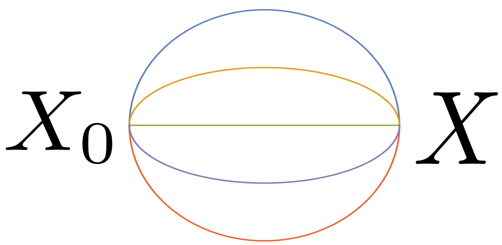
Loops (Banana) integrals in de Sitter (and anti de Sitter)



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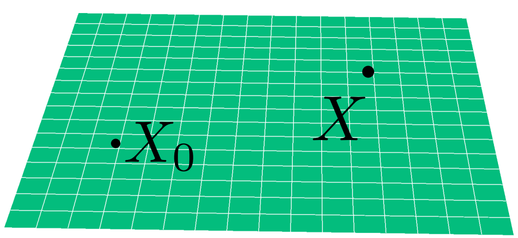
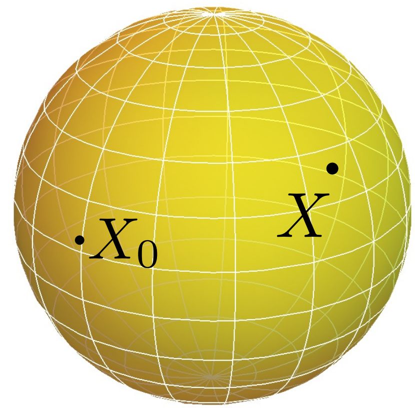
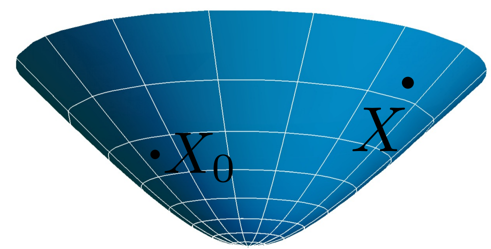






Lemaître's prophecy

The introduction of such a constant implies a considerable renunciation of the logical simplicity of the theory [...]
Since I introduced this term, I had always a bad conscience [...]
I am unable to believe that such an ugly thing should be realized in nature.

Albert Einstein,
Contribution to the book
"Albert Einstein: Philosopher–Scientist", 1949

The history of science provides many instances of discoveries which have been made for reasons which are no longer considered satisfactory [...]
It may be that the discovery of the cosmological constant is such a case.

George E. Lemaître,
Contribution to the book
"Albert Einstein: Philosopher–Scientist", 1949

…Einstein's prophetic outlook

Λ

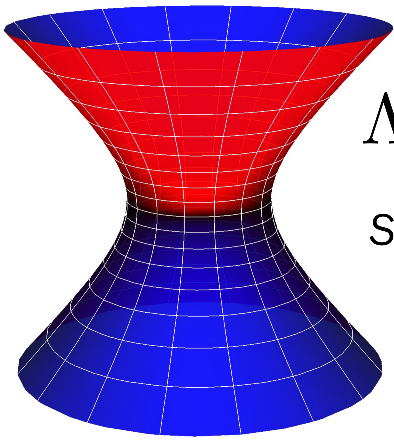
In any case, one thing is clear. The theory of general relativity allows adding the term $\Lambda g_{\mu\nu}$ in the equations.

One day, our real knowledge of the composition of the sky of fixed stars, the apparent motions of the fixed stars and the position of spectral lines as a function of distance, will probably be sufficient to decide empirically whether or not Λ is equal to zero.

Conviction is a good motive, but a bad judge

Albert Einstein
Letter to Willelm de Sitter
April 13, 1917

The cosmological constant at work



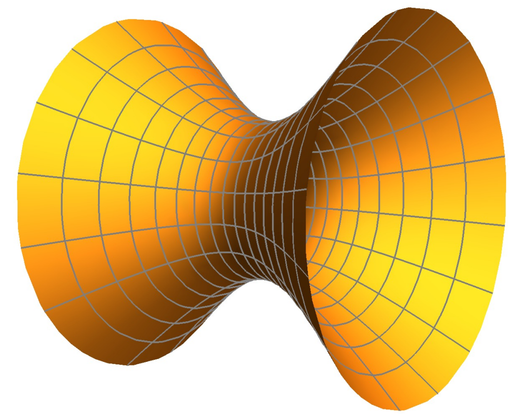
$\Lambda > 0$

$SO(1,d)$

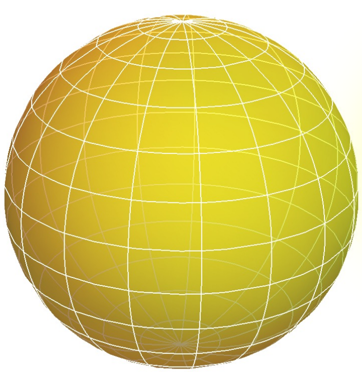
$$X_0^2 - X_1^2 - \ldots - X_{d-1}^2 - X_d^2 = -R^2$$

$\Lambda < 0$

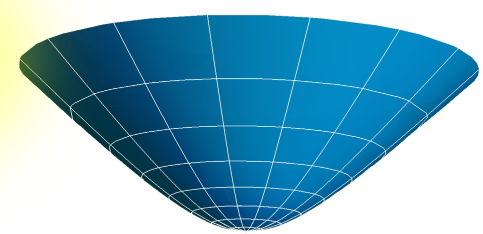
$SO(2,d-1)$



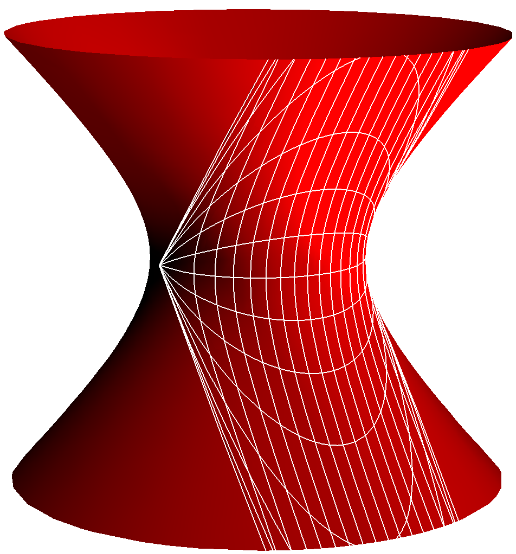
$$X_0^2 - X_1^2 - \ldots - X_{d-1}^2 + X_d^2 = +R^2$$

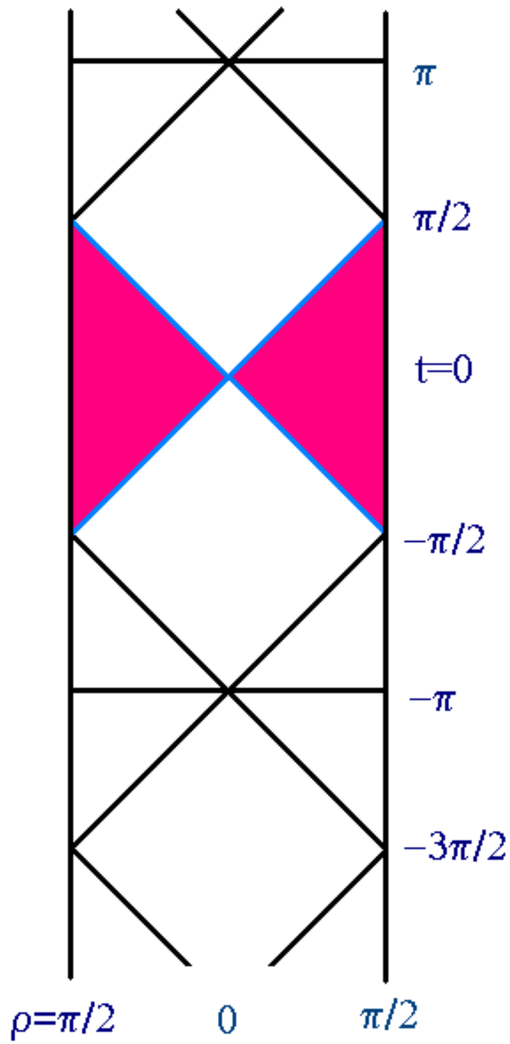


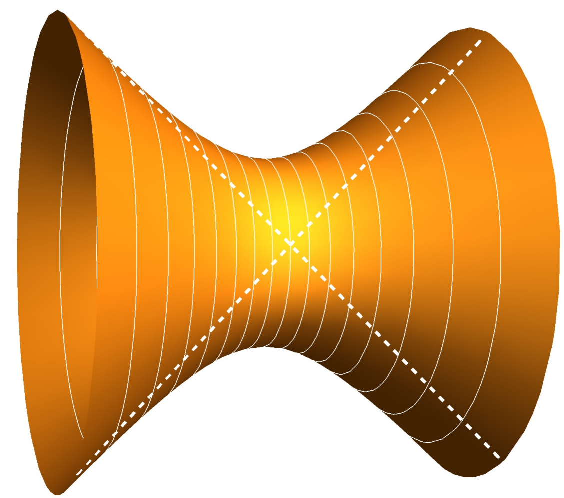
$$Z_0^2 + Z_1^2 + \ldots + Z_{d-1}^2 + Z_d^2 = R^2$$



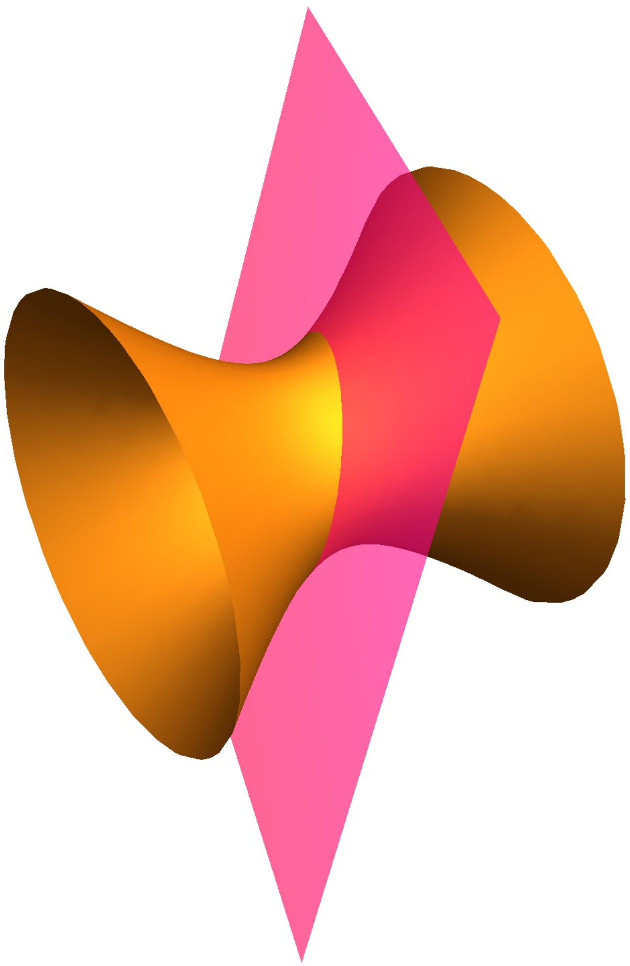
(A)dS QFT....?

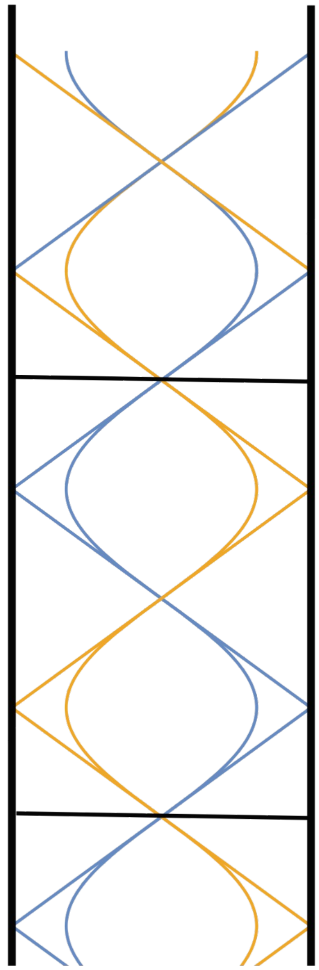


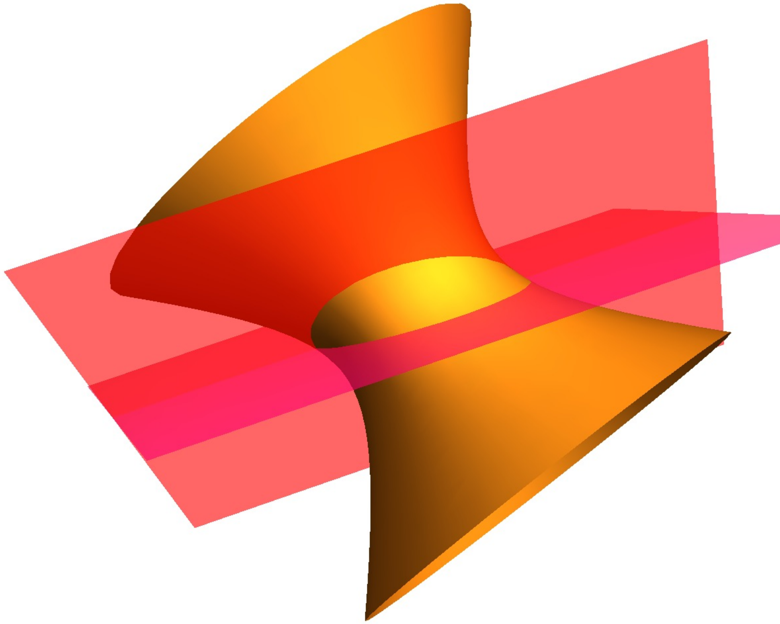




AdS covering does not really help







Prelude

A SOMETIMES FORGOTTEN KEY PROPERTY OF QUANTUM FIELD THEORY

The spectral property of QFT

1) Translations are represented by unitary operators $U(a,1) = \exp(ia_\mu \hat{P}^\mu)$.
There exists a unique translation invariant state Ψ_0.
2) The joint spectrum of the energy-momentum operator $\hat{P}^\mu$ is contained in the closed forward cone

$$\overline{V}^+ = \{p^2 \geq 0,\ p^0 \geq 0\}$$

The theory is completely encoded in the knowledge of the vacuum-to-vacuum transition amplitudes: $\mathcal{W}_n(x_1, \dots x_n) = \langle \Psi_0, \phi(x_1) \dots \phi(x_n) \Psi_0 \rangle$

