Naturalness, renormalization, and the cosmological constant problem

Arcangelo Pernace

University of Catania and INFN, Catania unit - Italy

September 10, 2024

4^{th} International FLAG Workshop : The Quantum and Gravity

September 9 - 11, 2024 - Catania

Carlo Branchina, Vincenzo Branchina, Filippo Contino, AP, in preparation

Introduction

Contribution to vacuum energy **from quantum fluctuations** $\sim M_P^4$

Value inferred **from observed accelerated expansion** of the universe $\rho_{\rm vac} \sim 10^{-123} M_P^4$

CC problem: most severe naturalness problem in physics

Several attempts towards its solution

* Polyakov ... and later Jackiw ... Moscow zero ...

* Taylor - Veneziano ... non-local terms : $V \log V$

... "Infinitely" many other attempts ...

Consider the one-loop VDW Effective Action

* Euclidean action - Einstein-Hilbert truncation

$$S_{\rm grav} = \frac{1}{16\pi G} \int d^4x \sqrt{g} \, (-R + 2\Lambda)$$

* Cosmological framework: manifolds with typical length scale $l \gg M_P^{-1}$

* Gauge-invariant one-loop effective action, $\Gamma^{1l}_{\rm grav} = S_{\rm grav} + \delta S^{1l}_{\rm grav}$
  geometrical approach, Vilkovisky-DeWitt

* Strategy put forward by Fradkin and Tseytlin / Taylor and Veneziano

* Particular attention to the role played by the **measure**

* Background field method: $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$

* When $\bar{g}_{\mu\nu}$ has spherical symmetry, one-loop VDW effective action
  coincides with the standard one calculated with gauge-fixing term

$$S_{\rm gf} = \frac{1}{32\pi G\xi} \int d^4x \sqrt{\bar{g}} \left[\nabla_\mu \left(h^\mu_\nu - \frac{1}{2}\delta^\mu_\nu \, h^\sigma_\sigma\right)\right]$$

after taking the limit $\xi \to 0$ at the end of the calculation

One-loop VDW Effective Action cont'd

Let us calculate the 1-loop correction $\delta S^{1l}_{\rm grav}$

Take spherical background $\bar{g}_{\mu\nu} = g^{(a)}_{\mu\nu}$ (a radius of the sphere)

(coordinates x angles ; $g^{(a)}_{\mu\nu}$ goes like a^2 ; $\int \mathrm{d}^4x \sqrt{g^{(a)}} = \frac{8\pi^2}{3} a^4$, $R(g^{(a)}) = \frac{12}{a^2}$)

Classical Action $\quad S^{(a)}_{\rm grav} = \frac{\pi\Lambda}{3G} a^4 - \frac{2\pi}{G} a^2$

Add to $S_{\rm grav} + S_{\rm gf}$ the corresponding ghost action (v_μ vector ghost fields)

$$S_{\rm ghost} = \frac{1}{32\pi G} \int \mathrm{d}^4x \sqrt{g^{(a)}}\, g^{(a)\,\mu\nu}\, v^*_\mu \left(-\nabla_\rho \nabla^\rho - \frac{3}{a^2}\right) v_\nu$$

Identify 1-loop corrections to $\frac{\Lambda}{G}$ and $\frac{1}{G}$ with coefficients of a^4 and a^2 in $\delta S^{1l}_{\rm grav}$

One-loop VDW Effective Action cont'd

VDW one-loop correction $\delta S^{1l}_{\rm grav}$ to $S^{(a)}_{\rm grav}$ given by

$$e^{-\delta S^{1l}_{\rm grav}} = \lim_{\xi \to 0} \int \left[\mathcal{D}u(h)\mathcal{D}v^*_\rho \, \mathcal{D}v_\sigma\right] e^{-\delta S^{(2)}}$$

where

$$\delta S^{(2)} \equiv S_2 + S_{\rm gf} + S_{\rm ghost}$$

S_2 quadratic term in the expansion of $S_{\rm grav}[g^{(a)}_{\mu\nu} + h_{\mu\nu}]$

$$S_2 \equiv \frac{1}{32\pi G} \int {\rm d}^4x \sqrt{g^{(a)}} \left[\frac{1}{2}\tilde{h}^{\mu\nu}\left(-\nabla_\rho\nabla^\rho - 2\Lambda + \frac{8}{a^2}\right) h_{\mu\nu} + \frac{h^2}{a^2} - \nabla^\rho \tilde{h}_{\rho\mu}\nabla^\sigma \tilde{h}^\mu_\sigma\right]$$

$h \equiv g^{(a)}_{\mu\nu} h^{\mu\nu}$, $\tilde{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2} g^{(a)}_{\mu\nu} h$

indexes raised with $g^{(a)\,\mu\nu}$; covariant derivatives in terms of $g^{(a)}_{\mu\nu}$

Measure $\left[\mathcal{D}u(h)\mathcal{D}v^*_\rho\,\mathcal{D}v_\sigma\right]$

$$\left[\mathcal{D}u(h)\mathcal{D}v^*_\rho\,\mathcal{D}v_\sigma\right] \equiv \prod_x \left[g^{(a)\,00}(x)\;\left(g^{(a)}(x)\right)^{-1} \left(\prod_{\alpha\leq\beta} \mathrm{d}h_{\alpha\beta}(x)\right)\left(\prod_\rho \mathrm{d}v^*_\rho(x)\right)\left(\prod_\sigma \mathrm{d}v_\sigma(x)\right)\right]$$

$g^{(a)\,00}(x)\;\left(g^{(a)}(x)\right)^{-1}$ from integration over conjugate momenta[1] (Fradkin - Vilkovisky)

Observe: $g^{(a)}_{\mu\nu}$ can be written as $g^{(a)}_{\mu\nu} = a^2 g^{(1)}_{\mu\nu}$

$g^{(1)}_{\mu\nu}$ metric of a sphere of unitary radius, $a = 1$

$$\Longrightarrow \qquad g^{(a)\,00}(x)\;\left(g^{(a)}(x)\right)^{-1} = a^{-10}\, g^{(1)\,00}(x)\;\left(g^{(1)}(x)\right)^{-1}$$

with $g^{(1)\,00}(x)\;\left(g^{(1)}(x)\right)^{-1}$ a-independent

[1] original expression in FV is $g^{(a)\,00}(x)\;\left(g^{(a)}(x)\right)^{-\frac{3}{2}}$. Difference due to the fact that here both v and v^* are world vectors, in FV different choice. $\sqrt{g^{(a)}}$ Jacobian due to the change between these two equivalent functional integration variables (Unz).

