New insights on the physical Riemann surfaces of the ratio $G_E^\Lambda / G_M^\Lambda$

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Palazzo Hercolani, Aula Poeti, Str. Maggiore, 45, Bologna

Francesco Rosini
Simone Pacetti

Overview

- Baryon form factors analyticity
- The Λ baryon special case
- Dispersion relations for the form factors' ratio
- Parametrization and χ^2 definition
- Results and discussions

Baryon - photon vertex

Given a baryon $\mathscr{B}$, the electromagnetic current is

$F_1^{\mathscr{B}}(q^2)$ and $F_2^{\mathscr{B}}(q^2)$ are the Dirac and Pauli form factors

$$\left\langle P_i \right| J^{\mu}_{\text{EM}}(0) \left| P_f \right\rangle = e\bar{u}(p_f)\left[\gamma^{\mu} F_1^{\mathscr{B}}(q^2) + \frac{i\sigma^{\mu\nu}q_{\nu}}{2M_{\mathscr{B}}} F_2^{\mathscr{B}}(q^2)\right] u(p_i)$$

$$F_1^{\mathscr{B}}(0) = Q_{\mathscr{B}} \qquad F_2^{\mathscr{B}}(0) = \kappa_{\mathscr{B}}$$

$Q_{\mathscr{B}}$ is the electric charge

$\kappa_{\mathscr{B}}$ is the anomalous magnetic moment

Breit frame

$$(p_f - p_i)^{\mu} = q^{\mu} = (0, \vec{q})$$

Sachs (electromagnetic) form factors

$$G_E^{\mathscr{B}}(q^2) = F_1^{\mathscr{B}}(q^2) + \frac{q^2}{4M_{\mathscr{B}}^2} F_2^{\mathscr{B}}(q^2)$$

$$G_M^{\mathscr{B}}(q^2) = F_1^{\mathscr{B}}(q^2) + F_2^{\mathscr{B}}(q^2)$$

$$G_E^{\mathscr{B}}(0) = Q_{\mathscr{B}} \qquad G_M^{\mathscr{B}}(0) = Q_{\mathscr{B}} + \kappa_{\mathscr{B}} = \mu_{\mathscr{B}}$$

$\mu_{\mathscr{B}}$ is the total magnetic moment

Asymptotic behaviour

The asymptotic form factors behaviour is given in pQCD by counting rules as $q^2 \to -\infty$

Helicity conservation

- $J^{\lambda,\lambda}(q^2) \propto G_M^{\mathscr{B}}(q^2)$
- 2 gluon propagators distributing the momentum transfer of the virtual photon
- $G_M^{\mathscr{B}}(q^2) \sim (q^2)^{-2}$

Helicity flip

- $J^{\lambda,-\lambda}(q^2) \propto G_E^{\mathscr{B}}(q^2)/\sqrt{-q^2}$
- [2 gluon propagators] / $\sqrt{-q^2}$
- $G_E^{\mathscr{B}}(q^2) \sim (q^2)^{-2}$

Dirac and Pauli Form Factors

$$F_1^{\mathscr{B}} \underset{q^2\to-\infty}{\sim} (q^2)^{-2}$$

$$F_2^{\mathscr{B}} \underset{q^2\to-\infty}{\sim} (q^2)^{-3}$$

Sachs Form Factors Ratio

$$\frac{G_E^{\mathscr{B}}(q^2)}{G_M^{\mathscr{B}}(q^2)} \underset{q^2\to-\infty}{\sim} \text{constant}$$

From factors in the time-like region

In the time-like region, $G_E^{\mathscr{B}}(q^2)$ and $G_M^{\mathscr{B}}(q^2)$ are complex functions

Crossing symmetry: $\left\langle P(p') \middle| J^\mu \middle| P(p) \right\rangle \to \left\langle \bar{P}(p')P(p) \middle| J^\mu \middle| 0 \right\rangle$

Optical theorem

$$\mathrm{Im}\left(\left\langle \bar{P}(p')P(p) \middle| J^\mu \middle| 0 \right\rangle\right) \approx \sum_n \left\langle \bar{P}(p')P(p) \middle| J^\mu \middle| n \right\rangle \left\langle n \middle| J^\mu \middle| 0 \right\rangle \Rightarrow \begin{cases} \mathrm{Im}\left(F_{1,2}^{\mathscr{B}}\right) \neq 0 \\ \text{for } q^2 > 4M_\pi^2 \end{cases}$$

Where $|n\rangle$ are intermediate states, i.e. $|n\rangle = 2\pi, 3\pi, \ldots$

Phragmén Lindelöf theorem

If $f(z) \to f_1$ as $z \to \infty$ along the straight line L_1 and $f(z) \to f_2$ as $z \to \infty$ along the straight line L_2, and $f(z)$ is regular and bounded in the angle between the lines, then $f_1 \equiv f_2 = f_{12}$ and $f(z) \to f_{12}$ in the region between L_1 and L_2

Asymptotic behaviour in the time-like region

$$\lim_{q^2 \to +\infty} G_M^{\mathscr{B}}(q^2) = \lim_{q^2 \to -\infty} G_M^{\mathscr{B}}(q^2)$$

Λ Form Factors

Theoretical threshold

$$q_{th}^2 = \left(2M_\pi + M_{\pi^0}\right)^2$$

$I(\Lambda\bar{\Lambda}) = 0$, and the lightest isoscalar hadronic state is $\pi^+\pi^-\pi^0$

Physical threshold

$$q_{phys}^2 = \left(2M_\Lambda\right)^2$$

Lowest center of mass energy to produce a $\Lambda\bar{\Lambda}$ couple

- Unphysical and space-like regions have no data
- The relative phase is measured through the weak decay $\Lambda \to p\pi^-$, $\bar{\Lambda} \to \bar{p}\pi^+$
- Form factors have nonzero imaginary parts for $q^2 \geq q_{\rm th}^2$
- $G_E^\Lambda(q^2)$ vanishes for $q^2 = 0$

Dispersion relations

The form factors $G^{\Lambda}_{E,M}$ are analytic functions on the q^2-complex plane with the cut $\left(q^2_{\text{th}}, \infty\right)$ on the real axis.

