Path integral measure & the RG equations of pure gravity

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CB, V. Branchina, F. Contino, A. Pernace, *To appear*

One-loop effective action

- Euclidean Einstein-Hilbert truncation

Cosmological framework: $l \gg M_P^{-1}$

$$S_{\rm grav} = \frac{1}{16\pi G}\int \mathrm{d}^4x\,\sqrt{g}\,\left(-R + 2\Lambda_{\rm cc}\right) \;\to\; S^{(a)}_{\rm grav} = \frac{\pi\Lambda_{\rm cc}}{3G}a^4 - \frac{2\pi}{G}a^2$$

- Gauge-invariant one-loop Vilkovisky-DeWitt effective action

$$\Gamma^{1l}_{\rm grav} = S_{\rm grav} + \delta S^{1l}_{\rm grav}.$$

- Measure $\left[\mathcal{D}u(h)\mathcal{D}v^*_\rho\,\mathcal{D}v_\sigma\right]$

Fradkin - Vilkovisky

$$\left[\mathcal{D}u(h)\mathcal{D}v^*_\rho\,\mathcal{D}v_\sigma\right] \equiv \prod_x \Bigg[\underbrace{g^{(a)\,00}(x)\,\left(g^{(a)}(x)\right)^{-1}}_{\propto a^{-10}}\Bigg(\prod_{\alpha\leq\beta}\mathrm{d}h_{\alpha\beta}(x)\Bigg)\Bigg(\prod_\rho \mathrm{d}v^*_\rho(x)\Bigg)\Bigg(\prod_\sigma \mathrm{d}v_\sigma(x)\Bigg)\Bigg]$$

$g^{(a)\,00}(x)\,\left(g^{(a)}(x)\right)^{-1}$ from integration over conjugate momenta

$$\delta S^{1l}_{\rm grav} = -\frac{1}{2}\log\frac{\det_1[-\Box^{(1)}_{a=1} - 3]\,\det_2[-\Box^{(0)}_{a=1} - 6]}{\det_0[-\Box^{(2)}_{a=1} - 2a^2\Lambda_{\rm cc} + 8]\,\det_2[-\Box^{(0)}_{a=1} - 2a^2\Lambda_{\rm cc}]}$$

One-loop effective action - 2

Dimensionless arguments

$\downarrow$

No need to introduce any arbitrary mass scale μ, det automatically dimensionless

Eigenvalues $\lambda_n^{(s)}$ and degeneracies $D_n^{(s)}$

$$\lambda_n^{(s)} = n^2 + 3n - s \qquad ; \qquad D_n^{(s)} = \frac{2s+1}{3}\left(n+\frac{3}{2}\right)^3 - \frac{(2s+1)^3}{12}\left(n+\frac{3}{2}\right)$$

$$n = s, s+1, \ldots$$

- Two methods: direct cut on sum over eigenvalues, proper time ... same result

$$\delta S^{1l}_{\mathrm{grav}} = -\left(\Lambda_{\mathrm{cc}}^2 \log N^2\right) a^4 + \Lambda_{\mathrm{cc}}\left(-N^2 + 8\log N^2\right) a^2 - \frac{N^4}{12} + \frac{17}{3}N^2 - \frac{1859}{90}\log N^2 + \mathcal{O}\left(N^{-2}\right)$$

$N \gg 1$ numerical cut on the number of eigenvalues

Connection to physical cutoff scale Λ_{cut}:

$$\Lambda_{\mathrm{cut}}(\sim M_P) = \frac{N}{a_{\mathrm{dS}}} = N\sqrt{\frac{\Lambda_{\mathrm{cc}}}{3}}$$

One-loop effective action - 3

$$\frac{\Lambda_{cc}^{1l}}{G^{1l}} = \frac{\Lambda_{cc}}{G}\left(1 - \frac{3G\Lambda_{cc}}{\pi}\log\frac{3\widetilde{\Lambda}^2}{\Lambda_{cc}}\right) + \text{finite}$$

$$\frac{1}{G^{1l}} = \frac{1}{G}\left[1 + \frac{G}{2\pi}\left(3\widetilde{\Lambda}^2 - 8\Lambda_{cc}\log\frac{3\widetilde{\Lambda}^2}{\Lambda_{cc}}\right)\right] + \text{finite}$$

- Taking $G \sim M_P^{-2} \to G \sim G^{1l} \sim \frac{1}{M_P^2}$: No naturalness problem with renorm. of G
- **Most importantly:** *only* logarithmic corrections to $\rho = \frac{\Lambda_{cc}}{8\pi G}$

In pure gravity **no naturalness problem arises**
We may **naturally** have $\Lambda_{cc} \ll M_P^2 \to \Lambda_{cc}^{1l} \sim \Lambda_{cc}$

Usual result re-obtained if one connects N and Λ_{cut} through: $\Lambda_{cut} = \frac{N}{a}$
i.e. cut imposed as $\sim n^2/a^2 \leq \Lambda_{cut}^2$

NOT THE SAME

Wilsonian RG equations

We again use two methods

- Direct sum over eigenvalues
- Proper time

Method 1: Sum over eigenvalues

$$S^{\rm UV}_{\rm grav}[g_{\mu\nu}] \equiv S_N[g_{\mu\nu}] = \frac{1}{16\pi G_N}\int {\rm d}^4x\,\sqrt{g}\,(-R+2\Lambda_N) \qquad (N \text{ integer})$$

As before: physical cutoff $\Lambda_{\rm cut}\,(\sim M_P) = N/a_{\rm dS}$

- Wilsonian action: $S_L[g^{(a)}_{\mu\nu}]$ $(L$ integer, $L < N$; $\delta L \ll L)$