Reabsorb $G^{-1/2}a^{-1}$ in $h_{\mu\nu}$ $\Longrightarrow$ $\widehat{h}_{\mu\nu} = (32\pi G)^{-1/2}\, a^{-1} h_{\mu\nu}$

$S_2 + S_{\rm gf}$ rewritten as

$$S_2 + S_{\rm gf} = \int \mathrm{d}^4x\, \sqrt{g^{(1)}}\left[\frac{1}{2}\bar{h}^{\mu\nu}\left(-\nabla_\rho\nabla^\rho - 2a^2\Lambda + 8\right)\widehat{h}_{\mu\nu} + \widehat{h}^2 - \left(1 - \frac{1}{\xi}\right)\nabla^\rho\bar{h}_{\rho\mu}\nabla^\sigma\bar{h}^\mu_\sigma\right]$$

$\widehat{h} \equiv g^{(1)}_{\mu\nu}\,\widehat{h}^{\mu\nu}$, $\bar{h}_{\mu\nu} \equiv \widehat{h}_{\mu\nu} - \frac{1}{2} g^{(1)}_{\mu\nu}\,\widehat{h}$

indexes raised with $g^{(1)\,\mu\nu}$; covariant derivatives in terms of $g^{(1)}_{\mu\nu}$

Clearly $\widehat{h}_{\mu\nu}$ defined on a sphere of unitary radius

Redefine $v_\mu \to (32\pi G)^{\frac{1}{2}}\, v_\mu$ (covariant derivatives in terms of $g^{(1)}_{\mu\nu}$)

$$S_{\rm ghost} = \int \mathrm{d}^4x\, \sqrt{g^{(1)}}\, g^{(1)\,\mu\nu}\, v^*_\mu\left(-\nabla_\rho\nabla^\rho - 3\right)v_\nu$$

Same as $\widehat{h}_{\mu\nu}$: v_μ defined on a sphere of unitary radius

$\Longrightarrow$ when written in terms of $\widehat{h}_{\mu\nu}$ and v_μ, $\delta S^{(2)} = S_2 + S_{\rm gf} + S_{\rm ghost}$ contains only **dimensionless fluctuation operators** ...

… and then …

$$\left[\mathcal{D}u(h)\mathcal{D}v_\rho^*\,\mathcal{D}v_\sigma\right] \equiv \prod_x \left[a^{-10} g^{(1)\,00}(x)\ \left(g^{(1)}(x)\right)^{-1} \left(\prod_{\alpha\leq\beta} \mathrm{d}h_{\alpha\beta}(x)\right)\left(\prod_\rho \mathrm{d}v_\rho^*(x)\right)\left(\prod_\sigma \mathrm{d}v_\sigma(x)\right)\right]$$

$$\oplus$$

$$\widehat{h}_{\mu\nu} = (32\pi G)^{-1/2}\, a^{-1} h_{\mu\nu} \quad\Longrightarrow\quad \prod_{\alpha\leq\beta} \mathrm{d}h_{\alpha\beta}(x) = (32\pi G)^5\, a^{10} \prod_{\alpha\leq\beta} \mathrm{d}\widehat{h}_{\alpha\beta}(x)$$

$$\Downarrow$$

$$\left[\mathcal{D}u(h)\mathcal{D}v_\rho^*\,\mathcal{D}v_\sigma\right] = \mathcal{N} \prod_x \left[\left(\prod_{\alpha\leq\beta} \mathrm{d}\widehat{h}_{\alpha\beta}(x)\right)\left(\prod_\rho \mathrm{d}v_\rho^*(x)\right)\left(\prod_\sigma \mathrm{d}v_\sigma(x)\right)\right]$$

a-independent terms as $\prod_x g^{(1)\,00}(x)\ \left(g^{(1)}(x)\right)^{-1}$ included in harmless constant $\mathcal{N}$

Since $\widehat{h}_{\mu\nu}$ and v_μ fields on a sphere of radius $a = 1 \Longrightarrow$

bases for symmetric tensors and vectors with eigenfunctions of the

Dimensionless Laplace-Beltrami operator $-\Box^{(s)}_{a=1} \equiv -a^2\,\Box^{(s)}_a$

$-\Box^{(s)}_a$ Laplace-Beltrami for sphere of radius a; s spins: $s = 0, 1, 2$

Dimensionless eigenvalues $\lambda^{(s)}_n$ and corresponding degeneracies $D^{(s)}_n$

$$\lambda^{(s)}_n = n^2 + 3n - s \qquad ; \qquad D^{(s)}_n = \frac{2s+1}{3}\left(n + \frac{3}{2}\right)^3 - \frac{(2s+1)^3}{12}\left(n + \frac{3}{2}\right)$$

where $n = s, s+1, \ldots$

Expanding $\widehat{h}_{\mu\nu}$, v^*_ρ and v_σ for $\delta S^{1l}_{\rm grav}$ we have (backup slides)

$$\delta S^{1l}_{\rm grav} = -\frac{1}{2}\log\frac{\det_1[-\Box^{(1)}_{a=1} - 3]\,\det_2[-\Box^{(0)}_{a=1} - 6]}{\det_0[-\Box^{(2)}_{a=1} - 2a^2\Lambda + 8]\,\det_2[-\Box^{(0)}_{a=1} - 2a^2\Lambda]} + \frac{1}{2}\log\left(2a^2\Lambda\right) + \mathcal{B}$$

$\mathcal{B}$ inessential a-independent term ; index i in $\det_i$ signals that product of eigenvalues starts from $\lambda^{(s)}_{\rm s+i}$

$$\delta S^{1l}_{\rm grav} = -\frac{1}{2}\log\frac{\det_1[-\Box^{(1)}_{a=1} - 3]\,\det_2[-\Box^{(0)}_{a=1} - 6]}{\det_0[-\Box^{(2)}_{a=1} - 2a^2\Lambda + 8]\,\det_2[-\Box^{(0)}_{a=1} - 2a^2\Lambda]} + \frac{1}{2}\log\left(2a^2\Lambda\right) + \mathcal{B}$$

$\frac{1}{2}\log\left(2a^2\Lambda\right)$ (from integration over one of the modes in $\widehat{h}_{\mu\nu}$ (backup slides)) and $\mathcal{B}$: irrelevant for our scopes

Truly important term: first one in the right hand side

* Peculiarity: written in terms of **automatically dimensionless determinants** $\Longrightarrow$

No need to introduce any arbitrary mass scale (μ)

In typical calculations of $\delta S^{1l}_{\rm grav}$: dimensionful determinants $\Longrightarrow$ arbitrary μ introduced

* Although calculation is performed for a **sphere of radius *a***, Laplace-Beltrami operators are those for sphere of unitary radius

* ***a*** only comes in the combination $\boldsymbol{a^2\Lambda}$

Calculation of the fluctuation determinants

Two different strategies

1 - direct calculation in terms of eigenvalues of Laplace-Beltrami ops.

