Physical and unphysical running of couplings

Roberto Percacci

SISSA, Trieste

4th int'l FLAG workshop: The quantum and gravity,
Catania, 11/09/2024

Based on

D. Buccio, J. F. Donoghue and R. P.
"Amplitudes and renormalization group techniques: A case study,"
Phys. Rev. D **109** (2024) no.4, 045008
arXiv:2307.00055 [hep-th].

D. Buccio, J. F. Donoghue, G. Menezes and R. P.,
"Physical Running of Couplings in Quadratic Gravity,"
Phys. Rev. Lett. **133** (2024) no.2, 021604
arXiv:2403.02397 [hep-th].

D. Buccio, J. F. Donoghue, G. Menezes and R. P.,
"Renormalization and running in the 2D $CP(1)$ model,"
arXiv:2408.13142 [hep-th].

Quadratic gravity

$$
\begin{aligned}
S &= \int d^4x\sqrt{-g}\Big[\frac{1}{16\pi G}R - \frac{1}{2\lambda}C^2 - \frac{1}{\xi}R^2\Big], \\
&= \int d^4x\,\sqrt{-g}\left[\frac{1}{16\pi G}R - \frac{1}{2\lambda}\left(C^2 - \frac{2\omega}{3}R^2\right)\right]
\end{aligned}
$$

Note: $S_E = -S_L$

Einstein–Hilbert GFP

Expanding $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$

$$\begin{aligned} S &= \frac{1}{G}\int d^d x \left[(\partial h)^2 + h(\partial h)^2 + h^2(\partial h)^2 + \dots\right] \\ &\quad + \frac{1}{\lambda}\int d^d x \left[(\Box h)^2 + h(\Box h)^2 + h^2(\Box h)^2 + \dots\right] \end{aligned}$$

then rescaling $h \to \sqrt{G}\, h$

$$\begin{aligned} S &= \int d^d x \left[(\partial h)^2 + \sqrt{G}\, h(\partial h)^2 + G\, h^2(\partial h)^2 + \dots\right] \\ &\quad + \frac{G}{\lambda}\int d^d x \left[(\Box h)^2 + \sqrt{G}\, h(\Box h)^2 + G\, h^2(\Box h)^2 + \dots\right] \end{aligned}$$

GFP for $\lambda \neq 0$ or $\lambda \to \infty$

Stelle GFP

Expanding $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$

$$\begin{aligned} S &= \frac{1}{G}\int d^dx \left[(\partial h)^2 + h(\partial h)^2 + h^2(\partial h)^2 + \dots\right] \\ &+\frac{1}{\lambda}\int d^dx \left[(\Box h)^2 + h(\Box h)^2 + h^2(\Box h)^2 + \dots\right] \end{aligned}$$

rescaling $h \to \sqrt{\lambda}\, h$

$$\begin{aligned} S &= \frac{\lambda}{G}\int d^dx \left[(\partial h)^2 + \sqrt{G}\, h(\partial h)^2 + G\, h^2(\partial h)^2 + \dots\right] \\ &+ \int d^dx \left[(\Box h)^2 + \sqrt{\lambda}\, h(\Box h)^2 + \lambda\, h^2(\Box h)^2 + \dots\right] \end{aligned}$$

GFP for $G \neq 0$ or even $G \to \infty$

This theory is renormalizable

K. S. Stelle,
"Renormalization of Higher Derivative Quantum Gravity,"
Phys. Rev. D **16** (1977), 953-969

It propagates a massless graviton, a massive spin 2 ghost and a massive (non-ghost) spin 0.

Maybe the issue of the ghost can be circumvented

D. Anselmi and M. Piva, JHEP 05 (2018), 027 [arXiv:1803.07777 [hep-th]].
A. Salvio, Front. in Phys. 6, 77 (2018) [arXiv:1804.09944 [hep-th]].
J. F. Donoghue and G. Menezes, Nuovo Cim. C 45, no.2, 26 (2022) [arXiv:2112.01974 [hep-th]].

L. Buoninfante, JHEP 12 (2023), 111 [arXiv:2308.11324 [hep-th]].

The massive spin 2 is a tachyon for $\lambda < 0$ and the massive spin 0 is a tachyon for $\xi > 0$.

