# Substructures of the Weyl group

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# Harmonic Weyl

[O'Raifeartaigh, Sachs and Wiesendanger, '96; Edery and Nakayama, 2014]

Consider the following structure:

$$g_{\mu\nu} \to g'_{\mu\nu} = \Omega^2 g_{\mu\nu} \,, \qquad \text{with}$$

$$H_g(\Omega) = \Box_g \Omega + \frac{d-4}{2\Omega} g^{\mu\nu} \partial_\mu \Omega \partial_\nu \Omega = 0$$

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$$\begin{cases} g'_{\mu\nu} = (\Omega_1)^2 g_{\mu\nu} \\ H_g(\Omega_1) = 0 \end{cases} \circ \begin{cases} g''_{\mu\nu} = (\Omega_2)^2 g'_{\mu\nu} \\ H_{g'}(\Omega_2) = 0 \end{cases} \Rightarrow \begin{cases} g''_{\mu\nu} = (\Omega_1 \cdot \Omega_2)^2 g_{\mu\nu} \\ H_g(\Omega_1 \cdot \Omega_2) = 0 \end{cases}$$

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A solution of $H_g(\Omega) = 0$ is inherently non-linear unless $d = 4$. Perturbatively we find

$\Omega = 1 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \dots$

$$\Box_g\omega_1 = 0\,, \qquad \Box_g\omega_2 = \frac{4-d}{2} g^{\mu\nu}\partial_\mu\omega_1\partial_\nu\omega_1 \dots$$

# Harmonic Weyl

- Ricci scalar transforms homogeneously under h.W. [Kühnel and Rademacher, '95]

$$R \to R' = \Omega^{-2}\Big(R - 2(d-1)\Omega^{-1}H_g(\Omega)\Big)$$

In $d = 4$ we have that $\sqrt{g}\,(\alpha R^2 + \beta W^2)$ is an invariant density.

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- We can think about $R$ as a partial gauge fixing [Oda, 2020; Edery, 2023]

  Consider the BRST action

$$\delta_B g_{\mu\nu} = 2 c g_{\mu\nu}\,, \qquad \delta_B c = 0\,, \qquad \delta_B \overline{c} = b\,, \qquad \delta_B b = 0\,,$$

Then, we can find a gauge fixing action can be find in the standard way

$$S_{\rm gf} + S_{\rm gh} = \delta_B \int \mathrm{d}^4x \sqrt{g}\,\overline{c}\left(R - \frac{\alpha}{2} b\right) \stackrel{!}{=} \frac{1}{2\alpha}\int \mathrm{d}^4x\sqrt{g} R^2 + 6\int \mathrm{d}^4x \sqrt{g}\,\overline{c}\,\Box_g c$$

We can see h.d. gravity [Stelle, '77] as gauge fixed agravity [Salvio and Strumia, 2014]

$$S = S_{\rm wg} + S_{\rm gf} + S_{\rm gh} = \int \mathrm{d}^4x \sqrt{g}\left\{\frac{1}{2\lambda} W^2 + \frac{1}{2\eta} E_4 + \frac{1}{2\alpha} R^2\right\} + 6\int \mathrm{d}^4x\sqrt{g}\overline{c}\Box_g c$$

# Invariant matter

- If we are give an invariant action of matter fields coupled to gravity

$$\Delta S[g,\psi] = S[\Omega^2 g, \Omega^{w_\psi}\psi] - S[g,\psi] = \int \mathrm{d}^d x \sqrt{g}\,\Phi H_g(\Omega)$$

At linear level

$$\frac{\delta}{\delta\omega_1}\int \mathrm{d}^d x\sqrt{g}\,\Phi\Box\omega_1 = 2\omega_1 g_{\mu\nu}\frac{\delta S}{\delta g_{\mu\nu}} + w_\psi\psi\frac{\delta S}{\delta\psi} \stackrel{!}{=} -\sqrt{g}\,\omega_1 T^\mu_\mu$$

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- e.g. scalar $\varphi$ with $w_\varphi = -1$

$$S[g,\varphi] = \int \mathrm{d}^4 x\sqrt{g}\Big\{\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi + \frac{\xi}{2}R\varphi^2 + \frac{1}{4!}\lambda\varphi^4\Big\}\,,$$

$$\Box_g\varphi = \xi R\varphi + \frac{\lambda}{6}\varphi^3\,, \qquad\qquad T^\mu{}_\mu = \Box_g\Big\{3\Big(\xi - \frac{1}{6}\Big)\varphi^2\Big\}$$

# General formalism

Generalize the framework. Consider a generic tensor $\mathcal{T}$ with the following transformation

$$\mathcal{T}(g) \xrightarrow{\Omega} \mathcal{T}(\Omega^2 g) = \Omega^{-\Delta} \left( \mathcal{T}(g) + O_g(\Omega) \right) \, ,$$