Analyticity of the n-point functions in the tubes and positivity of the energy spectrum in every Lorentz frame are equivalent properties

$$\mathcal{W}_n(x_1, \dots x_n) = \textit{boundary value of } W_n(z_1, \dots z_n)$$

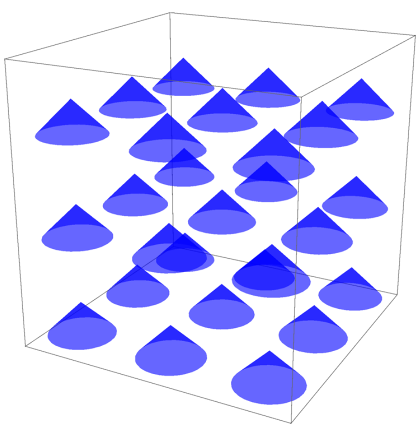
$$T_n = \{(z_1, \dots z_n) \ : \ \Im(z_{j+1} - z_j) \in V_+\}$$

Normal Analyticity

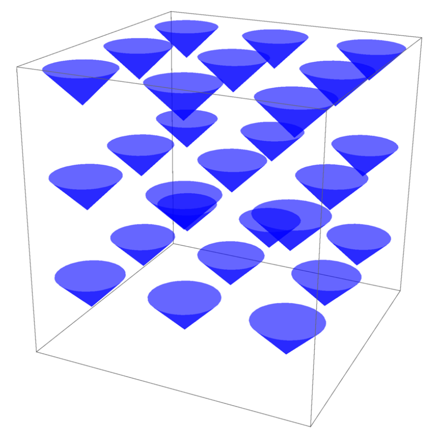
Spectral condition $\longleftrightarrow$ Any (general) two-point function is the boundary value of a function analytic in the tube of the complex Minkowski spacetime

$$\mathcal{W}(x_1, x_2) = boundary\ value\ W(z_1, z_2)$$

$$(z_1, z_2) \in T_- \times T_+$$



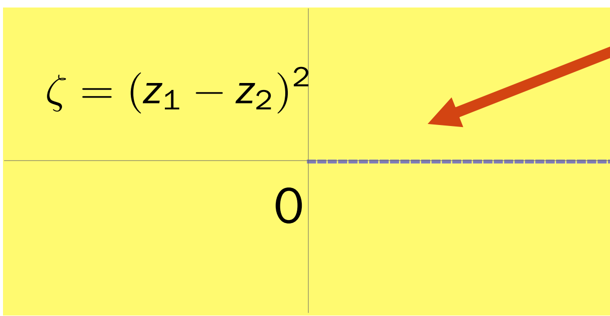
$$T_- = \{z = x + iy \in M^d,\ \ y^2 > 0,\ \ y^0 < 0\}$$



$$T_+ = \{z = x + iy \in M^d,\ \ y^2 > 0,\ \ y^0 > 0\}$$

Normal Analyticity + Complex Lorentz invariance =Maximal Analyticity

$$W(z_1, z_2) = W(\zeta) = \mathfrak{W}(z_1, z_2)$$



The cut reflects causality and QM

No cut $\longleftrightarrow$ No commutator

On the cut $\quad C(x,y) = \mathcal{W}(x,y) - \mathcal{W}(y,x) \neq 0$

The permuted extended tube contains the non coincident Euclidean points

$$S(x_1^E, x_2^E) = \mathfrak{W}(z_1, z_2)\big|_{E^d \times E^d \setminus \delta}$$

Klein-Gordon field : a crash review $(\Box + m^2)\phi = 0$

$$\mathcal{W}(x_1, x_2) = \mathcal{W}(x_1 - x_2) = \langle \Psi_0, \phi(x_1)\phi(x_2)\Psi_0 \rangle$$

$\xi = x_1 - x_2$ $\Rightarrow$ $(\Box_\xi + m^2)\mathcal{W}(\xi) = 0$ $\longleftrightarrow$ $(p^2 - m^2)\tilde{\mathcal{W}}(p) = 0$

Lorentz Invariance **Spectral Condition** **CCR**

$$\tilde{\mathcal{W}}(p) = A\theta(p^0)\delta(p^2 - m^2) + \xcancel{B\theta(-p^0)\delta(p^2 - m^2)}$$

$$\mathcal{W}(x_1 - x_2) = \frac{1}{(2\pi)^d} \int e^{-ip(x_1 - x_2)} \theta(p^0)\delta(p^2 - m^2) d^d p$$

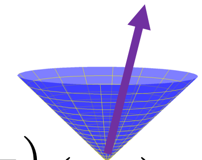
$$= \frac{1}{2(2\pi)^{d-1}} \int \frac{e^{-i\omega(x_1^0 - x_2^0) + i\vec{p}\cdot(\vec{x_1} - \vec{x_2})}}{\sqrt{|\vec{p}|^2 + m^2}} d\vec{p} = \int \psi_{\vec{p}}^{(-)}(x_1)\psi_{\vec{p}}^{(+)}(x_2) d\vec{p}$$

Wightman ➤ Schwinger ➤ Feynman

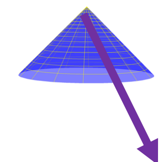
$$\psi_{\vec{p}}^{(\pm)}(x) = \frac{\exp(\pm ipx)}{2\sqrt{(2\pi)^{d-1}\omega}}$$

$$p^0 = \omega = \sqrt{|\vec{p}|^2 + m^2}$$

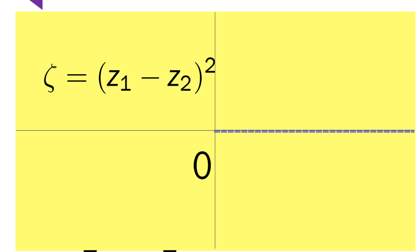
$$\mathcal{W}(x_1, x_2) = \int \psi_{\vec{p}}^{(-)}(x_1)\psi_{\vec{p}}^{(+)}(x_2)d\vec{p}$$



$$W(z_1, z_2) = \int \psi_{\vec{p}}^{(-)}(z_1)\psi_{\vec{p}}^{(+)}(z_2)d\vec{p}$$



$$\mathfrak{W}_m^d(z_1, z_2) = \frac{1}{(2\pi)^{\frac{d}{2}}}\left(\frac{\sqrt{-(z_1 - z_2)^2}}{m}\right)^{1-\frac{d}{2}} K_{\frac{d-2}{2}}(m\sqrt{-(z_1 - z_2)^2})$$



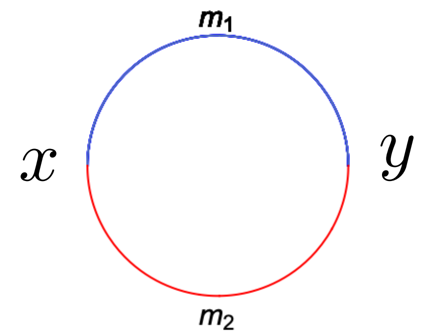
$$S_m^d(x_1^E, x_2^E) = \frac{1}{(2\pi)^{\frac{d}{2}}}\left(\frac{r}{m}\right)^{1-\frac{d}{2}} K_{\frac{d}{2}-1}(mr) = \frac{1}{(2\pi)^d}\int \frac{e^{-ip(x_1^E - x_2^E)}}{p^2 + m^2}dp$$

$$r = \sqrt{(x_1^E - x_2^E)^2}$$

$$S_m^d(x_1^E, x_2^E) = G_m^d(x_1^E, x_2^E)$$

An easy exercise: the bubble

$$G_m^d(x) = \frac{1}{(2\pi)^d} \int \frac{e^{-ipx}}{p^2 + m^2} dp$$



$$\int G_{m_1}(x-y) G_{m_2}(x-y) dx = \frac{1}{(2\pi)^d} \int \frac{1}{(p^2 + m_1^2)(p^2 + m_2^2)} d^d p$$

$$= \frac{1}{(2\pi)^d} \frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})} \frac{1}{m_1^2 - m_2^2} \int_0^\infty \left(\frac{1}{p^2 + m_2^2} - \frac{1}{p^2 + m_1^2} \right) p^{d-2} dp^2$$

$$\int G_{m_1}(x-y) G_{m_2}(x-y) dx = -\frac{\Gamma\left(1 - \frac{d}{2}\right)}{(4\pi)^{\frac{d}{2}}} \frac{(m_1^2)^{\frac{d}{2}-1} - (m_2^2)^{\frac{d}{2}-1}}{m_1^2 - m_2^2}$$

The bubble in x-space

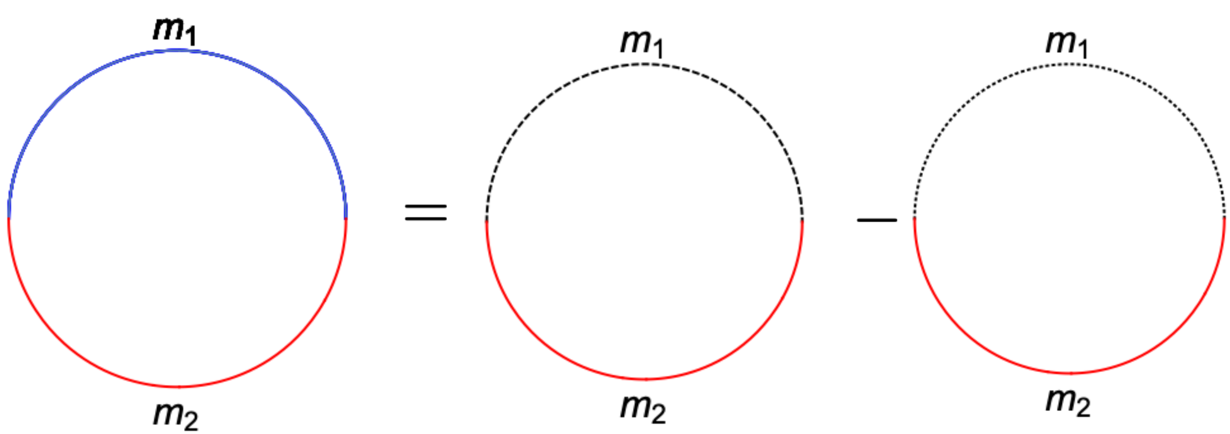
$$G_m^d(x) = \frac{1}{(2\pi)^{\frac{d}{2}}} \left(\frac{r}{m}\right)^{1-\frac{d}{2}} K_{\frac{d}{2}-1}(mr)\,, \quad r = \sqrt{x^2}$$

$$K_\nu(z) = \frac{\Gamma(1-\nu)\Gamma(\nu)}{2}(I_{-\nu}(z) - I_\nu(z)), \qquad I_\nu(z) = \sum_{n=0}^{\infty} \frac{1}{n!\,\Gamma(n+\nu+1)} \left(\frac{z}{2}\right)^{2n+\nu}$$

$$K_\nu(z) \sim e^{-z}\sqrt{\frac{2\pi}{z}}, \qquad I_\nu(z) \sim \frac{e^z}{\sqrt{2\pi z}}$$

$$\int G_{m_1}(x) G_{m_2}(x) dx = \frac{(m_1 m_2)^{\frac{d}{2}-1}}{2^{d-1}\pi^{\frac{d}{2}}\Gamma\left(\frac{d}{2}\right)} \int_0^\infty K_{\frac{d}{2}-1}(m_1 r)\, K_{\frac{d}{2}-1}(m_2 r)\, r dr$$

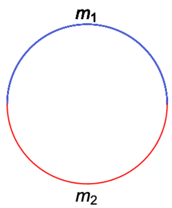
$$= \frac{\Gamma\left(1-\frac{d}{2}\right)(m_1 m_2)^{\frac{d}{2}-1}}{2^d \pi^{\frac{d}{2}}} \int_0^\infty \left(I_{\frac{d}{2}-1}(m_1 r) - I_{1-\frac{d}{2}}(m_1 r)\right) K_{\frac{d}{2}-1}(m_2 r)\, r dr$$



$$0 < m_1 < m_2$$

The bubble in x-space

$$\int G_{m_1}(x)G_{m_2}(x)dx = \frac{\Gamma\left(1-\frac{d}{2}\right)(m_1m_2)^{\frac{d}{2}-1}}{2^d\pi^{\frac{d}{2}}}\int_0^\infty \left(I_{\frac{d}{2}-1}(m_1r) - I_{1-\frac{d}{2}}(m_1r)\right)K_{\frac{d}{2}-1}(m_2r)\,rdr$$



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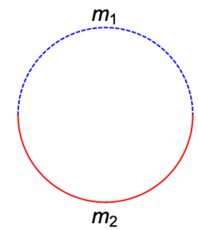

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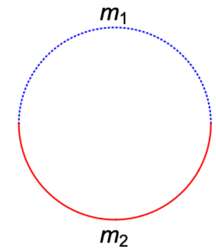


$0 < m_1 < m_2$

$$\int_0^\infty I_\nu(ar)K_\rho(br)\,rdr = \sum_{n=0}^\infty \frac{(a/2)^{2n+\nu}}{n!\,\Gamma(n+\nu+1)}\int_0^\infty r^{1+2n+\nu}K_\rho(br)\,dr$$