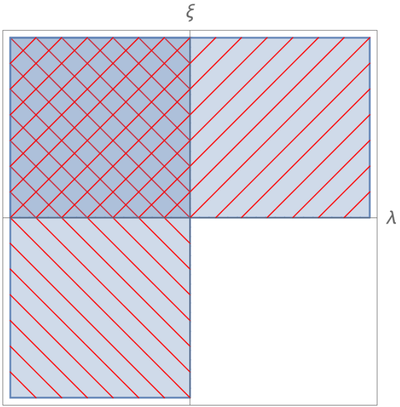


Figure: Left: spin 2 is a tachyon. Up: spin zero is a tachyon.

Linearization

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$$

One can choose the gauge and the normalization of the field so that the action can be rewritten as

$$S^{(2)} = \int d^4x \sqrt{|\bar{g}|} h_{\alpha\beta} \mathcal{O}^{\alpha\beta,\gamma\delta} h_{\gamma\delta} \ ,$$

where

$$\mathcal{O} = \bar{\Box}^2 \mathbb{I} + \mathbb{V}^{\mu\nu} \bar{\nabla}_\mu \bar{\nabla}_\nu + \mathbb{N}^\mu \bar{\nabla}_\mu + \mathbb{U} \ ,$$

with $\mathbb{V} \sim (\bar{R}, m_P^2)$, $\mathbb{N} \sim \bar{\nabla}\bar{R}$, $\mathbb{U} \sim (\bar{R}^2, \bar{\nabla}^2\bar{R}, m_P^2\bar{R}, m_P^2\Lambda)$.

Different ways of using the BF method

- choose a particular background (e.g. a sphere)
- the background is a small perturbation of flat space
- the background is a generic metric

Second method used by

J. Julve, M. Tonin, Nuovo Cim. B **46** (1978) 137.

Third method used by

E.S. Fradkin, A.A. Tseytlin, Phys. Lett. B **104** (1981) 377; Nucl. Phys. B **201** (1982) 469.

I.G. Avramidi, A.O. Barvinski, Phys. Lett. **159 B**, 269 (1985).

and everybody else since then

The logarithmic divergences or $1/\epsilon$ poles are proportional to the heat kernel coefficient

(A. O. Barvinsky and G. A. Vilkovisky, Phys. Rept. **119**, 1-74 (1985).)

$$b_4 = \frac{1}{32\pi^2}\int d^4x\, \mathrm{tr}\Big[\frac{\mathbb{I}}{90}\left(\bar{R}^2_{\rho\lambda\sigma\tau} - \bar{R}^2_{\rho\lambda} + \frac{5}{2}\bar{R}^2\right) + \frac{1}{6}\mathbb{R}_{\rho\lambda}\mathbb{R}^{\rho\lambda} - \frac{\bar{R}_{\rho\lambda}\mathbb{V}^{\rho\lambda} - \frac{1}{2}\bar{R}\mathbb{V}^{\rho}{}_{\rho}}{6} + \frac{\mathbb{V}_{\rho\lambda}\mathbb{V}^{\rho\lambda} + \frac{1}{2}\mathbb{V}^{\rho}{}_{\rho}\mathbb{V}^{\lambda}{}_{\lambda}}{24} - \mathbb{U}\Big],$$

where $\mathbb{R}_{\rho\lambda} = [\nabla_\rho, \nabla_\lambda]$ acting on symmetric tensors.

This leads to the beta functions

$$\beta_\lambda = -\frac{1}{(4\pi)^2}\frac{133}{10}\lambda^2$$

$$\beta_\omega = -\frac{1}{(4\pi)^2}\frac{25 + 1098\omega + 200\omega^2}{60}\lambda$$

$$\beta_\theta = \frac{1}{(4\pi)^2}\frac{7(56 - 171\theta)}{90}\lambda$$

[I.G. Avramidi, A.O. Barvinski, Phys. Lett. **159 B**, 269 (1985).]

Beta functions confirmed by several other calculations, also using the FRG in various approximations.

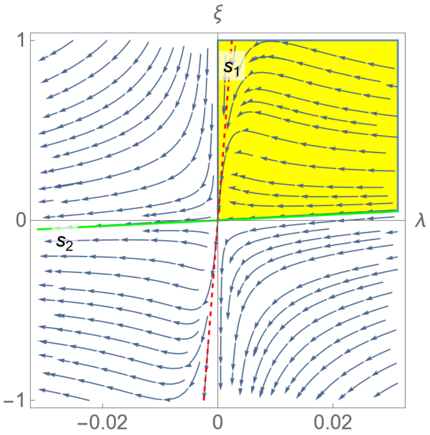
[G. de Berredo-Peixoto and I. L. Shapiro, Phys. Rev. D **71** (2005), 064005 [arXiv:hep-th/0412249 [hep-th]].]
[A. Codello, R. P., Phys.Rev.Lett. **97** 22 (2006).]