- Impose $O_g(1) = 0$
- Impose closure for two consecutive transformations $\Omega_1$ and $\Omega_2$

$$O_g(\Omega_1 \Omega_2) = O_g(\Omega_1) + \Omega_1^{\Delta} O_{\Omega_1^2 g}(\Omega_2) = O_g(\Omega_2) + \Omega_2^{\Delta} O_{\Omega_2^2 g}(\Omega_1) \, .$$

- Constrain the form of the trace of the EMT

$$\frac{\delta}{\delta \omega_1} \int \mathrm{d}^d x \sqrt{g} \, \Phi O_{\mathrm{lin}}(\omega_1) \stackrel{!}{=} -\sqrt{g} \, \omega_1 T^{\mu}_{\mu} \quad \Rightarrow \quad T^{\mu}_{\mu} = -O^{\dagger}_{\mathrm{lin}}(\Phi)$$

- Deduce the anomaly

$$\langle T^{\mu}_{\mu} \rangle = -\langle O^{\dagger}_{\mathrm{lin}}(\Phi) \rangle + \mathcal{A} + \beta\text{-terms}$$

# Non-local nature of restricted symmetries

Consider a 1-dimensional toy-model with scale invariance [Glavan, Noris and Zlosnik, 2024]

$$S[\sigma] \quad = \int_{t_a}^{t_b} \mathrm{d}t \left( \frac{\ddot{\sigma}}{\sigma} - \frac{\dot{\sigma}^2}{2\sigma^2} \right)^2 \qquad , \quad \begin{cases} \sigma(t) \to \Sigma(t) = \omega^{-2}(t)\sigma(t) \, , \\ \\ \mathcal{D}\omega = \left( \frac{\mathrm{d}}{\mathrm{d}t} + \frac{\dot{\Sigma}}{\Sigma} \right) \frac{\mathrm{d}}{\mathrm{d}t}\omega = 0 \end{cases}$$

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Note that even if the action is invariant, equations of motion do transform

$$0 = \mathcal{E}(\sigma) \quad \longrightarrow \quad \mathcal{E}(\Sigma) \quad = 2\dot{\omega}\Sigma \frac{\mathrm{d}}{\mathrm{d}t} A\left(\Sigma, \omega\right)$$

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$$\boxed{\delta S = \int_{t_a}^{t_b} \mathrm{d}t \frac{\mathrm{d}}{\mathrm{d}t}\left(\Sigma\frac{\dot{\omega}}{\omega}\Psi^2\right)}\,, 0 = \mathcal{E}(\sigma\,,\psi) \longrightarrow \quad \mathcal{E}(\Sigma\,,\Psi) = 2\dot{\omega}\Sigma\frac{\mathrm{d}}{\mathrm{d}t}A\left(\Sigma,\omega\right) + \dots$$

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Add an interaction with a matter field

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\sigma\dot{\psi}\right) = 0 \to \frac{\mathrm{d}}{\mathrm{d}t}\left(\Sigma\dot{\Psi}\right) = 0\,; \begin{cases} \delta\psi(t_a) = 0\,, \\ \delta\psi(t_b) = 0\,, \end{cases} \to \begin{cases} \delta\Psi(t_a) = 0\,, \\ \delta\Psi(t_b) = -\frac{\Psi_b}{\omega_b}\int_{t_a}^{tb}\mathrm{d}t\,\frac{\dot{\omega}(t)}{\Sigma(t)}\delta\Sigma(t)\,, \end{cases}$$

# Classification: Two derivatives

- Scalar constraints

  Most general constraint of degree two homogeneous in $\Omega$ and shift invariant

$$C_g(\Omega) = s_1 \Omega \Box_g \Omega + s_2 g^{\alpha\beta} \partial_\alpha \Omega \partial_\beta \Omega \,,$$

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  Closure and associativity enforce

  - $s_1 \neq 0$: $C_g(\Omega)$ proportional to $\Omega H_g(\Omega)$
  - $s_1 = 0$: $C_g(\Omega) = s_2 g^{\alpha\beta} \partial_\alpha \Omega \partial_\beta \Omega = 0$ light cone
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- Tensor constraints

  Most general tensorial structure

$$C_{g,\mu\nu}(\Omega) = c_1 \Omega \nabla_\mu \partial_\nu \Omega + c_2 g_{\mu\nu} \Omega \Box_g \Omega + c_3 \partial_\mu \Omega \partial_\nu \Omega + c_4 g_{\mu\nu} g^{\alpha\beta} \partial_\alpha \Omega \partial_\beta \Omega \,,$$

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 Closure and associativity enforce

  - $c_1 = 0$ implies $c_3 = 0 \Rightarrow$ scalar case
  - $c_1 \neq 0 \Rightarrow c_2 = 0, -\frac{1}{d}$ and the constraint can be written as

$$C_{g,\mu\nu}(\Omega) = -\Omega^3 L_{g,\mu\nu}(\Omega) + g_{\mu\nu} \Omega \left( \frac{1}{d} + c_2 \right) H_g(\Omega) \,,$$

$$L_{g,\mu\nu}(\Omega) = \Big( \nabla_\mu \nabla_\nu - \frac{1}{d} g_{\mu\nu} \Box_g \Big) \Omega^{-1} = 0$$