2 - proper-time, as usually done

Anticipating: both calculations show that quartically and quadratically divergent contributions to the vacuum energy usually present in the literature are actually absent

1 - Calculation in terms of the eigenvalues $\lambda_n^{(s)}$

$$\delta S^{1l}_{\rm grav} = \frac{1}{2}\sum_{n=2}^{N-2}\Big[D_n^{(2)}\log\left(\lambda_n^{(2)} - 2a^2\Lambda + 8\right) + D_n^{(0)}\log\left(\lambda_n^{(0)} - 2a^2\Lambda\right)$$
$$-D_n^{(1)}\log\left(\lambda_n^{(1)} - 3\right) - D_n^{(0)}\log\left(\lambda_n^{(0)} - 6\right)\Big] + \frac{1}{2}\log\left(2a^2\Lambda\right) + \mathcal{B}$$

* UV cutoff introduced as a numerical cut $^{(*)}$ N on the number of eigenvalues ($N-2$ rather than N simplifies the expression)

* De Sitter solution for the classical action

$$a_{\rm dS} = \sqrt{\frac{3}{\Lambda_{\rm cc}}}$$

$a_{\rm dS}$ size of the universe $\Longrightarrow$ connection between N and physical cutoff scale $\Lambda_{\rm cut} \sim M_P$ given by

$$\Lambda_{\rm cut} \sim M_P = \frac{N}{a_{\rm dS}} = N\sqrt{\frac{\Lambda_{\rm cc}}{3}}$$

$^{(*)}$ Numerical cut also introduced in Becker, Reuter, Phys.Rev.D 102 (2020) 12, 125001 and Phys.Rev.D 104 (2021) 12, 125008 ;

Ferrero, Percacci, e-Print: 2404.12357, but: different perspective, different conclusions

Calculation in terms of the $\lambda_n^{(s)}$ cont'd

* $N-2$: number of modes retained in the calculation of the determinants

* Since the eigenvalues $\widetilde{\lambda}_n^{(s)}$ of $-\Box_a^{(s)}$ go like $\widetilde{\lambda}_n^{(s)} \equiv \frac{\lambda_n^{(s)}}{a^2} \sim \frac{n^2}{a^2}$, the requirement $n \leq N-2$ **is not equivalent** to require $\widetilde{\lambda}_n^{(s)} \leq \Lambda_{\rm cut}^2$

* This latter choice might seem natural, since it would amount to require that the maximal eigenvalue $\widetilde{\lambda}_{\rm max}^{(s)}$ is $\widetilde{\lambda}_{\rm max}^{(s)} \sim \Lambda_{\rm cut}^2$

* But misleading reasoning: since $\widetilde{\lambda}_n^{(s)}$ go like $a^{-2} \implies$ introduction of **unphysical a-dependence** (on the BG $g_{\mu\nu}^{(a)}$) in the implementation of the cutoff

Fundamental observation to get the correct result for $\delta S_{\rm grav}^{1l}$, and to see that there are

No quartic and quadratic divergences in the vacuum energy

Calculation in terms of the $\lambda_n^{(s)}$ cont'd

Remarkably, sum in $\delta S^{1l}_{\rm grav}$ obtained in closed form (backup slides)

Expanding for $N \gg 1$

$$\delta S^{1l}_{\rm grav} = -\left(\Lambda_{\rm cc}^2 \log N^2\right) a^4 + \Lambda_{\rm cc}\left(-N^2 + 8\log N^2\right) a^2 + \frac{N^4}{24}\left(-1+2\log N^2\right) + \frac{N^2}{36}\left(203 - 75\log N^2\right) - \frac{779}{90}\log N^2 + \mathcal{B} + \frac{1}{2}\log\left(2a^2\Lambda_{\rm cc}\right) + \mathcal{F}(a^2\Lambda_{\rm cc}) + \mathcal{O}\left(N^{-2}\right)$$

where $\mathcal{F}(a^2\Lambda)$ contains only UV-finite terms (no dependence on N)

Using $\Lambda_{\rm cut} \sim M_P = \frac{N}{a_{\rm dS}} = N\sqrt{\frac{\Lambda_{\rm cc}}{3}}$

$$\delta S^{1l}_{\rm grav} = -\left(\Lambda_{\rm cc}^2 \log\frac{3\Lambda_{\rm cut}^2}{\Lambda_{\rm cc}}\right) a^4 + \left(-3\Lambda_{\rm cut}^2 + 8\Lambda_{\rm cc}\log\frac{3\Lambda_{\rm cut}^2}{\Lambda_{\rm cc}}\right) a^2 + \frac{3\Lambda_{\rm cut}^4}{8\Lambda_{\rm cc}^2}\left(-1+2\log\frac{3\Lambda_{\rm cut}^2}{\Lambda_{\rm cc}}\right) + \frac{\Lambda_{\rm cut}^2}{12\Lambda_{\rm cc}}\left(203-75\log\frac{3\Lambda_{\rm cut}^2}{\Lambda_{\rm cc}}\right) - \frac{779}{90}\log\frac{3\Lambda_{\rm cut}^2}{\Lambda_{\rm cc}} + \mathcal{B} + \frac{1}{2}\log\left(2a^2\Lambda_{\rm cc}\right) + \mathcal{F}(a^2\Lambda_{\rm cc}) + \mathcal{O}\left(\Lambda_{\rm cut}^{-2}\right)$$

2 - Calculation with proper-time

Being $(-\Box^{(s)}_{a=1} - \alpha)$ dimensionless $\Longrightarrow$ determinants regularized in terms of a **dimensionless** proper-time τ (lower cut: **number** $N_{\rm pt} \gg 1$)

$$\det_i(-\Box^{(s)}_{a=1} - \alpha) = e^{-\int_{1/N_{\rm pt}^2}^{+\infty} \frac{d\tau}{\tau} \mathrm{K}_i^{(s)}(\tau)} \quad ; \quad \mathrm{K}_i^{(s)}(\tau) = \sum_{n=s+i}^{+\infty} D_n^{(s)}\, e^{-\tau\left(\lambda_n^{(s)} - \alpha\right)}$$

After integration over τ, sum over n performed with EML sum formula

$$\sum_{n=n_i}^{n_f} f(n) = \int_{n_i}^{n_f} dx\, f(x) + \frac{f(n_f) + f(n_i)}{2} + \sum_{k=1}^{p} \frac{B_{2k}}{(2k)!}\left(f^{(2k-1)}(n_f) - f^{(2k-1)}(n_i)\right) + R_{2p}$$

p is an integer, B_m are Bernoulli numbers, R_{2p} is the rest given by

$$R_{2p} = \sum_{k=p+1}^{\infty} \frac{B_{2k}}{(2k)!}\left(f^{(2k-1)}(n_f) - f^{(2k-1)}(n_i)\right) = \frac{(-1)^{2p+1}}{(2p)!} \int_{n_i}^{n_f} dx\, f^{(2p)}(x) B_{2p}(x - [x])$$

$B_n(x)$ are the Bernoulli polynomials, $[x]$ the integer part of x, and $f^{(i)}$ the i-th derivative of f with respect to its argument

Calculation with proper-time cont'd

Expanding for $N_{\rm pt} \gg 1$

$$\begin{aligned}\delta S^{1l}_{\rm grav} = &-\left(\Lambda_{\rm cc}^2 \log N_{\rm pt}^2\right) a^4 + \Lambda_{\rm cc}\left(-N_{\rm pt}^2 + 8\log N_{\rm pt}^2\right) a^2 \\ &- \frac{N_{\rm pt}^4}{12} + \frac{17}{3} N_{\rm pt}^2 - \frac{1859}{90}\log N_{\rm pt}^2 + \mathcal{B} \\ &+ \frac{1}{2}\log\left(2a^2\Lambda_{\rm cc}\right) + \mathcal{G}(a^2\Lambda_{\rm cc}) + \mathcal{O}\left(N_{\rm pt}^{-2}\right)\end{aligned}$$