Dispersion relations are based only on unitarity and analyticity $\Rightarrow$ **model independent approach**

Dispersion relation for the imaginary part ($q^2 < 0$):

$$G(q^2) = \frac{1}{\pi}\int_{q^2_{\text{th}}}^{\infty} \frac{\text{Im}(G(s))}{s - q^2} ds$$

Dispersion relation for the logarithm ($q^2 < 0$):

$$\ln\left(G(q^2)\right) = \frac{\sqrt{q^2_{\text{th}} - q^2}}{\pi}\int_{q^2_{\text{th}}}^{\infty} \frac{\ln\left|G(s)\right|}{\left(s - q^2\right)\sqrt{s - q^2_{\text{th}}}} ds$$

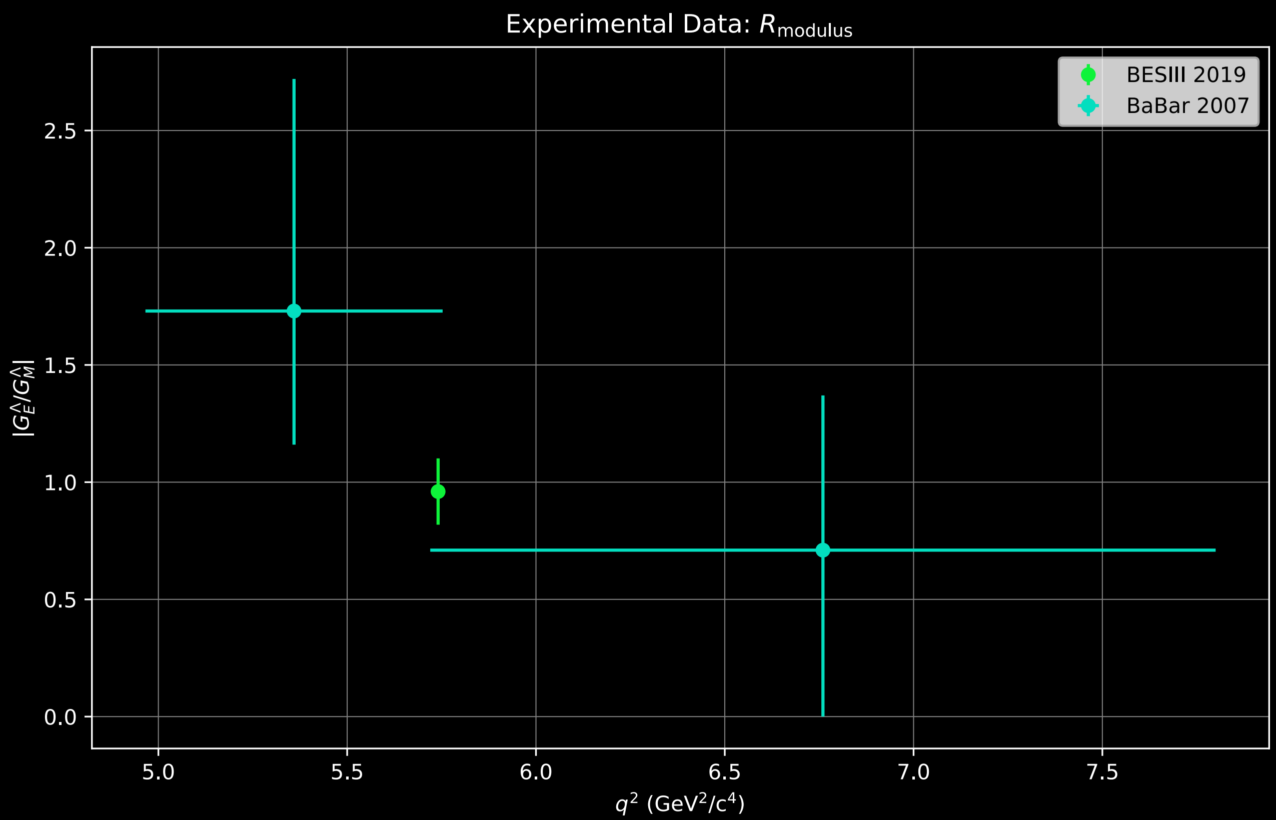
Experimental Inputs

- Time-like data for form factor's moduli from $e^+e^- \leftrightarrow \mathscr{B}\bar{\mathscr{B}}$
- Time-like data for the relative phase from $e^+e^- \leftrightarrow \mathscr{B}^{\uparrow}\bar{\mathscr{B}}$

