GRAVITATIONAL COLLAPSE IN SCALE-DEPENDENT GRAVITY

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Scale-dependent (Asymptotically safe) gravity

- In many QG approaches, the **basic parameters** that enter the action, such as Newton's constant, electromagnetic coupling, cosmological constant, etc, **become scale dependent quantities**.
- **Scale dependence** at the level of the effective action is a **generic feature** of ordinary quantum field theories.

1. A typical Scale dependent effective gravitational action

$$\Gamma_k[\mathfrak{g}] = \frac{1}{16\pi G(k)} \int \mathrm{d}^4x \sqrt{\mathfrak{g}} \Big(-R(\mathfrak{g}) + 2\bar{\lambda}(k) \Big).$$

$\mathfrak{g}$ = metric, k = energy/momentum scale, G(k) and λ(k) are the running Newton's and cosmological constants. Cosmological constant λ(k)~O in local problems. Dimensionless Newton's constant $g(k) = k^2 G(k)/\hbar$.

The evolution of scale-dependent quantities is governed by a differential equation, from which we derive an evolution equation for g(k)

$$\frac{\mathrm{d}g(t)}{\mathrm{d}t} \equiv \beta(g(t)) = \left[2 + \frac{B_1 g(t)}{1 - B_2 g(t)}\right] g(t)$$

where t=log(k)

Integrating for g(k), and using $g(k) = k^2 G(k)/\hbar$ we get the dimensionful Newton's coupling

$$G(k) = \frac{G_0}{1 + \tilde{\omega} G_0 k^2/\hbar}$$

Deviations at high energy: $G(k) \to 0$. Classical space-time recovered for $k \to 0$.

The ***SIGN*** of $\tilde{\omega}$ is ***crucial*** for the physics of the theory.

Asymptotically Safe Gravity (ASG) assumes $\tilde{\omega} > 0$. So that gravity can be "asymptotically safe" from divergences.

If instead $\tilde{\omega} < 0 \Longrightarrow G(k) \to \infty$ for $k \sim k_{Planck}$

When $\tilde{\omega} > 0$, it is a proper ASG theory. When $\tilde{\omega} < 0$, it is a general SDG theory

2. Link between energy scale k and radial coordinate r, namely k = k(r):

$$k(r) \equiv \hbar \left(\frac{r + \gamma G_0 M}{r^3} \right)^{1/2}$$

The energy scale k(r) is a modified proper distance, as *k(r) ~ 1/d(r)*. $\gamma = 9/2$ by identification infrared cutoff

3. Finally the running Newton coupling constant is

$$G(r) = \frac{G_0 r^3}{r^3 + \tilde{\omega} G_0 \hbar (r + \gamma G_0 M)}$$

$G_0 \hbar = \ell_p^2$ = quantum effects!

$\tilde{\omega}, \gamma$ numerical constants

In the SD (AS) gravity approach, in order to get scale-dependent solutions (Newtonian or general relativistic), we replace everywhere the Newton constant G_0 with the running constant G(r).

For a spherically symmetric Lorentzian metric

$$d\tilde{s}^2 = -f(\tilde{r})d\tilde{t}^2 + \frac{d\tilde{r}^2}{f(\tilde{r})} + \tilde{r}^2 d\Omega^2$$

the lapse function of the SD/AS-Schwarzschild metric reads

$$f(\tilde{r}) = 1 - \frac{\tilde{r}^2}{\tilde{r}^3 + \tilde{\omega}(\tilde{r} + \frac{\gamma}{2})},$$

$G_0 = c = \hbar = 1$ $\tilde{\tau} = \tau/2M$, $\tilde{r} = r/2M$, and $\tilde{\omega} = \omega/(2M)^2$

Important limits: large r, low energy k

$$f(\tilde{r}) = 1 - \frac{1}{\tilde{r}}$$

Small r, high energy k

$\gamma > 0$

$$f(\tilde{r}) = 1 - \frac{2\tilde{r}^2}{\gamma\tilde{\omega}}$$

$\gamma = 0$

$$f(r \to 0) \simeq 1 - \frac{2Mr}{\tilde{\omega}\hbar}$$

$\tilde{\omega}\gamma > 0$ DeSitter; $\tilde{\omega}\gamma < 0$ Anti-DeSitter.

Conic singularity

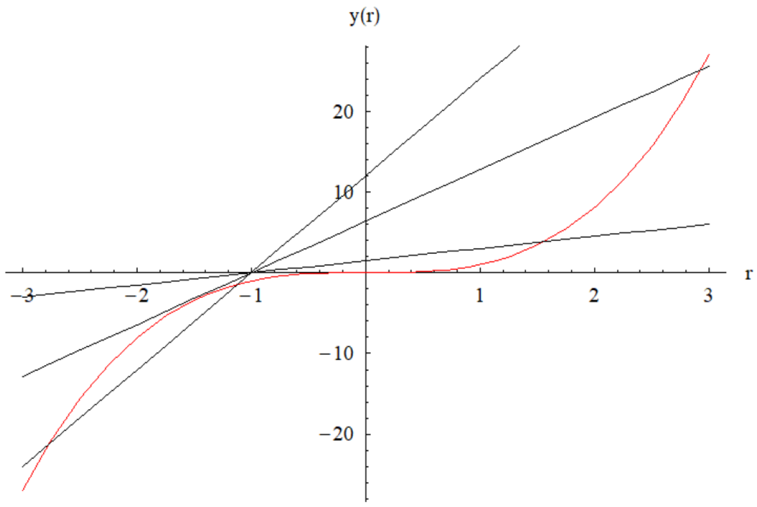


FIG. 1: Analysis of the denominator of (9). The plots of $y_1(\tilde{r}) = \tilde{r}^3$ and $y_2(\tilde{r}) = -\tilde{\omega}(\tilde{r} + \gamma/2)$ show that for any $\tilde{\omega} < 0$ there is always one single positive zero $\tilde{r}_0 > 0$ of the denominator of (9), namely a singularity of the metric.

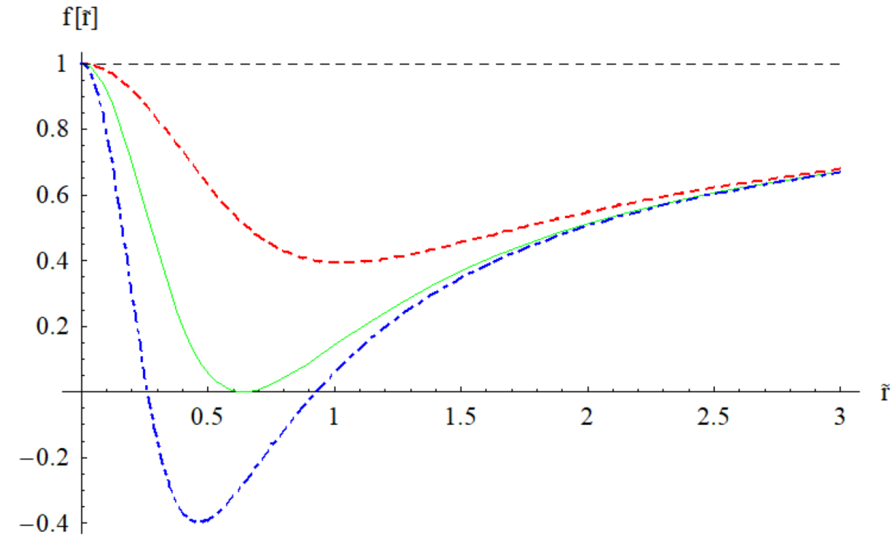


FIG. 5: Lapse function $f(\tilde{r})$ for $\tilde{\omega} > 0$. Horizons are the zeros of $f(\tilde{r})$, depicted for increasing values of $\tilde{\omega}$, $0 < \tilde{\omega}_{\rm blue} < \tilde{\omega}_{\rm green} \equiv \tilde{\omega}_c < \tilde{\omega}_{\rm red}$, i.e. for decreasing values of the mass M.

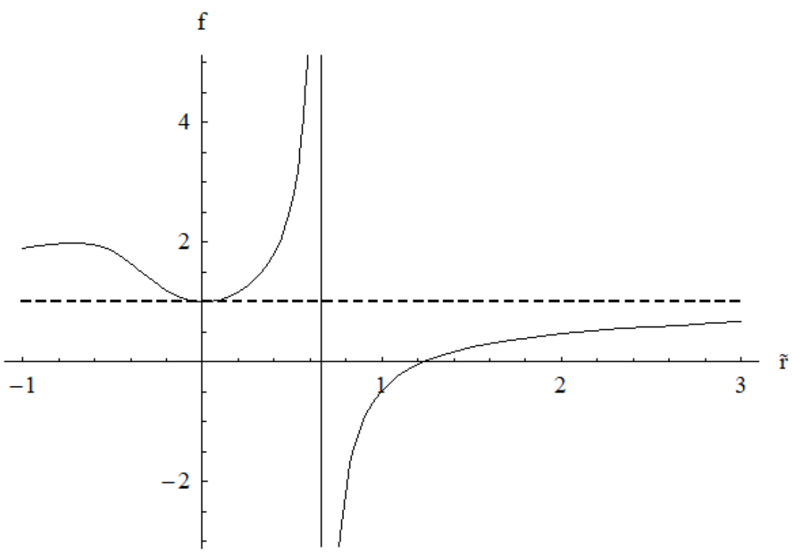


FIG. 6: Lapse function $f(\tilde{r})$ for $\tilde{\omega} < 0$ in the physical region $\tilde{r} > 0$: singularity at $\tilde{r} = \tilde{r}_0$, and horizon at $\tilde{r} = \tilde{r}_+$, where $f(\tilde{r}_+) = 0$.