[D. Benedetti, P. F. Machado, F. Saueressig, Mod. Phys. Lett. A **24** (2009) 2233 [arXiv:0901.2984 [hep-th]]
[M. Niedermaier, Nucl. Phys. B833, 226-270 (2010).]

[G. Narain and R. Anishetty, J. Phys. Conf. Ser. **405** (2012), 012024 [arXiv:1210.0513 [hep-th]].]
[K. Groh, S. Rechenberger, F. Saueressig, O. Zanusso, PoS EPS **-HEP2011** (2011) 124 [arXiv:1111.1743 [hep-th]].]
[N. Ohta, R.P. Class. Quant. Grav. **31** 015024 (2014); arXiv:1308.3398]

[K. Falls, N. Ohta and R. Percacci, Phys. Lett. B **810** (2020), 135773 [arXiv:2004.04126 [hep-th]].]
[S. Sen, C. Wetterich, M. Yamada, JHEP 03 (2022) 130, arXiv:2111.04696 [hep-th]

RG flow



AF in a subset of first quadrant, where spin 0 is a tachyon.
$s_1 : \omega = -0.0233$, $s_2 : \omega = -5.3588$

Gravity/QCD analogy

- weakly coupled in IR limit
- AF in the UV limit
- strongly coupled in intermediate regime

B. Holdom and J. Ren, “QCD analogy for quantum gravity,” Phys. Rev. D **93** (2016) no.12, 124030 [arXiv:1512.05305 [hep-th]].

A. Salvio and A. Strumia, Agravity, JHEP 06 (2014) 080, arXiv: 1403.4226 [hep-ph]

Summary

- EH FP describes gravity in the IR
- Stelle FP (FP_1) possible UV completion
- there may be other UV completions related to nontrivial FP's

Important question:

- does this flow reflect the momentum dependence of amplitudes?

Physical running in QG

For that we return to the second way of using the BF method

Remember

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$$

Expand the background field

$$\bar{g}_{\mu\nu} = \eta_{\mu\nu} + \bar{h}_{\mu\nu}$$

Then

$$\begin{aligned} \mathcal{O} \quad &\equiv \quad \Box^2 \mathbb{I} + \mathcal{D}^{\mu\nu\rho\sigma}\partial_\mu\partial_\nu\partial_\rho\partial_\sigma + \mathcal{C}^{\mu\nu\rho}\partial_\mu\partial_\nu\partial_\rho \\ &+ \quad \mathcal{V}^{\mu\nu}\partial_\mu\partial_\nu + \mathcal{N}^\mu\partial_\mu + \mathcal{U} \ , \end{aligned}$$

where $\mathcal{D}$, $\mathcal{C}$, $\mathcal{V}$, $\mathcal{N}$, $\mathcal{U}$ are polynomials in $\bar{h}$

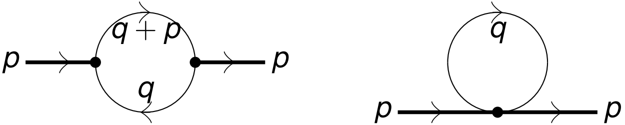


Figure: Diagrams contributing to the two-point function of $\bar{h}$: bubbles (left) and tadpoles (right). The thin line can be the h propagator or one of the ghosts, the thick line is the $\bar{h}$ propagator, with momentum p. The vertices can come from expanding any one among $\mathcal{D}$, $\mathcal{C}$, $\mathcal{V}$, $\mathcal{N}$, $\mathcal{U}$.

The physical running of λ and ξ can be read off the two point function, which from terms in the EA

$$b_\lambda \bar{C}^{\mu\nu\rho\sigma} \log \bar{\Box} \bar{C}_{\mu\nu\rho\sigma} + b_\xi \bar{R} \log \bar{\Box} \bar{R}$$

in the effective action, and the beta functions are

$$\beta_\lambda = -4 b_\lambda \lambda^2 \ , \ \ \beta_\xi = -2 b_\xi \xi^2 \ .$$

With Feynman diagrams we compute the linearized form of these expressions and determine the coefficients b_λ and b_ξ.