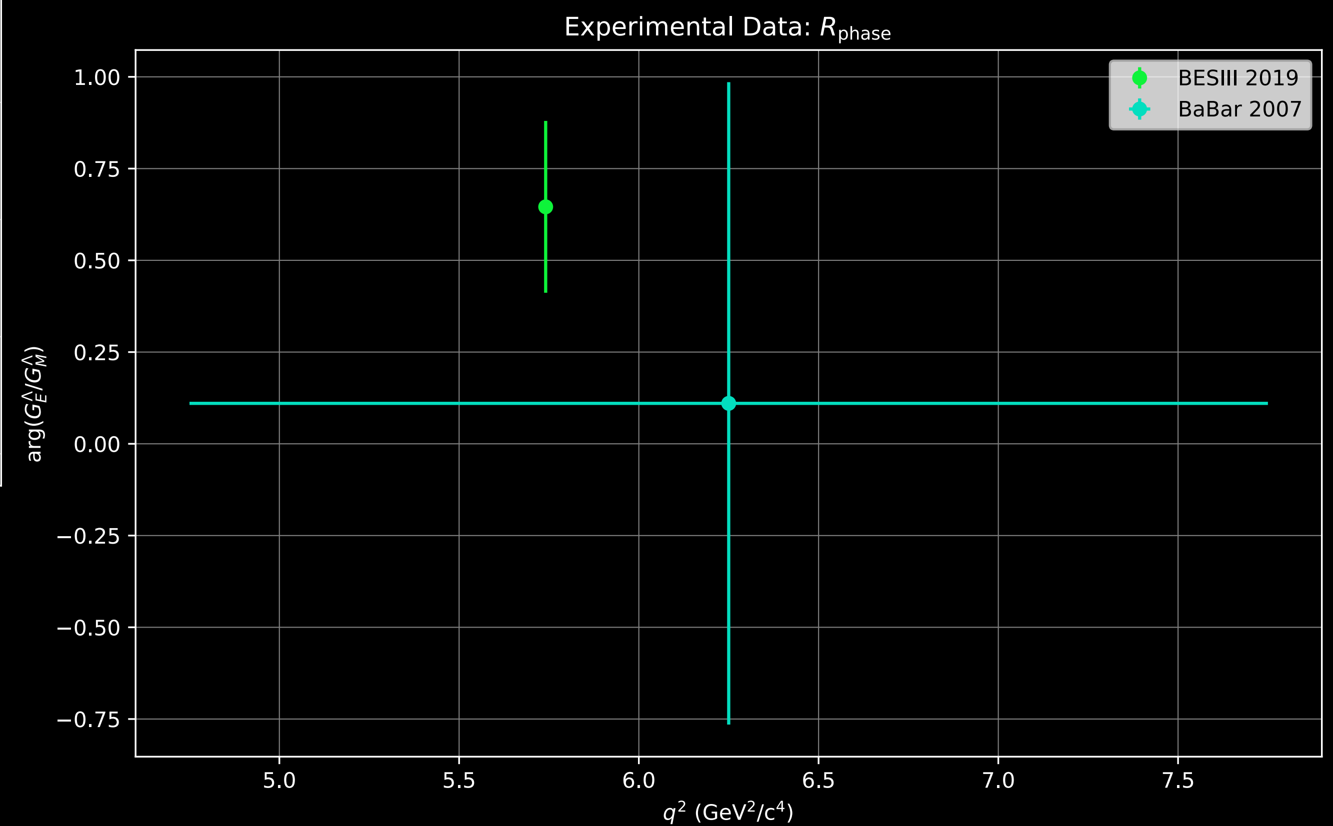
Theoretical Inputs

- Analyticity
- Threshold values
- Asymptotic behaviour

Data for modulus and phase of $G_E^\Lambda / G_M^\Lambda$



BaBar collaboration, Study of $e^+e^- \to \Lambda\bar{\Lambda}, \Lambda\bar{\Sigma}^0, \Sigma^0\bar{\Sigma}^0$ using initial state radiation with BaBar, Phys. Rev. D 76 (2007) 092006 [0709.1988]



M. Ablikim et al. (BESIII Collaboration), Complete Measurement of the Λ Electromagnetic Form Factors, Phys. Rev. Lett. 123, 122003

More data coming soon!

- Sine of the relative phase accessible through polarization
- No hints on the determination of the relative phase

$$\mathscr{P}_y = -\frac{2M_\Lambda\sqrt{q^2}\sin(2\theta)\left|G_E^\Lambda/G_M^\Lambda\right| \sin\left(\arg(G_E^\Lambda/G_M^\Lambda)\right)}{q^2\left(1+\cos^2(\theta)\right)+4M_\Lambda^2\left|G_E^\Lambda/G_M^\Lambda\right|\sin^2(\theta)}$$

Dispersive procedure

We define the ratio $R(q^2) = \dfrac{G_E^\Lambda(q^2)}{G_M^\Lambda(q^2)} \Rightarrow \begin{cases} G_E^\Lambda(0) = 0 \\ G_E^\Lambda(q^2_{\text{phy}}) = G_M^\Lambda(q^2_{\text{phy}}) \end{cases} \Rightarrow \begin{cases} R(0) = 0 \\ R(q^2_{\text{phy}}) = 1 \end{cases}$

The asymptotic behaviour

$$\lim_{q^2 \to \pm\infty} R(q^2) = \frac{G_E^\Lambda(q^2)}{G_M^\Lambda(q^2)} = \mathcal{O}(1)$$

Subtracted dispersion relations for real and imaginary part

$$R(q^2) = R(0) + \frac{q^2}{\pi} \int_{q^2_{\text{th}}}^{\infty} \frac{\mathrm{Im}(R(s))}{s(s-q^2)} ds, \quad \forall q^2 \notin \left[q^2_{\text{th}}, \infty\right)$$

$$\mathrm{Re}\left(R(q^2)\right) = \overset{Q_\Lambda = 0}{\cancel{R(0)}} + \frac{q^2}{\pi} \mathrm{Pr} \int_{q^2_{\text{th}}}^{\infty} \frac{\mathrm{Im}(R(s))}{s(s-q^2)} ds, \quad \forall q^2 \in \left[q^2_{\text{th}}, \infty\right)$$

The subtracted dispersion relations ensure the normalization at $q^2 = 0$

Parametrization for the form factors ratio

Parametrization through the set of **Chebyshev polynomials** $\left\{ T_j(x) \right\}_{j=0}^{N}$

$$\mathrm{Im(R(q^2))} \equiv Y\left(q^2; \overrightarrow{C}, q^2_{\mathrm{asy}}\right) = \begin{cases} \sum_{j=0}^{N} C_j T_j(x(q^2)), & q^2_{\mathrm{th}} < q^2 < q^2_{\mathrm{asy}} \\ 0, & q^2 \geq q^2_{\mathrm{asy}} \end{cases} \qquad x(q^2) = 2\frac{q^2 - q^2_{\mathrm{th}}}{q^2_{\mathrm{asy}} - q^2_{\mathrm{th}}} - 1$$

$$q^2 \in [q^2_{\mathrm{th}}, q^2_{\mathrm{asy}}] \Rightarrow x(q^2) \in [-1,1]$$

Theoretical constraints on $Y(q^2; \overrightarrow{C}, q^2_{\mathrm{asy}})$

$R(q^2_{\mathrm{th}})$ is real $\Rightarrow Y(q^2_{\mathrm{th}}; \overrightarrow{C}, q^2_{\mathrm{asy}}) = 0$

$R(q^2_{\mathrm{phy}})$ is real $\Rightarrow Y(q^2_{\mathrm{phy}}; \overrightarrow{C}, q^2_{\mathrm{asy}}) = 0$

$R(q^2 \geq q^2_{\mathrm{asy}})$ is real $\Rightarrow Y(q^2 \geq q^2_{\mathrm{asy}}; \overrightarrow{C}, q^2_{\mathrm{asy}}) = 0$

Theoretical constraints on $\mathrm{Re}(R(q^2))$

$$\mathrm{Re}\left(R(q^2_{\mathrm{phy}})\right) = \frac{q^2_{\mathrm{phy}}}{\pi}\mathrm{Pr}\int_{q^2_{\mathrm{th}}}^{q^2_{\mathrm{asy}}} \frac{Y(s; \overrightarrow{C}, q^2_{\mathrm{asy}})}{s(s - q^2_{\mathrm{th}})} ds = 1$$

$$\left|\mathrm{Re}\left(R(q^2_{\mathrm{asy}})\right)\right| = \frac{q^2_{\mathrm{asy}}}{\pi}\left|\mathrm{Pr}\int_{q^2_{\mathrm{th}}}^{q^2_{\mathrm{asy}}} \frac{Y(s; \overrightarrow{C}, q^2_{\mathrm{asy}})}{s(s - q^2_{\mathrm{asy}})} ds\right| = 1$$

Experimental constraints for the time-like region $(q^2 > q^2_{\mathrm{phy}})$

3 experimental points for the modulus and 2 for the phase from Babar (2007) and BESIII (2019).