Lapse f(r) for $\widetilde{\omega} < 0$: the denominator can develop zeros at $\boldsymbol{r = r_o > 0}$

- Essential singularity at $\boldsymbol{r = r_o > 0}$ (instead of $\mathrm{r} = 0$ as in Schwarzschild).
- Always a single positive zero $f(r_h) = 0$

SDG-BH Horizon at $\boldsymbol{r = r_h}$

Always $\boldsymbol{r_o < r_h}$ for any **M > o**.

No naked singularity in accordance with Cosmic Censorship Conjecture of Penrose.

Possible values of the parameters $\tilde{\omega}$ and γ

A SDG modified Newtonian potential can be obtained from the standard Newton formula by simply replacing the experimentally observed Newton constant G with the running coupling G(r)

$$V^{SDG}(r) = -\frac{G(r)Mm}{r} = -\frac{G\,M\,m\,r^2}{r^3 + \tilde{\omega}\,G\,\hbar\,(r + \gamma GM)}\,,$$

By expanding for *large r* we get

$$V^{SDG}(r) = -\frac{GMm}{r}\left[1 - \frac{\tilde{\omega}G\hbar}{r^2} - \frac{\gamma\tilde{\omega}G^2\hbar M}{r^3} + \mathcal{O}\left(\frac{G^2\hbar^2}{r^4}\right)\right]$$

- The corrections predicted by the SDG (ASG) approach are of genuine quantum nature (presence of ħ).
- To fix the value of $\tilde{\omega}$, people in ASG usually compared this potential with the Quantum corrected Newtonian Potential obtained via GR-EFT by Donoghue, Khriplovich, Shapiro, etc.
- Even if, recently (~2020 and later), there are doubts on the comparability between ASG/GR-EFT approaches (different classes of Feynman diagrams).

Possible $\tilde{\omega} < 0$ values from quantum corrections to the Newtonian potential

Computed by John Donoghue (PRD 1994, PRD 2003). GR is seen as a low energy Effective Field Theory. At ordinary energies GR gravity is a well-behaved QFT. The quantum corrections at low energy, and dominant effects at large distances can be isolated. **Potential energy between two heavy objects close to rest is (to the lowest order)**

$$V^{QGR}(r) = -\frac{GMm}{r}\left[1 + \frac{41}{10\pi}\frac{G\hbar}{r^2} + \ldots\right]$$

The last term is a true long distance quantum effect linear in ħ.
(We neglect post newtonian classical correction terms). By comparing with

$$V^{SDG}(r) = -\frac{GMm}{r}\left[1 - \frac{\tilde{\omega}G\hbar}{r^2} - \frac{\gamma\tilde{\omega}G^2\hbar M}{r^3} + \mathcal{O}\left(\frac{G^2\hbar^2}{r^4}\right)\right]$$

we get at the first order in ħ

$$\tilde{\omega} = -\frac{41}{10\pi}$$

Ilya Shapiro:

$\tilde{\omega}$ = - 17 / 20 π

Main message:

$\tilde{\omega}$ can be Negative!

The γ parameter is not fixed by previous considerations.

- $\gamma = 9/2$ by Gen.Rel. arguments (Bonanno-Reuter 2000): correct identification of the infrared cutoff
- $\gamma = 0$ by GUP considerations

Anyway $\gamma \geq 0$

The ***SIGN*** of $\tilde{\omega}$ is ***crucial*** for the physics of SDG/ASG-black hole.

Since

$$G(k) = \frac{G_0}{1 + \tilde{\omega} G_0 k^2/\hbar}$$

gravity is "asymptotically safe" from divergences only when $\tilde{\omega} > 0$.

$\tilde{\omega} > 0$ → singularity-free BH metric, with DeSitter or Anti DeSitter core

Instead when $\tilde{\omega} < 0$ ⟹ $G(k) \to \infty$ for $k \sim k_{Planck}$ (SDG !)

- GR as Effective QFT [Donoghue, Krilipovich, Froeb, Shapiro] (2003, 2005, 2022): all results point to a ***negative value of*** $\tilde{\omega}$.

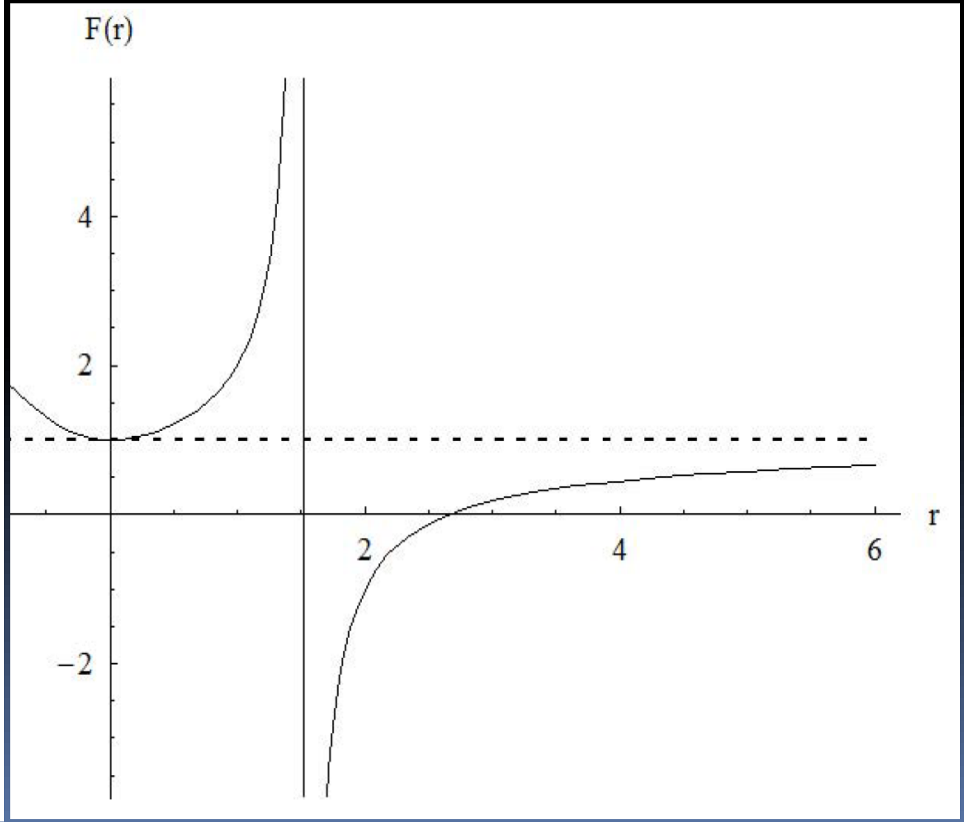
$\tilde{\omega} < 0$: deep consequences on the metric structure of the black hole.

THE NEW SDG-SCHWARZSCHILD METRIC

Under the condition $\tilde{\omega} < 0 \;\Rightarrow\; \tilde{\omega} = -|\tilde{\omega}|\,; \qquad \gamma > 0\,.$ the lapse F(r) is

$$F(r) = 1 - \frac{2GMr^2}{r^3 - |\tilde{\omega}|G\hbar(r + \gamma GM)}$$

Since $\widetilde{\omega} < 0$ the denominator can develop zeros, and hence the metric can have singularities.