The term $\mathrm{tr}\mathbb{U}$ in the heat kernel must come from a tadpole. Also some of the $\mathrm{tr}R\mathbb{R}$ terms

If one removes those terms, the rest is a bilinear form in $\bar{h}$ that is not the linearization of a covariant expression in $\bar{g}$.

However, there are also infrared contributions to the $log(-p^2)$

No IR divergences in the real world because of m_P.

At high energy one assumes that m_P can be neglected then there would be IR divergences.

IR contributions to $\log p^2$ are not the linearization of a covariant expression in $\bar{g}$ but summing them to the rest we get again a covariant expression.

The terms with $\log \mu^2$ only come from the UV and reproduce the old beta functions.

New beta functions

$$\beta_\lambda = -\frac{1}{(4\pi)^2}\frac{(1617\lambda - 20\xi)\lambda}{90},$$

$$\beta_\xi = -\frac{1}{(4\pi)^2}\frac{\xi^2 - 36\lambda\xi - 2520\lambda^2}{36},$$

Abstract definitions of RG

Various definitions of running couplings

$$g = g(\Lambda)\ , \qquad g = g(k)\ , \qquad g = g(\mu)\ , \qquad \text{etc.}$$

so that

$$\beta_g = \Lambda \frac{\partial g}{\partial \Lambda}\ , \qquad \beta_g = k \frac{\partial g}{\partial k}\ , \qquad \beta_g = \mu \frac{\partial g}{\partial \mu}\ , \qquad \text{etc.}$$

Acquire physical meaning in particular situations.

In perturbative evaluation of scattering amplitudes

Typical situation in $d = 4$: for $p^2 \gg m^2$, dimreg+$\overline{\text{MS}}$ give

$$\mathcal{M}(p) = \lambda + b\lambda^2 \log\left(\frac{p^2}{\mu^2}\right)$$

From the μ-independence

$$\mu \frac{d}{d\mu}\mathcal{M}(p) = 0$$

we get

$$\beta_\lambda \equiv \mu \frac{d}{d\mu}\lambda(\mu) = 2b\lambda^2$$

What is this good for?

- solves the problem of the large logarithms
- the beta functions gives us information on the behavior of the scattering amplitude at high energy

$$\frac{\partial \mathcal{M}}{\partial p} = 2b\lambda^2$$

Is this the case for the beta functions of quadratic gravity?

More general

In general, in the presence of a mass

$$\mathcal{M}(p) = \lambda + a\lambda^2 \log\left(\frac{m^2}{\mu^2}\right) + b\lambda^2 \log\left(\frac{p^2}{\mu^2}\right) + c\lambda^2 \log\left(\frac{p^2}{m^2}\right)$$

that can also be rewritten

$$\mathcal{M}(p) = \lambda + (a-c)\lambda^2 \log\left(\frac{m^2}{\mu^2}\right) + (b+c)\lambda^2 \log\left(\frac{p^2}{\mu^2}\right)$$

Blindly following the preceding steps, from the μ-independence

$$\mu\frac{d}{d\mu}\mathcal{M}(p) = 0$$

we get

$$\beta_\lambda \equiv \mu\frac{d}{d\mu}\lambda(\mu) = 2(a+b)\lambda^2$$

This beta function

- does not solve the problem of the large logarithms
- does not gives us correct information on the behavior of the scattering amplitude at high energy

$$\frac{\partial \mathcal{M}}{\partial p} = 2(b + c)\lambda^2$$

This can be seen as a shortcoming of $\overline{\text{MS}}$

Absorbing the 2nd term in the renormalized coupling

$$\mathcal{M}(p) = \lambda + (b+c)\lambda^2 \log\left(\frac{p^2}{\mu^2}\right)$$

From the requirement that this be μ-independent

$$\beta_\lambda \equiv \mu \frac{d}{d\mu}\lambda(\mu) = 2(b+c)\lambda^2$$

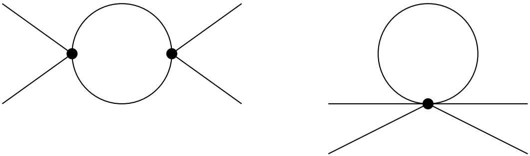
We call this the physical beta function. It solves the problem of the large logs and it faithfully reproduces the p-dependence of the amplitude at high energy.