The χ^2 definition

$$\chi^2\left(\vec{C}, q^2_{\mathrm{asy}}\right) = \chi^2_{|R|} + \chi^2_{\phi} + \tau_{\mathrm{phy}}\chi^2_{\mathrm{phys}} + \tau_{\mathrm{asy}}\chi^2_{\mathrm{asy}} + \tau_{\mathrm{curv}}\chi^2_{\mathrm{curv}}$$

$$\chi^2_{|R|} = \sum_{j=1}^{8}\left(\frac{\sqrt{X^2(q_j^2) + Y^2(q_j^2)} - \left|R_j\right|}{\delta\left|R_j\right|}\right)^2$$

$$X(q^2) \equiv \mathrm{Re}(R(q^2))$$

$$\chi^2_{\phi} = \sum_{k=1}^{7}\left(\frac{\sin\left(\arctan(Y(q_k^2)/X(q_k^2)\right) - \sin(\phi_k)}{\delta\sin(\phi_k)}\right)^2$$

Constraint at $q^2 = q^2_{\mathrm{phy}}$ $\longrightarrow$ $\chi^2_{\mathrm{phy}} = \left(1 - X(q^2_{\mathrm{phy}})\right)^2$

Constraint at $q^2 = q^2_{\mathrm{asy}}$ $\longrightarrow$ $\chi^2_{\mathrm{asy}} = \left(1 - X^2(q^2_{\mathrm{asy}})\right)^2$

Oscillation damping $\longrightarrow$ $\chi^2_{\mathrm{curv}} = \int_{q^2_{\mathrm{th}}}^{q^2_{\mathrm{asy}}}\left(\frac{d^2Y(s)}{ds^2}\right)^2 ds$

The values of $\boldsymbol{\tau}_{\mathbf{phys}}$ and $\boldsymbol{\tau}_{\mathbf{asy}}$ are chosen so that the theoretical conditions are exactly fulfilled.

The minimization procedure implies the solution of an ill-posed problem which has to be regularized.

The parametrization

The theoretical constraints $Y(q^2_{\rm th}; \overrightarrow{C}, q^2_{\rm asy}) = Y(q^2_{\rm phy}; \overrightarrow{C}, q^2_{\rm asy}) = Y(q^2_{\rm asy}; \overrightarrow{C}, q^2_{\rm asy}) = 0$ remove three degrees of freedom, allowing to determine three coefficients, i.e. C_0, C_1, C_2.

The asymptotic threshold $q^2_{\rm asy}$ is used as a free parameter.

If we consider $(N+1)$ Chebyshev polynomials, we are left with $(N-2)$ free coefficients.

We used $N = 5$, so we have four free parameters C_3, C_4, C_5 and $q^2_{\rm asy}$.

- $\tau_{\rm phy} = 10^4 \Rightarrow$ The real part of the ratio is forced to the unity at $q^2 = q^2_{\rm phy}$.
- $\tau_{\rm asy} = 0 \Rightarrow$ No constraint for the real part at $q^2 = q^2_{\rm asy}$.
- $\tau_{\rm curv} = 0.05 \Rightarrow$ Dumping relevant only for high degree polynomials.

  If $\tau_{\rm curv}$ is too large physical information are canceled.

  If $\tau_{\rm curv}$ is too small the solution has too much noise.

Results & discussion

At the thresholds $q^2_{\rm th}$ and $q^2_{\rm asy}$ the values of the ratio are real, so the relative phases are integer multiples of of π radians.

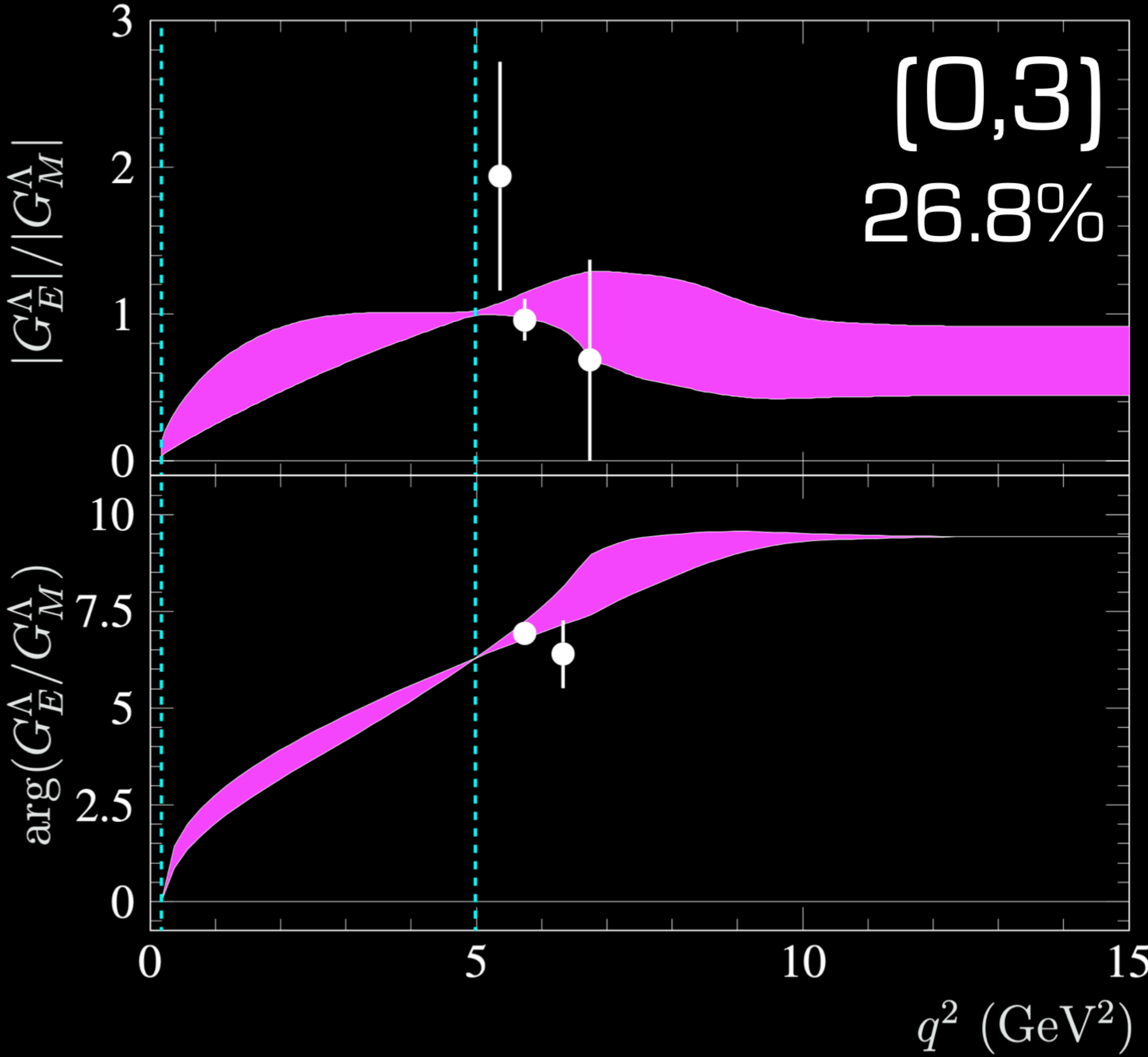
$$N_{\rm th,asy} = \frac{1}{\pi} \arg\left(\frac{G_E^\Lambda(q^2_{\rm th,asy})}{G_M^\Lambda(q^2_{\rm th,asy})} \right) \in \mathbb{Z}$$