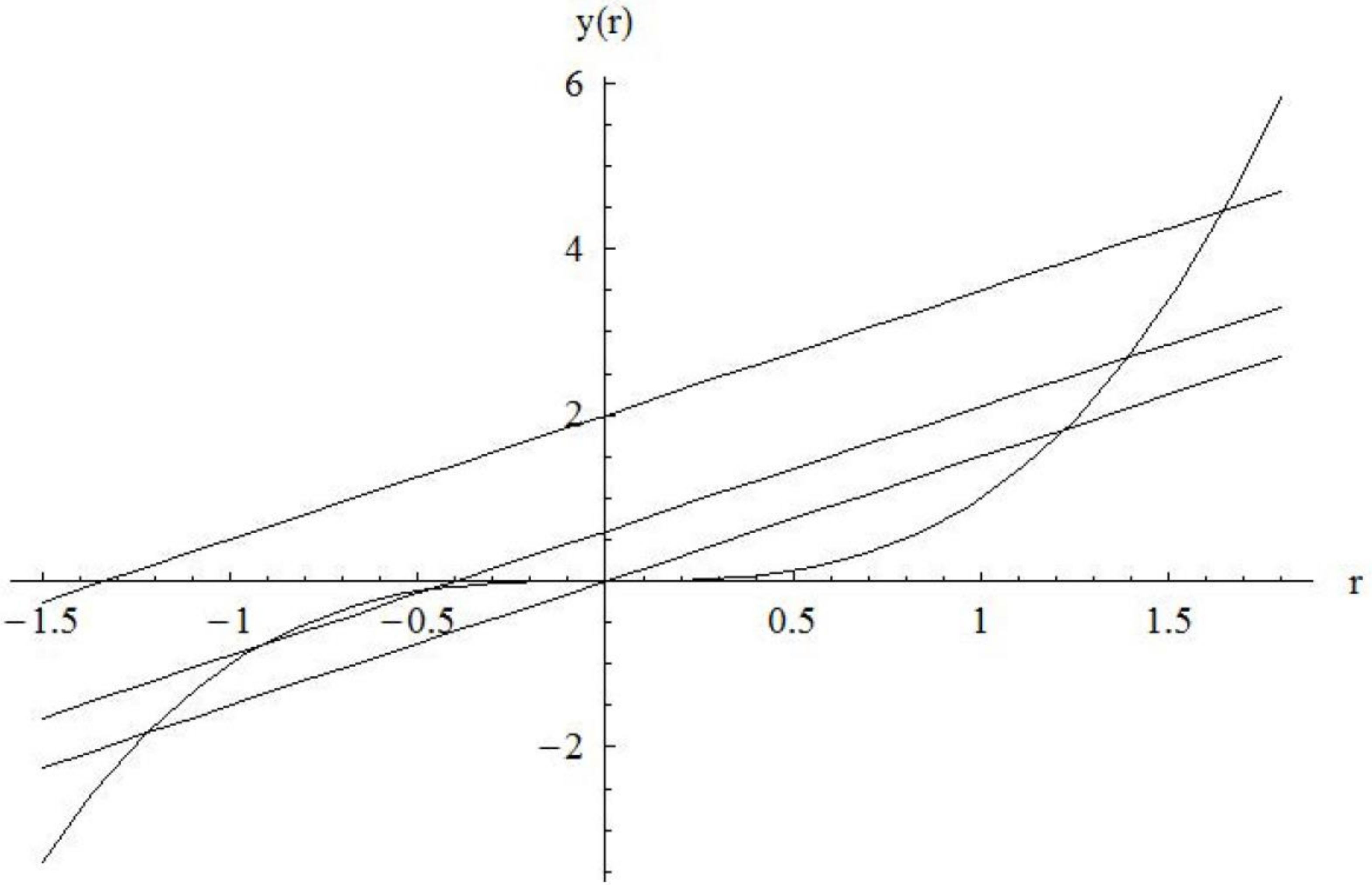


Lapse F(r) for any **r > 0, γ > 0, M > 0**.

Essential singularity at $\boldsymbol{r = r_o > 0}$
(instead of $\mathrm{r} = 0$ as in Schwarzschild).

Always one single positive zero $F(r_h) = 0$
SDG-BH Horizon at $\boldsymbol{r = r_h}$

Always $\boldsymbol{r_o < r_h}$ for any **M > o**.
No naked singularity in full accordance with the Cosmic Censorship Conjecture of Penrose.


y(r)
6
4
2
−2
r
−1.5
−1
−0.5
0.5
1
1.5

Horizon $r = r_h$, Singularity $r = r_o$ for large or small M

$$r_h \simeq 2GM + \frac{(2+\gamma)|\tilde{\omega}|\hbar}{4M}$$

for $M \to \infty$

(Schwarzschild recovered)

$$r_h \simeq \sqrt{|\tilde{\omega}|G\hbar} + \left(1 + \frac{\gamma}{2}\right) GM$$

for $M \to 0$

(essentially, $r_h \sim \ell_p$ which is reasonable, since any length below Planck length is physically meaningless)

$$r_0 \simeq \sqrt{|\tilde{\omega}|G\hbar} + \frac{\gamma}{2} GM$$

for $M \to 0$

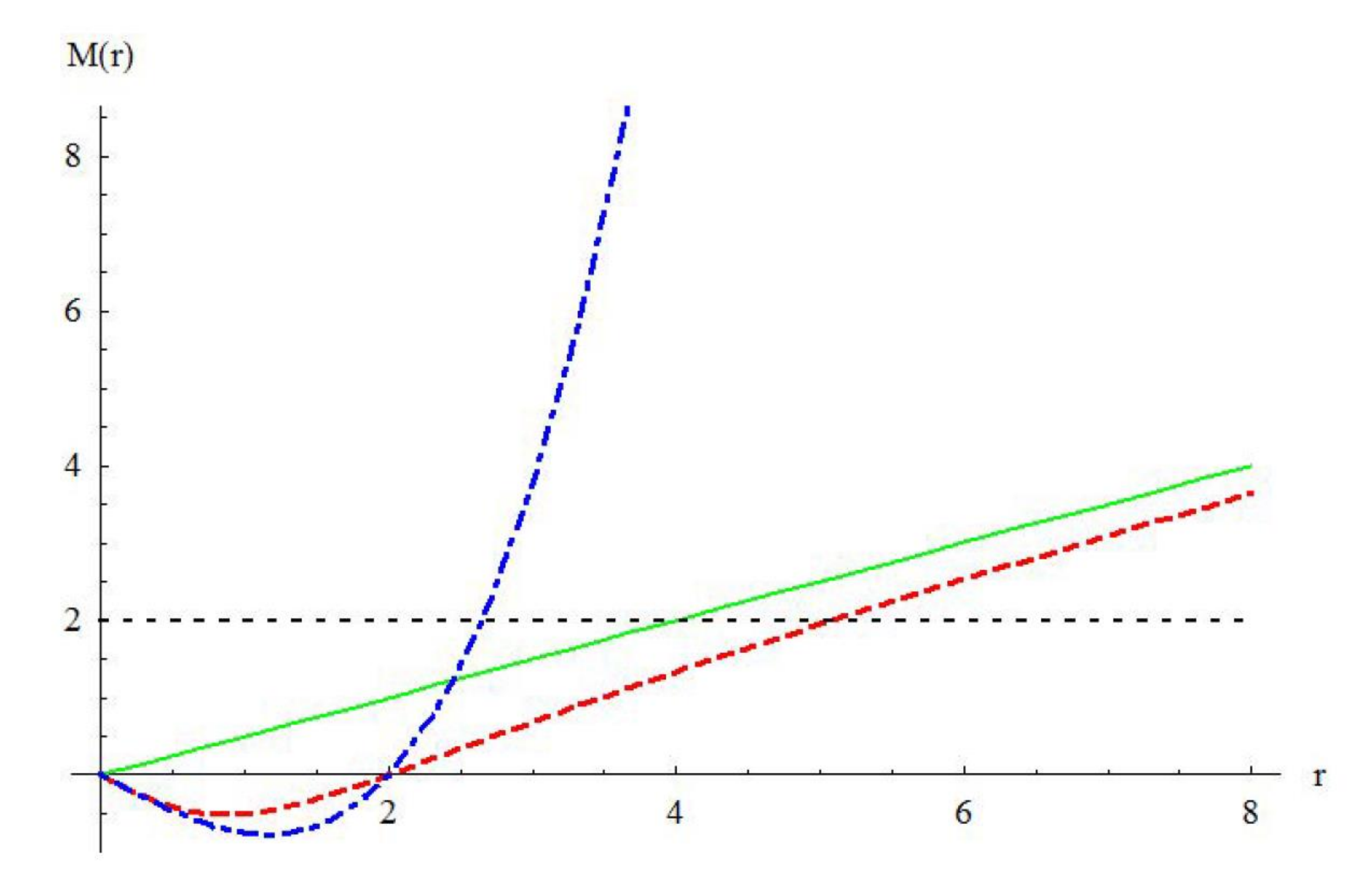
$$r_0 \simeq (\gamma|\tilde{\omega}|G^2\hbar M)^{1/3} + \frac{|\tilde{\omega}|G\hbar}{3(\gamma|\tilde{\omega}|G^2\hbar M)^{1/3}}$$

for $M \to \infty$

Note that, for both limits $M \to \infty$ or $M \to 0$, we always have $r_o < r_h$

So the ***singularity results always to be protected by the horizon***.

Horizon & Singularity Mass Functions



$$GM(r_h) = \left. \frac{r^3 - |\tilde{\omega}|\ell_P^2 r}{2r^2 + |\tilde{\omega}|\ell_P^2 \gamma} \right|_{r=r_h}$$

RED LINE

$$GM(r_0) = \left. \frac{r^3 - |\tilde{\omega}|\ell_P^2 r}{|\tilde{\omega}|\ell_P^2 \gamma} \right|_{r=r_0}$$

BLUE LINE

A METRIC FOR THE PLANCK STARS

The central hard singularity is located at

$$r = r_o = [\,\gamma\,|\widetilde{\omega}|\,\ell_p^2\,G\,M\,]^{1/3} > 0$$

(Schwarzschild: $r_o = 0$). Finite size of QUANTUM ORIGIN: $r_o \to 0$ for $\hbar \to 0$

Consider the whole collapsing mass M as concentrated into the central hard sphere of radius r_o. Its (non covariant) volume, and hence the density of this matter (as seen by an observer at infinity), results to be finite:

$$\varrho = \frac{M}{V_{core}} = \frac{3}{2\pi}\frac{m_p}{\gamma|\tilde{\omega}|\ell_P^3} \simeq \frac{m_p}{2\gamma|\tilde{\omega}|\ell_P^3} = \frac{\varrho_{Planck}}{2\gamma|\tilde{\omega}|}$$

$$\varrho_{Planck} = \frac{m_p}{\ell_P^3}$$

Our BH has a central hard kernel of finite size,

with density ~ Planck density.

These are exactly the characteristics of Planck stars, proposed by Rovelli in 2014, on the ground of general qualitative considerations.

The finite size of the central core, being of pure quantum origin, is presumably due to the action of the Heisenberg uncertainty principle, which prevents matter to be arbitrarily concentrated into a geometrical point of size zero.

The central kernel can in principle keep trace of the information swallowed by the black hole: a possible way out of the information paradox? (Of course $\gamma > 0$ strictly.)

Gravitational collapse to a scale-dependent BH

Oppenheimer-Snyder (OS) gravitational collapse of a star described by a homogeneous and isotropic perfect fluid. External metric rewritten as:

$$d\tilde{s}^2 = -\left(1 - \frac{1}{\tilde{r}\alpha(\tilde{r})}\right)d\tilde{t}^2 + \left(1 - \frac{1}{\tilde{r}\alpha(\tilde{r})}\right)^{-1} d\tilde{r}^2 + \tilde{r}^2 d\Omega^2$$

$$\alpha(\tilde{r}) = 1 + \frac{\tilde{\omega}(\tilde{r} + \frac{\gamma}{2})}{\tilde{r}^3}$$

Interior geometry: described by a spatially flat Robertson-Walker (RW) metric

$$ds^2 = -d\tau^2 + a^2(\tau)(dr_c^2 + r_c^2 d\Omega^2)$$

in co-moving coordinates.

