Reviewing the Role of Lattice QCD for Rare Decays: Achievements and Future Challenges

Giuseppe Gagliardi
Università di Roma Tre and INFN Sezione di Roma Tre

WIFAI 2024, 12-15 November 24, Bologna

Content

In this talk I will review the status of Lattice QCD calculations of the following rare decays:

Rare FCNC B-decays:

- $B_{q=d,s} \to \ell^+\ell^-$
- $B_s \to \mu^+\mu^-\gamma$
- $B \to K^{(*)}\ell^+\ell^-$ $(B_s \to \phi\ell^+\ell^-)$

Rare FCNC Kaon decays:

- $K \to \pi\nu\bar{\nu}$
- $K \to \pi\ell^+\ell^-$

$(W+\gamma)$-mediated decays into dilepton+lepton pair

- $K \to \ell'^+\ell'^-\ell\nu$
- heavier charged pseudoscalars $(D_s, \ldots, B)$?

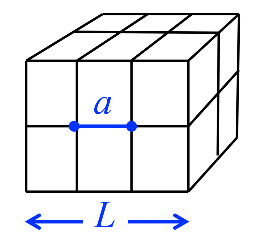
...many more rare processes are currently being investigated on the lattice (e.g., $\Sigma^+ \to p\ell^+\ell^-$ [Erben et al., PoS LATTICE2022 (2023) 315] , $K_L \to \mu^+\mu^-$ N.H. Christ et al. PRD 110 (2024), ...). Due to time constraints, unfortunately, will not be discussed.

Lattice QCD, a short guide

QFT in Euclidean time (obtained through Wick-rotation $t \to -i\tau$)

$$\boxed{\langle \phi(x_1)\phi(x_2)\ldots\phi(x_n)\rangle = \frac{1}{\mathcal{Z}}\int [d\phi]\ \phi(x_1)\phi(x_2)\ldots\phi(x_n)\exp(-S_E[\phi])}$$

Infinite-dimensional path integral (PI) discretized on a 4-dimensional grid (the lattice) : $x_\mu \to n_\mu a$, which provides an UV ($1/a$) and IR ($1/L$) cut-off.



- PI evaluated using MC methods, generating a stream of gauge configurations $\{U_1, \ldots, U_N\}$ distributed according to $e^{-S_E[U]}$, then...

$$\langle \bar{\mathcal{O}} \rangle = \frac{1}{N}\sum_{i=1}^{N}\mathcal{O}[U_i] \implies \sigma_{\bar{\mathcal{O}}} \propto \frac{1}{\sqrt{N}}$$

- Repeat the calculation for different lattice size L and lattice spacings a and extrapolate to $a, 1/L \to 0$.

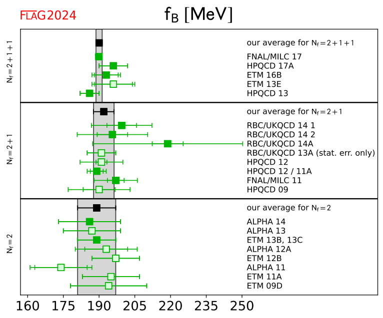
Challenges: calculations of decay rates are more complex for large recoiling daughter hadrons (large stat. noise), propagating intermediate states with lesser energy than external states (lack of analytic continuation to Euclidean spacetime), or for heavy quarks of mass m_q where $am_q < 1$ is hard to achieve on current lattices.

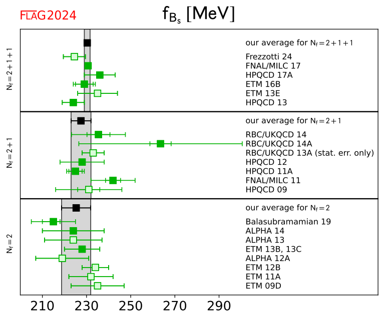
$B_q \to \ell^+\ell^-$ FYI: FLAG '24 is out $\Longrightarrow$ [arXiv:2411.04268]