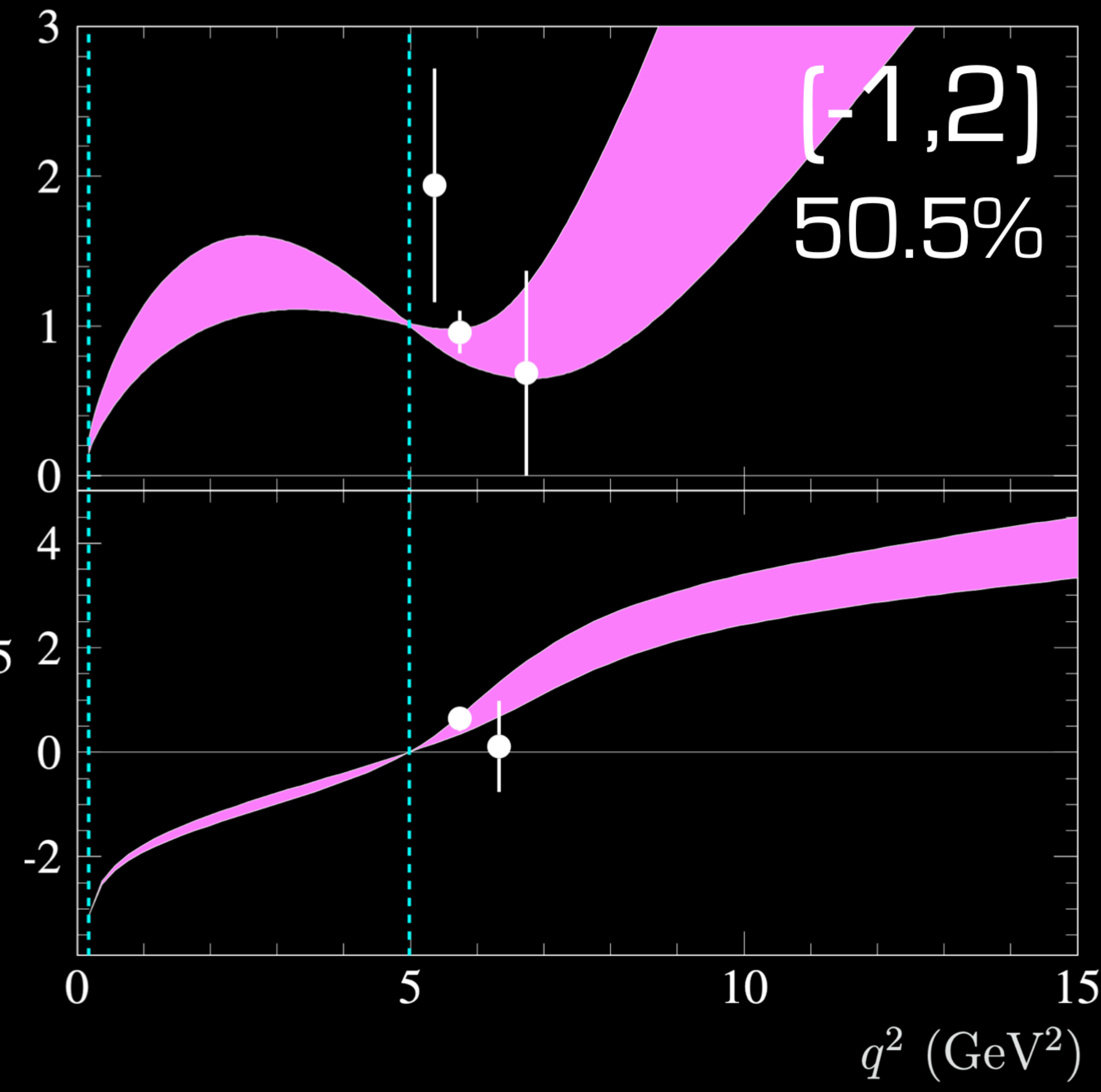
The χ^2 minimization alongside with the theoretical constraints allows to produce the $(N_{\rm th}, N_{\rm asy})$ possible pairs compatible with the data points.

A Monte Carlo procedure allows to obtain the probability of occurrence of each pair $(N_{\rm th}, N_{\rm asy})$.

$N_{\rm th}$	$N_{\rm asy}$	%
-1	2	50.5%
-1	1	16.0%
0	3	26.8%
-1	0	4%

Moduli and relative phases





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Charge radius of a neutral baryon

The charge radius squared $\langle r_E \rangle^2$ of an extended particle is proportional to the first derivative of the electric form factor $G_E\left(q^2\right)$ at $q^2 = 0$.

$$\langle r_E \rangle^2 = 6 \left. \frac{dG_E(q^2)}{dq^2} \right|_{q^2=0}$$

In the Breit frame, $q = \left(0, \vec{q}\right)$, the electric form factor is the Fourier transform of the spacial charge distribution.