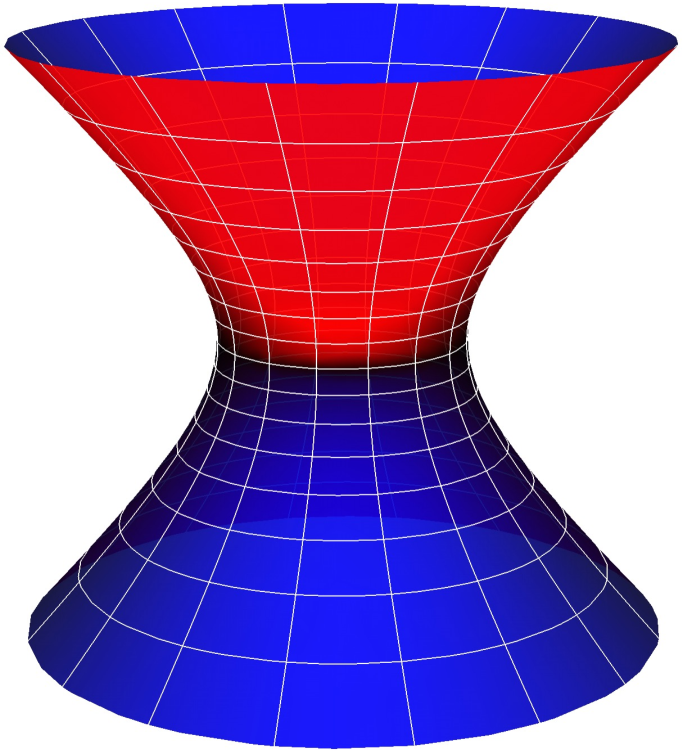
$$= \sum_{n=0}^\infty 2^{\nu+2n}b^{-\nu-2n-2}\Gamma\left(n+\frac{\nu}{2}-\frac{\rho}{2}+1\right)\Gamma\left(n+\frac{\nu}{2}+\frac{\rho}{2}+1\right)$$

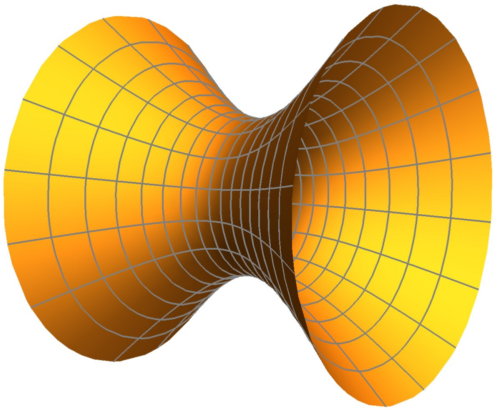
$$= \frac{a^\nu\Gamma\left(\frac{\nu+\rho}{2}+1\right)\Gamma\left(\frac{\nu+\rho}{2}-1\right)\ {}_2F_1\left(\frac{\nu+\rho}{2}+1, \frac{\nu-\rho}{2}+1; \nu+1; \frac{a^2}{b^2}\right)}{b^{\nu+2}\Gamma(\nu+1)}$$

$$\int G_{m_1}(x)G_{m_2}(x)dx = -\frac{\Gamma\left(1-\frac{d}{2}\right)}{(4\pi)^{\frac{d}{2}}}\,\frac{(m_1^2)^{\frac{d}{2}-1}-(m_2^2)^{\frac{d}{2}-1}}{m_1^2-m_2^2}$$

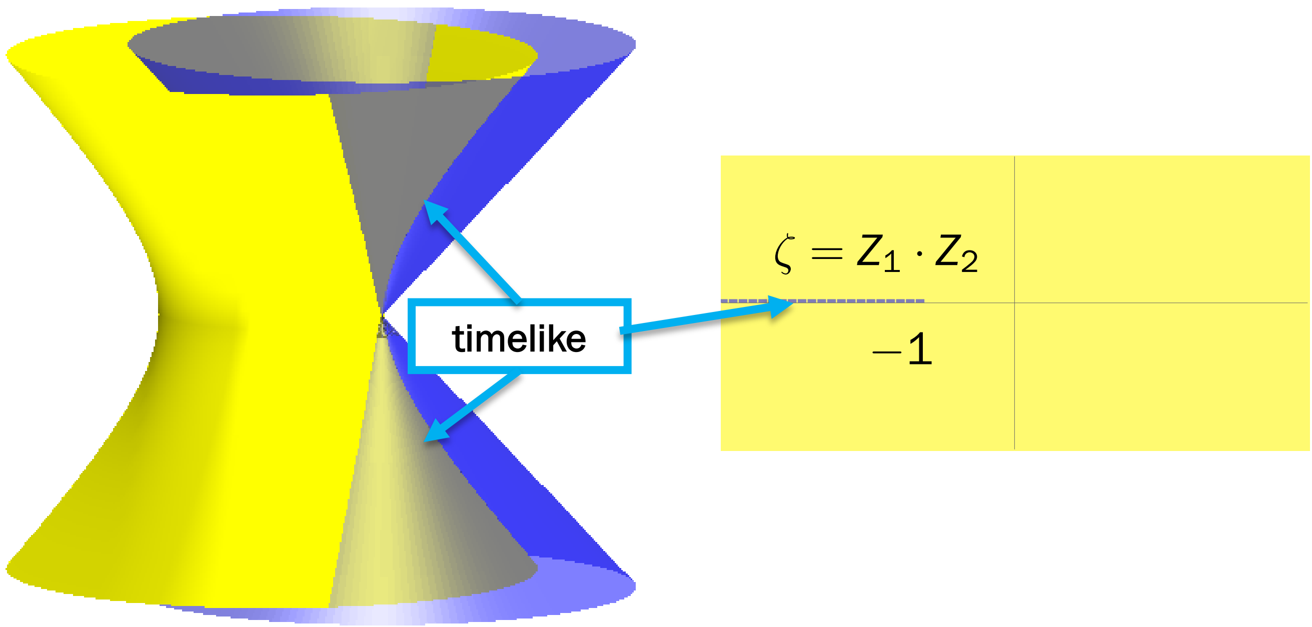
Main Course

BACK TO DE SITTER





The asymptotic cone: causal structure



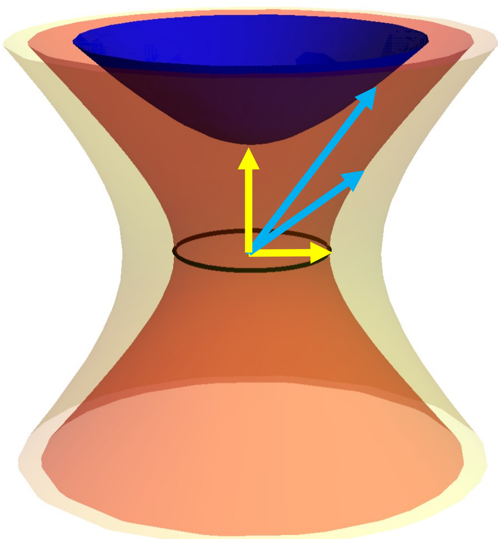
X and Y are spacelike separated iff $(X-Y)^2 < 0$

timelike $(X-Y)^2 = -2 - 2X \cdot Y > 0$

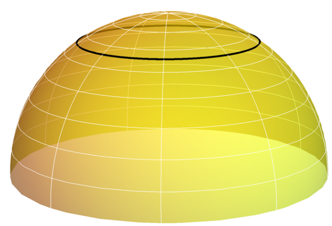
Forward de Sitter tubes

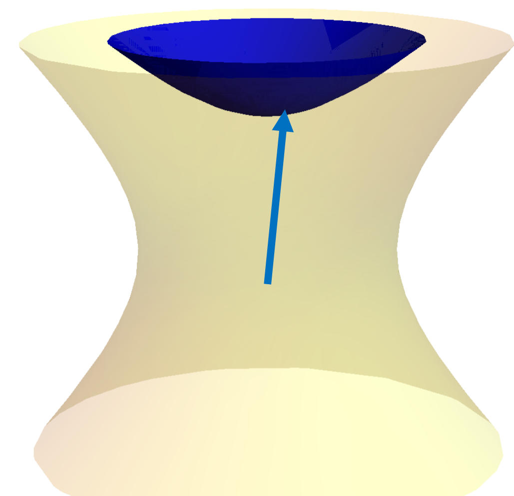
$$Z^2 = -R^2 \qquad X^2 - Y^2 = -R^2 \qquad X \cdot Y = 0$$

$$\mathcal{T}^{\pm} = \{Z = X + iY : Y^2 > 0, \ \ \pm Y^0 > 0\}$$



$$X^2 = -R^2 + Y^2$$



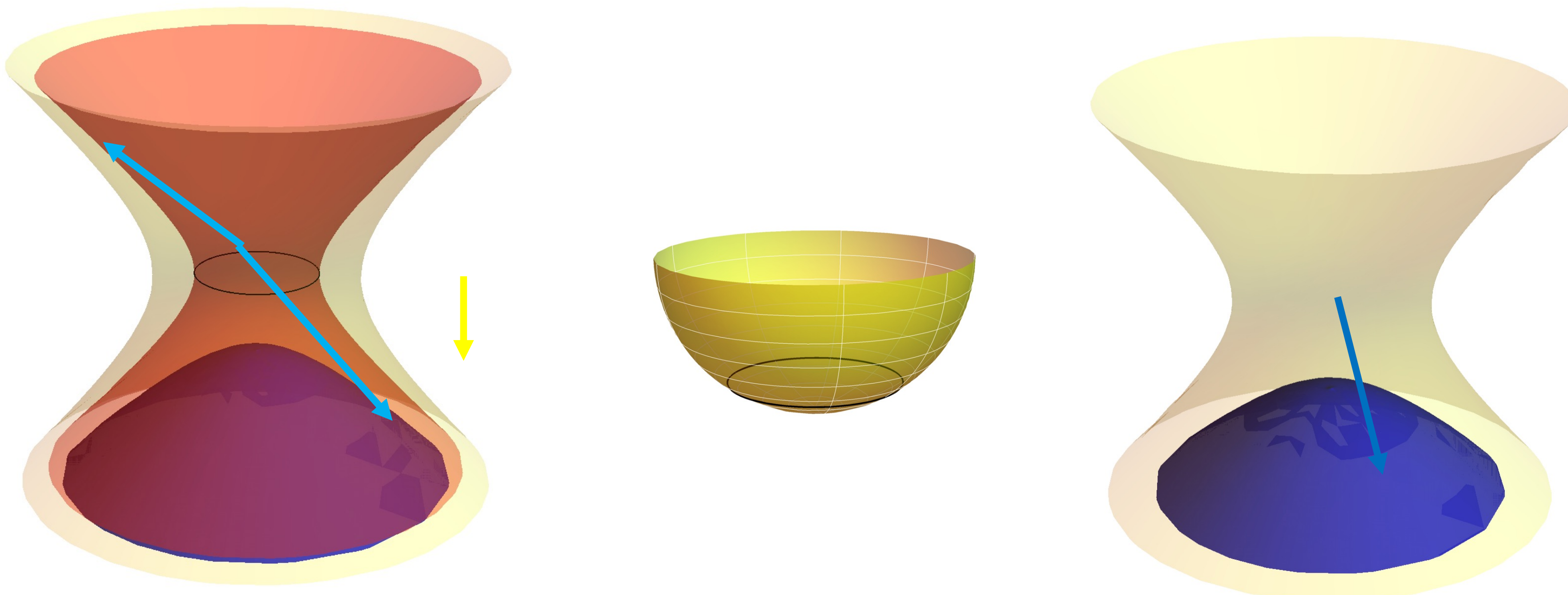


$$0 = -R^2 + Y^2$$

$$H_d^+ = \{X = 0, \ \ Y^2 = R^2, Y^0 > 0\}$$

Backward de Sitter tube

$$\mathcal{T}^{\pm} = \{Z = X + iY : Y^2 > 0, \quad \pm Y^0 > 0\}$$



$$X^2 - Y^2 = -R^2$$
$$X \cdot Y = 0$$

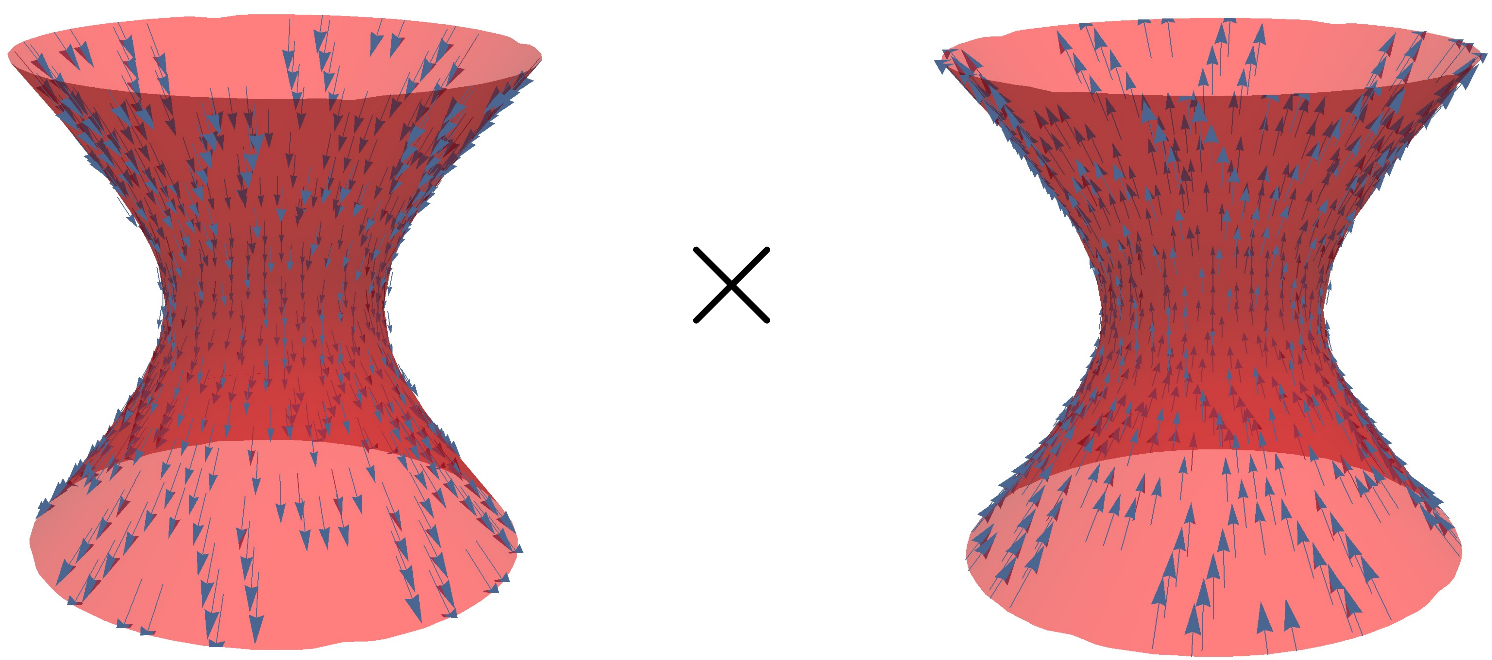
$$H_d^- = \{X = 0, \quad Y^2 = R^2, Y^0 < 0\}$$

dS-QFT : Normal Analiticity Hypothesis

$$\mathcal{W}(x_1, x_2) = \textit{boundary value of } W(z_1, z_2)$$

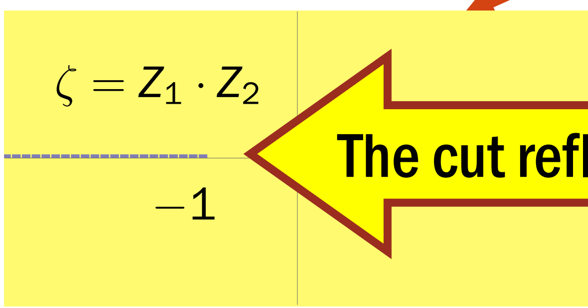
$$W(z_1, z_2) \text{ holomorphic in } \mathcal{T}_- \times \mathcal{T}_+,$$

$$\mathcal{T}_\pm = \{Z = X + iY : Y^2 > 0, \ \pm Y^0 > 0\}$$



Normal Analyticity +complex de Sitter invariance =Maximal Analyticity

$$W(z_1, z_2) = W(\zeta) = \mathfrak{W}(z_1, z_2)$$



The cut reflects causality and QM

No cut $\longleftrightarrow$ No commutator

On the cut $\quad C(x_1, x_2) = \mathcal{W}(x_1, x_2) - \mathcal{W}(x_2, x_1) \neq 0$