The branching for $B_q \to \ell^+\ell^-$ is given by

$$B(B_q \to \ell^+\ell^-) = \tau_{B_q} \frac{G_F^2}{\pi} \left(\frac{\alpha_s}{4\pi \sin^2 \Theta_W} \right)^2 m_{B_q} f_{B_q}^2 |V_{tb}^* V_{tq}|^2 m_\ell^2 \sqrt{1 - \frac{4m_\ell^2}{m_{B_q}^2}}$$

- Function Y includes QCD+EW NLO corrections.
- f_{B_q} is the B_q-meson decay constant $\langle 0|A_\mu|B_q(p)\rangle = ip_\mu f_{B_q}$





$B(B \to \mu^+\mu^-) < 1.9 \times 10^{-10}$ at 95% CL

$B(B_s \to \mu^+\mu^-) = \left(2.69^{+0.37}_{-0.35}\right) \times 10^{-9}$

$\Longleftarrow$ Experimental results compatible with SM predictions at $\simeq 2\sigma$ level.

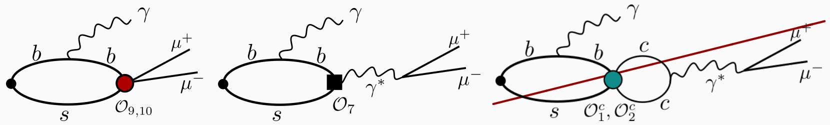
Then...let's go to the radiative $B_s \to \mu^+\mu^-\gamma$

$b \to s$ effective Hamiltonian (neglecting doubly Cabibbo-suppressed terms):

$$\mathcal{H}_{\rm eff}^{b\to s} = 2\sqrt{2} G_F V_{tb} V_{ts}^* \left[\sum_{i=1,2} C_i(\mu)\mathcal{O}_i^c + \sum_{i=3}^{6} C_i(\mu)\mathcal{O}_i + \frac{\alpha_{\rm em}}{4\pi} \sum_{i=7}^{10} C_i(\mu)\mathcal{O}_i \right]$$

Despite the $\mathcal{O}(\alpha_{\rm em})$-suppression w.r.t. $B_s \to \mu^+\mu^-$, removal of helicity-suppression makes the two decay rates comparable in magnitude.

- Within the SM, to good approximation $B_s \to \mu^+\mu^-$ only sensitive to Wilson operator (coefficient) $\mathcal{O}_{10}$ (C_{10})
- $B_s \to \mu^+\mu^-\gamma$ offers sensitivity to C_{10}, C_9, C_7. However, for $q^2 \lesssim (4~\mathrm{GeV})^2$ also resonant charming penguin diagrams, difficult on the lattice, contribute...
- Focusing on large dimuon invariant masses $\sqrt{q^2} > 4.2~\mathrm{GeV}$ can be a way out!

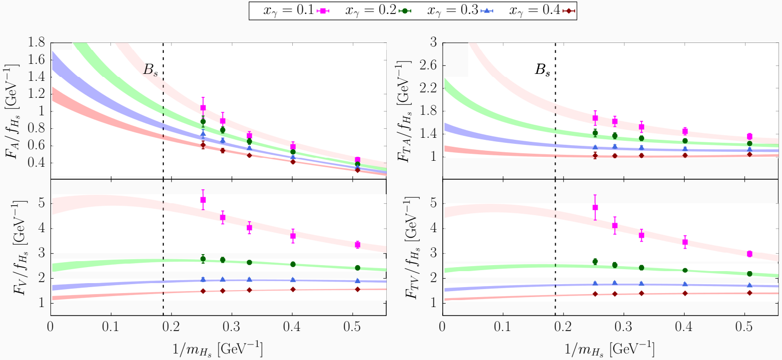


$+$ [γ emitted by s] $+$ [exchange $\gamma \to \gamma^*$]

