needed foil thickness.

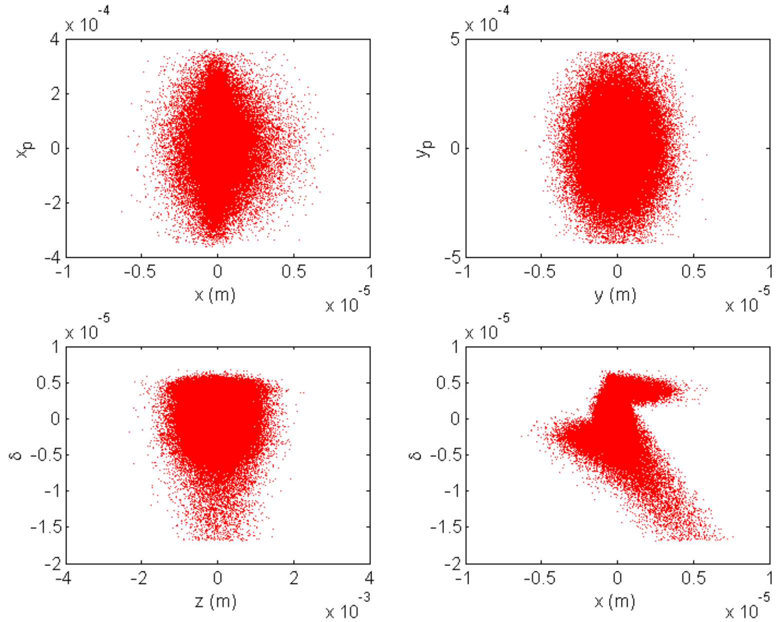
It can be shown analytically that the best one can achieve by a tapered foil is $\varepsilon_{x,eig}$ = 63% ε_{x0} [Jiao 2011].

3D simulations with foil:

[Jiao 2011, G4Beamline, Muons Inc.]

For this simulated case, analytic model gives final x eigen-emittance = 0.5 μm. Simulation after the beam is diagonalized (adapter 15 m, 4 dipoles + 4 crab cavities):





Final x emittance ~0.47 μm. Usefulness of tapered foil for emittance partitioning seems rather limited. More work is needed.

IV. Summary

1. Phase space gymnastics is a powerful technique:
   - Adapters
   - Emittance exchangers
   - Phase space exchangers
   - Emittance partitioning
     a) by magnetizing the cathode
     b) by tilting the photocathode laser
     c) by tapered foil

2. This is still on-going R&D. This talk wishes to generate more interest:
   - applications
   - more ideas
   - design optimization
   - space charge and IBS effects
   - nonlinearities in lattice optics
   - emittance preservation
   - tolerance simulations
   - experimental demonstration of large aspect ratios