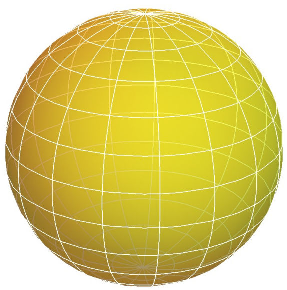
$$S(x_1^E, x_2^E) = \mathfrak{W}(z_1, z_2)\big|_{E^d \times E^d \setminus \delta}$$

(Jacques Bros and UM 1994)

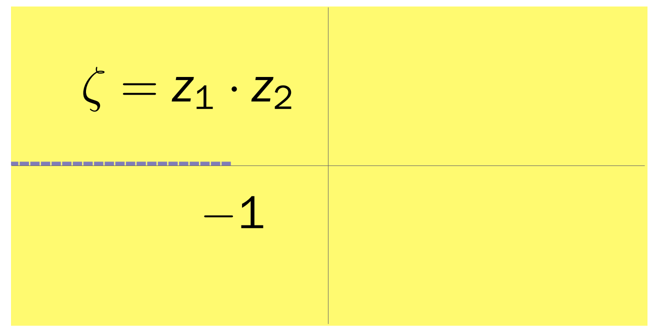
Maximally analytic two-point function (aka BD)

$$W_\nu(Z_1,Z_2) = \frac{\Gamma\left(\frac{d-1}{2}+i\nu\right)\Gamma\left(\frac{d-1}{2}-i\nu\right)}{2(2\pi)^{d/2}}(\zeta^2-1)^{-\frac{d-2}{4}}\,P^{-\frac{d-2}{2}}_{-\frac{1}{2}+i\nu}(\zeta)$$

$$= \frac{\Gamma\left(\frac{d-1}{2}+i\nu\right)\Gamma\left(\frac{d-1}{2}-i\nu\right)}{(4\pi)^{d/2}\Gamma\left(\frac{d}{2}\right)}\,{}_2F_1\left(\frac{d-1}{2}+i\nu,\frac{d-1}{2}-i\nu;\frac{d}{2};\frac{1-\zeta}{2}\right)$$



$z_1\cdot z_2 = -\cos(s)$

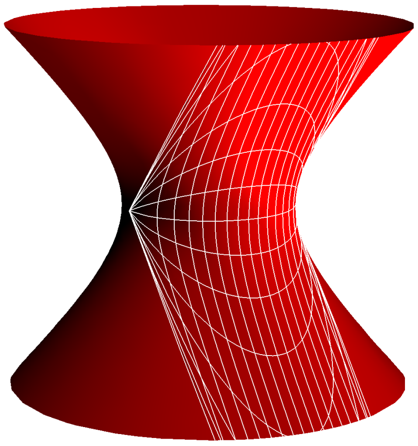


The Schwinger function (the Euclidean propagator) is the restriction of the maximally analytic two-point function to the Euclidean sphere

$$G_\nu(-\cos s) = \frac{\Gamma(\frac{d-1}{2}+i\nu)\Gamma(\frac{d-1}{2}-i\nu)}{2(2\pi)^{d/2}}(\sin s)^{-\frac{d-2}{2}}\,e^{\frac{i\pi}{2}(d-2)}\mathrm{P}^{-\frac{d-2}{2}}_{-\frac{1}{2}+i\nu}(-\cos s)$$

Physical interpretation: temperature

$$X(t,r) = \begin{cases} X^0 = \sqrt{R^2 - r^2}\,\sinh\left(\frac{t}{R}\right) \\ X^i = r^i \quad (r^2 < R^2) \\ X^d = \sqrt{R^2 - r^2}\,\cosh\left(\frac{t}{R}\right) \end{cases}$$

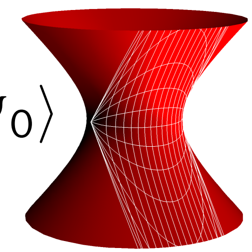


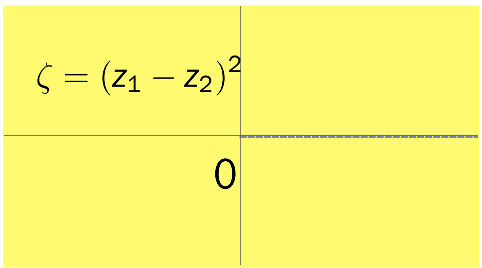
Time translations:

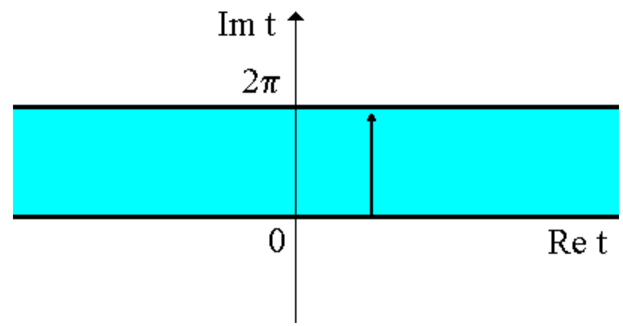
$$\alpha(s)X(t,r) = X(t+s,r)$$

Maximal analyticity ==I> KMS condition

$$W(X_1, X_2(t)) = \langle \Psi_0, \phi(X_1) U(t) \phi(X_2) U(-t) \Psi_0 \rangle$$





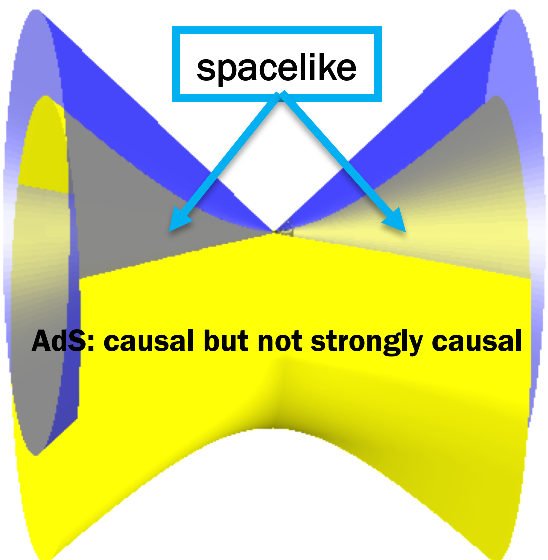


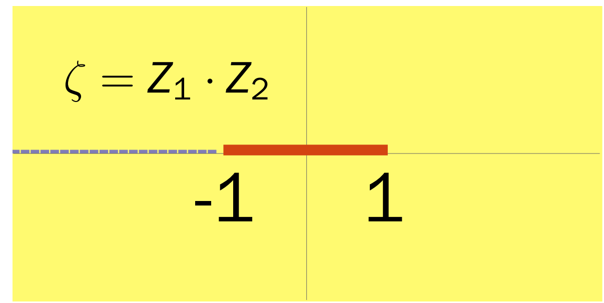
1) $W(X_1, X_2(t))$ is analytic in the strip $0 < Im\ t < 2\pi R$

2) For t real $W(X_1, X_2(t + 2\pi i R)) = W(X_2(t), X_1)$

KMS condition at inverse temperature 2πR

The asymptotic cone: causal structure



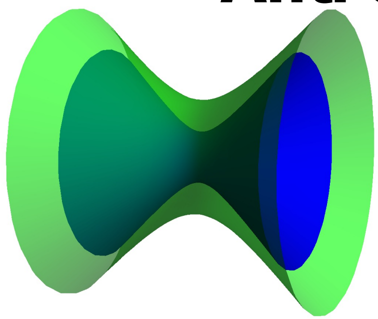


X and Y are spacelike separated iff $(X - Y)^2 < 0$

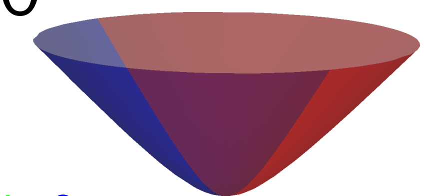
$$(X - Y)^2 = 2R^2 - 2X \cdot Y = 2R^2 - 2(X^0Y^0 + X^dY^d - (X^1Y^1 + \ldots + X^{d-1}Y^{d-1})) < 0$$

$$X \cdot Y > 1$$

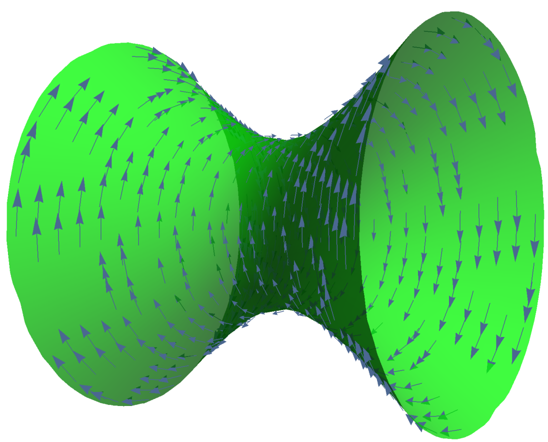
Anti de Sitter chiral tubes

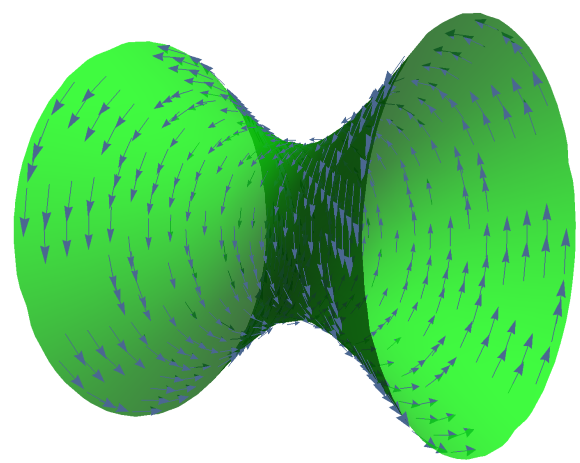


$$Z^2 = R^2 \qquad \begin{cases} X^2 - Y^2 = R^2 \\ X \cdot Y = 0 \end{cases}$$



$$\mathcal{Z}_\pm = \{Z = X + iY : Y^2 > 0, \;\; \pm(X^0 Y^d - X^d Y^0 > 0\}$$



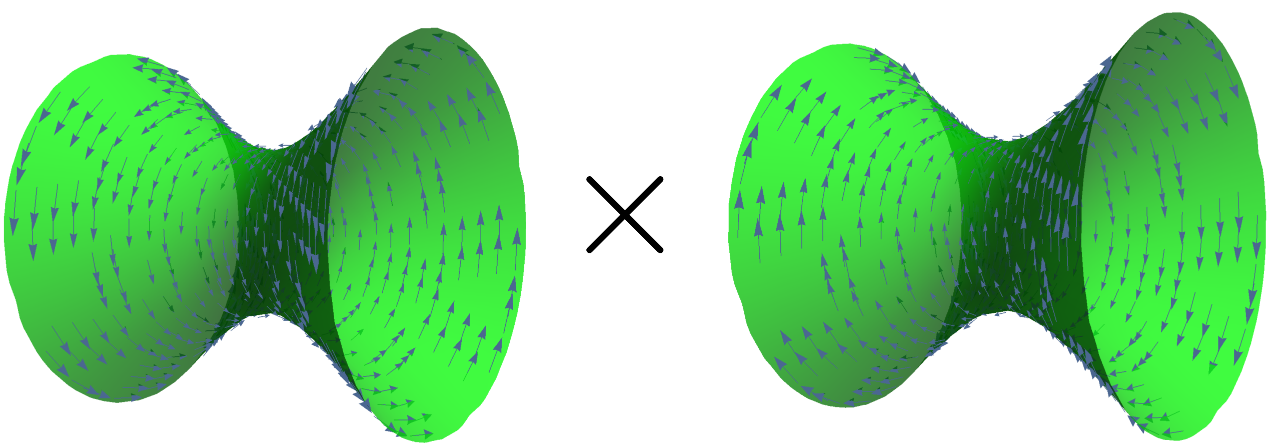


dS-QFT : Normal Analiticity

The AdS spectral condition implies that

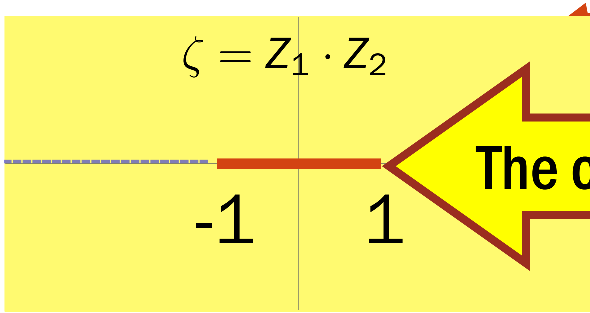
$$\mathcal{W}(x_1, x_2) = \textit{boundary value of } W(z_1, z_2)$$