For a neutral baryon the Sachs form factors are normalized as $G_E(0) = 0$, $G_M(0) = \mu \neq 0$, then taking the derivative of the ratio $R(q^2) = G_E(q^2)/G_M(q^2)$

$$\left. \frac{dR(q^2)}{dq^2} \right|_{q^2=0} = \frac{1}{G_M(q^2)} \left(\frac{dG_E\left(q^2\right)}{dq^2} - \overbrace{\frac{G_E(q^2)}{G_M(q^2)}}^{=0 \text{ at } q^2=0} \frac{dG_M(q^2)}{dq^2} \right) \Bigg|_{q^2=0} = \frac{1}{G_M(q^2)} \left. \frac{dG_E(q^2)}{dq^2} \right|_{q^2=0} = \frac{1}{\mu} \frac{\langle r_E \rangle^2}{6}$$

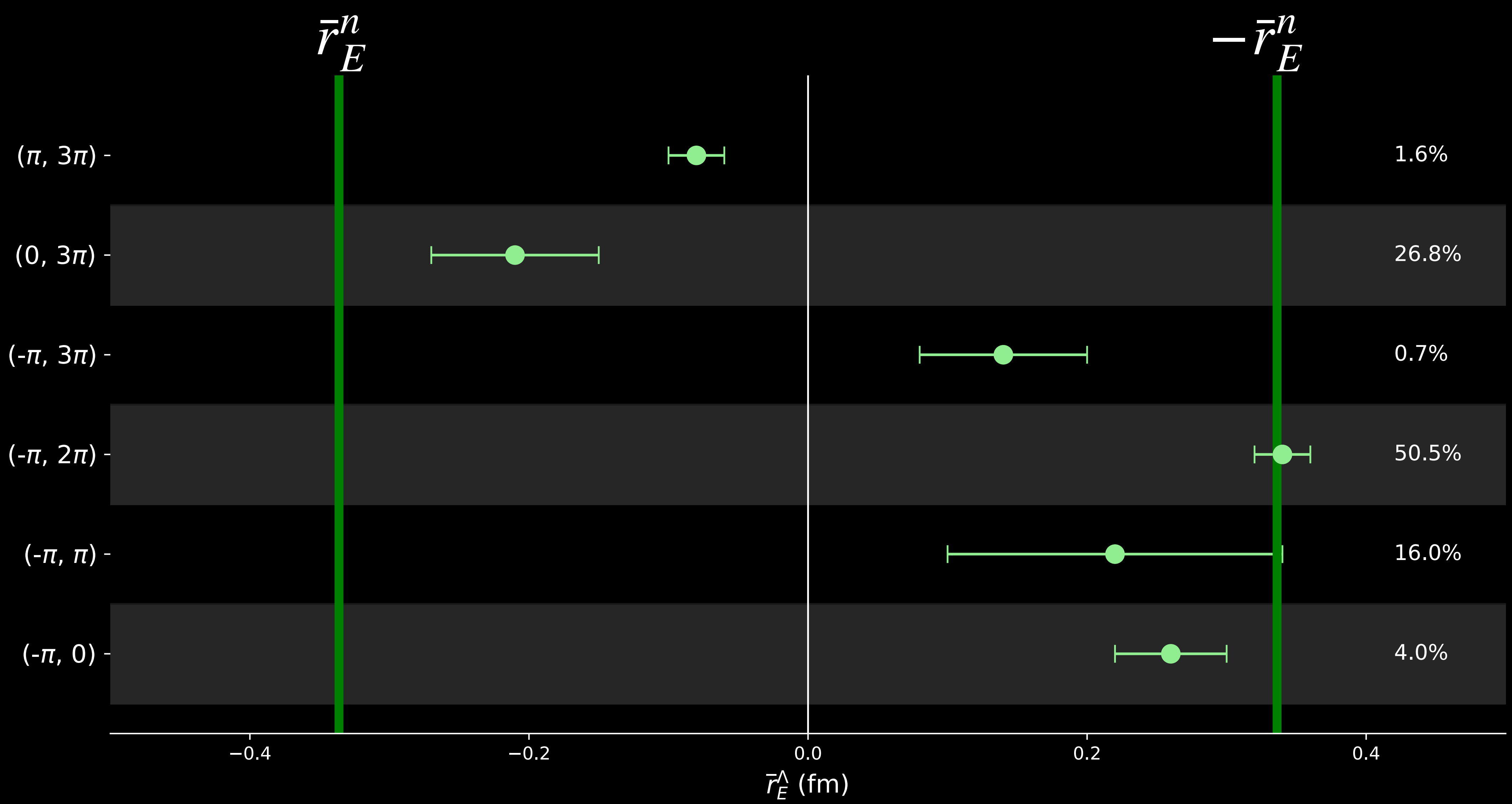
In terms of the dispersion relations for the imaginary part, the first derivative of the ratio $R(q^2)$ at $q^2 = 0$ is computed as

$$\langle r_E \rangle^2 = 6\mu \left. \frac{dR(q^2)}{dq^2} \right|_{q^2=0} = \frac{6\mu}{\pi} \int_{q_{th}^2}^{\infty} \frac{\mathrm{Im}\left(R(s)\right)}{s^2} ds = \frac{6\mu}{\pi \Delta q^2} \sum_{j=0}^{N} C_j \int_{-1}^{1} \frac{T_j(x) dx}{(x + 1 + q_{th}^2/\Delta q^2)^2}, \qquad \Delta q^2 = \frac{q_{asy}^2 - q_{th}^2}{2}$$

Charge radius of a neutral baryon

$$\bar{r}_E^{\mathscr{B}} \equiv \text{Sign}\left(\left\langle r_E^{\mathscr{B}} \right\rangle^2\right)\sqrt{\left|\left\langle r_E^{\mathscr{B}} \right\rangle^2\right|}$$

The Λ baryon charge radius is comparable with the neutron one $\bar{r}_E^n$.



The $\bar{r}_E^\Lambda$ values suggest that the negative charge of the Λ baryon's s quark lies further to the center than the d quark of the neutron.

Final considerations

The bands represent the one-sigma-error computed with statistical analysis of the Monte Carlo procedure.

The dispersive procedure, connecting time-like experimental values and theoretical constraints, allows to assign different determinations to the phase, and hence to the measured values of the phase. This gives information about the space-like behaviour of the form factors ratio.