Are there other examples of this phenomenon?

Example: the $O(3)$ NLSM

$$L = -\frac{g^2}{2}\frac{(\partial_\mu\varphi)^2}{1+\frac{\varphi_1^2}{4}+\frac{\varphi_2^2}{4}}$$

the $2 \to 2$ amplitude is



$$\mathcal{M} = g_0^2 s - \frac{g_0^4}{4}\left[I(t)(s+t+u) + I(u)(s-t+u)\right]$$

where

$$I(p^2) = 2T - p^2 B(p^2)$$

is the unique IR finite combination of

$$T = -i\int \frac{d^2q}{(2\pi)^2}\frac{1}{q^2+i\epsilon}$$

$$B(p^2) = -i\int \frac{d^2q}{(2\pi)^2}\frac{1}{(q^2+i\epsilon)((p-q)^2+i\epsilon)}$$

$$g_R^2(E^2) = g_0^2 + \frac{g_0^4}{2} I(E^2)$$

$$I(p^2) - I(E^2) = \log\left(E^2/p^2\right)$$

Then

$$\mathcal{M} = g^2(E^2)s + \frac{g_R^4}{8\pi}\left(\log\frac{-t^2}{E^2} + \log\frac{-u}{E^2}\right) - \frac{g_R^4}{8\pi}(t-u)\log\frac{t}{u}$$

giving

$$\beta_g = E\frac{\partial g_R}{\partial E} = -\frac{g^3}{4\pi}$$

With UV and IR cutoff

$$T = \frac{1}{2\pi}\log(\Lambda^2/k^2)$$

$$p^2 B(p^2) = \frac{1}{2\pi}\log(-p^2/k^2)$$

With dimreg at both ends

$$T = 0$$

$$p^2 B(p^2) = \frac{1}{2\pi}\left[\frac{1}{\epsilon} - \log(-p^2/\mu^2)\right]$$

with dimreg in UV and cutoff in IR

$$T = \frac{1}{2\pi}\left[\frac{1}{\epsilon} - \log(k^2/\mu^2)\right]$$

$$p^2 B(p^2) = \frac{1}{2\pi}\log(-p^2/k^2)$$

in each case the p-dependence is only in the finite bubble diagram

Shift-invariant scalar

$$\mathcal{L} = -\frac{Z_1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}Z_2\Box\phi\Box\phi - \frac{1}{4}Z_2^2 g(\partial_\mu\phi\partial^\mu\phi)(\partial_\nu\phi\partial^\nu\phi)$$

with $Z_2 = \frac{Z_1}{m^2}$. ($[Z_1] = [g] = 0$, $[Z_2] = -2$)

Characteristic scales:

- ghost mass m
- interaction scale: $m/\sqrt[4]{g}$

In order for ghosts to be propagating and weakly coupled need $g \ll 1$

General 4 point amplitude

$$\frac{5g^2\left(s^2+t^2+u^2\right)}{64\pi^2 m^4\epsilon}+\frac{g^2}{5760\pi^2 m^8}\left\{\frac{m^4}{s^2}\left[-6m^4\left(s^2+t^2+u^2\right)+3sm^2\left(-31s^2+9\left(t^2+u^2\right)\right)\right.\right.$$

$$\left.+2s^2\left((352-195\gamma_E)s^2-(15\gamma_E-37)\left(t^2+u^2\right)\right)\right]$$

$$+6s^{-1/2}m^4\sqrt{4m^2-s}\left[16m^4(6s^2+t^2+u^2)-8sm^2(16s^2+t^2+u^2)+s^2(41s^2+t^2+u^2)\right]\mathrm{arccot}\sqrt{\frac{4m^2}{s}-1}$$

$$+3s^2\left(41s^2+t^2+u^2\right)\log\left(-\frac{m^2}{s}\right)$$

$$+\frac{6(s-m^2)^3}{s^3}\log\left(\frac{m^2}{m^2-s}\right)\left[m^4\left(s^2+t^2+u^2\right)-2sm^2\left(-9s^2+t^2+u^2\right)+s^2\left(41s^2+t^2+u^2\right)\right]$$

$$+(\text{same with } u\to s\to t)+(\text{samewith } t\to u\to s)$$

$$\left.+450m^4\left(s^2+t^2+u^2\right)\log\left(\frac{4\pi\mu^2}{m^2}\right)\right\}$$