We choose the time coordinate of the exterior (interior) metric to be the proper time of freely falling (co-moving) particles. ⇒ Therefore we use

$r > R(\tau)$ External region

Painleve'-Gullstrand (PG) coord.

$$ds^2 = -d\tau^2 + \left(dr + \sqrt{\frac{2M}{r\alpha(r)}}d\tau\right)^2 + r^2 d\Omega^2$$

$r < R(\tau)$ Internal region

Since $r_i(\tau) = a(\tau) r_c$

$$ds^2 \;=\; -d\tau^2 + (dr - rH(\tau)d\tau)^2 + r^2 d\Omega^2$$

Imposing the continuity of the metric and of its first derivatives (Israel JC) yields

EoM of star 's surface R(τ)

$$\dot{R}(\tau) = -\sqrt{\frac{2M}{R(\tau)\alpha(R)}}$$

with

$$H(\tau) = \frac{\dot{a}(\tau)}{a(\tau)} = \frac{\dot{a}(\tau)R_c}{a(\tau)R_c} = \frac{\dot{R}(\tau)}{R(\tau)}$$

In dimensionless variables $\tilde{R} = R/2M,$ $\tilde{\tau} = \tau/2M,\ \tilde{\omega} = \omega/(2M)^2$

the EoM of the star's surface is

$$\dot{\tilde{R}}(\tilde{\tau}) = -\frac{\tilde{R}}{\sqrt{\tilde{R}^3 + \tilde{\omega}(\tilde{R} + \frac{\gamma}{2})}}$$

EoM of Internal radial outgoing null geodesics (=internal radial light rays)(from the inner metric):

$$dr = (+1 + rH(\tau))d\tau$$

Equation of APPARENT HORIZON inside the star ($dr > 0$):

$$r_{ah} = -\frac{1}{H(\tau)}$$

Equations for outward null geodesics (outward light rays)

$$\frac{d\tilde{r}}{d\tilde{\tau}} = 1 - \frac{\tilde{r}}{\tilde{R}}\sqrt{\frac{1}{\tilde{R}\alpha(\tilde{R})}} \qquad \tilde{r} < \tilde{R}$$

$$\frac{d\tilde{r}}{d\tilde{\tau}} = 1 - \sqrt{\frac{1}{\tilde{r}\alpha(\tilde{r})}} \qquad \tilde{r} > \tilde{R}$$

$$H(\tau) = \frac{\dot{a}R_c}{aR_c} = \frac{\dot{R}(\tau)}{R(\tau)}$$

Equation of evolving EVENT HORIZON, i.e. the last light ray (=null geodesic) able to leave the star surface from the internal region. It satisfies the Cauchy problem:

$$\frac{dr_{eh}}{d\tau} = 1 + r_{eh} H(\tau)$$
$$R(\tau_f) = r_{eh}(\tau_f) = r_+$$

with r_+= outer horizon

$$H(\tau) = \frac{\dot{a}R_c}{aR_c} = \frac{\dot{R}(\tau)}{R(\tau)}$$

Since also $\tilde{R}$**(τ)** is evolving, the two variables $\tilde{R}$ and $\tilde{r}_{eh}$ should satisfy the first order non linear coupled differential system [in dimensionless variables]

$$\dot{\tilde{R}} = -\frac{\tilde{R}}{\sqrt{\tilde{R}^3 + \tilde{\omega}(\tilde{R} + \frac{\gamma}{2})}}\,; \qquad \tilde{R}(\tilde{\tau}_f) = \tilde{r}_+$$

$$\dot{\tilde{r}}_{eh} = 1 - \frac{\tilde{r}_{eh}}{\sqrt{\tilde{R}^3 + \tilde{\omega}(\tilde{R} + \frac{\gamma}{2})}}\,; \qquad \tilde{r}_{eh}(\tilde{\tau}_f) = \tilde{r}_+$$

Case $\tilde{\omega} = 0$:

Standard OS collapse in a Schwarzschild metric. Everything can be explicitly integrated easily.

Star's surface:

$$\tilde{R} = \left(\tilde{R}_0^{3/2} - \frac{3}{2}\tilde{\tau} \right)^{2/3}$$

Apparent horizon:

$$\tilde{r}_{ah} = \tilde{R}(\tilde{\tau})^{3/2} = \tilde{R}_0^{3/2} - \frac{3}{2}\tilde{\tau}$$

Event horizon:

$$\tilde{r}_{eh} = -2\left(\tilde{R}_0^{3/2} - \frac{3}{2}\tilde{\tau} \right) + 3\left(\tilde{R}_0^{3/2} - \frac{3}{2}\tilde{\tau} \right)^{2/3}$$

Case $\widetilde{\omega} = 0$

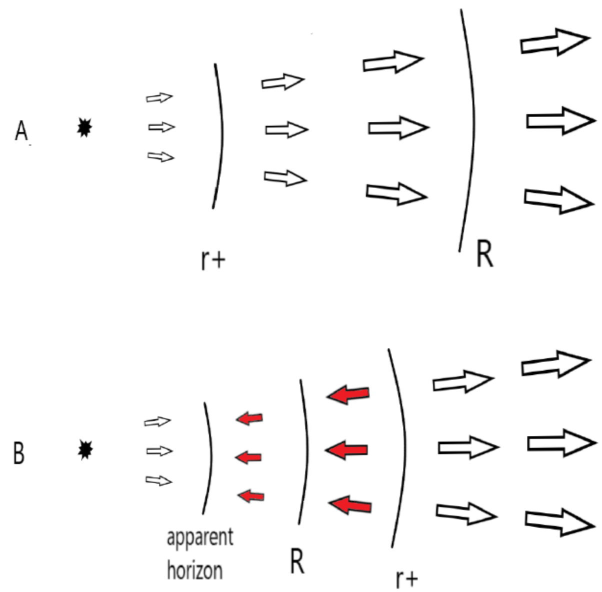


FIG. 8. Birth and behavior of the apparent horizon for $\tilde{\omega} = 0$. Outward null geodesics (i.e. light rays) travel from left to right (diagrams A and B). The apparent horizon (ah) comes to exist only when the surface of the star falls below the outer horizon, $\tilde{R} < \tilde{r}_+$ (diagram B). Then, in the region $\tilde{r}_{ah} < \tilde{r}_+$ even outward light rays are forced to travel inward. Any sphere with $\tilde{r}_{ah} < \tilde{r} < \tilde{r}_+$ is a trapped surface.

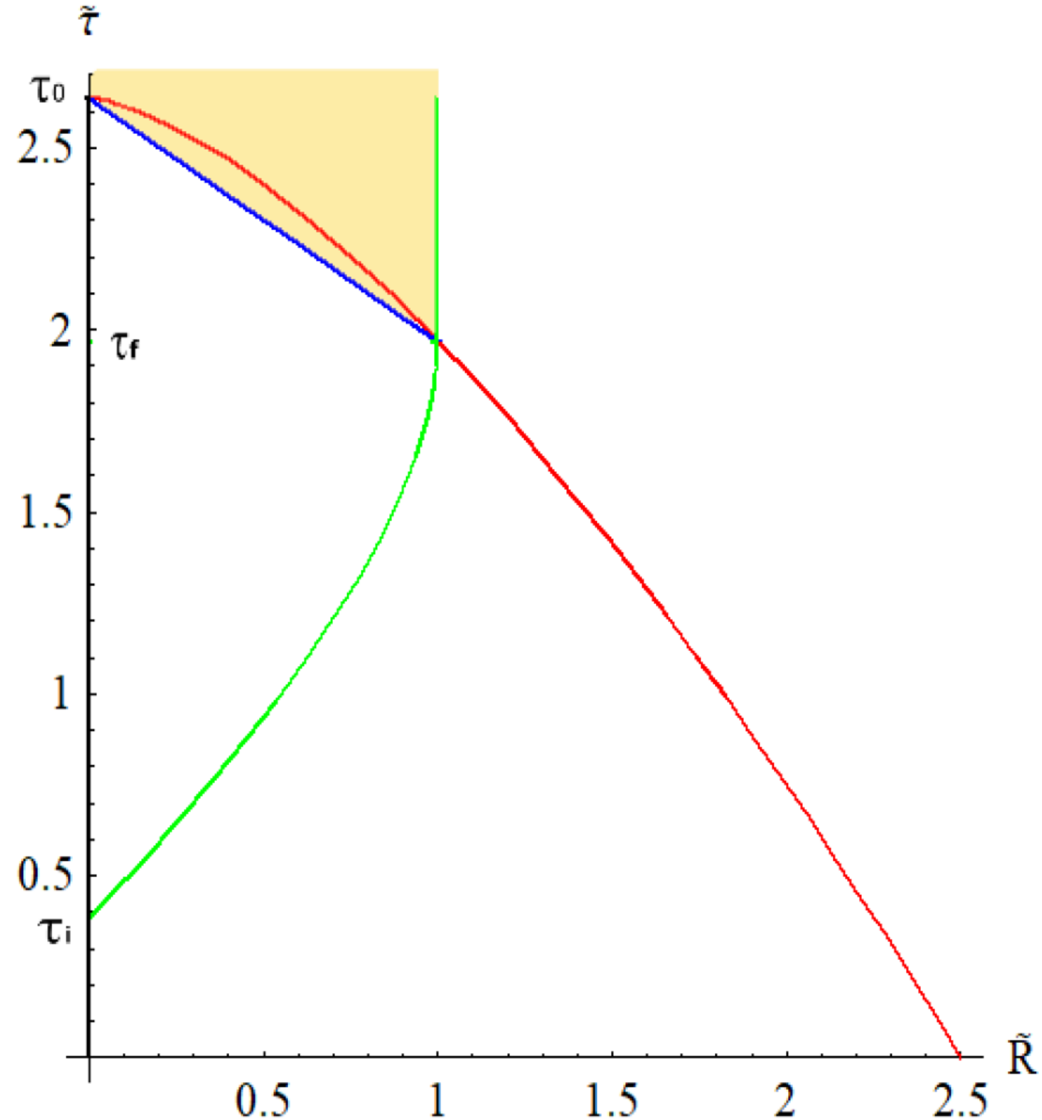


FIG. 9: Evolution of the star surface (red), of the event horizon (green), and of the apparent horizon (blue). At any time $\tilde{\tau} > \tilde{\tau}_f$, the spherical surfaces with $\tilde{r}$ in $\tilde{r}_{ah}(\tilde{\tau}) < \tilde{r} < 1 = \tilde{r}_+$ are trapped surfaces (yellow area).