Assuming no zeroes for the magnetic form factor, the Levinson's Theorem allows to count the number of zeroes of the electric form factor, aside from the theoretical one at $q^2 = 0$

$$\Delta\phi = \phi(\infty) - \phi(q^2_{\mathrm{th}}) = \pi\left(N_{\mathrm{asy}} - N_{\mathrm{th}}\right) \geq \pi$$

The most probable value for $N_{\mathrm{asy}} - N_{\mathrm{th}}$ is 3, hence there are at least two additional zeroes for $G_E^\Lambda(q^2)$.

To do list:

- Update the plots with the new data from BESIII collaboration.
- Unravel the systematic uncertainty given by the degree of the polynomial used for the fit.

Thank you for your attention!

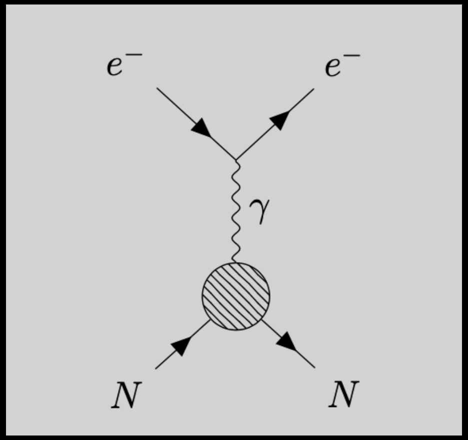
Backup Slides

Cross section

Scattering cross section ($q^2 < 0$)

Real valued

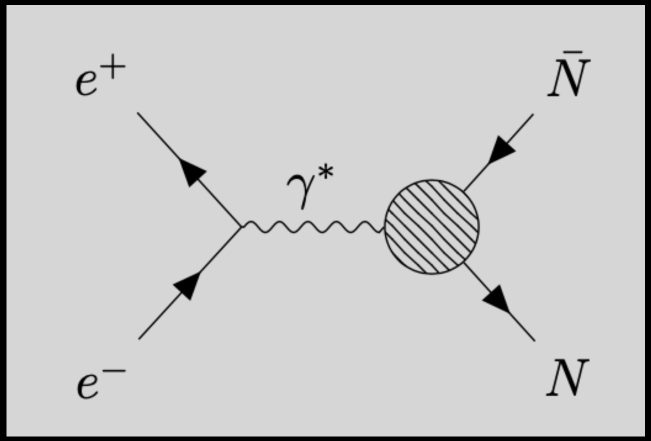
$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2 E_e' \cos^2(\theta/2)}{4E_e^3 \sin^4(\theta/2)} \left[\left(G_E^{\mathscr{B}}\right)^2 - \tau \left(1 + 2(1-\tau)\tan^2(\theta/2)\right) \left(G_M^{\mathscr{B}}\right)^2 \right] \frac{1}{1-\tau}$$



Annihilation cross section ($q^2 \geq (2M_{\mathscr{B}})^2$)

Complex valued

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2 \beta \mathscr{C}}{16E^2} \left[\left(1 + \cos^2(\theta)\right) \left| G_E^{\mathscr{B}} \right|^2 + \frac{1}{\tau} \sin^2(\theta) \left| G_M^{\mathscr{B}} \right|^2 \right]$$



Coulomb correction

$$\mathscr{C} = \frac{\pi\alpha}{\beta} \frac{1}{1 - e^{-\pi\alpha/\beta}}$$

$\mathscr{C}$ is a final state interaction effect

Analyticity of form factors

Spacelike region

$$q^2 < 0$$

$$e\mathscr{B} \to e\mathscr{B}$$

$$G_E^{\mathscr{B}}(q^2), G_M^{\mathscr{B}}(q^2)$$

Unphysical region

$$q_{th}^2 < q^2 \leq q_{phys}^2$$

$$\mathscr{B}\bar{\mathscr{B}} \to e^+e^-\mathscr{M}_0$$

$$\left|G_E^{\mathscr{B}}(q^2)\right|, \left|G_M^{\mathscr{B}}(q^2)\right|$$

Timelike region

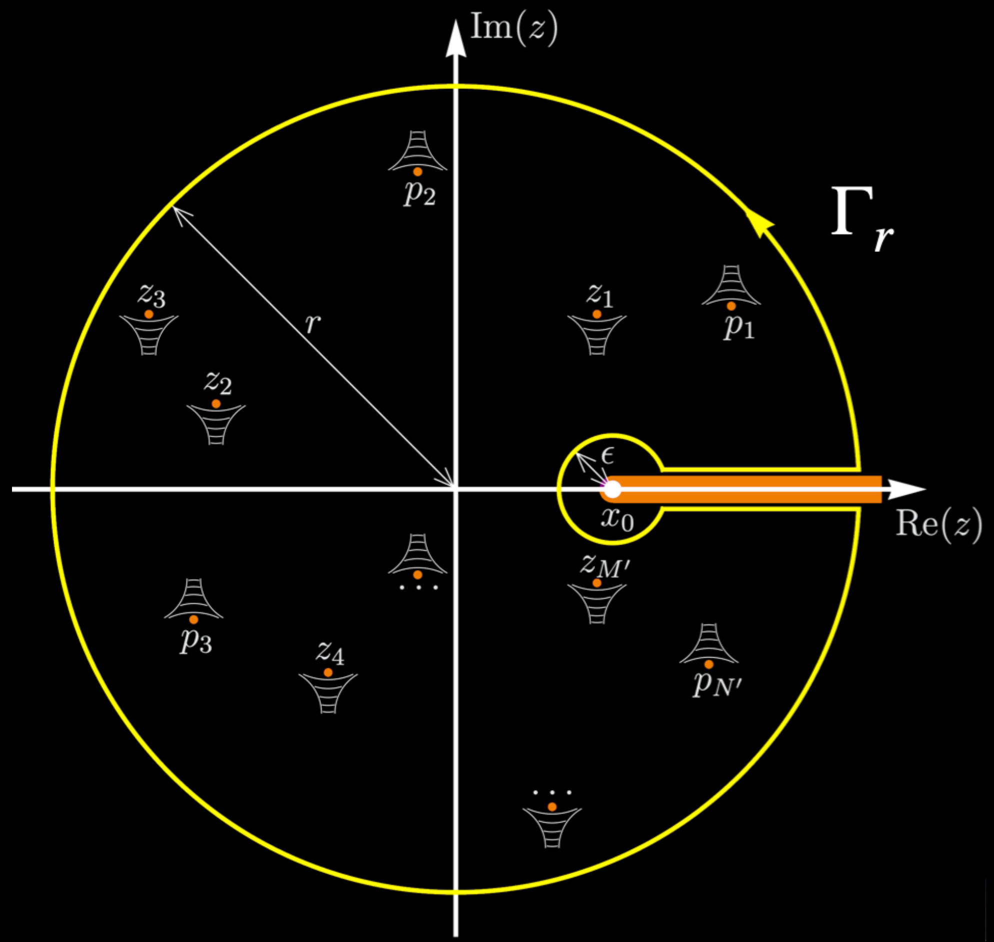
$$q^2 > q_{phys}^2$$

$$e^+e^- \leftrightarrow \mathscr{B}\bar{\mathscr{B}}$$

$$\begin{cases} \left|G_E^{\mathscr{B}}(q^2)\right|, \left|G_M^{\mathscr{B}}(q^2)\right| \\ \arg\left(G_E^{\mathscr{B}}/G_M^{\mathscr{B}}\right)^* \end{cases}$$