RG equation: $S_{L-\delta L}[g^{(a)}_{\mu\nu}] = S_L[g^{(a)}_{\mu\nu}] + \delta S_L \equiv S_L[g^{(a)}_{\mu\nu}] + \sum_{n=L-\delta L}^{L} f_L(n)$

- Einstein-Hilbert truncation ($S_L[g^{(a)}_{\mu\nu}] = \frac{\pi\Lambda_L}{3G_L}a^4 - \frac{2\pi}{G_L}a^2$)

$$\boxed{\delta S_L = -\frac{1}{2}\log\frac{\det_1[-\Box^{(1)}_{a=1}-3]\,\det_2[-\Box^{(0)}_{a=1}-6]}{\det_0[-\Box^{(2)}_{a=1}-2a^2\Lambda_L+8]\,\det_2[-\Box^{(0)}_{a=1}-2a^2\Lambda_L]}}$$

$$f_L = D^{(2)}_n\log\left(\lambda^{(2)}_n - 2a^2\Lambda_L + 8\right) + D^{(0)}_n\log\left(\lambda^{(0)}_n - 2a^2\Lambda_L\right) - D^{(1)}_n\log\left(\lambda^{(1)}_n - 3\right) - D^{(0)}_n\log\left(\lambda^{(0)}_n - 6\right)$$

- Differential form: $\frac{\partial}{\partial L}S_L = -\left(\frac{\partial}{\partial L}\sum_{n=2}^{L} f_L(n)\right)_{\Lambda_L,\,G_L}$

Method 2: Proper time

- Writing the determinants as

$$\det_i(-\Box^{(s)}_{a=1}-\alpha) = e^{-\int_{1/L^2}^{+\infty} \frac{d\tau}{\tau}\, \mathrm{K}_i^{(s)}(\tau)} \quad ; \quad \mathrm{K}_i^{(s)}(\tau) = \sum_{n=s+i}^{+\infty} D_n^{(s)}\, e^{-\tau\left(\lambda_n^{(s)}-\alpha\right)}$$

- Deriving w.r.t. L we get the proper-time Wilsonian RG equation

$$\frac{\partial}{\partial L} S_L = -\left(\frac{\partial}{\partial L}\delta S_{\mathrm{pt}}\right)_{\Lambda_L,\, G_L}$$

$\longrightarrow$ **same result** as the one obtained with direct sum $\longleftarrow$

Expanding the r.h.s. for $L \gg 1$ any of the two

$$L\frac{\partial}{\partial L} S_L = 2\Lambda_L^2\, a^4 + 2\Lambda_L\left(L^2-8\right) a^2 + \frac{L^4}{3} - \frac{34L^2}{3} + \frac{1859}{45} + \mathcal{O}\left(\frac{1}{L^2}\right)$$

$$\downarrow$$

$$L\frac{\partial}{\partial L}\left(\frac{\Lambda_L}{G_L}\right) = \frac{6}{\pi}\Lambda_L^2 \qquad\qquad L\frac{\partial \Lambda_L}{\partial L} = \frac{G_L\Lambda_L^2}{\pi}\left(L^2-2\right)$$

$$\Longleftrightarrow$$

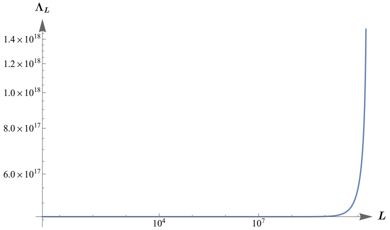
$$L\frac{\partial}{\partial L}\left(\frac{1}{G_L}\right) = -\frac{\Lambda_L}{\pi}\left(L^2-8\right) \qquad\qquad L\frac{\partial G_L}{\partial L} = \frac{G_L^2\Lambda_L}{\pi}\left(L^2-8\right)$$

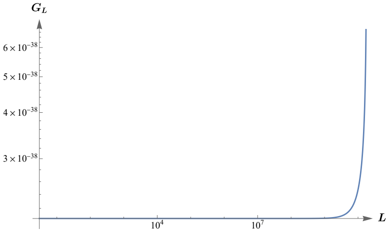
Solutions to the RG equations

$$(\Lambda_{\rm cc} \equiv \Lambda_N \;\; ; \;\; G \equiv G_N)$$

$$\Lambda_L = \frac{\Lambda_{\rm cc}}{\sqrt{1+\frac{G\,\Lambda_{\rm cc}}{\pi}(N^2-L^2)}} \qquad \xleftrightarrow{\Lambda_{\rm cut}=\frac{N}{\sqrt{3}}\,\Lambda_{\rm cc}^{1/2}} \qquad \Lambda_L = \frac{\Lambda_{\rm cc}}{\sqrt{1+\frac{G}{\pi}(3\Lambda_{\rm cut}^2-\Lambda_{\rm cc}L^2)}}$$

$$G_L = \frac{G}{\sqrt{1+\frac{G\,\Lambda_{\rm cc}}{\pi}(N^2-L^2)}} \qquad\qquad G_L = \frac{G}{\sqrt{1+\frac{G}{\pi}(3\Lambda_{\rm cut}^2-\Lambda_{\rm cc}L^2)}}$$





$$\frac{\Lambda_3}{\Lambda_{\rm cc}} = \frac{G_3}{G} \sim 0.31$$

$$\frac{\Lambda_L}{\Lambda_{\rm cc}} = \frac{G_L}{G} \text{ at this order}$$

Solutions to the RG equations - 2

$$\Lambda_L = \frac{\Lambda_{cc}}{\sqrt{1+\frac{G}{\pi}(3\Lambda_{cut}^2 - \Lambda_{cc}L^2)}} \qquad G_L = \frac{G}{\sqrt{1+\frac{G}{\pi}(3\Lambda_{cut}^2 - \Lambda_{cc}L^2)}}$$

$$\Lambda_{IR} = \frac{\Lambda_{cc}}{\sqrt{1+\frac{3G}{\pi}(\Lambda_{cut}^2 - 3\Lambda_{cc})}} \qquad G_{IR} = \frac{G}{\sqrt{1+\frac{3G}{\pi}(\Lambda_{cut}^2 - 3\Lambda_{cc})}}$$