$\mathcal{G}(a^2\Lambda)$ contains UV-finite terms (no dependence on $N_{\rm pt}$). As before, connection between $N_{\rm pt}$ and dimensionful cutoff $\Lambda_{\rm pt}$ given by

$$\Lambda_{\rm pt} \equiv \frac{N_{\rm pt}}{a_{\rm dS}} = \sqrt{\frac{\Lambda_{\rm cc}}{3}}\, N_{\rm pt} \qquad \Longrightarrow$$

$$\begin{aligned}\delta S^{1l}_{\rm grav} = &-\left(\Lambda_{\rm cc}^2 \log \frac{3\Lambda_{\rm pt}^2}{\Lambda_{\rm cc}}\right) a^4 + \left(-3\Lambda_{\rm pt}^2 + 8\Lambda_{\rm cc}\log\frac{3\Lambda_{\rm pt}^2}{\Lambda_{\rm cc}}\right) a^2 \\ &- \frac{3\Lambda_{\rm pt}^4}{4\Lambda_{\rm cc}^2} + \frac{17\Lambda_{\rm pt}^2}{\Lambda_{\rm cc}} - \frac{1859}{90}\log\frac{3\Lambda_{\rm pt}^2}{\Lambda_{\rm cc}} + \mathcal{B} \\ &+ \frac{1}{2}\log\left(2a^2\Lambda_{\rm cc}\right) + \mathcal{G}(a^2\Lambda_{\rm cc}) + \mathcal{O}\left(\Lambda_{\rm pt}^{-2}\right)\end{aligned}$$

The two methods give the same result

... to be compared with ...

Nuclear Physics B234 (1984) 509–523

ON THE NEW DEFINITION OF OFF-SHELL EFFECTIVE ACTION

E.S. FRADKIN and A.A. TSEYTLIN

Department of Theoretical Physics, P.N. Lebedev Physical Institute, USSR Academy of Sciences, Leninsky Pr. 53, Moscow 117924, USSR

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...

$$\Gamma_\infty = -\tfrac{1}{2}\left(\tfrac{1}{2}L^4 B_0 + L^2 B_2 + B_4 \log\left(L^2/\mu^2\right)\right), \qquad L \to \infty,$$

$$B_p = \frac{1}{(4\pi)^2}\int \mathrm{d}^4x\sqrt{g}\, b_p, \qquad b_0 = 2, \qquad b_2 = \rho_1 R + \rho_2 \Lambda_0,$$

Nuclear Physics B345 (1990) 210–230
North-Holland

QUANTUM GRAVITY AT LARGE DISTANCES AND THE COSMOLOGICAL CONSTANT

T.R. TAYLOR

Fermi National Accelerator Laboratory, PO Box 500, Batavia, IL 60510, USA
and
*Department of Physics, Northeastern University, Boston, MA 02115, USA**

G. VENEZIANO

CERN, 1211 Geneva 23, Switzerland

Received 8 December 1989

$$\tilde{\Gamma}_1 = r^4\left[-\tfrac{1}{12}(\alpha')^{-2} - \Lambda(\alpha')^{-1} + \Lambda^2 \log(\alpha' M^2)\right] + r^2\left[\tfrac{17}{3}(\alpha')^{-1} - 8\Lambda \log(\alpha' M^2)\right] + \tfrac{1679}{90}\log(\alpha'/r^2) + \tfrac{257}{15}\log(M^2 r^2) + \tfrac{1}{2}\log(r^2\Lambda) + \mathcal{O}(1)\,, \tag{3.35}$$

quartically divergent contribution to the cosmological constant {of order $\mathcal{O}[(\alpha')^{-2}]$} has to be disregarded in a theory where there is a protection (e.g. from broken supersymmetry) of the cosmological constant making $\Lambda \ll (\alpha')^{-1}$.

Coefficients of a^4 and a^2 identify the one-loop corrections to $\frac{\Lambda_{cc}}{G}$ and $\frac{1}{G}$

$$\frac{\Lambda_{cc}^{1l}}{G^{1l}} = \frac{\Lambda_{cc}}{G}\left(1 - \frac{3G\Lambda_{cc}}{\pi}\log\frac{3\widetilde{\Lambda}^2}{\Lambda_{cc}}\right) + \text{finite}$$

$$\frac{1}{G^{1l}} = \frac{1}{G}\left[1 + \frac{G}{2\pi}\left(3\widetilde{\Lambda}^2 - 8\Lambda_{cc}\log\frac{3\widetilde{\Lambda}^2}{\Lambda_{cc}}\right)\right] + \text{finite}$$

* $\widetilde{\Lambda}$ is equivalently either Λ_{cut} or Λ_{pt} $\quad(\sim M_P)$

* **Unexpected result:** *only* logarithmic corrections to $\rho = \frac{\Lambda_{cc}}{8\pi G}$

* Taking for G the natural value $G \sim M_P^{-2}$ we see that quantum corrections **do not spoil the naturalness** of this relation

No naturalness problem with the renormaliz. of the Newton constant

$$G \sim G^{1l} \sim \frac{1}{M_P^2}$$

Vacuum energy

$$\frac{\Lambda_{cc}^{1l}}{G^{1l}} = \frac{\Lambda_{cc}}{G}\left(1 - \frac{3G\Lambda_{cc}}{\pi}\log\frac{3\widetilde{\Lambda}^2}{\Lambda_{cc}}\right)$$

Quantum correction to the vacuum energy $\rho = \frac{\Lambda_{cc}}{8\pi G}$ goes like

$\log M_P$ rather than M_P^4

* Usual result: $\rho \sim M_P^4 \quad \Longrightarrow \quad$ bare value of $\rho \sim M_P^4$ with a coefficient that must be enormously fine-tuned for it to cancel (quite exactly) the one-loop generated M_P^4 correction

* Our result: loop corrections $\rightarrow$ only mild (log) correction to $\rho \quad \Longrightarrow$

In **pure gravity no naturalness problem arises**: the bare cosmological constant Λ_{cc} does not need to be $\sim M_P^2$. We may **naturally** have $\Lambda_{cc} \ll M_P^2$, and so

$$\boldsymbol{\Lambda_{cc}^{1l} \sim \Lambda_{cc}}$$

Incorrect identification of the cutoff

What is at the origin of our unexpected result?