Soton-RM123 results for $B_s \to \mu^+\mu^-\gamma$

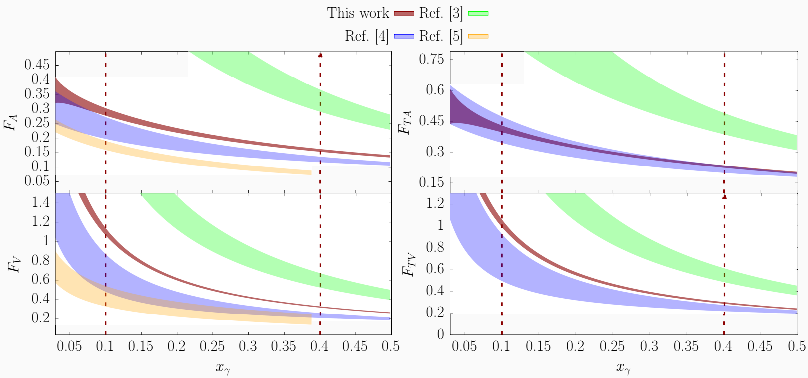
We (RM123-Soton Collaboration), using gauge configurations generated by the ETM Collaboration, have recently published the results of a lattice calculation of $B_s \to \mu^+\mu^-\gamma$ [PRD 109 (2024)]

- Mainly, four non-perturbative form factors contribute: F_V, F_A, F_{TV}, F_{TA}



- B_s-meson too heavy to be directly simulated on current lattices. We simulate lighter H_s-mesons with $m_h < m_b$ and extrapolate to the physical B_s employing pole-like + HQET scaling relations. In the figure, $x_\gamma = 2E_\gamma/M_{H_s}$.

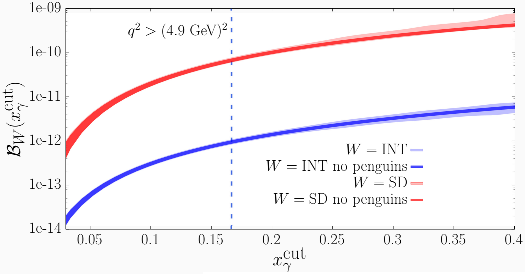
Comparison with model/effective-theory calculations



- Ref. [3] = Janowski, Pullin , Zwicky , JHEP '21 , light-cone sum rules.
- Ref. [4] = Kozachuk, Melikhov, Nikitin , PRD '18 , relativistic dispersion relations.
- Ref. [5] = Guadagnoli, Normand, Simula, Vittorio, JHEP '23, VMD/quark-model/lattice.

With a few exceptions, our results for the form factors differ significantly from the earlier estimates (which also differ from each other).

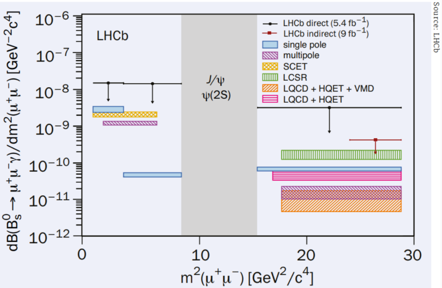
The branching fractions



At $E_\gamma^{\mathrm{cut}} = x_\gamma^{\mathrm{cut}} M_{B_s}/2 \gtrsim 1$ GeV systematic uncertainty due to resonant charming-penguin contributions is around 30% (much larger than estimated in previous works).

LHCb

LHCb targets rare radiative decay



Best LHCb bound is

$$B(B_s \to \mu^+\mu^-\gamma) < 2.0 \times 10^{-9} \quad \text{at 95\% CL,}$$

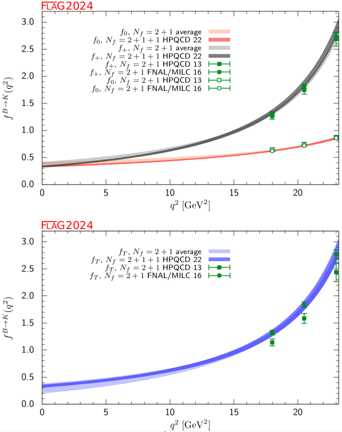
in the kinematic region $\sqrt{q^2} > 4.9\,\mathrm{GeV}$

. . . one order of magnitude larger than our prediction!