The above equations for Λ_{IR} and G_{IR} contain important messages

- Sign of couplings is fixed
- Taking for G the natural value $G \sim M_P^{-2}$ [1]

For $\sim$ any value of Λ_{cc}, including $\Lambda_{cc} \sim \Lambda_{cut}^2$

$$\boldsymbol{G \sim G_{IR} \sim M_P^{-2}}, \qquad \boldsymbol{\Lambda_{IR} \sim \Lambda_{cc}}$$

No naturalness problem arises: the bare cosmological constant Λ_{cc} does *not* need (!) to be $\sim M_P^2$. We may **naturally** have

$$\boldsymbol{\Lambda_{cc} \ll M_P^2}$$

[1] Actually, for any value: $\Lambda_{cut}^2 - 3\Lambda_{cc}$ must be $\Lambda_{cut}^2 - 3\Lambda_{cc} \geq 0$

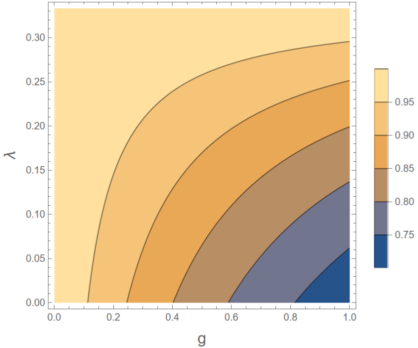
Solutions to the RG equations - 3

$$\Lambda_L = \frac{\Lambda_{\rm cc}}{\sqrt{1+\frac{G}{\pi}(3\Lambda_{\rm cut}^2-\Lambda_{\rm cc}L^2)}} \qquad G_L = \frac{G}{\sqrt{1+\frac{G}{\pi}(3\Lambda_{\rm cut}^2-\Lambda_{\rm cc}L^2)}}$$

$$\Lambda_{\rm IR} = \frac{\Lambda_{\rm cc}}{\sqrt{1+\frac{3G}{\pi}(\Lambda_{\rm cut}^2-3\Lambda_{\rm cc})}} \qquad G_{\rm IR} = \frac{G}{\sqrt{1+\frac{3G}{\pi}(\Lambda_{\rm cut}^2-3\Lambda_{\rm cc})}}$$

$$\lambda = \Lambda_{cc}/\Lambda_{\rm cut}^2,\ g = G\cdot\Lambda_{\rm cut}^2$$

$$\lambda_{\rm IR}/\lambda = g_{\rm IR}/g$$



RG equations in terms of the physical running scale k

$$L\frac{\partial \Lambda_L}{\partial L} = \frac{G_L \Lambda_L^2}{\pi}\left(L^2 - 2\right)$$
$$L\frac{\partial G_L}{\partial L} = \frac{G_L^2 \Lambda_L}{\pi}\left(L^2 - 8\right)$$

Similar to one-loop: $L \quad \Longrightarrow$ physical running scale k

$$k \equiv \frac{L}{\bar{a}_L} = L\sqrt{\frac{\Lambda_L}{3}} \qquad \left(k_{\rm IR} \leq k \leq \Lambda_{\rm cut}\ ;\ k_{\rm IR} = (3\Lambda_3)^{1/2}\right)$$

The RG equations become

$$k\frac{\partial \Lambda_k}{\partial k} = \frac{3G_k}{\pi}\,\frac{\Lambda_k\left(k^2 - \frac{2}{3}\Lambda_k\right)}{1 + \frac{3G_k}{2\pi}\left(k^2 - \frac{2}{3}\Lambda_k\right)}$$
$$k\frac{\partial G_k}{\partial k} = \frac{3G_k^2}{\pi}\,\frac{k^2 - \frac{8}{3}\Lambda_k}{1 + \frac{3G_k}{2\pi}\left(k^2 - \frac{2}{3}\Lambda_k\right)}$$

Solutions for G_k and Λ_k

(same as before in terms of k)

$$k\frac{\partial \Lambda_k}{\partial k} = \frac{3G_k}{\pi}\frac{\Lambda_k\left(k^2-\frac{2}{3}\Lambda_k\right)}{1+\frac{3G_k}{2\pi}\left(k^2-\frac{2}{3}\Lambda_k\right)}$$

$$k\frac{\partial G_k}{\partial k} = \frac{3G_k^2}{\pi}\frac{k^2-\frac{8}{3}\Lambda_k}{1+\frac{3G_k}{2\pi}\left(k^2-\frac{2}{3}\Lambda_k\right)}$$

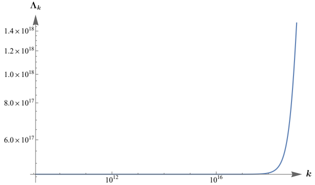
$$\xRightarrow{\text{since } \frac{\Lambda_k}{k^2}=\frac{3}{L^2}\ll 1}$$