Why usually quartic$^{(*)}$ and quadratic divergences found?

$$\delta S^{1l}_{\text{grav}} = -\left(\Lambda^2_{\text{cc}} \log N^2_{\text{pt}}\right) a^4 + \Lambda_{\text{cc}}\left(-N^2_{\text{pt}} + 8\log N^2_{\text{pt}}\right) a^2$$
$$-\frac{N^4_{\text{pt}}}{12} + \frac{17}{3} N^2_{\text{pt}} - \frac{1859}{90}\log N^2_{\text{pt}}$$

Connect for a moment N_{pt} and Λ_{pt} through

$$\Lambda_{\text{pt}} = \frac{N_{\text{pt}}}{a} \quad \left(\text{rather than through } \Lambda_{\text{pt}} = \frac{N_{\text{pt}}}{a_{\text{dS}}}\right)$$

which corresponds to the (incorrect) identification of Λ_{pt} with the maximal eigenvalue $\widetilde{\lambda}^{(s)}_{\text{max}}$ ... then for $\delta S^{1l}_{\text{grav}}$ we obtain

$$\delta S^{1l}_{\text{grav}} = -\left[\Lambda^2_{\text{cc}} \log\left(\Lambda^2_{\text{pt}}\, a^2\right)\right] a^4 + \Lambda_{\text{cc}}\left[-\Lambda^2_{\text{pt}}\, a^2 + 8\log\left(\Lambda^2_{\text{pt}}\, a^2\right)\right] a^2$$
$$-\frac{\Lambda^4_{\text{pt}}}{12}\, a^4 + \frac{17}{3}\Lambda^2_{\text{pt}}\, a^2 - \frac{1859}{90}\log\left(\Lambda^2_{\text{pt}}\, a^2\right)$$

$^{(*)}$ absence of quartic divergences also noted in Donoghue, Phys.Rev.D 104 (2021) 4, 045005

Trivially rewritten as

$$\delta S^{1l}_{\rm grav} = -\left[\frac{\Lambda^4_{\rm pt}}{12} + \Lambda_{\rm cc}\Lambda^2_{\rm pt} + \Lambda^2_{\rm cc}\log\left(\Lambda^2_{\rm pt}\, a^2\right)\right] a^4 + \left[\frac{17}{3}\Lambda^2_{\rm pt} + 8\Lambda_{\rm cc}\log\left(\Lambda^2_{\rm pt}\, a^2\right)\right] a^2 - \frac{1859}{90}\log\left(\Lambda^2_{\rm pt}\, a^2\right) .$$

known result found with heat-kernel (Taylor, Veneziano ; Fradkin, Tseytlin)

* Implementing the cut in the fluctuation determinants taking as physical cutoff the maximal eigenvalues $\widetilde{\lambda}^{(s)}_{\rm max}$ introduces in $\delta S^{1l}_{\rm grav}$ spurious, unphysical dependence on $g^{(a)}_{\mu\nu}$

* Connection between $N_{\rm pt}$ and $\Lambda_{\rm pt}$ must be realised through $a_{\rm dS}$ (size of the universe)

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quartically divergent contribution to the cosmological constant {of order $\mathcal{O}[(\alpha')^{-2}]$} has to be disregarded in a theory where there is a protection (e.g. from broken supersymmetry) of the cosmological constant making $\Lambda \ll (\alpha')^{-1}$.

Matter contribution

Free real scalar field ϕ of mass m on the grav. background $g^{(a)}_{\mu\nu}$ (sphere of radius a)

$$S = \frac{\pi\Lambda}{3G}a^4 - \frac{2\pi}{G}a^2 + \int d^4x\,\sqrt{g^{(a)}}\left[\frac{1}{2}\,g^{(a)\,\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{1}{2}\,m^2\phi^2\right]$$

* Write $\phi(x) = \Phi + \eta(x)$, Φ constant background. Effective gravitational action $S^{\rm eff}_{\rm grav}$ with quantum fluctuations of ϕ included

$$S^{\rm eff}_{\rm grav} = \frac{\pi\Lambda}{3G}a^4 - \frac{2\pi}{G}a^2 + \delta S_{\rm grav}$$

with $\delta S_{\rm grav}$ given by

$$e^{-\delta S_{\rm grav}} = \int\prod_x\left[\left(g^{(a)\,00}(x)\right)^{\frac{1}{2}}\left(g^{(a)}(x)\right)^{\frac{1}{4}}d\eta(x)\right]e^{-\int d^4x\,\sqrt{g^{(a)}}\left[-\frac{1}{2}\eta\Box\eta+\frac{1}{2}m^2\eta^2\right]}$$

* As before $\left(g^{(a)\,00}(x)\right)^{\frac{1}{2}}\left(g^{(a)}(x)\right)^{\frac{1}{4}}$ from integration over conjugate momenta (Fradkin, Vilkovisky). Since $g^{(a)}_{\mu\nu} = a^2 g^{(1)}_{\mu\nu} \implies$

$\left(g^{(a)\,00}(x)\right)^{\frac{1}{2}}\left(g^{(a)}(x)\right)^{\frac{1}{4}} = a\left(g^{(1)\,00}(x)\right)^{\frac{1}{2}}\left(g^{(1)}(x)\right)^{\frac{1}{4}}$, no dimensionful parameter in

$$e^{-\delta S_{\rm grav}} = \mathcal{N}\int\prod_x\left[d\,\widehat{\eta}(x)\right]e^{-\int d^4x\,\sqrt{g^{(1)}}\left[-\frac{1}{2}\,\widehat{\eta}\left(\Box^{(0)}_{a=1}\right)\widehat{\eta}+\frac{1}{2}a^2m^2\,\widehat{\eta}^{\,2}\right]}$$

Matter contribution cont'd

* $-\Box^{(0)}_{a=1}$: Laplace-Beltrami operator for sphere of unitary radius

* $\widehat{\eta} \equiv a\eta$: dimensionless fluctuation field

* $\mathcal{N}$: inessential a-independent constant

Expanding $\widehat{\eta}(x)$ in terms of the eigenfunctions[2] $\phi^{(i)}_n(x)$ (i degeneracy index and $n = 0, 1, \dots$) of $-\Box^{(0)}_{a=1}$: $\widehat{\eta} = \sum_{n,i} a^{(i)}_n \phi^{(i)}_n$

$$e^{-\delta S_{\rm grav}} = \mathcal{N} \int \prod_{n,i} \mathrm{d}a^{(i)}_n \, e^{-\frac{1}{2}\sum_{n,i}\left[a^{(i)}_n\right]^2\left(\lambda^{(0)}_n + a^2 m^2\right)}$$

and then ($\mathcal{C}$ inessential a-independent constant)

$$S^{\rm eff}_{\rm grav} = \frac{\pi\Lambda}{3G}a^4 - \frac{2\pi}{G}a^2 + \frac{1}{2}\log\left[\det\left(-\Box^{(0)}_{a=1} + a^2 m^2\right)\right] + \mathcal{C}\,.$$

[2]The $\phi^{(i)}_n$ are normalized as $\int \mathrm{d}^4x \sqrt{g^{(1)}}\, \phi^{(i)}_n(x)\, \phi^{(j)}_m(x) = \delta^{ij}\delta_{nm}$.

Matter contribution cont'd

* Calculate determinant with direct product of $\lambda_n^{(0)}$ up to $n = N - 2$ as before, and expand for $N \gg 1$

$$S_{\text{grav}}^{\text{eff}} = \frac{\pi}{3}\left(\frac{\Lambda}{G} - \frac{m^4}{8\pi}\log N^2\right)a^4 - 2\pi\left[\frac{1}{G} - \frac{m^2}{24\pi}\left(N^2 + 2\log N^2\right)\right]a^2$$
$$+ \frac{N^4}{48}\left(-1 + 2\log N^2\right) - \frac{N^2}{72}\left(13 + 3\log N^2\right) - \frac{29}{180}\log N^2 + \mathcal{C}$$
$$+ \mathcal{H}(a^2 m^2) + \mathcal{O}\left(N^{-2}\right)$$

* Similarity with the result obtained in the pure gravity case : evident

* Consider the vacuum energy term. Once again : if N *correctly* related to $\Lambda_{\text{cut}} \sim M_P$ through $\Lambda_{\text{cut}} \sim M_P = \frac{N}{a_{\text{dS}}} = N\sqrt{\frac{\Lambda_{\text{cc}}}{3}} \implies \rho = \frac{\Lambda_{\text{cc}}}{8\pi G}$ receives only

mild logarithmically divergent correction

$$\delta\left(\frac{\Lambda_{\text{cc}}}{G}\right) = -\frac{m^4}{8\pi}\log\frac{3\Lambda_{\text{cut}}^2}{\Lambda_{\text{cc}}}$$

... But ...