$W(z_1, z_2)$ holomorphic in $\mathcal{Z}_- \times \mathcal{Z}_+$,



Normal Analyticity +Anti de Sitter invariance = Maximal Analyticity

$$W(Z_1, Z_2) = W(\zeta) = \mathfrak{W}(Z_1, Z_2)$$



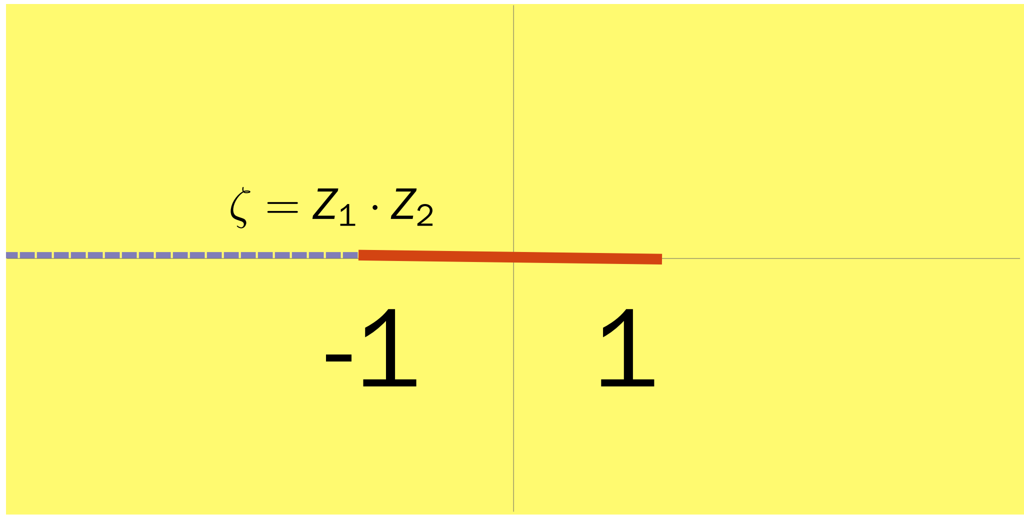
The cut reflects causality and QM

On the cut $C(X_1, X_2) = \mathcal{W}(X_1, X_2) - \mathcal{W}(X_2, X_1) \neq 0$

$$S(X_1^E, X_2^E) = \mathfrak{W}(Z_1, Z_2)\big|_{H^d \times H^d \setminus \delta}$$

(Jacques Bros, Henri Epstein and UM 1999)

KG - Two-point function



$$W^d_\nu(z_1, z_2) = \frac{1}{(2\pi)^{\frac{d}{2}}} (\zeta^2 - 1)^{-\frac{d-2}{4}} e^{-i\pi\frac{d-2}{2}} Q^{\frac{d-2}{2}}_{-\frac{1}{2}+\nu}(\zeta)$$

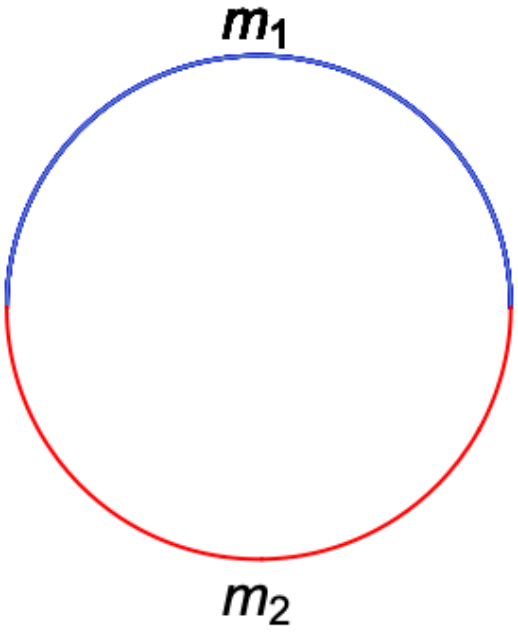
$$= \frac{\Gamma\left(\frac{d-1}{2} + \nu\right)}{2\pi^{\frac{d-1}{2}} (2\zeta)^{\frac{d-1}{2}+\nu} \Gamma(\nu+1)} \, {}_2F_1\left(\frac{d-1}{4} + \frac{\nu}{2}, \frac{d+1}{4} + \frac{\nu}{2}; \nu + 1; \frac{1}{\zeta^2}\right)$$

$$\zeta = Z_1 \cdot Z_2$$

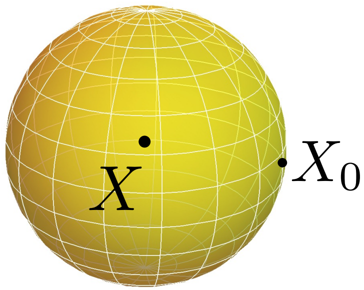
$$m^2 = \nu^2 - \frac{(d-1)^2}{4}$$

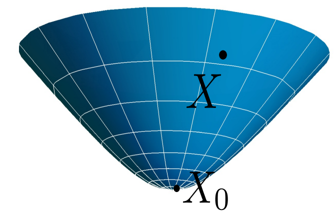
THE BUBBLE

$$\int_{E_d} G^d_{m_1}(x_1, x_2) G^d_{m_2}(x_1, x_2) dx_1$$



The bubble in dS e AdS





$$I_2(\lambda,\nu,d) = \frac{\Gamma(\frac{d-1}{2}-i\lambda)\Gamma(\frac{d-1}{2}+i\lambda)\Gamma(\frac{d-1}{2}-i\nu)\Gamma(\frac{d-1}{2}+i\nu)}{2\left(2\sqrt{\pi}\right)^d\Gamma\left(\frac{d}{2}\right)}\int_{-1}^{1}\mathrm{P}^{-\frac{d-2}{2}}_{-\frac{1}{2}+i\lambda}(u)\mathrm{P}^{-\frac{d-2}{2}}_{-\frac{1}{2}+i\nu}(u)du$$

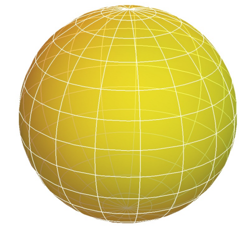
$$I_2(\lambda,\nu,d) = \frac{e^{-i\pi(d-2)}}{2^{d-1}\pi^{\frac{d}{2}}\Gamma\left(\frac{d}{2}\right)}\int_{1}^{\infty}Q^{\frac{d-2}{2}}_{-\frac{1}{2}+\lambda}(u)Q^{\frac{d-2}{2}}_{-\frac{1}{2}+\nu}(u)du$$

The above integrals may be computed by using the Wronskian relations among solutions of the Legendre differential equation:

$$(1-z^2)\frac{d^2f}{dz^2}-2z\frac{df}{dz}+\left(\nu(\nu+1)-\frac{\mu^2}{1-z^2}\right)f(z)=0$$

$$\int_a^b f^{\mu}_{\nu}(x)g^{\mu}_{\sigma}(x)dx = \frac{1}{(\nu-\sigma)(\sigma+\nu+1)}\Big[(1-x^2)^{\frac{1}{2}}(\sigma+\mu)(\sigma-\mu+1)f^{\mu}_{\nu}(x)g^{\mu-1}_{\sigma}(x)$$
$$-(1-x^2)^{\frac{1}{2}}(\nu+\mu)(\nu-\mu+1)f^{\mu-1}_{\nu}(x)g^{\mu}_{\sigma}(x)\Big]_a^b.$$

1-loop - two masses



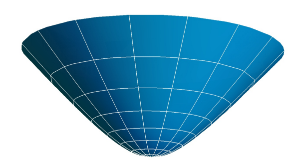
$$I_2(\lambda,\nu,d) = \frac{\Gamma\left(1-\frac{d}{2}\right)}{2^d\pi^{\frac{d}{2}}(\lambda^2-\nu^2)}\left(\frac{\Gamma\left(\frac{d-1}{2}-i\nu\right)\Gamma\left(\frac{d-1}{2}+i\nu\right)}{\Gamma\left(\frac{1}{2}-i\nu\right)\Gamma\left(\frac{1}{2}+i\nu\right)}-\frac{\Gamma\left(\frac{d-1}{2}-i\lambda\right)\Gamma\left(\frac{d-1}{2}+i\lambda\right)}{\Gamma\left(\frac{1}{2}-i\lambda\right)\Gamma\left(\frac{1}{2}+i\lambda\right)}\right).$$

$$I_2(\nu,\nu,d) = 2^{-d-1}\pi^{-\frac{d}{2}-1}\Gamma\left(1-\frac{d}{2}\right)\Gamma\left(\frac{d-1}{2}-i\nu\right)\Gamma\left(\frac{d-1}{2}+i\nu\right)\times$$

$$\times\frac{\left(i\cosh(\pi\nu)\,\psi(d/2-1/2-i\nu)-i\cosh(\pi\nu)\,\psi(d/2-1/2+i\nu)+\pi\sinh(\pi\nu)\right)}{\nu}$$

$$\begin{aligned} I_2(\lambda,\nu,4) \;\simeq\; & -\frac{1}{8\pi^2(d-4)}+\frac{1-\gamma+\log(4\pi)}{16\pi^2} \\ & -\frac{\left(4\lambda^2+1\right)\left(\psi\left(\frac{3}{2}-i\lambda\right)+\psi\left(\frac{3}{2}+i\lambda\right)\right)-\left(4\nu^2+1\right)\left(\psi\left(\frac{3}{2}-i\nu\right)+\psi\left(\frac{3}{2}+i\nu\right)\right)}{64\pi^2\left(\lambda^2-\nu^2\right)}+\mathrm{O}(d-4). \end{aligned}$$

$$I_2(\lambda,\nu,d) = \frac{\Gamma\left(1-\frac{d}{2}\right)}{2^d\pi^{\frac{d}{2}}(\lambda^2-\nu^2)}\left(\frac{\Gamma\left(\frac{d-1}{2}+\nu\right)}{\Gamma\left(\frac{3-d}{2}+\nu\right)}-\frac{\Gamma\left(\frac{d-1}{2}+\lambda\right)}{\Gamma\left(\frac{3-d}{2}+\lambda\right)}\right)$$

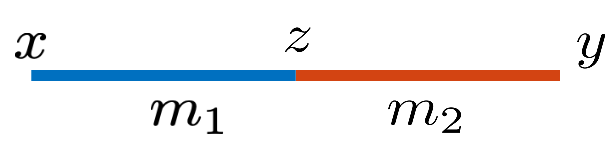


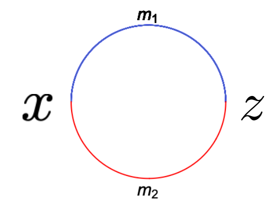
A nicer elementary derivation

$$F_{m_1 m_2}(x,y) = \int_{E_d} G^d_{m_1}(x,z) G^d_{m_2}(y,z)\sqrt{g(z)}\,dz = \frac{G^d_{m_2}(x,y) - G^d_{m_1}(x,y)}{m_1^2 - m_2^2}$$

$$\int_{E_d} G^d_m(x,z) G^d_m(y,z)\sqrt{g(z)}\,dz = -\frac{\partial}{\partial m^2} G^d_m(x,y)$$

$$bubble = lim_{y \to x} \frac{G^d_{m_2}(x,y) - G^d_{m_1}(x,y)}{m_1^2 - m_2^2}.$$





Examples: Minkowski and AdS (same in dS)

$$G^d_m(r) \simeq \frac{r^{2-d}}{4\pi^{\frac{d}{2}}}\Gamma\left(\frac{d}{2}-1\right) + \frac{m^{d-2}}{(4\pi)^{\frac{d}{2}}}\Gamma\left(1-\frac{d}{2}\right)$$

$$\int_{E_d} G^d_{m_1}(x,z) G^d_{m_2}(x,z)\,dz = -\frac{\Gamma\left(1-\frac{d}{2}\right)}{(4\pi)^{\frac{d}{2}}} \frac{(m_2)^{\frac{d}{2}-1} - (m_1^2)^{\frac{d}{2}-1}}{m_1^2 - m_2^2}$$

$$\lim_{\zeta \to 1} G^{(AdS)}_\nu(\zeta) = \frac{\Gamma\left(1-\frac{d}{2}\right)\Gamma\left(\frac{d-1}{2}+\nu\right)}{2^d \pi^{\frac{d}{2}}\Gamma\left(\frac{3-d}{2}+\nu\right)} \quad (d<2).$$

$$bubble_{AdS} = F_{\lambda\,\nu}(x,x) = \frac{\Gamma\left(1-\frac{d}{2}\right)}{(4\pi)^{\frac{d}{2}}(\lambda^2-\nu^2)}\left(\frac{\Gamma\left(\frac{d+1}{2}+\nu\right)}{\Gamma\left(\frac{3-d}{2}+\nu\right)} - \frac{\Gamma\left(\frac{d+1}{2}+\lambda\right)}{\Gamma\left(\frac{3-d}{2}+\lambda\right)}\right)$$

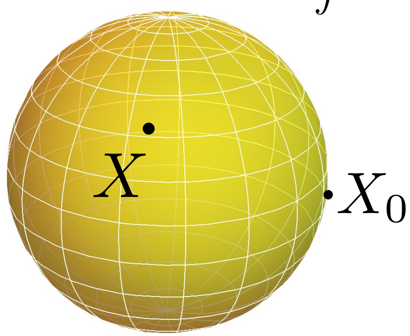
Cosmological corrections to the effective potential in the O(N) model