Case $0 < \tilde{\omega} < \tilde{\omega}_c$

$$\tilde{r}_{ah}(\tilde{\tau}) = \sqrt{\tilde{R}(\tilde{\tau})^3 + \tilde{\omega}(\tilde{R}(\tilde{\tau}) + \gamma/2)}$$

$$\tilde{r}_- < \tilde{r}_{ah} < \tilde{R} < \tilde{r}_+$$

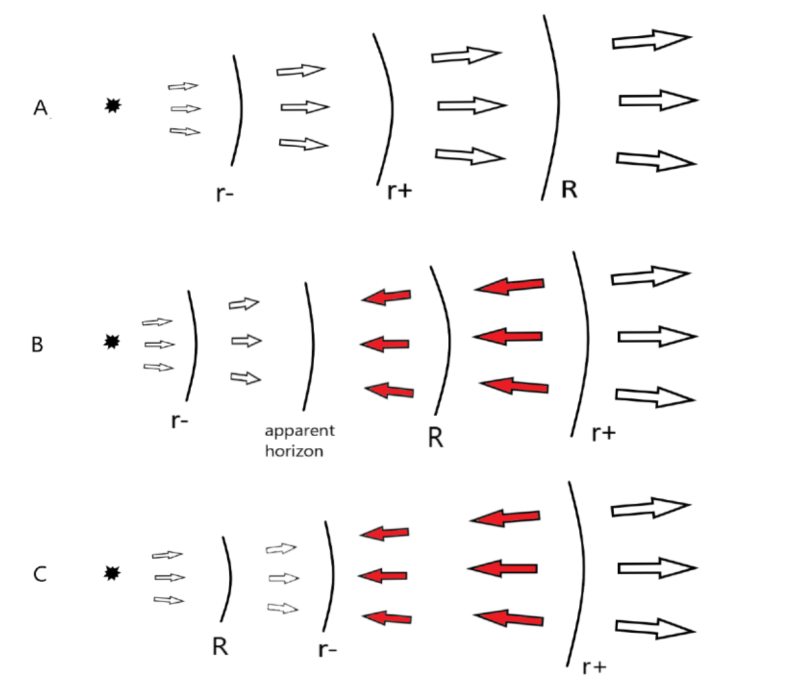


FIG. 11. Birth and behavior of the apparent horizon for $0 < \tilde{\omega} < \tilde{\omega}_c$. Outward null geodesics (i.e. light rays) travel from left to right. The apparent horizon comes to exist only when the surface of the star falls below the outer horizon, $\tilde{R} < \tilde{r}_+$. Then, in the region $\tilde{r}_{ah} < \tilde{r}_+$ even outward light rays are forced to travel inward. Any sphere with $\tilde{r}_{ah} < \tilde{r} < \tilde{r}_+$ is a trapped surface. Once the star surface crosses the inner horizon $\tilde{r}_-$, the apparent horizon ceases to exist, and any sphere with $\tilde{r}_- < \tilde{r} < \tilde{r}_+$ is a trapped surface.

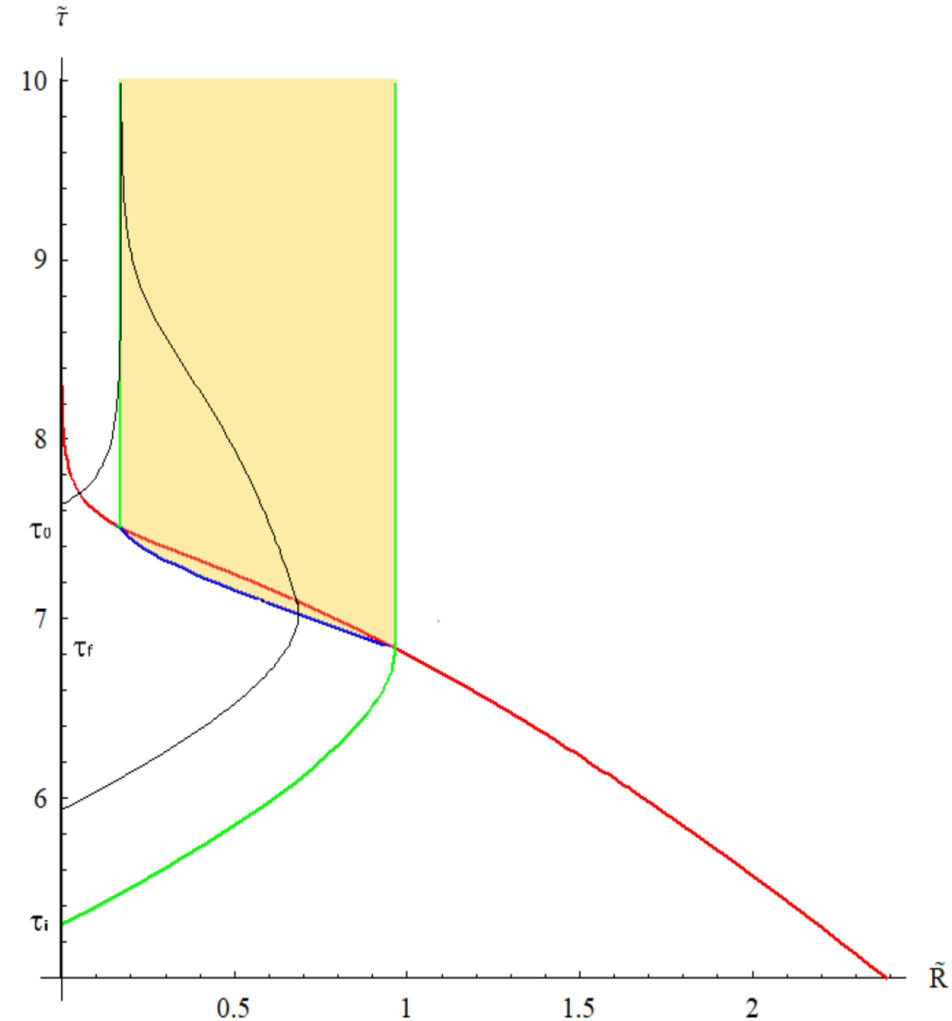


FIG. 14: For $0 < \tilde{\omega} < \tilde{\omega}_c$, here we represent the evolution of the star surface (red), the event/outer horizon $\tilde{r}_+$ (green), the inner horizon $\tilde{r}_-$ (green), the apparent horizon (blue). At any time $\tilde{\tau}_f < \tilde{\tau} < \tilde{\tau}_0$ the surfaces $\tilde{r}_{ah}(\tilde{\tau}) < \tilde{r} < \tilde{r}_+$ are trapped. Then, for $\tilde{\tau} > \tilde{\tau}_0$, all the surfaces $\tilde{r}_- < \tilde{r} < \tilde{r}_+$ are trapped (yellow area). Notice that the surfaces $0 < \tilde{r} < \tilde{r}_-$ are *not trapped*. Trajectories of light rays inside $\tilde{r}_+$ and inside $\tilde{r}_-$ are depicted in thin black.

Light rays leaving the star surface at times $\tilde{\tau}_f < \tilde{\tau} < \tilde{\tau}_0$ are confined in the spherical shell $S(\tilde{r}_-, \tilde{r}_+)$, namely $\tilde{r}(\tilde{\tau}) \to \tilde{r}_-$ when $\tau \to +\infty$; null trajectories leaving the star surface when $\tilde{\tau} > \tilde{\tau}_0$, can move outward, but only until approaching smoothly, from inside, the inner horizon surface $\tilde{r}_-$. The surfaces inside the inner horizon, $0 < \tilde{r} < \tilde{r}_-$, are **not trapped surfaces**. Our black hole consists of a **luminous core surrounded by a thick spherical black shell.**

Cases $\widetilde{\omega} \neq 0$.

Equations of motion can be explicitly integrated in terms of Elliptic Integrals. We get exact expressions but not appealing or clarifying. Look at *asymptotic behavior*, or to **numerical integrations**.