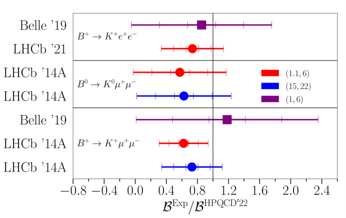
FCNC semileptonic decays

$B \to K\ell^+\ell^-$

Main NP contribution to the branching comes from QCD matrix elements $\langle K|J|B\rangle$, with $J = V, A, T$, which are parameterized by the form factors $f_+(q^2)$, $f_0(q^2)$ and $f_T(q^2)$. Three lattice results available: HPQCD '22, HPQCD '13, FNAL/MILC '16



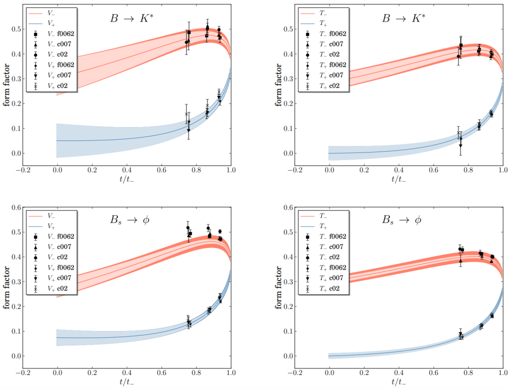
- In HPQCD '22 heavy quark simulated without relying on the non-relativistic action.
- Low-q^2 extrapolation performed using modified z-expansion.
- In PRD 107 (2023), HPQCD compares with exp. results:



As in $B_s \to \mu^+\mu^-\gamma$ charming penguin contributions may play a role! Focusing on the low (large) q^2-region only, avoiding main charmonium resonances, reduces their impact (in the future, penguins on the lattice?)

$P \to V$ FCNC semileptonic decays

So far only one lattice calculation published, where K^* and ϕ are treated as stable states (for the ϕ expected to be a good approximation) [Horgan et al., PRD 89 (2014), PRL 112 (2014)]



- Seven form factors contribute $T_\pm, V_\pm, A_0, A_{12}, T_{23}$ plus, as usual, (neglected) long-distance effects of which charming penguin contributions are expected to be the most important!
- MILC gauge configurations + non-relativistic heavy quark action employed.

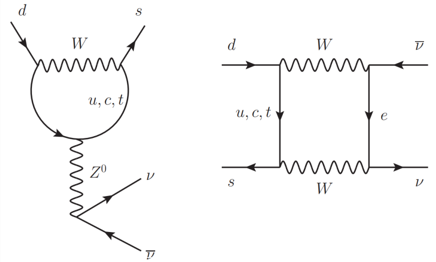
- New calculations, going beyond the stable-hadron approximation for the K^*, are ongoing [Leskovec et al, arXiv:2403.19543].

Rare FCNC kaon decays

The golden mode $K \to \pi\nu\bar{\nu}$

The $K \to \pi\nu\bar{\nu}$ decay is one of the cleanest mode where to look for signals of New Physics, owing to the essentially short-distance nature (top-dominated) of this FCNC decay. The branching, for the charged mode, can be cast in the form

$$B(K^+ \to \pi^+\nu\bar{\nu}) = \kappa^+(1+\Delta_{\text{EM}})\left[\left(\frac{\text{Im}\,\lambda_t}{\lambda^5}X\right)^2 + \left(\frac{\text{Re}\,\lambda_t}{\lambda^5}X + \frac{\text{Re}\,\lambda_c}{\lambda}(P_c + \delta P_{c,u})\right)^2\right]$$



- Long-distance (LD) effects in the charged mode, due to charm- and up-quark loops, parametrized by $\delta P_{c,u}$.
- ChPT + OPE [Isidori et al, Nucl. Phys B B718 (200 5)] predicts $\delta P_{c,u} = 0.04 \pm 0.02$, which shifts the SM value by $\simeq 1\sigma$ (6%).
- OPE+ChPT predictions for $\delta P_{c,u}$, can be replaced by a first-principle lattice QCD calculation!
- Neutral mode, $K_L \to \pi^0\nu\bar{\nu}$, is instead unaffected by LD effects.