If again we perform the *incorrect* replacement of N as $N = a\,\Lambda_{\rm cut}$
$\Longrightarrow$ spurious quartically and quadratically divergent terms generated

For instance:

$$-\frac{N^4}{48} \longrightarrow -\frac{\Lambda_{\rm cut}^4}{48}\,a^4$$

Quartically divergent contribution to $\frac{\Lambda_{\rm cc}}{G}$

$$\frac{N^2}{12}\,m^2\,a^2 \longrightarrow \frac{\Lambda_{\rm cut}^2}{12}\,m^2\,a^4$$

Quadratically divergent contribution to $\frac{\Lambda_{\rm cc}}{G}$

Conclusions

The **absence** of quartic and quadratic divergences in

our result for the vacuum energy

even when the presence of matter is taken into account

possibly a progress towards the solution of the CC problem

Naturally

the question of how to dispose of the terms $m^4 \log \Lambda_{\rm cut}$

needs to be further investigated

maybe along the lines put forward in the present work

THANKS FOR YOUR ATTENTION

ADDITIONAL SLIDES

Comparison with Becker-Reuter, PRD 102 (2020) 12 - PRD 104 (2021) 12 and Ferrero-Percacci, e-Print: 2404.12357

Consider the **modified Einstein equation** at one-loop (Becker, Reuter PRD 104 (2021) 12)

$$\frac{3G}{\pi}\, a \frac{\mathrm{d}}{\mathrm{d}a}\Gamma^{1l}_{\text{grav}}(a) = 4\Lambda_{\text{cc}} a^4 - 12a^2 + \frac{3G}{\pi}\, a\frac{\mathrm{d}}{\mathrm{d}a}\delta S^{1l}_{\text{grav}}(a) = 0 \qquad (g_{\mu\nu} = g^{(a)}_{\mu\nu})$$

* **BRST-invariant Fujikawa measure** (or measure not fully considered)

$$\left[\mathcal{D}u(h)\mathcal{D}v^*_\rho\ \mathcal{D}v_\sigma\right] \equiv \prod_x \Big[\underbrace{\left(g^{(a)}(x)\right)^{-2}}_{\propto\, a^{-16}} \Big(\prod_{\alpha\leq\beta} \mathrm{d}h_{\alpha\beta}(x)\Big)\Big(\prod_\rho \mathrm{d}v^*_\rho(x)\Big)\Big(\prod_\sigma \mathrm{d}v_\sigma(x)\Big)\Big]$$

$\Longrightarrow \quad \delta S^{1l}_{\text{grav}}(a) \sim N^4 \log(a\mu) \quad \Longrightarrow \quad$ **solution** $\bar{a}_N \sim N\sqrt{M_P^{-1}\Lambda_{\text{cc}}^{-1}}$

* **Fradkin-Vilkovisky measure**

$$\left[\mathcal{D}u(h)\mathcal{D}v^*_\rho\ \mathcal{D}v_\sigma\right] \equiv \prod_x \Big[\underbrace{g^{(a)\,00}(x)\ \left(g^{(a)}(x)\right)^{-1}}_{\propto\, a^{-10}} \Big(\prod_{\alpha\leq\beta} \mathrm{d}h_{\alpha\beta}(x)\Big)\Big(\prod_\rho \mathrm{d}v^*_\rho(x)\Big)\Big(\prod_\sigma \mathrm{d}v_\sigma(x)\Big)\Big]$$

$\Longrightarrow \quad \delta S^{1l}_{\text{grav}}(a)$ does **not** contain $N^4 \log(a\mu) \quad \Longrightarrow \quad$ **NO** $\bar{a}_N$ **solution**

Similar considerations apply to the results found in Becker-Reuter, PRD 102 (2020) 12, and Ferrero-Percacci, e-Print: 2404.12357

Why Fradkin-Vilkovisky measure?

Free real, massless scalar field ϕ in grav. background $g_{\mu\nu} = (1 + h(x))\,\eta_{\mu\nu}$

$$\sqrt{-g}\mathcal{L} = \frac{1}{2}\,(1 + h(x))\,\partial_\mu\phi\,\partial^\mu\phi$$

(Donoghue, PRD 104 (2021) 4)

Of the kind

$$\mathcal{L} = \frac{1}{2}\left[\delta^{ab} + \bar{G}^{ab}(\pi)\right]\partial_\mu\pi^a\,\partial^\mu\pi^b$$

(Gerstein, Jackiw, Lee and Weinberg, "Chiral loops", PRD 3, 2486 (1971))

*** Canonical quantization** (using GJLW result)

1. $\mathcal{H}_{\rm I} = -\frac{1}{2}h(x)\partial_\mu\tilde{\phi}\,\partial^\mu\tilde{\phi} + \frac{1}{2}\partial_0\tilde{\phi}\left[\frac{h^2}{1+h}\right]\partial_0\tilde{\phi}\;\oplus$ non-std prop. $\Delta_{\mu\nu}(q) = \frac{iq_\mu q_\nu}{q^2 + i\epsilon} - i\eta_{\mu 0}\eta_{\nu 0}$

2. $\mathcal{H}_{\rm I} = -\frac{1}{2}h(x)\partial_\mu\tilde{\phi}\,\partial^\mu\tilde{\phi} + \frac{i}{2}\delta^{(4)}(0)\,\log\left(1 + h(x)\right)\;\oplus$ std Feynman rules

*** Functional methods** (Honerkamp and Meetz, "Chiral-invariant perturbation theory", PRD 3, 1996 (1971))

path integral measure from integration over conjugate momenta

$$\mathcal{D}u(\phi) = \prod_x (1 + h(x))^{1/2}\,\mathrm{d}\phi(x) \quad\Longleftrightarrow\quad \Delta\mathcal{L} = -\frac{i}{2}\delta^{(4)}(0)\,\log\left(1 + h(x)\right)$$

same as that in canonical quantization

Why Fradkin-Vilkovisky measure? Cont'd

For generic metric $g_{\mu\nu}$ (Donoghue, PRD 104 (2021) 4)

$$\mathcal{D}u(\phi) = \prod_x \left(-\det g_{\mu\nu}(x)\right)^{1/8} \mathrm{d}\phi(x)$$

Measure we use (from integration over conjugate momenta) (Fradkin-Vilkovisky, Unz)

$$\mathcal{D}u(\phi) = \prod_x \left[\left(g^{00}(x)\right)^{1/2} \left(-\det g_{\mu\nu}(x)\right)^{1/4} \mathrm{d}\phi(x) \right]$$