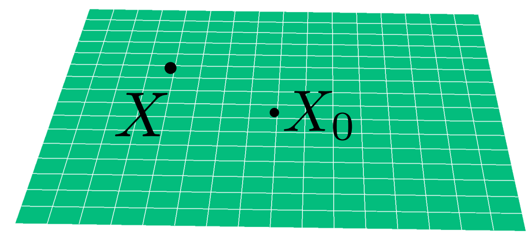
$$S[\phi] = \int_{S_d} \left[\Lambda_0 + \frac{1}{2}\langle \partial_\mu \phi \mid \partial^\mu \phi \rangle + \frac{m_0^2}{2}\langle \phi \mid \phi \rangle + \frac{c_0}{4}\langle \phi \mid \phi \rangle^2 \right] \sqrt{g}\, d^d x,$$

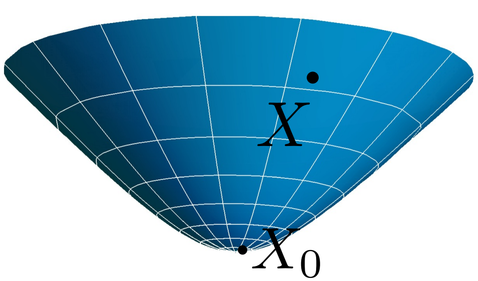
$$\begin{aligned}
\mathcal{V}_\Lambda = & \frac{m^2}{2}\varphi^2 + \frac{c_g}{4}\varphi^4 + \frac{1}{64\pi^2}\left[(m^2 + 3c_g\varphi^2)^2 \log\frac{m^2 + 3c_g\varphi^2}{\mu^2} + (N-1)(m^2 + c_g\varphi^2)^2 \log\frac{m^2 + c_g\varphi^2}{\mu^2}\right] \\
& - \frac{\Lambda}{48\pi^2}\left[(m^2 + 3c_g\varphi^2)\left(\log\frac{m^2 + 3c_g\varphi^2}{\mu^2} + \frac{1}{6}\right)\right. \\
& \qquad \left. + (N-1)(m^2 + c_g\varphi^2)\left(\log\frac{m^2 + c_g\varphi^2}{\mu^2} + \frac{1}{6}\right)\right] \\
& + \frac{\Lambda^2}{144\pi^2}\left[-\frac{1}{30}\log\frac{3(m^2 + 3c_g\varphi^2)}{\Lambda} + \log\frac{m^2 + 3c_g\varphi^2}{\mu^2} + \frac{49}{60} - \frac{3\zeta(3)}{4\pi^3}\right. \\
& \left. + (N-1)\left(-\frac{1}{30}\log\frac{3(m^2 + c_g\varphi^2)}{\Lambda} + \log\frac{m^2 + c_g\varphi^2}{\mu^2} + \frac{49}{60} - \frac{3\zeta(3)}{4\pi^3}\right)\right] + O(\Lambda^3)
\end{aligned}$$

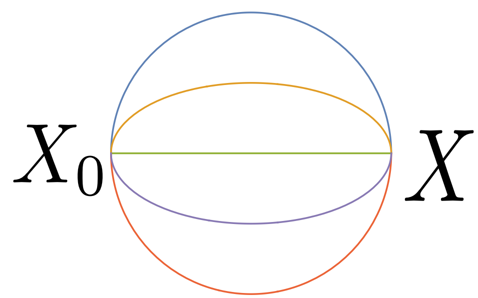
2-loop banana integrals

$$I_n(\nu_1, \ldots, \nu_{n+1}, d) = \int G^d_{\nu_1}(X_0 \cdot X) G^d_{\nu_2}(X_0 \cdot X) \ldots G^d_{\nu_{n+1}}(X_0 \cdot X)\, d\mu(X)$$





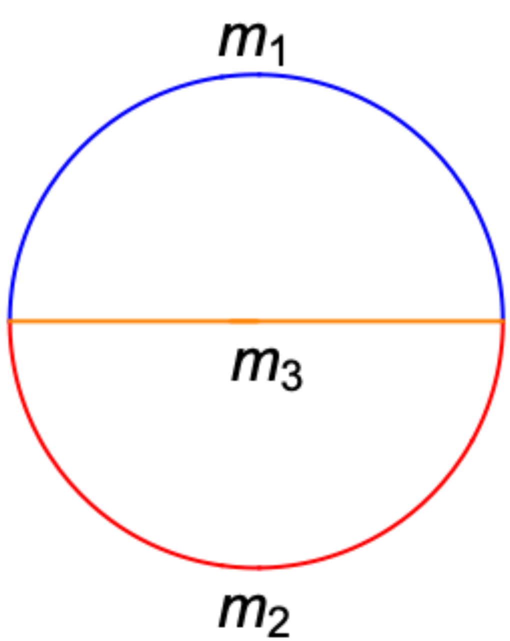




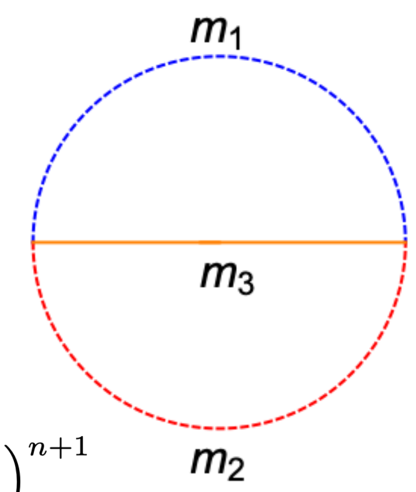
(Sergio Cacciatori, Henri Epstein and UM 2024)

Two-loop : PDE's

$$F(u,v,z,d) = \frac{1}{(2\pi)^{3d}} \int \frac{e^{-ikx}}{k^2+u} \frac{e^{-iqx}}{q^2+v} \frac{e^{-ipx}}{p^2+z} dk dq dp dx$$
$$= \frac{1}{(2\pi)^{2d}} \int \frac{dq dk}{(k^2+u)(q^2+v)((q+k)^2+z)}$$



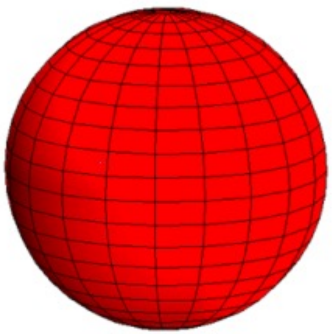
But in x-space still elementary (ourselves, 2023)



$$
\begin{aligned}
R_{++}(m_1,m_2,m_3,d) &= \int_0^\infty r^{2-\frac{d}{2}} I_{\frac{d}{2}-1}(m_1 r)\, I_{\frac{d}{2}-1}(m_2 r)\, K_{\frac{d}{2}-1}(m_3 r)\, dr \\
&= \sum_{n=0}^{\infty} \frac{2^{1-\frac{d}{2}} m_1^{\frac{d}{2}-1} m_2^{\frac{d}{2}-1}\, {}_2F_1\left(n+1, \frac{d}{2}+n; \frac{d}{2}; \frac{m_1^2}{m_3^2}\right)}{c^{\frac{d}{2}+1}\Gamma\left(\frac{d}{2}\right)} \frac{m_2^{2n}}{m_3^{2n}} \\
&= \sum_{n=0}^{\infty} \frac{2^{1-\frac{d}{2}} m_1^{\frac{d}{2}-1} m_2^{\frac{d}{2}-1}\, {}_2F_1\left(n+1, -n; \frac{d}{2}; \frac{m_1^2}{m_1^2-m_3^2}\right)}{m_3^{\frac{d}{2}+1}\Gamma\left(\frac{d}{2}\right)} \frac{m_2^{2n}}{m_3^{2n}} \left(1-\frac{m_1^2}{m_3^2}\right)^{-n-1} \\
&= \frac{2^{1-\frac{d}{2}} m_1^{\frac{d}{2}-1} m_2^{\frac{d}{2}-1}}{m_3^{\frac{d}{2}+1}} \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{\Gamma(k-n)\Gamma(k+n+1)\left(\frac{m_1^2}{m_1^2-m_3^2}\right)^k \left(\frac{m_2}{m_3}\right)^{2n} \left(\frac{m_3^2}{m_3^2-m_1^2}\right)^{n+1}}{\Gamma(k+1)\Gamma(-n)\Gamma(n+1)\Gamma\left(\frac{d}{2}+k\right)} \\
&= \sum_{k=0}^{\infty}\sum_{n=k}^{\infty} \frac{2^{1-\frac{d}{2}} m_1^{\frac{d}{2}-1} m_2^{\frac{d}{2}-1}}{m_3^{\frac{d}{2}+1}} \frac{m_2^{2n}}{m_3^{2n}} \left(\frac{m_3^2}{m_3^2-m_1^2}\right)^{n+1} \frac{\left(\frac{m_1^2}{m_3^2-m_1^2}\right)^k}{\Gamma(k+1)\Gamma\left(\frac{d}{2}+k\right)} \frac{\Gamma(k+n+1)}{\Gamma(-k+n+1)} \\
&= \left(\frac{m_1 m_2}{2m_3}\right)^{\frac{d}{2}-1} \frac{{}_2F_1\left(\frac{1}{2}, 1; \frac{d}{2}; \frac{4m_1^2 m_2^2}{\left(m_3^2-m_1^2-m_2^2\right)^2}\right)}{\Gamma\left(\frac{d}{2}\right)\left(m_3^2-m_1^2-m_2^2\right)}. \qquad (1)
\end{aligned}
$$

dS two-loop banana

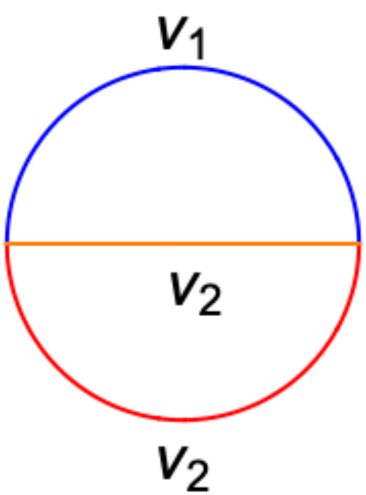
$$G_\nu^d(-\cos s) = \frac{\Gamma(\frac{d-1}{2}+i\nu)\Gamma(\frac{d-1}{2}-i\nu)}{2(2\pi)^{d/2}}(\sin s)^{-\frac{d-2}{2}}\, P^{-\frac{d-2}{2}}_{-\frac{1}{2}+i\nu}(-\cos s).$$



$$I(\nu_1,\nu_2,\nu_3,d) = \int G_{\nu_1}(x_0\cdot x)G_{\nu_2}(x_0\cdot x)G_{\nu_3}(x_0\cdot x) =$$

$$= \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}\int_0^\pi G_{\nu_1}(-\cos s)G_{\nu_2}(-\cos s)G_{\nu_3}(-\cos s)(\sin s)^{d-1}ds$$

$$= K_d(\nu_1,\nu_2,\nu_3)\int_{-1}^1 P^{-\frac{d-2}{2}}_{-\frac{1}{2}+i\nu_1}(u)P^{-\frac{d-2}{2}}_{-\frac{1}{2}+i\nu_2}(u)P^{-\frac{d-2}{2}}_{-\frac{1}{2}+i\nu_3}(u)(1-u^2)^{-\frac{d-2}{4}}du$$



$$K_d(\nu_1,\nu_2,\nu_3) = \frac{\prod_{j=1}^3\Gamma(\frac{d-1}{2}-i\nu_j)\Gamma(\frac{d-1}{2}+i\nu_j))}{2^{2+\frac{3d}{2}}\pi^d\Gamma\left(\frac{d}{2}\right)}$$

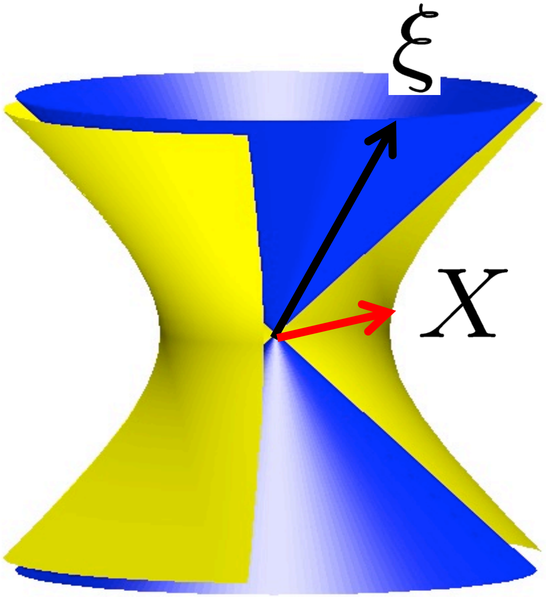
de Sitter plane waves

$$\psi_\lambda(X,\xi) = (X\cdot\xi)^\lambda$$

$$\lambda \in \mathbf{C}, \quad \xi^2 = 0$$

$$\Box\psi_\lambda(X,\xi) = \lambda(\lambda + d - 1)\psi_\lambda(X,\xi)$$

Involution: $\lambda \to \bar\lambda = -\lambda - d + 1$
Complex (squared) mass $m^2 = \bar\lambda\lambda$



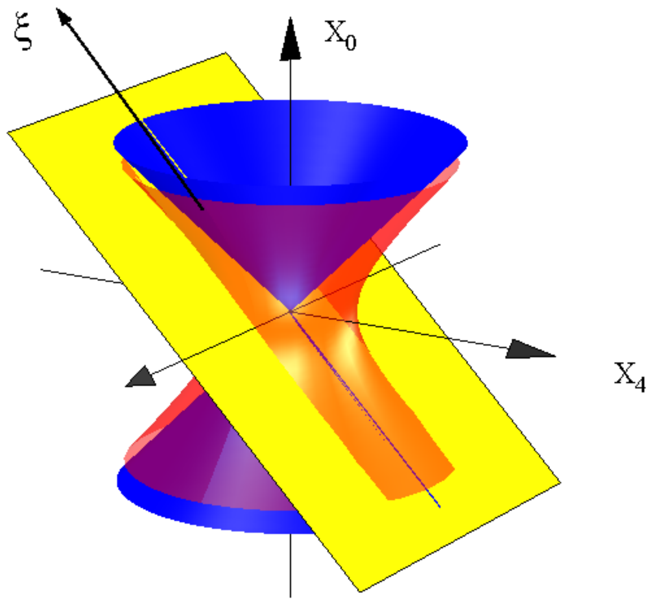
$$\psi_\lambda(X,\xi) = e^{\lambda \log(X\cdot\xi)}$$

$$\psi_m(x,\hat p) = e^{i\,p\cdot x} = e^{i\,m\,\hat p\cdot x}$$

$$\lambda = -\tfrac{d-1}{2} + i\nu, \quad \nu \in \mathbf{R}$$

$$\lambda = -\tfrac{d-1}{2} + \nu, \quad \nu \in \mathbf{R}, \ |\nu| < \tfrac{d-1}{2},$$