 $c_2 = 0$: tensor harmonic, $c_2 = -\frac{1}{d}$ Liouville-Weyl: $\tilde{K}_{\mu\nu} \to \tilde{K}'_{\mu\nu} = \tilde{K}_{\mu\nu} + \Omega L_{g,\mu\nu}(\Omega)$

 [Kühnel and Rademacher, '95; Shaposhnikov and A. Tokareva, 2023]

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Harmonic Weyl invariant theories

- scalar with $w_\varphi = -1$

$$(4\pi)^2 \mathcal{A} = \frac{1}{120} W^2_{\alpha\beta\mu\nu} - \frac{1}{360} E_4 + \frac{1}{2}\left(\xi - \frac{1}{6}\right)^2 R^2 \,,$$

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- higher derivatives scalar with $w_\varphi = 0$

$$S[\varphi, g] = \int \mathrm{d}^4x \sqrt{g} \Big\{ \frac{1}{2} \Box_g \varphi \Box_g \varphi - R^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi + \xi R \partial_\mu \varphi \partial^\mu \varphi + \lambda \partial_\mu \varphi \partial^\mu \varphi \partial_\nu \varphi \partial^\nu \varphi \Big\}$$

$$T^\mu_\mu \overset{!}{=} 6 \left(\xi - \frac{1}{3}\right) \Box \left(\partial^\mu \varphi \partial_\mu \varphi\right) \quad \Rightarrow \quad (4\pi)^2 \mathcal{A} = -\frac{1}{30} W^2_{\alpha\beta\mu\nu} + \frac{7}{180} E_4 + \left(\tilde{\xi} - \frac{1}{3}\right)^2 R^2$$

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Liouville-Weyl invariant theories: $T^\mu_\mu \overset{!}{=} \nabla_\mu\nabla_\nu X^{\mu\nu}\,, \quad X^\mu_\mu = 0$

- scalar with $w_\varphi = 0$

$$S[\varphi, g] = \int \mathrm{d}^4x\sqrt{g}\Big\{\frac{1}{2}\Box_g\varphi\Box_g\varphi - \frac{1}{6}R\partial_\mu\varphi\partial^\mu\varphi + \tilde{\zeta}\tilde{R}^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi + \tilde{\lambda}\partial_\mu\varphi\partial^\mu\varphi\partial_\nu\varphi\partial^\nu\varphi\Big\}$$

$$X_{\mu\nu} = 2(\tilde{\zeta} + 2)\Big\{\partial_\mu\varphi\partial_\nu\varphi - \frac{1}{4}g_{\mu\nu}\partial_\rho\varphi\partial^\rho\varphi\Big\}\,,$$

$$(4\pi)^2\mathcal{A} = -\frac{1}{30}W^2_{\alpha\beta\mu\nu} + \frac{7}{180}E_4 + \frac{1}{12}(2 + \tilde{\zeta})^2\tilde{R}^2_{\mu\nu}$$

## Higher derivatives, higher dimensions ($d \geq 4$)

Use the properties of $Q$-curvatures to construct constraints and invariant actions

$$Q_{4,g} = \frac{d}{2}\mathcal{K}^2 - 2K^{\mu\nu}K_{\mu\nu} - \Box_g \mathcal{K}\,, \qquad Q_{4,\Omega^2 g} = \Omega^{-4}\left[Q_{4,g} + O_{Q_{4,g}}(\Omega)\right]$$

with $O_{Q_{4,g}}(\Omega) = \ldots$ *a very long expression* $\ldots$

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$$O_{\text{lin},g}(\omega_1) = \left[\Box_g^2 - \frac{d^2-4d+8}{2(d-2)(d-1)}R\,\Box_g + \frac{4}{d-2}R^{\mu\nu}\nabla_\mu\nabla_\nu - \frac{d-6}{2(d-1)}\nabla^\mu R\nabla_\mu\right]\omega_1$$

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Construct a quadratic invariant action for a scalar with $w_\varphi = \frac{4-d}{2}$

$$S[\varphi, g] = \frac{1}{2}\int \mathrm{d}^d x\sqrt{g}\,\varphi\left(\Delta_4 + \xi Q_4\right)\varphi\,,$$

$$\Delta_4 = \Box_g^2 + \nabla_\mu\left[4K^{\mu\nu} - (d-2)g^{\mu\nu}\mathcal{J}\right]\nabla_\nu + \frac{d-4}{2}Q_4$$

for which

$$T^\mu_\mu = \frac{\xi}{2}O_{\text{lin},g}(\varphi^2) + \frac{d-4}{2}\varphi\left(\Delta_4 + \xi Q_4\right)\varphi$$

# Conclusions and outlook

## Summary

- We classified groupoids with two derivatives
- Gave examples of higher derivative constraints
- Found the expression for the classical trace of the EMT and its 1-loop anomaly

## Outlook

- Understand the geometry the metric space constrained by the restricted symmetries
- Clarify the role of such symmetries for gravitational theories
- Clarify the issues due to non-localities