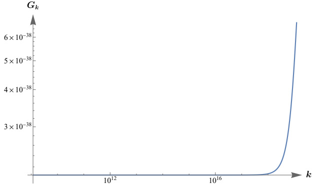
$$k\frac{\partial \Lambda_k}{\partial k} = \frac{3G_k}{\pi}\frac{k^2\Lambda_k}{1+\frac{3G_k}{2\pi}k^2}$$

$$k\frac{\partial G_k}{\partial k} = \frac{3G_k^2}{\pi}\frac{k^2}{1+\frac{3G_k}{2\pi}k^2}$$

$$\Lambda_k = \frac{k^2\Lambda_{cc}}{2\left(\Lambda_{cut}^2+\frac{\pi}{3G}\right)}\left[1+\sqrt{1+\frac{4\pi}{3G}\left(\Lambda_{cut}^2+\frac{\pi}{3G}\right)\frac{1}{k^4}}\right]$$

$$G_k = \frac{k^2 G}{2\left(\Lambda_{cut}^2+\frac{\pi}{3G}\right)}\left[1+\sqrt{1+\frac{4\pi}{3G}\left(\Lambda_{cut}^2+\frac{\pi}{3G}\right)\frac{1}{k^4}}\right]$$





FIXED POINTS

Fixed points

$$t = \ln \frac{k}{k_0} \qquad \lambda_t = \frac{\Lambda_k}{k^2} \qquad g_t = k^2 G_k$$

$$\frac{\partial \lambda_t}{\partial t} = -2\lambda_t + \frac{3g_t}{\pi} \frac{\lambda_t \left(1 - \frac{2}{3}\lambda_t\right)}{1 + \frac{3g_t}{2\pi}\left(1 - \frac{2}{3}\lambda_t\right)}$$

$$\frac{\partial g_t}{\partial t} = 2g_t + \frac{3g_t^2}{\pi} \frac{1 - \frac{8}{3}\lambda_t}{1 + \frac{3g_t}{2\pi}\left(1 - \frac{2}{3}\lambda_t\right)}$$

Fixed points from

$$-2\lambda + \frac{3g}{\pi} \frac{\lambda \left(1 - \frac{2}{3}\lambda\right)}{1 + \frac{3g}{2\pi}\left(1 - \frac{2}{3}\lambda\right)} = 0$$

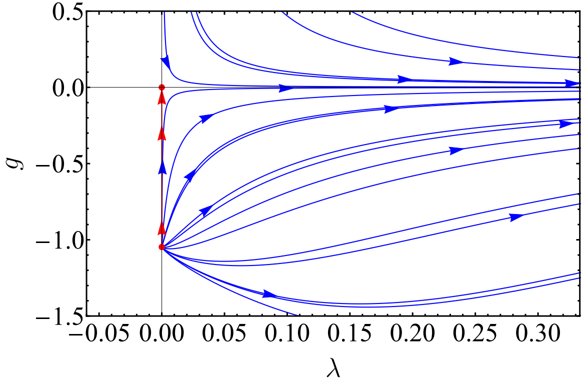
$$2g + \frac{3g^2}{\pi} \frac{1 - \frac{8}{3}\lambda}{1 + \frac{3g}{2\pi}\left(1 - \frac{2}{3}\lambda\right)} = 0$$

Fixed points

$$(\lambda, g)_1 = (0, 0)$$

$$(\lambda, g)_2 = (0, -\pi/3)$$

RG flows



$(\lambda, g)_1 = (0, 0)$	axes $\lambda = 0$, $g = 0$ UV-repulsive/attractive respectively
$(\lambda, g)_2 = (0, -\pi/3)$	unphysical UV-attractive fixed point

... We do not see any physical UV-attractive fixed point (AS) ...

What generates the AS behaviour?

Same starting point: $$L\frac{\partial}{\partial L}S_L = 2\Lambda_L^2\, a^4 + 2\Lambda_L \left(L^2 - 8\right) a^2 + \frac{L^4}{3} - \frac{34L^2}{3} + \frac{1859}{45}$$

Identify the running scale k as $\boxed{k = L/a}$ (rather than $k = L/\bar{a}_L$)

$$\to k\frac{\partial}{\partial k}S_k = \left[\frac{k^4}{3} + 2\Lambda_k\left(k^2 + \Lambda_k\right)\right] a^4 - \left(\frac{34k^2}{3} + 16\Lambda_k\right) a^2 + \frac{1859}{45}$$

$$\downarrow$$

$$k\frac{\partial}{\partial k}\Lambda_k = \frac{G_k}{\pi}\left[k^4 + 6\Lambda_k\left(k^2 + \Lambda_k\right) - \Lambda_k\frac{34k^2 + 48\,\Lambda_k}{6}\right]$$

$$k\frac{\partial}{\partial k}G_k = -G_k^2\,\frac{34k^2 + 48\,\Lambda_k}{6\pi}$$

$\Lambda_L \equiv \Lambda_k,\ G_L \equiv G_k$

What generates the AS behaviour? - 2

$$t = \ln \frac{k}{k_0} \qquad \lambda_t = \frac{\Lambda_k}{k^2} \qquad g_t = k^2 G_k$$

$$\frac{\partial \lambda_t}{\partial t} = -2\lambda_t + \frac{g_t}{\pi}\left[1 + 6\lambda_t\left(1+\lambda_t\right) - \lambda_t \frac{34+48\lambda_t}{6}\right] \equiv \beta_\lambda\left(\lambda_t, g_t\right)$$

$$\frac{\partial g_t}{\partial t} = 2g_t - g_t^2 \frac{34+48\lambda_t}{6\pi} \equiv \beta_g\left(\lambda_t, g_t\right)$$