Fujikawa measure

$$\mathcal{D}u(\phi) = \prod_x \left[\left(-\det g_{\mu\nu}(x)\right)^{1/4} \mathrm{d}\phi(x) \right]$$

* $g_{\mu\nu} = (1 + h(x))\,\eta_{\mu\nu} \Rightarrow \left(-\det g_{\mu\nu}(x)\right)^{1/8} = \left(g^{00}(x)\right)^{1/2} \left(-\det g_{\mu\nu}(x)\right)^{1/4} = (1+h(x))^{1/2}$

$\left(-\det g_{\mu\nu}(x)\right)^{1/4} = 1 + h(x)$ **DIFFERENT!**

* Sphere $g_{\mu\nu} = g^{(a)}_{\mu\nu} \implies \left(-\det g^{(a)}_{\mu\nu}(x)\right)^{1/8}$ and $\left(g^{(a)\,00}(x)\right)^{1/2} \left(-\det g^{(a)}_{\mu\nu}(x)\right)^{1/4}$ both $\sim a$

$\left(-\det g_{\mu\nu}(x)\right)^{1/4} \sim a^2$

Additional comments

$\frac{1}{2}\log\left(2a^2\Lambda_{\rm cc}\right)$ and $\mathcal{G}(a^2\Lambda_{\rm cc})$ are negligible $\mathcal{O}(1)$ contributions to $\delta S^{1l}_{\rm grav}$

The constant terms (proportional to a^0) in principle could be interpreted as corrections to $\int \mathrm{d}^4x\,\sqrt{g}\,R^2$ rather then as constants to be discarded

... Due to the high symmetry of the background considered (sphere), it is impossible to distinguish between constant terms and corrections to R^2

... since our universe seems to be well described by the Einstein-Hilbert action (with cosmological constant) even at large energy scales, we rather expect these terms to be interpreted as inessential constants ...

This question should be further investigated ...

Expansion of $\widehat{h}_{\mu\nu}$, v^*_ρ and v_σ

We indicate with $h_n^{\mu\nu(i)}$ (transverse-traceless), $\xi_n^{\mu(i)}$ (transverse) and $\phi_n^{(i)}$ the pure spin-2, spin-1 and spin-0 eigenfunctions of the Laplace-Beltrami operator on the sphere of unitary radius that are normalized as

$$\delta^{ij}\delta_{nm} = \int \mathrm{d}^4x\,\sqrt{g^{(1)}}\,h_n^{\mu\nu(i)}(x)h_{\mu\nu}^{m(j)}(x) = \int \mathrm{d}^4x\,\sqrt{g^{(1)}}\,\xi_n^{\mu(i)}(x)\xi_\mu^{m(j)}(x) = \int \mathrm{d}^4x\,\sqrt{g^{(1)}}\,\phi_n^{(i)}(x)\,\phi_m^{(j)}(x \tag{1}$$

corresponding to the eigenvalues $\lambda_n^{(2)}$, $\lambda_n^{(1)}$ and $\lambda_n^{(0)}$ respectively. The modes $\{h_n^{\mu\nu},\, v_n^{\mu\nu},\, w_n^{\mu\nu},\, z_n^{\mu\nu}\}$, with

$$\begin{aligned}
v_n^{\mu\nu} &= \left[\frac{1}{2}\left(\lambda_n^{(1)} - 3\right)\right]^{-\frac{1}{2}} \nabla^{(\mu}\xi_n^{\nu)}\,, \quad n = 2, \ldots\,, \\
w_n^{\mu\nu} &= \left[\lambda_n^{(0)}\left(\frac{3}{4}\lambda_n^{(0)} - 3\right)\right]^{-\frac{1}{2}} \left(\nabla^\mu\nabla^\nu - \frac{1}{4}g^{(1)\,\mu\nu}\Box\right)\phi_n\,, \quad n = 2, \ldots\,, \\
z_n^{\mu\nu} &= \frac{1}{2}g^{(1)\,\mu\nu}\phi_n\,, \quad n = 0, 1, 2, \ldots\,,
\end{aligned} \tag{2}$$

of which we do not write explicitly the degeneracy indexes form the orthonormal basis for symmetric tensors.

Moreover, defining the longitudinal vector modes

$$l_n^{\mu} = \left(\lambda_n^{(0)}\right)^{-\frac{1}{2}} \nabla^{\mu} \phi_n \, , \quad n = 1, 2, \ldots \, , \tag{3}$$

the latter, together with the transverse modes ξ_n^{μ}, form the orthonormal basis for vectors.

Expand the graviton field $\widehat{h}^{\mu\nu}$ as [8]

$$\widehat{h}^{\mu\nu} = \sum_{n=2}^{\infty} a_n h_n^{\mu\nu} + \sum_{n=2}^{\infty} b_n v_n^{\mu\nu} + \sum_{n=2}^{\infty} c_n w_n^{\mu\nu} + \sum_{n=0}^{\infty} e_n z_n^{\mu\nu} \tag{4}$$

$$\widehat{h} \equiv g_{\mu\nu}^{(1)} \widehat{h}^{\mu\nu} = 2 \sum_{n=0}^{\infty} e_n \phi_n \, , \tag{5}$$

and the ghost field v^{μ} as

$$v^{\mu} = \sum_{n=1}^{\infty} g_n \, \xi_n^{\mu} + \sum_{n=1}^{\infty} f_n \, l_n^{\mu} \tag{6}$$

so that we have

$$64\pi G\,(S_2 + S_{\rm gf}) = \sum_{n=2}^{\infty} a_n^2 \left[\lambda_n^{(2)} - 2a^2\Lambda + 8\right]$$
$$+ \sum_{n=2}^{\infty} b_n^2 \left[\xi^{-1}\left(\lambda_n^{(1)} - 3\right) - 2a^2\Lambda + 6\right]$$
$$+ \sum_{n=2}^{\infty} c_n^2 \left[\xi^{-1}\left(\frac{3}{4}\lambda_n^{(0)} - 6\right) - \frac{\lambda_n^{(0)}}{2} - 2a^2\Lambda + 6\right]$$
$$+ \sum_{n=0}^{\infty} e_n^2 \left[\frac{-3+\xi^{-1}}{2}\lambda_n^{(0)} + 2a^2\Lambda\right]$$
$$+ \sum_{n=2}^{\infty} 2e_n c_n (\xi^{-1} - 1)\left[\lambda_n^{(0)}\left(\frac{3}{4}\lambda_n^{(0)} - 3\right)\right]^{\frac{1}{2}} \tag{7}$$

$$32\pi G\,S_{\rm ghost} = \sum_{n=1}^{\infty} g_n^*\, g_n \left(\lambda_n^{(1)} - 3\right) + \sum_{n=1}^{\infty} f_n^*\, f_n \left(\lambda_n^{(0)} - 6\right)\ . \tag{8}$$

Therefore, the functional measure in (**??**) can be written as (defined as)

$$\mathcal{D}\widehat{h}_{\mu\nu}\,\mathcal{D}v_\rho^*\,\mathcal{D}v_\sigma \equiv \frac{1}{V_{SO(5)}}\prod_{n=2}^{\infty}\mathrm{d}a_n\prod_{n=2}^{\infty}\mathrm{d}b_n\prod_{n=2}^{\infty}\mathrm{d}c_n\prod_{n=0}^{\infty}\mathrm{d}e_n\prod_{n=2}^{\infty}\mathrm{d}g_n^*\prod_{n=2}^{\infty}\mathrm{d}g_n\prod_{n=1}^{\infty}\mathrm{d}f_n^*\prod_{n=1}^{\infty}\mathrm{d}f_n\,, \tag{9}$$

Notice that there is no integration over the zero modes g_1^* and g_1 of $S_{\rm ghost}$ [16]. The corresponding ghost fields are proportional to the ten Killing vectors ξ_1^μ. These zero eigenmodes correspond to residual gauge invariances which are not eliminated by gauge fixing in the presence of an $SO(5)$ spherical symmetry. Overcounting has been compensated by inserting the explicit group-volume factor $V_{SO(5)}$ in Eq. (9) (see, e.g., [17]).