* Sine of the argument measurable in polarized cross section only

Dispersion relations



- Consider the complex function $R(z)$ with N poles $\{p_j\}_{j=1}^{N}$ and M zeroes $\{z_k\}_{k=1}^{M}$ and a branch cut $\left(x_0, \infty\right)$

- Taking the integral over the contour Γ_r gives the Cauchy's argument principle

$$\lim_{r\to\infty} \frac{1}{2i\pi} \oint_{\Gamma_r} \frac{d \ln \left(R(z)\right)}{dz} dz = M - N$$

- By taking each contribution into account

$$\lim_{r\to\infty} \frac{1}{2i\pi} \oint_{\Gamma_r} \frac{d \ln \left(R(z)\right)}{dz} dz = \frac{1}{\pi} \left(\arg(R(\infty)) - \arg(R(x_0))\right)$$

$$\left(\arg(R(\infty)) - \arg(R(x_0))\right) = \pi \left(M - N\right)$$

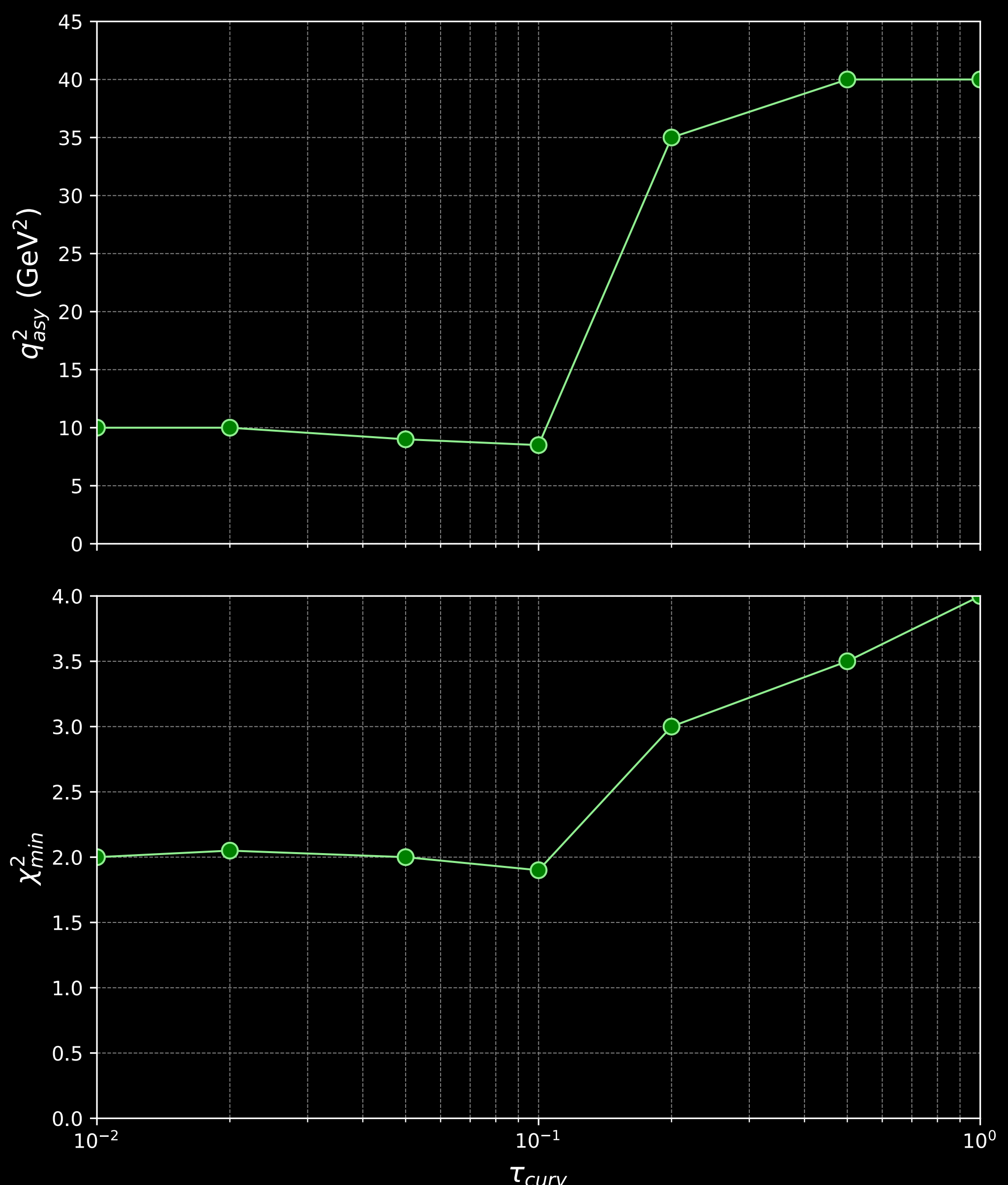
Levinson's Theorem

The curvature weight

The curvature weight $\boldsymbol{\tau}_{\mathbf{curv}}$ regularises the fit function behaviour.

If $\boldsymbol{\tau}_{\mathbf{curv}}$ is too large physical information are canceled.

If $\boldsymbol{\tau}_{\mathbf{curv}}$ is too small the solution has too much noise.



The polynomial degree $\boldsymbol{N}$ and the curvature weight $\boldsymbol{\tau}_{\mathbf{curv}}$ are mutually dependent.

The value of $\boldsymbol{\tau}_{\mathbf{curv}}$ at a given polynomial degree is given by a “phase transition” of the asymptotic threshold.