The PDG quotes for the branching: $B(K^+ \to \pi^+\nu\bar{\nu}) = 1.14^{+0.40}_{-0.33} \times 10^{-10}$.

$K^+ \to \pi^+ \nu \bar{\nu}$ on the lattice

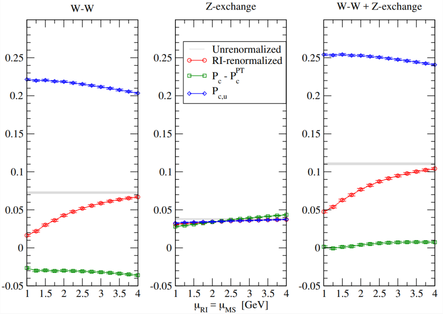
The RBC/UKQCD Collaboration, has faced the problem of computing on the lattice the LD contribution to $K^+ \to \pi^+ \nu \bar{\nu}$ in a series of papers.

[N.H. Christ et al., PRD 93 (2016), PRL 118 (2017), PRD 100 (2019)]

Many issues to be addressed:

- How to handle the contribution from 2π intermediate states with lesser energy than the kaon (contr. found of $\mathcal{O}(1\%)$ in N.H. Christ et al, PRD 100 (2019)).
- Evaluate finite-size effects from simulations on a finite spatial box $V = L^3$.
- Renormalisation of the four-fermion operators.

From C.T. Sachrajda talk at Kaon@CERN



- RBC/UKQCD results obtained on a domain-wall ensemble with lattice spacing $a^{-1} = 1.73$ GeV, $m_\pi \sim 420$ MeV, $m_K \sim 563$ MeV, $m_c^{\overline{\text{MS}}}(2\text{ GeV}) \sim 863$ MeV.
- On this unphysical ensemble they obtained:

$$P_c = 0.2529 \pm (13) \pm (32) \pm (-45)_{\text{FV}}$$

$$\delta P_c = 0.0040 \pm (13) \pm (32) \pm (-45)_{\text{FV}}$$

Calculations at the phyisical point are ongoing!

The rare $K^+ \to \pi^+ \ell\ell$ decay

Hadronic amplitude A_μ given by (H_W is the weak $s \to d$ effective Hamiltonian)

$$A_\mu(q^2) = \int d^4x \, \langle \pi^+(p) | T \left[j_\mu^{\text{em}}(0) \, H_W(x) \right] | K^+(k) \rangle$$

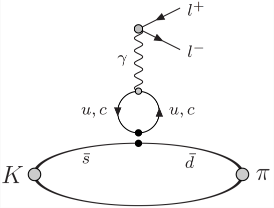
$$= -i \frac{G_F}{16\pi^2} V(z) \left(q^2 (k+p)_\mu - (m_K^2 - m_\pi^2) q_\mu \right) \, , \qquad z = q^2/m_K^2$$

$V(z) = a_+ + b_+ z + V^{\pi\pi}(z)$

Measurement	a_+	b_+
E865 - $K_{\pi ee}$	-0.587 ± 0.010	-0.655 ± 0.044
NA48/2 - $K_{\pi ee}$	-0.578 ± 0.016	-0.779 ± 0.066
NA48/2 - $K_{\pi\mu\mu}$	-0.575 ± 0.039	-0.813 ± 0.145
NA62 - $K_{\pi\mu\mu}$	-0.575 ± 0.013	-0.722 ± 0.043