Case $\widetilde{\omega} > \widetilde{\omega}_c$

No horizons at all,

no apparent horizon,

no trapped surfaces

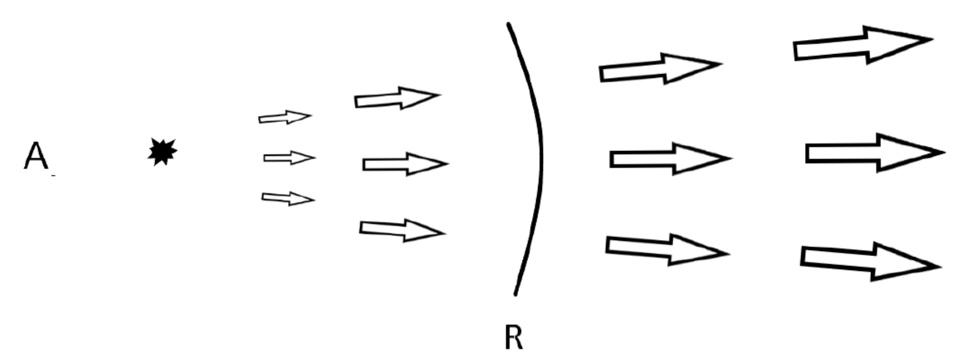
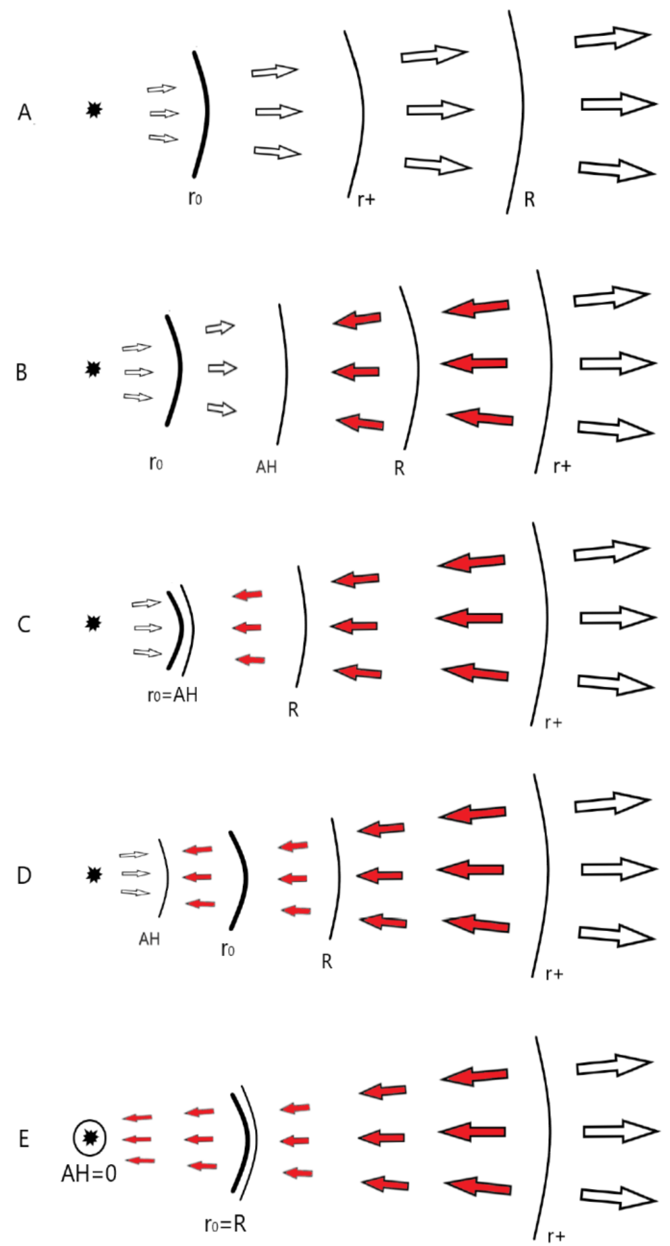


FIG. 13. No apparent horizon and no trapped surfaces appear when $\tilde{\omega} > \tilde{\omega}_c$.

Case $\tilde{\omega} < 0$



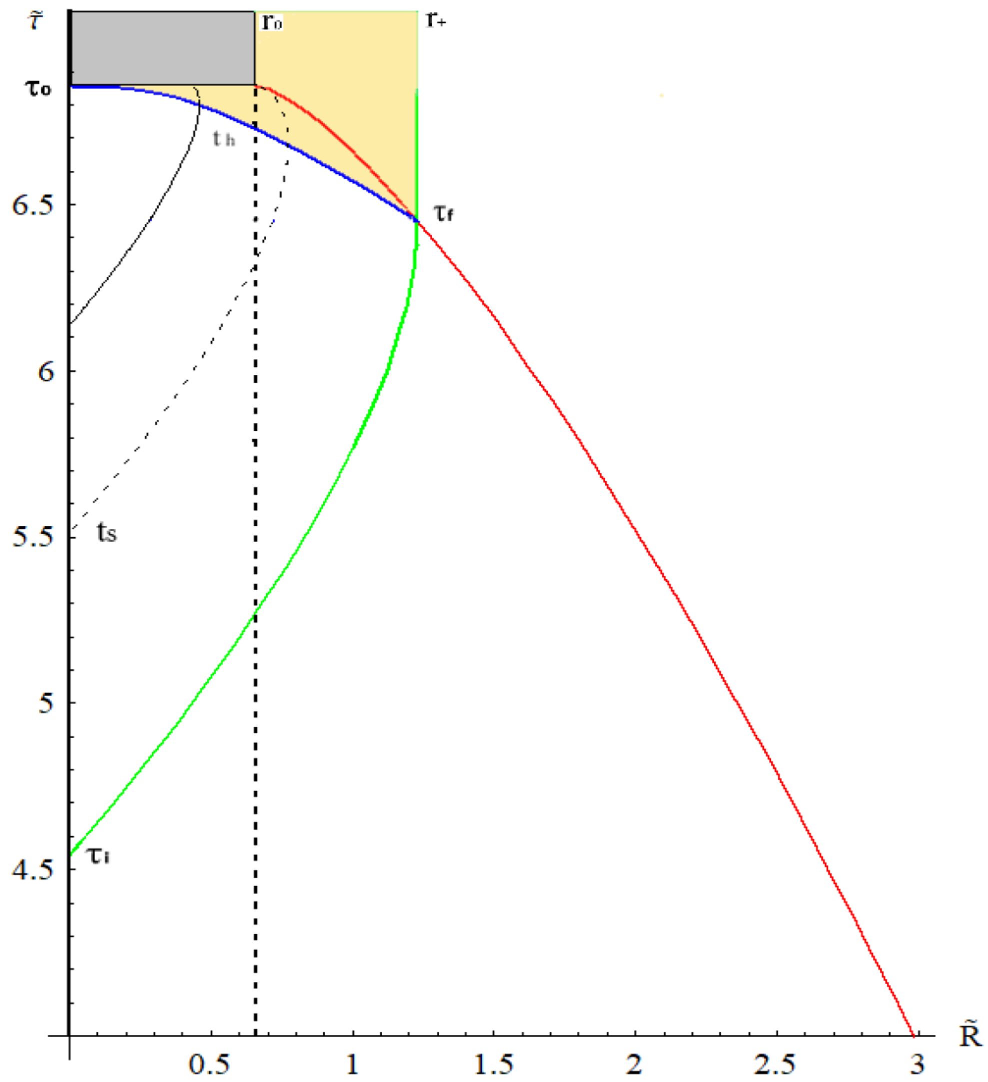


FIG. 17: Evolution of the star surface (red), of the event horizon (green), of the apparent horizon (blue) during the collapse. The black dashed curved line represents the last light ray able to leave the star surface. The grey area is the extended "central" singularity. All the surfaces with $\tilde{r}_{ah}(\tilde{\tau}) < \tilde{r} < \tilde{r}_+$ are *trapped surfaces* (yellow and grey areas).

Notice that

$$\tilde{r}_{ah} < \tilde{R} < \tilde{r}_+$$

$$\tilde{R}(\tilde{\tau}) \geq \tilde{r}_0 \quad \Longleftrightarrow \quad \tilde{R}^3 + \tilde{\omega}(\tilde{R} + \frac{\gamma}{2}) \geq 0$$

The spherical surface at $\tilde{r} = \tilde{r}_0$ remains a virtual singular surface, but until when the collapsing stellar surface $\tilde{R}$ hits upon it. After that instant, $\tilde{\tau} > \tilde{\tau}_0$, the whole region $S(0, \tilde{r}_0)$ becomes actually singular (and, obviously, trapped).