Fixed points from

$$-2\lambda + \frac{g}{\pi}\left[1 + 6\lambda\left(1+\lambda\right) - \lambda\frac{34+48\lambda}{6}\right] = 0$$

$$2g - g^2\frac{34+48\lambda}{6\pi} = 0$$

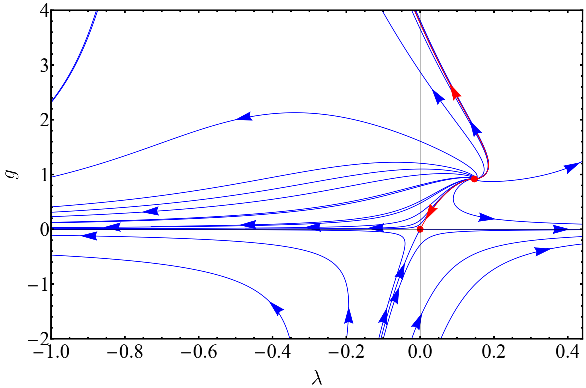
Fixed points

$$(\lambda, g)_1 = (0, 0)$$

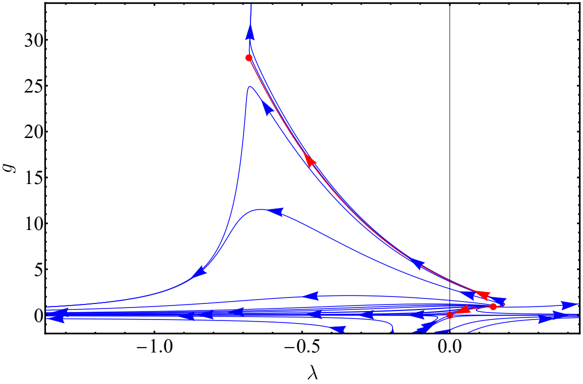
$$(\lambda, g)_2 = \left(-\frac{8-\sqrt{154}}{30}, \frac{2\pi}{23}\left(53 - 4\sqrt{154}\right)\right)$$

$$(\lambda, g)_3 = \left(-\frac{8+\sqrt{154}}{30}, \frac{2\pi}{23}\left(53 + 4\sqrt{154}\right)\right)$$

What generates the AS behaviour? - 3



$(\lambda, g)_1 = (0, 0)$	axes $g = 4\pi\lambda$, $g = 0$ UV-repulsive/attractive respectively
$(\lambda, g)_2 = (0.147, 0.918)$	UV-attractive fixed point (**AS**)
$(\lambda, g)_3 = (-0.680, 28.039)$	axes $g = -65.741\lambda$, $g = 627.551\lambda$ UV-repulsive/attractive respectively


30
25
20
15
10
5
0
g
−1.0
−0.5
0.0
λ

Functional RG

$\Gamma_k[g,\bar{g}]$ Effective Average Action ; $\kappa \equiv (32\pi G)^{-1/2}$; $\bar{g}_{\mu\nu}$ fixed background

$$k\,\partial_k\Gamma_k[g,\bar{g}] = \frac{1}{2}\,\mathrm{Tr}\left[\left(\kappa^{-2}\Gamma_k^{(2)}[g,\bar{g}] + R_k^{\mathrm{grav}}[\bar{g}]\right)^{-1} k\,\partial_k R_k^{\mathrm{grav}}[\bar{g}]\right] - \mathrm{Tr}\left[\left(-\mathcal{M}[g,\bar{g}] + R_k^{\mathrm{gh}}[\bar{g}]\right)^{-1} k\,\partial_k R_k^{\mathrm{gh}}[\bar{g}]\right]$$

M. Reuter, C. Wetterich

$\mathcal{M}[g,\bar{g}]$ classical kinetic term of the ghosts

$$\mathcal{M}[g,\bar{g}]^{\mu}{}_{\nu} = \bar{g}^{\mu\rho}\bar{g}^{\sigma\lambda}\bar{D}_\lambda\left(g_{\rho\nu}D_\sigma + g_{\sigma\nu}D_\rho\right) - \bar{g}^{\rho\sigma}\bar{g}^{\mu\lambda}\bar{D}_\lambda g_{\sigma\nu}D_\rho$$

$\bar{D}_\mu$ covariant derivative ; Christoffel symbols from $\bar{g}_{\mu\nu}$

Choice: $\bar{g}_{\mu\nu}$ of sphere radius a. Regulators $R_k^{\mathrm{grav}}[\bar{g}]$ and $R_k^{\mathrm{gh}}[\bar{g}]$ have the form

$$R_k[\bar{g}] = \mathcal{Z}_k k^2 R^{(0)}(-\Box/k^2) \quad , \quad \Box \equiv \bar{g}^{\mu\nu}\bar{D}_\mu\bar{D}_\nu$$

Modes w/ $p^2 > k^2$: integrated out ; modes w/ $p^2 < k^2$: suppressed by R_k:
$\Longrightarrow$ "Tr" effectively contains only eigenmodes of $-\Box$ with eigenvalues

$$p^2 \sim \frac{n^2}{a^2} \sim k^2$$

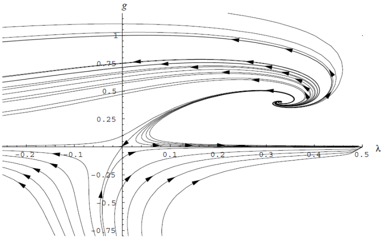
Eigenvalues are used as discriminant to introduce sliding scale k

RG flows from Wetterich/Reuter equation

$$\boldsymbol{p^2 \sim \frac{n^2}{a^2} \sim k^2}$$

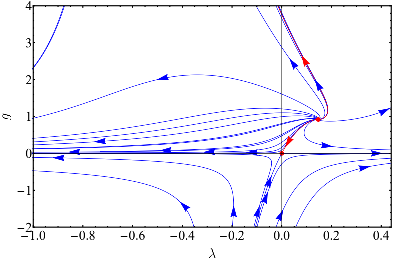
Introduction of running scale k through eigenvalues $\lambda_n \sim \frac{n^2}{a^2}$