Sum over the eigenvalues in closed form

$$
\begin{aligned}
F(a^2\Lambda) = {} & 9\Lambda a^2 - \frac{1}{6}\Lambda\sqrt{8\Lambda a^2+9}\log\Gamma\left(\frac{7}{2}-\frac{1}{2}\sqrt{8\Lambda a^2+9}\right)a^2 - 5\Lambda\psi^{(-2)}\left(\frac{1}{2}\left(\sqrt{8a^2\Lambda-15}+7\right)\right)a^2 \\
& - 5\Lambda\psi^{(-2)}\left(\frac{7}{2}-\frac{1}{2}\sqrt{8a^2\Lambda-15}\right)a^2 - \Lambda\psi^{(-2)}\left(\frac{1}{2}\left(\sqrt{8\Lambda a^2+9}+7\right)\right)a^2 \\
& - \Lambda\psi^{(-2)}\left(\frac{7}{2}-\frac{1}{2}\sqrt{8\Lambda a^2+9}\right)a^2 + \frac{1}{6}\Lambda\log\Gamma\left(\frac{1}{2}\left(\sqrt{8\Lambda a^2+9}+7\right)\right)\sqrt{8\Lambda a^2+9}a^2 \\
& - 5\log(120) + \frac{49\log(A)}{3} - 2\sqrt{\frac{11}{3}}\log\Gamma\left(\frac{1}{2}\left(\sqrt{33}+7\right)\right) \\
& - \frac{5}{6}\left(a^2\Lambda-5\right)\sqrt{8a^2\Lambda-15}\log\Gamma\left(\frac{7}{2}-\frac{1}{2}\sqrt{8a^2\Lambda-15}\right) \\
& - \frac{1}{6}\sqrt{8\Lambda a^2+9}\log\Gamma\left(\frac{7}{2}-\frac{1}{2}\sqrt{8\Lambda a^2+9}\right) + 3\psi^{(-4)}(1) + 3\psi^{(-4)}(6) + \psi^{(-4)}\left(\frac{7}{2}-\frac{\sqrt{33}}{2}\right) \\
& + \psi^{(-4)}\left(\frac{1}{2}\left(\sqrt{33}+7\right)\right) - 5\psi^{(-4)}\left(\frac{1}{2}\left(\sqrt{8a^2\Lambda-15}+7\right)\right) - 5\psi^{(-4)}\left(\frac{7}{2}-\frac{1}{2}\sqrt{8a^2\Lambda-15}\right) \\
& - \psi^{(-4)}\left(\frac{1}{2}\left(\sqrt{8\Lambda a^2+9}+7\right)\right) - \psi^{(-4)}\left(\frac{7}{2}-\frac{1}{2}\sqrt{8\Lambda a^2+9}\right) + \frac{15\psi^{(-3)}(1)}{2} - \frac{15\psi^{(-3)}(6)}{2} \\
& - \frac{1}{2}\sqrt{33}\psi^{(-3)}\left(\frac{1}{2}\left(\sqrt{33}+7\right)\right) - \frac{5}{2}\sqrt{8a^2\Lambda-15}\psi^{(-3)}\left(\frac{7}{2}-\frac{1}{2}\sqrt{8a^2\Lambda-15}\right)
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{2}\sqrt{8\Lambda a^2+9}\,\psi^{(-3)}\left(\frac{7}{2}-\frac{1}{2}\sqrt{8\Lambda a^2+9}\right)+\frac{33\psi^{(-2)}(1)}{4}+\frac{33\psi^{(-2)}(6)}{4}\\
&+\frac{49}{12}\psi^{(-2)}\left(\frac{7}{2}-\frac{\sqrt{33}}{2}\right)+\frac{49}{12}\psi^{(-2)}\left(\frac{1}{2}\left(\sqrt{33}+7\right)\right)+\frac{175}{12}\psi^{(-2)}\left(\frac{1}{2}\left(\sqrt{8a^2\Lambda-15}+7\right)\right)\\
&+\frac{175}{12}\psi^{(-2)}\left(\frac{7}{2}-\frac{1}{2}\sqrt{8a^2\Lambda-15}\right)-\frac{13}{12}\psi^{(-2)}\left(\frac{1}{2}\left(\sqrt{8\Lambda a^2+9}+7\right)\right)\\
&-\frac{13}{12}\psi^{(-2)}\left(\frac{7}{2}-\frac{1}{2}\sqrt{8\Lambda a^2+9}\right)+\frac{1}{2}\psi^{(-3)}\left(\frac{7}{2}-\frac{\sqrt{33}}{2}\right)\sqrt{33}+2\log\Gamma\left(\frac{7}{2}-\frac{\sqrt{33}}{2}\right)\sqrt{\frac{11}{3}}\\
&+\frac{5}{6}\left(a^2\Lambda-5\right)\log\Gamma\left(\frac{1}{2}\left(\sqrt{8a^2\Lambda-15}+7\right)\right)\sqrt{8a^2\Lambda-15}\\
&+\frac{5}{2}\psi^{(-3)}\left(\frac{1}{2}\left(\sqrt{8a^2\Lambda-15}+7\right)\right)\sqrt{8a^2\Lambda-15}\\
&+\frac{1}{6}\log\Gamma\left(\frac{1}{2}\left(\sqrt{8\Lambda a^2+9}+7\right)\right)\sqrt{8\Lambda a^2+9}\\
&+\frac{1}{2}\psi^{(-3)}\left(\frac{1}{2}\left(\sqrt{8\Lambda a^2+9}+7\right)\right)\sqrt{8\Lambda a^2+9}+\frac{7\zeta(3)}{4\pi^2}-\frac{2}{3}\zeta'(-3)-\frac{20801}{1080}
\end{aligned}
$$

$$g^{(a)}_{\mu\nu} = \begin{pmatrix} a^2 & 0 & 0 & 0 \\ 0 & a^2 \sin^2\theta_1 & 0 & 0 \\ 0 & 0 & a^2 \sin^2\theta_1 \sin^2\theta_2 & 0 \\ 0 & 0 & 0 & a^2 \sin^2\theta_1 \sin^2\theta_2 \sin^2\theta_3 \end{pmatrix}$$

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