A sewing is necessary around $(X \cdot \xi) = 0$



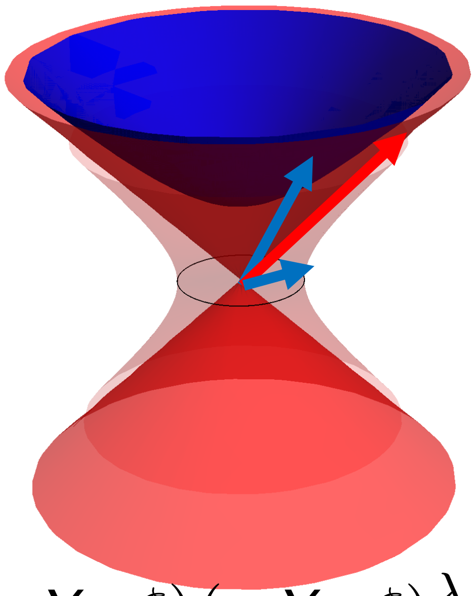
$$(X \cdot \xi)^{\lambda} \to \theta(X \cdot \xi)(X \cdot \xi)^{\lambda} + a(\lambda)\theta(-X \cdot \xi)(-X \cdot \xi)^{\lambda}$$

Plane waves are holomorphic in the tubes

$$\psi_\lambda^{\pm}(Z,\xi) = (X \pm iY\cdot\xi)^\lambda = ((X\cdot\xi) + e^{\pm\frac{i\pi}{2}}(Y\cdot\xi))^\lambda$$

$(Y\cdot\xi) > 0$ for $Z \in \mathcal{T}^+$

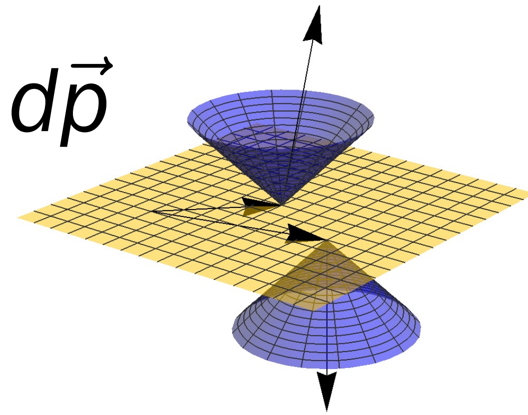
$(Y\cdot\xi) < 0$ for $Z \in \mathcal{T}^-$



$$(X\cdot\xi)_\pm^\lambda \to \theta(X\cdot\xi)(X\cdot\xi)^\lambda + e^{\pm i\pi\lambda}\theta(-X\cdot\xi)(-X\cdot\xi)^\lambda$$

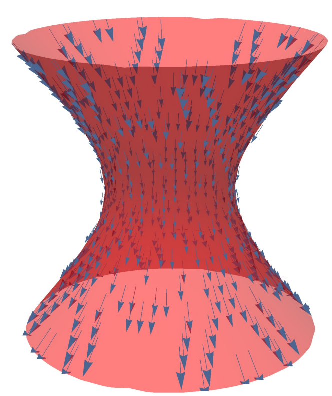
Fourier - type representation of the two-point functions

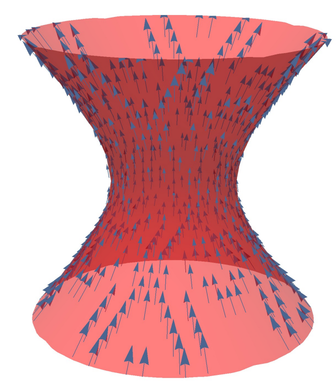
$$W(z_1, z_2) = \int \psi^{(-)}_{\vec{p}}(z_1)\psi^{(+)}_{\vec{p}}(z_2)d\vec{p}$$

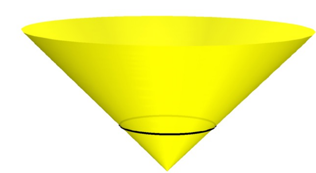


$$W_\nu(Z_1, Z_2) = c(\nu)\int_\gamma (\xi \cdot Z_1)^{-\frac{d-1}{2}-i\nu}(\xi \cdot Z_2)^{-\frac{d-1}{2}+i\nu}\, d\mu_\gamma(\xi)$$

ν is any complex number







Kallen Lehmann expansions - Linearization

$$W_{m_1}(z_1, z_2) \dots W_{m_1}(z_1, z_2) = \int_0^\infty \rho(\mu, m_1, \dots, m_n) W_\mu(z_1, z_2) d\mu^2$$

$$W_m^2(z_1, z_2) = \int_0^\infty \rho(\mu, m) W_\mu(z_1, z_2) d\mu^2$$

dS: Mehler –Fock transform

$$W_\nu(Z_1,Z_2) = \frac{\Gamma\left(\frac{d-1}{2}+i\nu\right)\Gamma\left(\frac{d-1}{2}-i\nu\right)}{2(2\pi)^{d/2}}(\zeta^2-1)^{-\frac{d-2}{4}}\,P^{-\frac{d-2}{2}}_{-\frac{1}{2}+i\nu}(\zeta)$$

Evaluate the Mehler-Fock transform

$$h_d(\kappa,\lambda,\nu) = \int_1^\infty P^{-\frac{d-2}{2}}_{-\frac{1}{2}+i\kappa}(u)\,P^{-\frac{d-2}{2}}_{-\frac{1}{2}+i\lambda}(u)P^{-\frac{d-2}{2}}_{-\frac{1}{2}+i\nu}(u)(u^2-1)^{-\frac{d-2}{4}}\;du$$

$$\rho(\kappa,\lambda,\nu) = \frac{\Gamma\left(\frac{d-1}{2}+i\nu\right)\Gamma\left(\frac{d-1}{2}-i\nu\right)\Gamma\left(\frac{d-1}{2}+i\lambda\right)\Gamma\left(\frac{d-1}{2}-i\lambda\right)\sinh(\pi\kappa)\;h_d(\kappa,\lambda,\nu)}{2(2\pi)^{1+\frac{d}{2}}}.$$

$$W_\lambda(Z_1,Z_2)W_\nu(Z_1,Z_2) = \int_{-\infty}^{\infty}\rho(\kappa,\lambda,\nu)W_\kappa(Z_1,Z_2)d\kappa^2$$

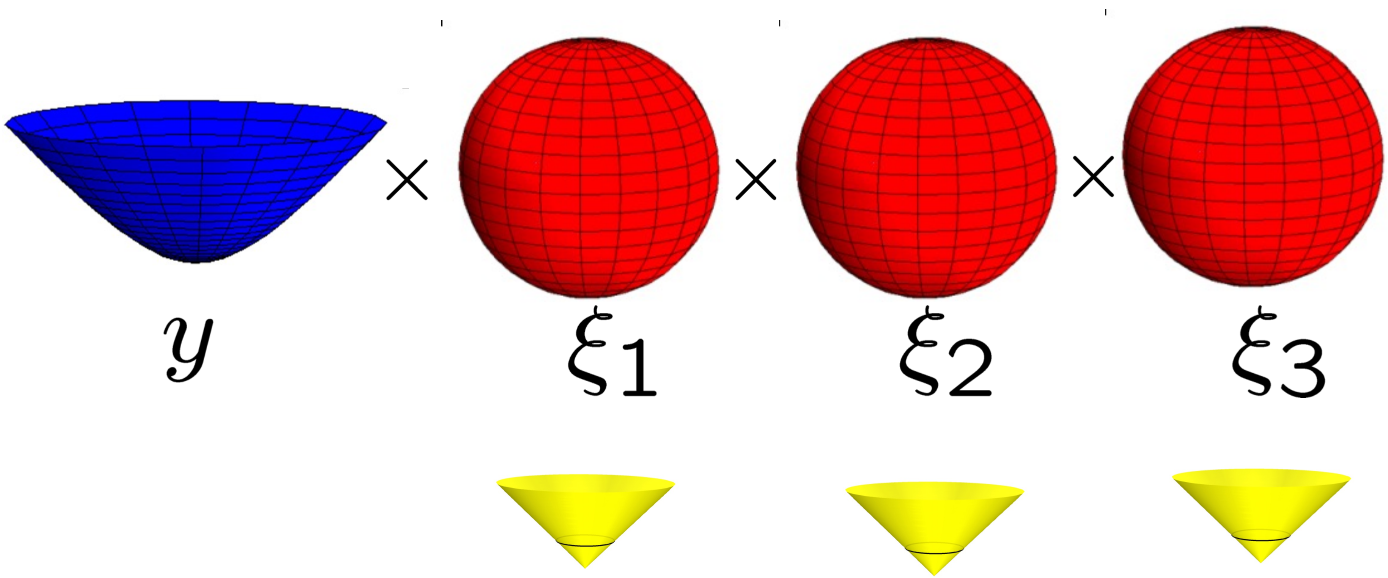
Trivial case : $d=3$ $\qquad P^{-\frac{1}{2}}_{-\frac{1}{2}+i\nu}(\cosh u) = \sqrt{\frac{2}{\pi\sinh u}}\,\frac{\sin\nu u}{\nu}.$

$$h_3(\kappa,\nu,\lambda) = \frac{1}{\sqrt{8\pi}\kappa\nu\lambda}\,\frac{\sinh(\pi\kappa)\sinh(\pi\lambda)\sinh(\pi\nu)}{\cosh\frac{\pi(\kappa-\lambda-\nu)}{2}\cosh\frac{\pi(\kappa+\lambda-\nu)}{2}\cosh\frac{\pi(\kappa-\lambda+\nu)}{2}\cosh\frac{\pi(\kappa+\lambda+\nu)}{2}}.$$

Using the Fourier-like representation

$$h_d(\kappa, \lambda, \nu) = \int_1^\infty P^{-\frac{d-2}{2}}_{-\frac{1}{2}+i\kappa}(u)\, P^{-\frac{d-2}{2}}_{-\frac{1}{2}+i\lambda}(u) P^{-\frac{d-2}{2}}_{-\frac{1}{2}+i\nu}(u)(u^2-1)^{-\frac{d-2}{4}}\ du$$

$$\int_{H_d}\int_{S^3_{d-1}} (y\cdot\xi_1)^{-\frac{d-1}{2}-i\kappa}\,(y\cdot\xi_2)^{-\frac{d-1}{2}-i\nu}\,(y\cdot\xi_3)^{-\frac{d-1}{2}-i\lambda}\,dy\,d\Omega_1\,d\Omega_2\,d\Omega_3$$



The formula

$$\int_1^\infty P^{-\frac{d-2}{2}}_{-\frac{1}{2}+i\kappa}(u)\, P^{-\frac{d-2}{2}}_{-\frac{1}{2}+i\nu}(u) P^{-\frac{d-2}{2}}_{-\frac{1}{2}+i\lambda}(u)(u^2-1)^{-\frac{d-2}{4}}\, du =$$

$$= \frac{\prod_{\epsilon,\epsilon',\epsilon''=\pm 1} \Gamma\left(\frac{d-1}{4} + \frac{i\epsilon\kappa + i\epsilon'\nu + i\epsilon''\lambda}{2}\right)}{\left[\prod_{\epsilon=\pm 1}\Gamma\left(\frac{d-1}{2} + i\epsilon\kappa\right)\right]\left[\prod_{\epsilon'=\pm 1}\Gamma\left(\frac{d-1}{2} + i\epsilon'\nu\right)\right]\left[\prod_{\epsilon''=\pm 1}\Gamma\left(\frac{d-1}{2} + i\epsilon''\lambda\right)\right]}$$

(Jacques Bros, Henri Epstein, Michel Gaudin, UM, Vincent Pasquier 2009)

J. Penedones, 2011

KL AdS

$$W^d_\nu(z_1, z_2) = w^d_\nu(\zeta) = \frac{1}{(2\pi)^{\frac{d}{2}}}(\zeta^2 - 1)^{-\frac{d-2}{4}} e^{-i\pi\frac{d-2}{2}} Q^{\frac{d-2}{2}}_{-\frac{1}{2}+\nu}(\zeta)$$

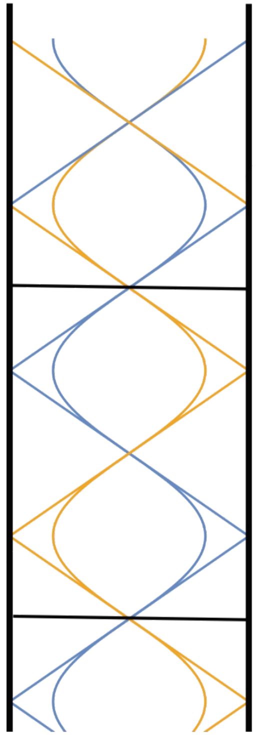
$$w^d_\lambda(\zeta)\, w^d_\nu(\zeta) = \sum_{k=0}^{\infty} \rho(k; \lambda, \nu)\, w^d_{\delta+2k+\lambda+\nu}(\zeta)$$

$$\rho_d(k; \lambda, \nu) = \frac{(2\delta + 4k + 2\lambda + 2\nu)\Gamma(1 + 2k + \lambda + \nu)}{4\pi^\delta \Gamma(\delta)\Gamma(2\delta + 2k + \lambda + \nu)} \times$$

$$\times \frac{\Gamma(\delta + k)\Gamma(\delta + \lambda + k)\Gamma(\delta + \nu + k)\Gamma(\delta + \lambda + \nu + k)}{\Gamma(1 + k)\Gamma(1 + \lambda + k)\Gamma(1 + \nu + k)\Gamma(1 + \lambda + \nu + k)}\,.$$

$$\delta = \frac{d-1}{4}$$

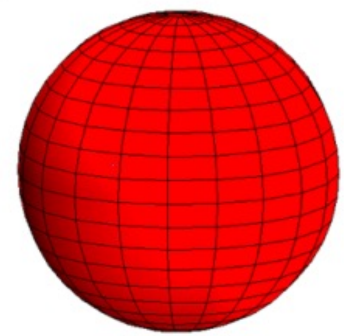
(Jacques Bros, Henri Epstein, Michel Gaudin, UM, Vincent Pasquier CMP, 2012 *[Submitted on 26 Jul 2011]***)**