Notice that

for $\tilde{\tau} \rightarrow \tilde{\tau}_0$ we have $\tilde{R}(\tilde{\tau}) \rightarrow \tilde{r}_0$, and

$$\tilde{r}_{ah}(\tilde{\tau}) = \sqrt{\tilde{R}(\tilde{\tau})^3 + \tilde{\omega}(\tilde{R}(\tilde{\tau}) + \gamma/2)} \rightarrow 0$$

Equation of state of the collapsing star

Three unknowns: $a(\tau),\ \varrho(\tau),\ p(\tau)$

Three Equations: 1. (00) component of Einstein eq. $G_{\mu\nu} = 8\pi G(k) T_{\mu\upsilon}$

2. Energy-momentum conservation eq. $[8\pi G(k) T_{\mu\upsilon}]_{;\upsilon} = 0$

3. Equation of state of the "matter": $p = p(\varrho)$

Since $G(\tilde{r}) = \dfrac{\tilde{r}^3}{\tilde{r}^3 + \tilde{\omega}(\tilde{r} + \frac{\gamma}{2})}$ & $\alpha(\tilde{r}) = 1 + \dfrac{\tilde{\omega}(\tilde{r} + \frac{\gamma}{2})}{\tilde{r}^3}$ → $G(\tilde{R}) = \dfrac{1}{\alpha(\tilde{R})}$

(00) Einstein eq. $\left(\dfrac{\dot{a}}{a}\right)^2 = \left(\dfrac{\dot{a}\tilde{R}_c}{a\tilde{R}_c}\right)^2 = \left(\dfrac{\dot{\tilde{R}}}{\tilde{R}}\right)^2 = \dfrac{8\pi}{3} G(\tilde{R})\tilde{\rho}$

Junction conditions → EoM $\dot{\tilde{R}}(\tilde{\tau})^2 = \dfrac{1}{\tilde{R}\,\alpha(\tilde{R})}$

Density

$$\tilde{\rho}(\tilde{\tau}) = \frac{3}{8\pi \tilde{R}(\tilde{\tau})^3}$$

Continuity equation

$$\dot{\tilde{\rho}} + 3\left(\frac{\dot{\tilde{R}}}{\tilde{R}}\right)(\tilde{\rho} + \tilde{p}) = -\frac{\tilde{\rho}\dot{G}(\tilde{R})}{G(\tilde{R})}$$

→ surface pressure

$$\tilde{p}(\tilde{\tau}) = -\frac{\tilde{\omega}(2\tilde{R} + 3\gamma/2)}{8\pi \tilde{R}^3 \left(\tilde{R}^3 + \tilde{\omega}(\tilde{R} + \gamma/2)\right)}$$

Eliminating $\tilde{R}$ from the two expressions we get **the equation of state**

$$\tilde{p}(\tilde{\rho}) = -\frac{4\pi\tilde{\omega}\tilde{\rho}^2}{3}\left\{\frac{4\left(\frac{3}{8\pi\tilde{\rho}}\right)^{1/3} + 3\gamma}{3 + 4\pi\tilde{\omega}\tilde{\rho}\left[2\left(\frac{3}{8\pi\tilde{\rho}}\right)^{1/3} + \gamma\right]}\right\}$$

Notice:

- For $\tilde{\omega} = 0$ ⇒ Standard OS collapse in Schwarzschild background

  Eq. of state $\tilde{p} = 0$ i.e. DUST

- For $\tilde{\omega} > 0$ ⇒ Pressure **negative** $\tilde{p} < 0$ ⇒ **anti-gravitational effect** resist against the gravitational collapse. Exotic matter with negative pressure to avoid the singularity [Gravastar, Collapse with minimal length].

  ⇒ **Violation of energy condition(s)**

- **For** $\tilde{\omega} < 0$ ⇒ $\tilde{R} > \tilde{r}_0$

  ⇒ When $\tilde{R} \to \tilde{r}_0$ then $\tilde{\rho} \to \tilde{\rho}_c = \frac{3}{8\pi\tilde{r}_0^3}$

  and $\tilde{p}(\tilde{\rho}) \to +\infty$

  Therefore $0 < \tilde{\rho} < \tilde{\rho}_c$

  ⇒ Violation of DEC

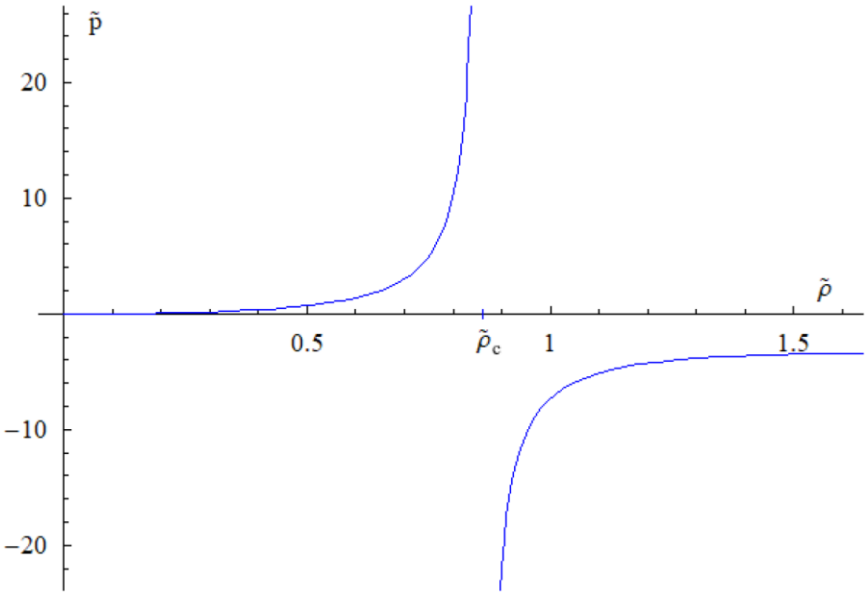


FIG. 18: For $\tilde{\omega} < 0$, the behavior of the star pressure $\tilde{p}(\tilde{\rho})$, Eq.(92) is plotted against density. The density cannot be larger than $\tilde{\rho}_c$, since $\tilde{R} > \tilde{r}_0$. For $0 < \tilde{\rho} < \tilde{\rho}_c$ the pressure is always positive. When $\tilde{\rho} \to \tilde{\rho}_c$, where there is a curvature singularity, the density is finite and the pressure goes to infinity. The region $\tilde{\rho} > \tilde{\rho}_c$ is unphysical.

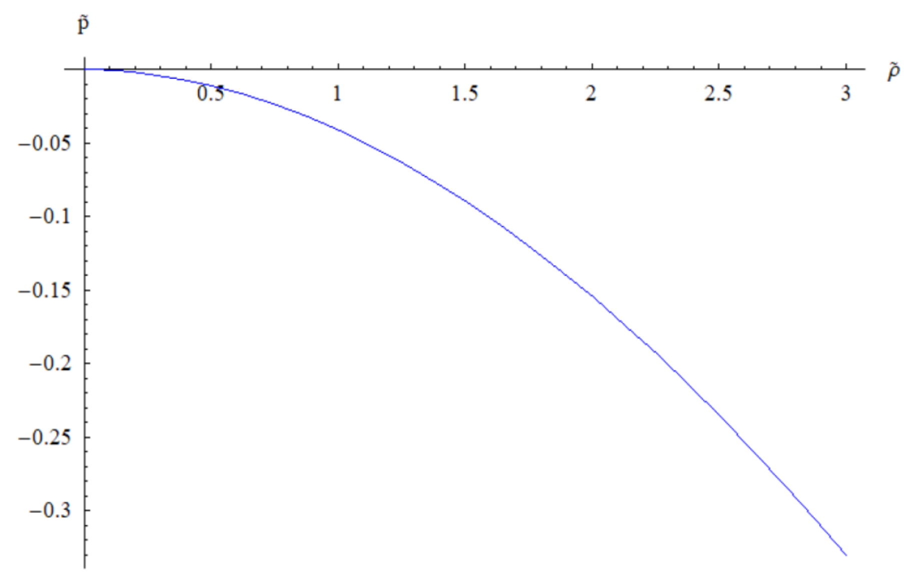


FIG. 19: For $\tilde{\omega} > 0$, the behavior of the star pressure $\tilde{p}(\tilde{\rho})$, Eq.(92), is plotted against density. The pressure is always negative, for any $\tilde{\rho} > 0$ and any $\tilde{\omega} > 0$.

Energy conditions

Energy-momentum tensor for a perfect fluid

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu} \qquad g_{\mu\nu}u^\mu u^\nu = -1$$

NEC (Null Energy Condition): $T_{\mu\nu}k^\mu k^\nu \geq 0$ $\quad k_\mu k^\mu = 0$ ⇒ $\rho + p \geq 0$

WEC (Weak Energy Cond.): $T_{\mu\nu}t^\mu t^\nu \geq 0$ $\quad t_\mu t^\mu \leq 0$ ⇒ $\rho \geq 0$, $\rho + p \geq 0$

DEC (Dominant En. Cond.): E-M current density should not exceed the speed of light ⇒ $\rho \geq |p|$

SEC (Strong Energy Cond.): Focussing effect on timelike geodesics ⇒ $\rho + p \geq 0$, $\rho + 3p \geq 0$

Clearly:

$$\begin{array}{lllll} \text{DEC} & \Rightarrow & \text{WEC} & \Rightarrow & \text{NEC} \\ \text{and} & & & & \\ \text{SEC} & \Rightarrow & \text{NEC} & & \end{array}$$

Case $\tilde{\omega} > 0$:

DEC: $\rho \geq |p|$ → $\dfrac{3}{8\pi\tilde{R}^3} \geq \dfrac{1}{8\pi\tilde{R}^3} \cdot \dfrac{\tilde{\omega}(2\tilde{R}+3\gamma/2)}{\tilde{R}^3+\tilde{\omega}(\tilde{R}+\gamma/2)}$ → $3\tilde{R}^3 + \tilde{\omega}\tilde{R} \geq 0$

ALWAYS TRUE.

Therefore WEC & NEC are obeyed too.