$\Longrightarrow$ spurious k^4 terms $\Longrightarrow$ "**AS flow**"



M.Reuter, Phys.Rev.D 57 (1998) 971-985

M.Reuter and F.Saueressig, Phys.Rev.D 65 (2002) 065016



Our result with the identification $k = \frac{n}{a}$

Conclusions

- Fradkin - Vilkovisky measure in the path integral $\rightarrow$ dimless determinants
- Cutoff / sliding cutoff scale imposed as a cut on the number of modes
- One-loop effective action / Wilsonian RG equations: absence of quartic and quadratic contributions to ρ
- Solutions to the RG equations are multiplicative: no naturalness issue arises in pure gravity!
- Renormalization pattern considerably different: no physical UV-attractive fixed point
- Quartic dependence & AS fixed point generated if we impose the cutoff as a cut on $\sim n^2/a^2$ rather than n ... spurious powers of a

... Is Gravity non-perturbatively renormalizable? ...

ADDITIONAL SLIDES

The case $\Lambda = 0$

$$L\frac{\partial}{\partial L}S_L = 2\Lambda_L^2\, a^4 + 2\Lambda_L\left(L^2 - 8\right)a^2 + \frac{L^4}{3} - \frac{34L^2}{3} + \frac{1859}{45}$$

Using our equations with the identification $k = L/a$

$$\to k\frac{\partial}{\partial k}S_k = \left[\frac{k^4}{3} + 2\Lambda_k\left(k^2 + \Lambda_k\right)\right]a^4 - \left(\frac{34k^2}{3} + 16\Lambda_k\right)a^2 + \frac{1859}{45}$$

$$\frac{\partial g_t}{\partial t} = 2g_t - g_t^2\frac{34}{6\pi} \quad\longrightarrow\quad g^* = \frac{12\pi}{34} \qquad \text{UV-attractive fixed point}$$

Match results in the literature Souma

Generation of g^*: $2\Lambda_L L^2 a^2 \to 2\Lambda_k k^2 a^4$, $-34/3L^2 \to -34/3k^2a^2$

Change sign of a^2 term!

$$\delta S_L = -\frac{1}{2}\log\frac{\det_1[-\Box^{(1)}_{a=1} - 3]\,\det_2[-\Box^{(0)}_{a=1} - 6]}{\det_0[-\Box^{(2)}_{a=1} - 2a^2\Lambda_L + 8]\,\det_2[-\Box^{(0)}_{a=1} - 2a^2\Lambda_L]}$$

When $\Lambda_L = 0 \to$ no powers of a in fluctuation determinants

Renormalization of G_L must be $\propto \Lambda_L$: g^* cannot exist

The case $\Lambda = 0$ Cont'd

$$\partial_t g_k = \beta_g = [d - 2 + \eta(k)] g_k,$$

$$\partial_t \lambda_k = \beta_\lambda = -(2-\eta)\lambda_k + \frac{1}{2} g_k (4\pi)^{1-d/2} \left\{ 2d(d+1)\Phi^1_{d/2}(-2\lambda_k) - d(d+1)\eta(k)\widetilde{\Phi}^1_{d/2}(-2\lambda_k) - 8d\Phi^1_{d/2}(0) \right\}.$$

Souma

$$\frac{\partial \lambda_t}{\partial t} = -2\lambda_t + \frac{g_t}{\pi}\left[1 + 6\lambda_t(1+\lambda_t) - \lambda_t \frac{34 + 48\lambda_t}{6}\right] \equiv \beta_\lambda(\lambda_t, g_t)$$

$$\frac{\partial g_t}{\partial t} = 2g_t - g_t^2 \frac{34 + 48\lambda_t}{6\pi} \equiv \beta_g(\lambda_t, g_t)$$

Our equations with the identification $k = L/a$

λ is generated ... inconsistent to study g equation alone

Comparison with Becker-Reuter, PRD 102 (2020) 12 - PRD 104 (2021) 12 and Ferrero-Percacci, e-Print: 2404.12357

Consider the modified Einstein equation at one-loop

Becker, Reuter PRD 104 (2021) 12

$$\frac{3G}{\pi}\, a \frac{\mathrm{d}}{\mathrm{d}a}\Gamma^{1l}_{\mathrm{grav}}(a) = 4\Lambda_{\mathrm{cc}} a^4 - 12 a^2 + \frac{3G}{\pi}\, a \frac{\mathrm{d}}{\mathrm{d}a}\delta S^{1l}_{\mathrm{grav}}(a) = 0 \qquad (g_{\mu\nu} = g^{(a)}_{\mu\nu})$$

* **BRST-invariant Fujikawa measure** (or measure not fully considered)

$$\left[\mathcal{D}u(h)\mathcal{D}v^*_\rho\,\mathcal{D}v_\sigma\right] \equiv \prod_x \Big[\underbrace{\left(g^{(a)}(x)\right)^{-2}}_{\propto a^{-16}}\Big(\prod_{\alpha\le\beta}\mathrm{d}h_{\alpha\beta}(x)\Big)\Big(\prod_\rho \mathrm{d}v^*_\rho(x)\Big)\Big(\prod_\sigma \mathrm{d}v_\sigma(x)\Big)\Big]$$

$\Longrightarrow \quad \delta S^{1l}_{\mathrm{grav}}(a) \sim N^4 \log(a\mu) \quad \Longrightarrow \quad$ **solution** $\bar{a}_N \sim N\sqrt{M_P^{-1}\Lambda_{\mathrm{cc}}^{-1}}$