From C.T. Sachrajda talk at Kaon@CERN

From C.T. Sachrajda talk at Kaon@CERN



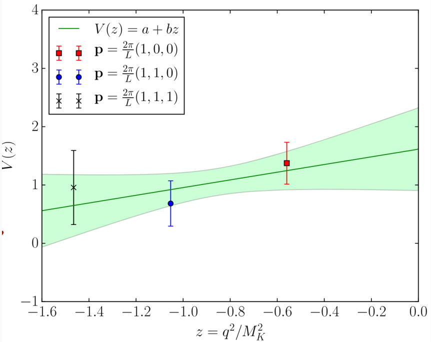
- No power divergencies as a consequence of e.m. gauge invariance [G. Isidori et al. Phys.Lett. B633 (2006)].
- Feasibility of a lattice calculation demonstrated in N.H. Christ et al. (RBC/UKQCD) PRD 92 (2015).
- Experimental results very precise, new NA62 results for the $K^+ \to \pi^+ \mu^+ \mu^-$ branching have less than 1% uncertainties.

Numerical studies by RBC/UKQCD

- First exploratory calculation carried by the RBC/UKQCD Collaboration in N.H. Christ et al PRD 94 (2016), using a single domain-wall ensemble at $a^{-1} \sim 1.78$ GeV and higher-than-physical pion mass $M_\pi \sim 430$ MeV.

- Three pion momenta $\mathbf{p}$ considered in the kaon rest frame $\Longrightarrow$

From C.T. Sachrajda talk at Kaon@CERN



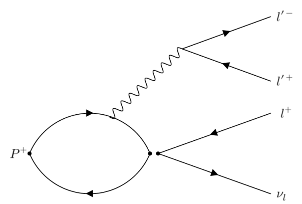
More recently RBC/UKQCD has published the results at $a^{-1} \sim 1.73$ GeV, physical quark masses and at a single pion momentum $\mathbf{p} = \frac{2\pi}{L}(1,0,0) \sim 225$ MeV [P.A. Boyle et al PRD 107 (2023)]

$$\text{obtaining:} \qquad V(z = 0.013(2)) \sim a_+ = -0.87 \pm 4.44$$

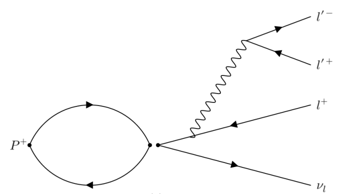
- Statistical uncertainties are presently large, but new numerical methods are under investigation to reduce the noise (split-even approach, R. Hodgson talk at LATTICE2024).

The rare decay $K \to \ell'^+ \ell'^- \ell \nu$

The $P \to \ell'^+\ell'^-\ell\nu$ decay



(a)



(b)

- $P \to \ell'^+\ell'^-\ell\nu$ is the electroweak decay of a flavoured and charged pseudoscalar meson P into a dilepton ($\ell'^+\ell'^-$) and a lepton pair ($\ell\nu$).

- It proceeds via

$$P \to \gamma^*\ell\nu \to \ell'^+\ell'^-\ell\nu$$

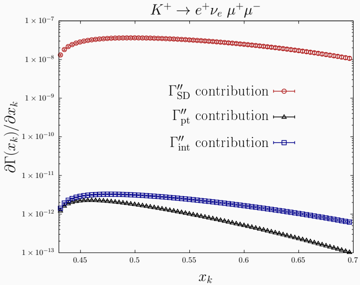
- The intermediate virtual photon γ^* can be either emitted from the final-state lepton ℓ (so-called Bremsstrahlung contribution, depends only on f_P) or from a quark (so-called structure-dependent contribution).

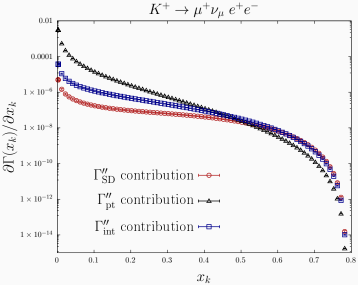
- They are rare decays with decay rates of order $\mathcal{O}(G_F^2\,\alpha_{\rm em}^2)$, which can thus be interesting probe of NP beyond the SM.

Results of an exploratory calculation

In PRD 105 (2022) we (RM123-Soton Coll.) presented the results for $K \to \ell'^{+}\ell'^{-}\ell\nu$, in an unphysical setup with $2M_\pi > M_K$.