High energy amplitude

$$\bar{g}(E) = g + \frac{5g^2}{32\pi^2}\left[\log\left(\frac{E^2}{m^2}\right) - \frac{17}{30}\right]$$

higher derivative terms cancel out

$$\begin{aligned}
&-\frac{\bar{g}(E)}{2m^4}(s^2+t^2+u^2) \\
&+\frac{\bar{g}^2}{192\pi^2 m^4}\left[\log\left(\frac{-s}{E^2}\right)(13s^2+t^2+u^2)\right. \\
&\qquad +\log\left(\frac{-t}{E^2}\right)(s^2+13t^2+u^2) \\
&\qquad \left. +\log\left(\frac{-u}{E^2}\right)(s^2+t^2+13u^2)\right]
\end{aligned}$$

High energy physical beta function

$$\beta_{\bar{g}} = \frac{5\bar{g}^2}{16\pi^2}$$

agrees with the μ-beta function and with the FRG

Beta functions

	Old	New
β_λ	$-\frac{1}{(4\pi)^2}\frac{133}{10}\lambda^2$	$-\frac{1}{(4\pi)^2}\frac{(1617\lambda-20\xi)\lambda}{90}$
β_ξ	$-\frac{1}{(4\pi)^2}\frac{5(\xi^2-36\lambda\xi+72\lambda^2)}{36}$	$-\frac{1}{(4\pi)^2}\frac{\xi^2-36\lambda\xi-2520\lambda^2}{36}$

Separatrices

Old flow

$$s_1 : \quad \xi = \frac{1291 + \sqrt{1637881}}{20}\lambda \approx 128.5\lambda \quad \Rightarrow \omega = -0.0233$$

$$s_2 : \quad \xi = \frac{1291 - \sqrt{1637881}}{20}\lambda \approx 0.5601\lambda \quad \Rightarrow \omega = -5.3558$$

New flow

$$s_1 : \quad \xi = \frac{569 + \sqrt{386761}}{15}\lambda \approx 79.4\lambda \quad \Rightarrow \omega = -0.03778$$

$$s_2 : \quad \xi = \frac{569 - \sqrt{386761}}{15}\lambda \approx -3.53\lambda \quad \Rightarrow \omega = 0.8506$$

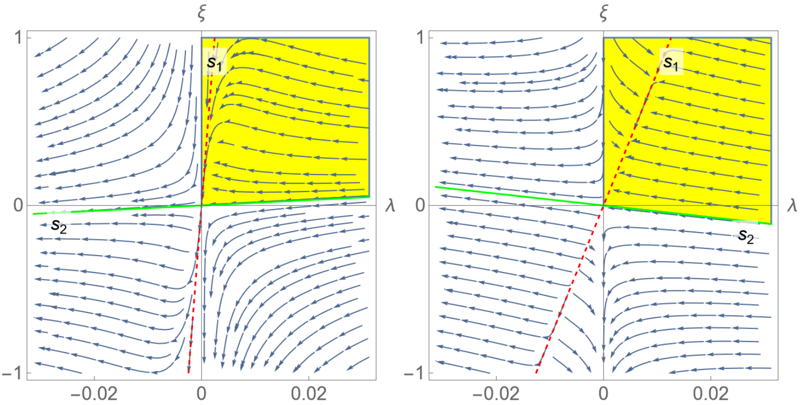


Figure: Left: old flow. Right: new flow.

Main new feature

Asymptotic freedom is possible without the tachyon.

High energy puzzle

Theory is asymptotically free, but it becomes strongly coupled at high energy

What is the meaning of asymptotic freedom in this case?

For inclusive processes, ghost and non-ghost contributions cancel and the cross sections are well behaved.

B. Holdom, “Running couplings and unitarity in a 4-derivative scalar field theory,” [arXiv:2303.06723 [hep-th]]

General summary

Whereas in theories with 2-derivative kinetic terms the μ-running and physical running agree in the high energy limit, in theories with 4 derivatives it is not necessarily so.

If there are IR divergences, the two definitions of running may differ. This does not happen in the O(3) model in 2d and the four-derivative scalar model but it happens in gravity.

AF does not yet mean that the theory is well behaved, because the amplitude still grows like E^4.

Meaning of AF in these theories has to be better understood.