* **Fradkin-Vilkovisky measure**

$$\left[\mathcal{D}u(h)\mathcal{D}v^*_\rho\,\mathcal{D}v_\sigma\right] \equiv \prod_x \Big[\underbrace{g^{(a)\,00}(x)\left(g^{(a)}(x)\right)^{-1}}_{\propto a^{-10}}\Big(\prod_{\alpha\le\beta}\mathrm{d}h_{\alpha\beta}(x)\Big)\Big(\prod_\rho \mathrm{d}v^*_\rho(x)\Big)\Big(\prod_\sigma \mathrm{d}v_\sigma(x)\Big)\Big]$$

$\Longrightarrow \quad \delta S^{1l}_{\mathrm{grav}}(a)$ does **not** contain $N^4 \log(a\mu) \quad \Longrightarrow \quad$ **NO** $\bar{a}_N$ **solution**

Similar considerations apply to the results found in Becker-Reuter, PRD 102 (2020) 12, and Ferrero-Percacci, e-Print: 2404.12357

Why Fradkin-Vilkovisky measure?

Free real, massless scalar field ϕ in grav. background $g_{\mu\nu} = (1 + h(x))\, \eta_{\mu\nu}$

$$\sqrt{-g}\mathcal{L} = \frac{1}{2}\,(1 + h(x))\, \partial_\mu \phi\, \partial^\mu \phi$$

(Donoghue, PRD 104 (2021) 4)

Of the kind

$$\mathcal{L} = \frac{1}{2}\left[\delta^{ab} + \bar{G}^{ab}(\pi)\right] \partial_\mu \pi^a\, \partial^\mu \pi^b$$

(Gerstein, Jackiw, Lee and Weinberg, "Chiral loops", PRD 3, 2486 (1971))

* **Canonical quantization** (using GJLW result)

1. $\mathcal{H}_I = -\frac{1}{2} h(x) \partial_\mu \tilde{\phi}\, \partial^\mu \tilde{\phi} + \frac{1}{2} \partial_0 \tilde{\phi} \left[\frac{h^2}{1+h}\right] \partial_0 \tilde{\phi} \;\oplus\;$ non-std prop. $\Delta_{\mu\nu}(q) = \frac{i q_\mu q_\nu}{q^2 + i\epsilon} - i\eta_{\mu 0}\eta_{\nu 0}$

2. $\mathcal{H}_I = -\frac{1}{2} h(x) \partial_\mu \tilde{\phi}\, \partial^\mu \tilde{\phi} + \frac{i}{2}\delta^{(4)}(0)\, \log\left(1 + h(x)\right) \;\oplus\;$ std Feynman rules

* **Functional methods** (Honerkamp and Meetz, "Chiral-invariant perturbation theory", PRD 3, 1996 (1971))

path integral measure from integration over conjugate momenta

$$\mathcal{D}u(\phi) = \prod_x (1 + h(x))^{1/2}\, \mathrm{d}\phi(x) \quad \Longleftrightarrow \quad \Delta\mathcal{L} = -\frac{i}{2}\delta^{(4)}(0)\, \log\left(1 + h(x)\right)$$

same as that in canonical quantization

Why Fradkin-Vilkovisky measure? Cont'd

For generic metric $g_{\mu\nu}$ (Donoghue, PRD 104 (2021) 4)

$$\mathcal{D}u(\phi) = \prod_x \left(-\det g_{\mu\nu}(x)\right)^{1/8} d\phi(x)$$

Measure we use (from integration over conjugate momenta) (Fradkin-Vilkovisky, Unz)

$$\mathcal{D}u(\phi) = \prod_x \left[\left(g^{00}(x)\right)^{1/2} \left(-\det g_{\mu\nu}(x)\right)^{1/4} d\phi(x)\right]$$

Fujikawa measure

$$\mathcal{D}u(\phi) = \prod_x \left[\left(-\det g_{\mu\nu}(x)\right)^{1/4} d\phi(x)\right]$$

* $g_{\mu\nu} = (1 + h(x))\,\eta_{\mu\nu} \Rightarrow \left(-\det g_{\mu\nu}(x)\right)^{1/8} = \left(g^{00}(x)\right)^{1/2} \left(-\det g_{\mu\nu}(x)\right)^{1/4} = (1 + h(x))^{1/2}$

$$\boldsymbol{\left(-\det g_{\mu\nu}(x)\right)^{1/4}} = 1 + h(x) \qquad \textbf{DIFFERENT!}$$

* Sphere $g_{\mu\nu} = g^{(a)}_{\mu\nu} \Longrightarrow \left(-\det g^{(a)}_{\mu\nu}(x)\right)^{1/8}$ and $\left(g^{(a)\,00}(x)\right)^{1/2} \left(-\det g^{(a)}_{\mu\nu}(x)\right)^{1/4}$ both $\sim \boldsymbol{a}$