SEC: $\tilde{\rho} + 3\tilde{p} \geq 0$ → $\tilde{R}^3 \geq \tilde{\omega}(\tilde{R}+\gamma)$ → SEC violated for $0 < \tilde{R} < \tilde{r}_2$

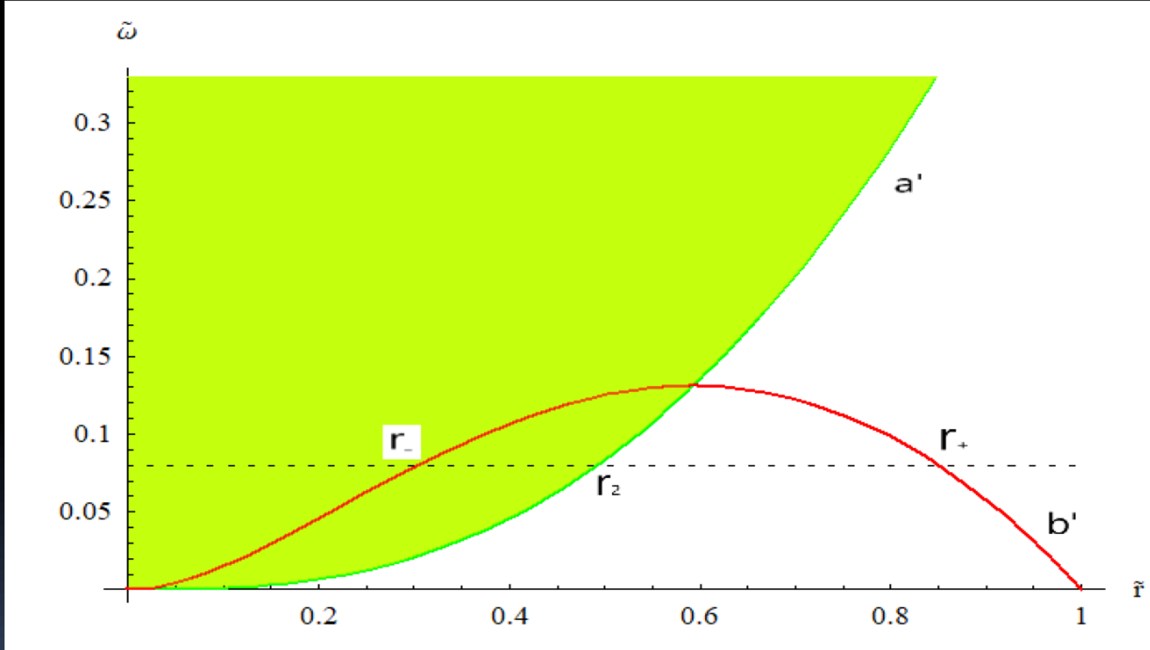


FIG. 20: For any given $\tilde{\omega} > 0$, SEC is violated for any $0 < \tilde{R} < \tilde{r}_2(\tilde{\omega})$, namely in the green region, whose boundary is line $a' \equiv \tilde{\omega}(\tilde{r}_2)$. For any given $0 < \tilde{\omega} < \tilde{\omega}_c$, we have two horizons $\tilde{r}_-$, $\tilde{r}_+$, and we always have $\tilde{r}_-(\tilde{\omega}) < \tilde{r}_2(\tilde{\omega}) < \tilde{r}_+(\tilde{\omega})$, so the region where SEC is violated is protected by the horizon $\tilde{r}_+$.

Case $\tilde{\omega} < 0$:

WEC:

$$\rho \geq 0$$

$$\rho + p \geq 0$$

$$\Rightarrow \quad \frac{3}{8\pi\tilde{R}^3} + \frac{1}{8\pi\tilde{R}^3} \cdot \frac{|\tilde{\omega}|(2\tilde{R} + 3\gamma/2)}{\tilde{R}^3 - |\tilde{\omega}|(\tilde{R} + \gamma/2)} \geq 0$$

which is always true for any $\tilde{\omega} < 0$ and any $\tilde{R} > \tilde{r}_0$. ⇒ WEC & NEC are always fulfilled for any $\tilde{\omega} < 0$.

SEC:

$$\rho + p \geq 0$$

$$\rho + 3p \geq 0$$

$$\Rightarrow \quad \frac{3}{8\pi\tilde{R}^3} \cdot \frac{\tilde{R}^3 - \tilde{\omega}(\tilde{R} + \gamma)}{\tilde{R}^3 + \tilde{\omega}(\tilde{R} + \gamma/2)} \geq 0$$

which is always true for any $\tilde{\omega} < 0$ and any $\tilde{R} > \tilde{r}_0$

Case $\tilde{\omega} < 0$:

DEC: $\tilde{\rho} \geq |\tilde{p}|$ ⇒ $\frac{3}{8\pi\tilde{R}^3} \geq \frac{1}{8\pi\tilde{R}^3} \cdot \frac{|\tilde{\omega}|(2\tilde{R}+3\gamma/2)}{\tilde{R}^3 - |\tilde{\omega}|(\tilde{R}+\gamma/2)}$ ⇒ $\tilde{R}^3 \geq \frac{5}{3}|\tilde{\omega}|\left(\tilde{R}+\frac{3}{5}\gamma\right)$

⇒ $0 < \tilde{r}_0(\tilde{\omega}) < \tilde{r}_1(\tilde{\omega})$ ⇒

For $\tilde{R} \geqslant \tilde{r}_1(\tilde{\omega})$ DEC is fulfilled;

For $\tilde{r}_0(\tilde{\omega}) \leqslant \tilde{R} < \tilde{r}_1(\tilde{\omega})$ DEC is violated.

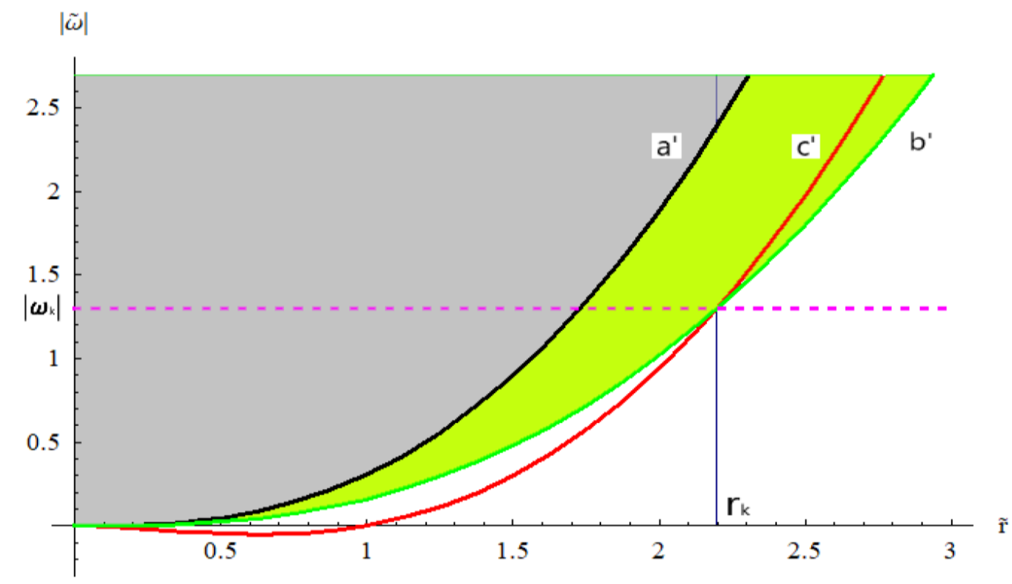


FIG. 22: Black line a': location of the singularity $\tilde{r}_0$. Green line b': border of the DEC violation green region $\tilde{r}_1$. Red line c': location of the event horizon $\tilde{r}_+$. When $0 < |\tilde{\omega}| < |\tilde{\omega}_k|$ (i.e. when $\tilde{\omega}_k < \tilde{\omega} < 0$) the green region, where DEC is violated, is actually protected by the outer horizon $\tilde{r}_+$. But for $|\tilde{\omega}| > |\tilde{\omega}_k|$ (i.e. for $\tilde{\omega} < \tilde{\omega}_k < 0$) the DEC violation can happen also in observable regions.

Violation of DEC not a surprise since $\tilde{R} \to \tilde{r}_0$

$\tilde{\rho} \to \tilde{\rho}_c$; $\tilde{p}(\tilde{\rho}) \to +\infty$

Speed of sound in a fluid

$$v_s^2 = \frac{\partial p}{\partial \rho}\bigg|_{\sigma\ const}$$

$v_s \leq 1$

$p \leq \rho$

Violation of DEC in a fluid signals speed of sound can exceed speed of light ⇒ lower bound for $\tilde{\omega}$

$$\tilde{\omega}_k \leq \tilde{\omega} < 0$$

CONCLUSIONS

- The SDG(ASG) approach suggests a general metric describing quantum effects (ħ)
- We explored both values $\widetilde{\omega} > 0$ and $\widetilde{\omega} < 0$ (the negative suggested by the matching of a SDG-metric with Donoghue quantum corrections to Schwarzschild metric).
- Gravitational collapse in the framework of an OS-like model. The inner geometry is described by a spatially flat FLRW metric, then joined smoothly to the outer AS/SD black hole metric. The star's density and pressure are determined by the smooth matching of the geometries at the surface of the star (junction conditions).
- For $\widetilde{\omega} > 0$ the BH is regular, the curvature scalar and metric remain finite in the limit of zero radius, and the gravitational collapse takes an infinite proper time.
- For $\widetilde{\omega} < 0$ there is a curvature singularity at a finite radial coordinate $\tilde{r}_0 > 0$.
- We studied the dynamics of interior apparent and event horizons and of the stellar surface during the collapse.
- The equation of state shows that when $\widetilde{\omega} > 0$ the pressure is always negative. For $\widetilde{\omega} < 0$ there exists a maximum value for the density. For values of the density smaller than ρ_c the pressure is positive. When $\rho \rightarrow \rho_c$, the pressure goes to infinity. This infinity for pressure happens at the when $R \rightarrow r_o$.
- Finally, the energy conditions. For $\widetilde{\omega} > 0$, a possible violation of SEC is anyway protected by the outer horizon r_+ , at least until horizons exist. For $\widetilde{\omega} < 0$, we found that a violation of DEC strongly suggests to impose a lower bound on the negative values of ω.