We computed the differential and total rate for different decay-channels $(x_k = \sqrt{k^2}/M_K)$:





This work	point-like	E865 exp.
$\mathrm{Br}\left[K^+ \to e^+\,\nu_e\,\mu^+\,\mu^-\right]$		
$0.762(49) \times 10^{-8}$	3.0×10^{-13}	$1.72(45) \times 10^{-8}$
$\mathrm{Br}\left[K^+ \to \mu^+\,\nu_\mu\,e^+\,e^-\right]$ for $x_k > 0.284$		
$8.26(13) \times 10^{-8}$	4.8×10^{-8}	$7.93(33) \times 10^{-8}$

- Having unphysical quark-masses is OK for semi-quantitative predictions.
- In '25 we plan to publish the results at the physical point!
- We would be very happy to see new experimental results in this channel.

A new tool in the lattice QCD arsenal

Over the past few years, significant advancements have been made in spectral reconstruction techniques from Euclidean correlators:

$$C_E(t) = \int dE\, e^{-Et} \rho(E), \qquad H(E) = \int dE\, K(E)\, \rho(E)$$

- **HLT method for spectral-density reconstruction**: M. Hansen et al, PRD 99 (2019).
- **Inclusive decay rates from Euclidean lattice correlators**: M. T. Hansen et al. PRD 96 (2017), Gambino et al. PRL 125 (2020), Bulava et al. JHEP 07(2022).
- **Scattering amplitudes from Euclidean correlators**: Bulava et al. PRD 100 (2019), Patella et al. arXiv:2407.02069 .
- **Complex electroweak amplitudes**: Frezzotti et al. PRD 108 (2023) .
- ... and many other

These techniques have already been successfully applied to the calculation of observables previously deemed impossible to compute in lattice QCD, e.g. :

- **Evaluation of the energy-smeared R-ratio**: ETM Collaboration, PRL 130 (2023) .
- **Inclusive hadronic τ decays**: ETM Collaboration, PRD 108 (2023), ETM Collaboration, PRL 132 (2024).
- **Inclusive semileptonic decays of heavy mesons**: Gambino et al, JHEP 07 (2022).

What's next?

This theoretical and numerical progress has also had an impact on the lattice QCD calculation of rare decay amplitudes. In fact, in some of the results presented here, e.g. in the calculation of the $B_s \to \mu^+\mu^-\gamma$ decay rate, these techniques have already been employed.

As demonstrated in Frezzotti et al. PRD 108 (2023), the HLT method can be used to evaluate complex electroweak amplitudes in those cases where intermediate states with lesser energy than the final states are present (standard approach fails due to the presence of unphysical diverging contributions in the Euclidean amplitude). E.g.:

- $K \to \ell'^+\ell'^-\ell\nu$ at the phys. point (ongoing calculation by RM123-Soton Coll.)...
- ...and extension to heavier pseudoscalar meson. A proof-of-principle calculation of $\phi \to \ell'^+\ell'^-\ell\nu$ already performed in Frezzotti et al. PRD 108 (2023).

Besides that, many new processes will be feasible in the next future:

- Radiative B-decays ($\propto |V_{ub}|^2$): $B \to \ell\nu\gamma$. Results for $\pi, K, D_{(s)}$ published.
- Many new results appearing for tree-level semileptonic B-decays ($\propto |V_{ub}|^2$): $B \to \pi\ell\nu, B \to \rho\ell\nu, B_s \to K\ell\nu$.

Conclusions

- I hope I conveyed the message that there are intense ongoing activities on the lattice to evaluate rare decay amplitudes.
- Many interesting results already published!
- A series of theoretical and numerical developments in the very last years will probably allow the calculation of many new interesting observables.
- Increasing statistical accuracy in the lattice results is to be expected with the next generation of high-performance machines.
- Synergy between the lattice and experimental communities is of utmost importance. Awareness of forthcoming measurements or lattice computations can guide efforts on both sides.

Thank you for the attention!