$$\boldsymbol{\left(-\det g_{\mu\nu}(x)\right)^{1/4}} \sim \boldsymbol{a^2}$$

Sum over the eigenvalues in closed form

$$
\begin{aligned}
F(a^2\Lambda) = {} & 9\Lambda a^2 - \frac{1}{6}\Lambda\sqrt{8\Lambda a^2+9}\log\Gamma\left(\frac{7}{2}-\frac{1}{2}\sqrt{8\Lambda a^2+9}\right)a^2 - 5\Lambda\psi^{(-2)}\left(\frac{1}{2}\left(\sqrt{8a^2\Lambda-15}+7\right)\right)a^2 \\
& - 5\Lambda\psi^{(-2)}\left(\frac{7}{2}-\frac{1}{2}\sqrt{8a^2\Lambda-15}\right)a^2 - \Lambda\psi^{(-2)}\left(\frac{1}{2}\left(\sqrt{8\Lambda a^2+9}+7\right)\right)a^2 \\
& - \Lambda\psi^{(-2)}\left(\frac{7}{2}-\frac{1}{2}\sqrt{8\Lambda a^2+9}\right)a^2 + \frac{1}{6}\Lambda\log\Gamma\left(\frac{1}{2}\left(\sqrt{8\Lambda a^2+9}+7\right)\right)\sqrt{8\Lambda a^2+9}a^2 \\
& - 5\log(120) + \frac{49\log(A)}{3} - 2\sqrt{\frac{11}{3}}\log\Gamma\left(\frac{1}{2}\left(\sqrt{33}+7\right)\right) \\
& - \frac{5}{6}\left(a^2\Lambda-5\right)\sqrt{8a^2\Lambda-15}\log\Gamma\left(\frac{7}{2}-\frac{1}{2}\sqrt{8a^2\Lambda-15}\right) \\
& - \frac{1}{6}\sqrt{8\Lambda a^2+9}\log\Gamma\left(\frac{7}{2}-\frac{1}{2}\sqrt{8\Lambda a^2+9}\right) + 3\psi^{(-4)}(1) + 3\psi^{(-4)}(6) + \psi^{(-4)}\left(\frac{7}{2}-\frac{\sqrt{33}}{2}\right) \\
& + \psi^{(-4)}\left(\frac{1}{2}\left(\sqrt{33}+7\right)\right) - 5\psi^{(-4)}\left(\frac{1}{2}\left(\sqrt{8a^2\Lambda-15}+7\right)\right) - 5\psi^{(-4)}\left(\frac{7}{2}-\frac{1}{2}\sqrt{8a^2\Lambda-15}\right) \\
& - \psi^{(-4)}\left(\frac{1}{2}\left(\sqrt{8\Lambda a^2+9}+7\right)\right) - \psi^{(-4)}\left(\frac{7}{2}-\frac{1}{2}\sqrt{8\Lambda a^2+9}\right) + \frac{15\psi^{(-3)}(1)}{2} - \frac{15\psi^{(-3)}(6)}{2} \\
& - \frac{1}{2}\sqrt{33}\psi^{(-3)}\left(\frac{1}{2}\left(\sqrt{33}+7\right)\right) - \frac{5}{2}\sqrt{8a^2\Lambda-15}\psi^{(-3)}\left(\frac{7}{2}-\frac{1}{2}\sqrt{8a^2\Lambda-15}\right)
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{2}\sqrt{8\Lambda a^2+9}\,\psi^{(-3)}\left(\frac{7}{2}-\frac{1}{2}\sqrt{8\Lambda a^2+9}\right)+\frac{33\psi^{(-2)}(1)}{4}+\frac{33\psi^{(-2)}(6)}{4}\\
&+\frac{49}{12}\psi^{(-2)}\left(\frac{7}{2}-\frac{\sqrt{33}}{2}\right)+\frac{49}{12}\psi^{(-2)}\left(\frac{1}{2}\left(\sqrt{33}+7\right)\right)+\frac{175}{12}\psi^{(-2)}\left(\frac{1}{2}\left(\sqrt{8a^2\Lambda-15}+7\right)\right)\\
&+\frac{175}{12}\psi^{(-2)}\left(\frac{7}{2}-\frac{1}{2}\sqrt{8a^2\Lambda-15}\right)-\frac{13}{12}\psi^{(-2)}\left(\frac{1}{2}\left(\sqrt{8\Lambda a^2+9}+7\right)\right)\\
&-\frac{13}{12}\psi^{(-2)}\left(\frac{7}{2}-\frac{1}{2}\sqrt{8\Lambda a^2+9}\right)+\frac{1}{2}\psi^{(-3)}\left(\frac{7}{2}-\frac{\sqrt{33}}{2}\right)\sqrt{33}+2\log\Gamma\left(\frac{7}{2}-\frac{\sqrt{33}}{2}\right)\sqrt{\frac{11}{3}}\\
&+\frac{5}{6}\left(a^2\Lambda-5\right)\log\Gamma\left(\frac{1}{2}\left(\sqrt{8a^2\Lambda-15}+7\right)\right)\sqrt{8a^2\Lambda-15}\\
&+\frac{5}{2}\psi^{(-3)}\left(\frac{1}{2}\left(\sqrt{8a^2\Lambda-15}+7\right)\right)\sqrt{8a^2\Lambda-15}\\
&+\frac{1}{6}\log\Gamma\left(\frac{1}{2}\left(\sqrt{8\Lambda a^2+9}+7\right)\right)\sqrt{8\Lambda a^2+9}\\
&+\frac{1}{2}\psi^{(-3)}\left(\frac{1}{2}\left(\sqrt{8\Lambda a^2+9}+7\right)\right)\sqrt{8\Lambda a^2+9}+\frac{7\zeta(3)}{4\pi^2}-\frac{2}{3}\zeta'(-3)-\frac{20801}{1080}
\end{aligned}
$$

$$g_{\mu\nu}^{(a)} = \begin{pmatrix} a^2 & 0 & 0 & 0 \\ 0 & a^2 \sin^2\theta_1 & 0 & 0 \\ 0 & 0 & a^2 \sin^2\theta_1 \sin^2\theta_2 & 0 \\ 0 & 0 & 0 & a^2 \sin^2\theta_1 \sin^2\theta_2 \sin^2\theta_3 \end{pmatrix}$$