L. Fitzpatrick and J. Kaplan, “Analyticity and the Holographic S-Matrix,” JHEP 10 (2012) *[Submitted on 29 Nov 2011]*

dS two-loop banana

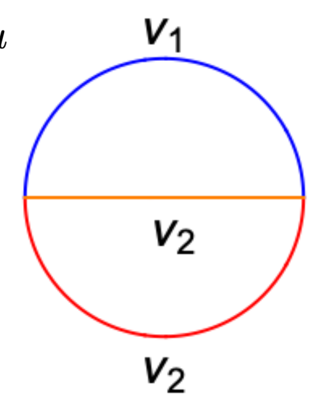
$$G^d_\nu(-\cos s) = \frac{\Gamma(\frac{d-1}{2} + i\nu)\Gamma(\frac{d-1}{2} - i\nu)}{2(2\pi)^{d/2}}(\sin s)^{-\frac{d-2}{2}}\, P^{-\frac{d-2}{2}}_{-\frac{1}{2}+i\nu}(-\cos s).$$



$$I(\nu_1,\nu_2,\nu_3,d) = \int G_{\nu_1}(x_0\cdot x)G_{\nu_2}(x_0\cdot x)G_{\nu_3}(x_0\cdot x) =$$

$$= \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}\int_0^\pi G_{\nu_1}(-\cos s)G_{\nu_2}(-\cos s)G_{\nu_3}(-\cos s)(\sin s)^{d-1}ds$$

$$= K_d(\nu_1,\nu_2,\nu_3)\int_{-1}^1 P^{-\frac{d-2}{2}}_{-\frac{1}{2}+i\nu_1}(u)P^{-\frac{d-2}{2}}_{-\frac{1}{2}+i\nu_2}(u)P^{-\frac{d-2}{2}}_{-\frac{1}{2}+i\nu_3}(u)(1-u^2)^{-\frac{d-2}{4}}du$$



$$K_d(\nu_1,\nu_2,\nu_3) = \frac{\prod_{j=1}^3\Gamma(\frac{d-1}{2} - i\nu_j)\Gamma(\frac{d-1}{2} + i\nu_j))}{2^{2+\frac{3d}{2}}\pi^d\Gamma\left(\frac{d}{2}\right)}$$

Using the KL and the bubble formulae

$$I_2(\lambda,\nu,d) = \frac{\Gamma\left(1-\frac{d}{2}\right)}{2^d\pi^{\frac{d}{2}}(\lambda^2-\nu^2)}\left(\frac{\Gamma\left(\frac{d-1}{2}-i\nu\right)\Gamma\left(\frac{d-1}{2}+i\nu\right)}{\Gamma\left(\frac{1}{2}-i\nu\right)\Gamma\left(\frac{1}{2}+i\nu\right)} - \frac{\Gamma\left(\frac{d-1}{2}-i\lambda\right)\Gamma\left(\frac{d-1}{2}+i\lambda\right)}{\Gamma\left(\frac{1}{2}-i\lambda\right)\Gamma\left(\frac{1}{2}+i\lambda\right)}\right).$$

$$I_3(\nu_1,\nu_2,\nu_3,d) = \int G_{\nu_1}(x_0\cdot x)G_{\nu_2}(x_0\cdot x)G_{\nu_3}(x_0\cdot x)dx =$$
$$= \int \rho(\nu_1,\nu_2,\kappa)I_2(\kappa,\nu_3,d)\kappa d\kappa = I_3^{(1)}(\nu_1,\nu_2,\nu_3,d) - I_3^{(2)}(\nu_1,\nu_2,\nu_3,d).$$

$$I_3^{(1)} = \sum_{\epsilon,\epsilon'=\pm 1} A_d(\epsilon x,\epsilon' y,w) \qquad x=\frac{i\nu_1}{2},\ y=\frac{i\nu_2}{2},\ z=\frac{i\nu_3}{2},\ \delta=\frac{d-1}{4}$$

$$A_d(x,y,w) = a_d(x,y,w)\times$$
$${}_9F_8\left(\begin{matrix}2\delta, u+\frac{1}{2}, u+1, \delta+u+1, 2\delta+2x, 2\delta+2y, 2\delta+2u, \delta-w+u, \delta+w+u \\ 2x+1, 2y+1, 2u+1, \delta+u, 2\delta+u, 2\delta+u+\frac{1}{2}, \delta-w+u+1, \delta+w+u+1\end{matrix};1\right)$$

$$a_d(x,y,w) = \frac{4^{-2\delta-3}\pi^{-4\delta-\frac{1}{2}}\Gamma\left(\frac{1}{2}-2\delta\right)\cos(2\pi w)\Gamma(2\delta-2w)\Gamma(2w+2\delta)\Gamma(2x+2\delta)\Gamma(2y+2\delta)}{\sin(2\pi x)\sin(2\pi y)\Gamma(2x+1)\Gamma(2y+1)\left(w^2-(\delta+u)^2\right)\Gamma(-2u-2\delta)\Gamma(2u+4\delta)}$$

Using the KL and the bubble formulae

$$I_2(\lambda,\nu,d)=\frac{\Gamma\left(1-\frac{d}{2}\right)}{2^d\pi^{\frac{d}{2}}(\lambda^2-\nu^2)}\left(\frac{\Gamma\left(\frac{d-1}{2}-i\nu\right)\Gamma\left(\frac{d-1}{2}+i\nu\right)}{\Gamma\left(\frac{1}{2}-i\nu\right)\Gamma\left(\frac{1}{2}+i\nu\right)}-\frac{\Gamma\left(\frac{d-1}{2}-i\lambda\right)\Gamma\left(\frac{d-1}{2}+i\lambda\right)}{\Gamma\left(\frac{1}{2}-i\lambda\right)\Gamma\left(\frac{1}{2}+i\lambda\right)}\right).$$

$$I_3(\nu_1,\nu_2,\nu_3,d)=\int G_{\nu_1}(x_0\cdot x)G_{\nu_2}(x_0\cdot x)G_{\nu_3}(x_0\cdot x)dx=$$
$$=\int\rho(\nu_1,\nu_2,\kappa)I_2(\kappa,\nu_3,d)\kappa d\kappa=I_3^{(1)}(\nu_1,\nu_2,\nu_3,d)-I_3^{(2)}(\nu_1,\nu_2,\nu_3,d).$$

$$I_3^{(2)}=\sum_{\epsilon,\epsilon'=\pm1}B_d(\epsilon x,\epsilon' y,w)\qquad x=\frac{i\nu_1}{2},\ y=\frac{i\nu_2}{2},\ z=\frac{i\nu_3}{2},\ \delta=\frac{d-1}{4}$$

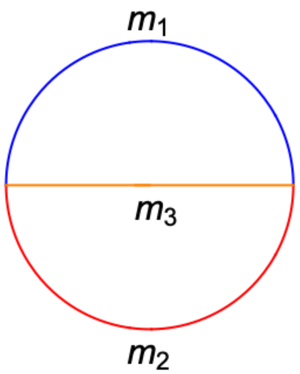
$$B_d(x,y,w)=b_d(x,y,w)$$
$$\times\ {}_7F_6\left(\begin{matrix}2\delta,\delta+u+1,\delta-w+u,\delta+w+u,2\delta+2x,2\delta+2y,2\delta+2u\\ 2x+1,2y+1,2x+2y+1,\delta+u,\delta-w+u+1,\delta+w+u+1\end{matrix};1\right)$$

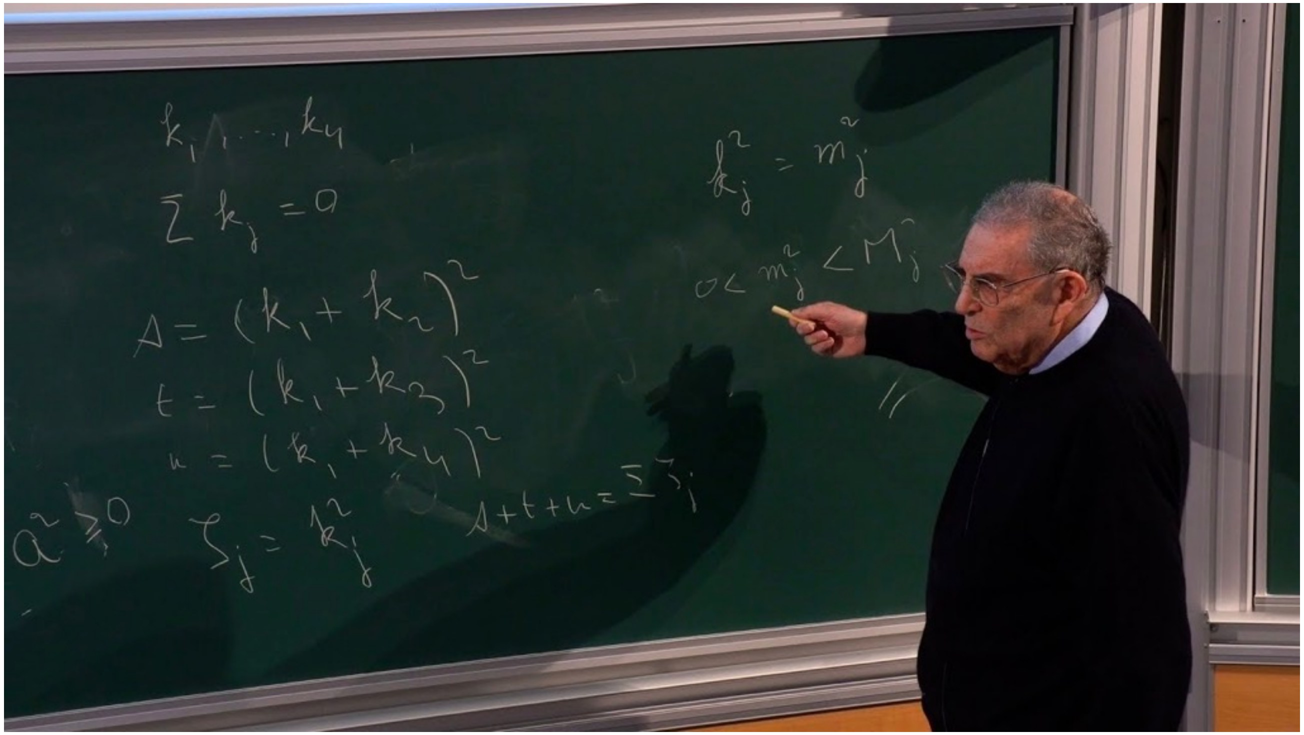
$$b_d(x,y,w)=\frac{2^{-8\delta-4u-7}\pi^{1-4\delta}\Gamma\left(\frac{1}{2}-2\delta\right)\csc(2\pi x)\csc(2\pi y)\csc(2\pi u)\Gamma(2x+2\delta)\Gamma(2y+2\delta)}{\left((\delta+u)^2-w^2\right)\Gamma(2x+1)\Gamma(2y+1)\Gamma(2u+1)\Gamma(-4u-4\delta)\Gamma\left(2u+2\delta+\frac{1}{2}\right)}.$$

Example : First Divergent case : d=3

$$I(m_1, m_2, m_3, d \sim 3) = -\frac{1}{32\pi^2}\frac{1}{d-3} - \frac{1}{16\pi^2}\log(m_1+m_2+m_3) + \frac{1-\gamma+\log(4\pi)}{32\pi^2}$$

$$\begin{aligned}
I(\nu_1,\nu_2,\nu_3,d) &= -\frac{1}{32\pi^2(d-3)} + \gamma \\
&- \frac{\psi\left(\frac{1}{2}+\frac{i(\nu_1-\nu_2-\nu_3)}{2}\right)+\psi\left(\frac{1}{2}-\frac{i(\nu_1-\nu_2-\nu_3)}{2}\right)}{128\pi^2\sinh(\pi\nu_1)\sinh(\pi\nu_2)\sinh(\pi\nu_3)}\sinh(\pi(\nu_1-\nu_2-\nu_3))+ \\
&- \frac{\psi\left(\frac{1}{2}+\frac{i(\nu_1+\nu_2+\nu_3)}{2}\right)+\psi\left(\frac{1}{2}-\frac{i(\nu_1+\nu_2+\nu_3)}{2}\right)}{128\pi^2\sinh(\pi\nu_1)\sinh(\pi\nu_2)\sinh(\pi\nu_3)}\sinh(\pi(\nu_1+\nu_2+\nu_3))+ \\
&+ \frac{\psi\left(\frac{1}{2}+\frac{i(\nu_1-\nu_2+\nu_3)}{2}\right)+\psi\left(\frac{1}{2}-\frac{i(\nu_1-\nu_2+\nu_3)}{2}\right)}{128\pi^2\sinh(\pi\nu_1)\sinh(\pi\nu_2)\sinh(\pi\nu_3)}\sinh(\pi(\nu_1-\nu_2+\nu_3))+ \\
&+ \frac{\psi\left(\frac{1}{2}+\frac{i(\nu_1+\nu_2-\nu_3)}{2}\right)+\psi\left(\frac{1}{2}-\frac{i(\nu_1+\nu_2-\nu_3)}{2}\right)}{128\pi^2\sinh(\pi\nu_1)\sinh(\pi\nu_2)\sinh(\pi\nu_3)}\sinh(\pi(\nu_1+\nu_2-\nu_3)). \qquad (1)
\end{aligned}$$





S.L. Cacciatori, H. Epstein and U. Moschella, “Loops in de Sitter space,” JHEP 07 (2024), 182 [arXiv:2403.13145 [hep-th]]

S. L. Cacciatori, H. Epstein and U. Moschella, “Loops in Anti de Sitter space,” JHEP 08 (2024) 109 [arXiv:2403.13142 [hep-th]]. Published August 15, 2024