# Cosmological constant and Dark Dimension scenario

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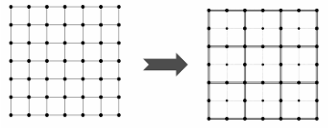
C. Branchina, V. Branchina, FP, PRD 108 (2023) 4, 045007

C. Branchina, V. Branchina, FP, A. Pernace, arXiv:2308.16548, accepted in JGMMP

C. Branchina, V. Branchina, FP, A. Pernace, arXiv:2404.10068, accepted in JGMMP

# Wilson - Effective Field Theory paradigm



Theory at $\Lambda$: $S_{\Lambda}$ $\rightarrow$ Theory at $\Lambda/2$: $S_{\Lambda/2}$ $\rightarrow$ ... $\rightarrow$ $\Gamma$

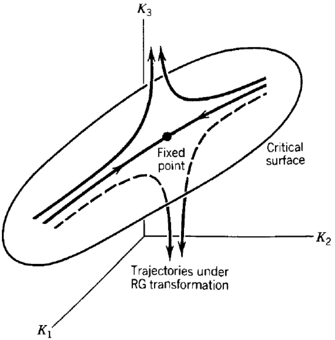
Progressive evaluation of fluctuations, physical running scale $\Lambda \rightarrow \Lambda/2 \rightarrow \Lambda/4 \rightarrow \Lambda/8 \rightarrow ...$

Piling up of fluctuations $\rightarrow$ Evolution of parameters

**Theoretical foundation of EFT paradigm: any QFT is an EFT**

- Contain an ultimate UV scale $\Lambda$
- $E > \Lambda$: UV completion (microscopic fluctuations)
- $E < \Lambda$: QFT effective, EFT (persistent fluctuations on all scales)

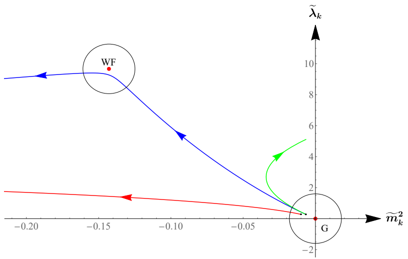
# Renormalized theory



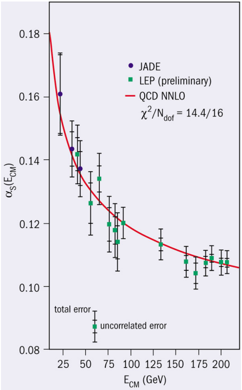

Renormalized theory: defined around a fixed point (critical surface)

# In any dimesion: ..., $D = 3$, $D = 4$, $D = 4 + d$ ...

$D = 3$ dimensions : Wilson-Fisher




$D = 4$ dimensions : AF




Also for theories with $D > 4$ dimesions ... in particular...

Kaluza-Klein theories: $D = 4 + d$

# EFTs with compact dimensions : $D = 4 + d$

- Field Theories with compact extra dimensions are ubiquitous
- Typically studied as $4D$ theories with infinite* towers of $4D$ states:

$$m_n = f_n \, m_{\text{KK}}$$

- Surprising UV-softness :

**Vacuum Energy / Effective Potential @ 1l $\sim m_{\text{KK}}^4$**

$V_{1l}$ with cutoff $\Lambda$ for the $5D$ momentum $\widehat{p}$ : independent mode approximation of $U_k(\phi)$ in LPA

## How is this possible? Why not $\sim \Lambda^4$?

* Sometimes truncated in a way that is equivalent (see later)

# Example : Scherk-Schwarz

$5D$ SUSY theory $\mathcal{S}_{(5)}$ defined on multiply connected spacetime $\mathcal{M}^4 \times S^1$

- Different R-charges for superpartners $(i = b, f)$

$$\Psi_i(x, z + 2\pi R) = e^{2\pi i q_i} \Psi_i(x, z) \;\Rightarrow\; \Psi_i(x, z) = \sum_{n=-\infty}^{+\infty} \frac{\psi_{i,n}(x) e^{i \frac{n+q_i}{R} z}}{\sqrt{2\pi R}}$$

$\int dz\, \mathcal{L}_{(5)} \to \mathcal{L}_{(4)}$ infinite tower of KK fields, $m_{i,n}^2 \propto \frac{(n+q_i)^2}{R^2} \equiv (n+q_i)^2\, m_{\mathrm{KK}}^2$

- 4D "masses" mismatch: effective 4D non-local soft SUSY breaking

**Higgs field** $\phi$ : $\phi_0$ , or $4D$ brane field , or ...

Effective 4D quadratic operator

$$M_{i,n}^2(\phi) = m^2(\phi) + \frac{(n+q_i)^2}{R^2}$$

$m$: same for boson and fermion superpartners, $q$: different

# One-loop Higgs Effective Potential (4D calculation)

$$V_{1l}^{(4)}(\phi) = \frac{1}{2}\sum_a \sum_{i_a} (-1)^{\delta_{i_a,f_a}} \sum_{n=-\infty}^{\infty} \int \frac{d^4p}{(2\pi)^4} \log\left(p^2 + m_a^2(\phi) + \left(\frac{n+q_{i_a}}{R}\right)^2\right)$$

One way of doing the calculation (not the only one)*:

- (First) infinite sum; (then) integrate $d^4p$ with cutoff $\Lambda$

Antoniadis, Dimopoulos, Pomarol, Quiros/Delgado, Pomarol, Quiros/Barbieri, Hall, Nomura/Arkani-Hamed, Hall, Nomura, Smith, Weiner

Each tower contributes :

$$V_{1l}^{(4)}(\phi) = R\left(\frac{m^2\Lambda^3}{48\pi} - \frac{m^4\Lambda}{64\pi} + \frac{m^5}{60\pi}\right) - \sum_{k=1}^{\infty} \frac{e^{-2\pi k m R}(2\pi k m R(2\pi k m R + 3) + 3)\cos(2\pi k q/R)}{64\pi^6 k^5 R^4}$$

* Other methods, Proper time (Antoniadis, Quiros), Pauli-Villars (Contino, Pilo), Thick brane (Delgado, von Gersdorff, John, Quiros), all give the same result, see later

# A closer look to this potential

From each tower the Higgs Potential receives the contribution

$$V_{1l}^{(4)}(\phi) = R\left(\frac{m^2\Lambda^3}{48\pi} - \frac{m^4\Lambda}{64\pi} + \frac{m^5}{60\pi}\right) - \sum_{k=1}^{\infty} \frac{e^{-2\pi k m R}(2\pi k m R(2\pi k m R + 3) + 3)\cos(2\pi k q)}{64\pi^6 k^5 R^4}$$

- Power UV-sensitivity through $m$ $\implies$ canceled by SUSY
- No UV-sensitivity through $q$

$\implies$ Finite Higgs potential

$$V_{1l}(\phi) \sim R^{-4} \equiv m_{\mathrm{KK}}^4$$

# Old Times ∼ 2000


- UV-insensitive Higgs mass!
- UV-insensitive Higgs potential!

Criticism : sum $[-L, L] \rightarrow$ UV-sensitive terms Ghilencea, Nilles/Kim

... Heated debate! ...

Calculations done in a different setup, proper time, thick brane, Pauli-Villars, dimensional regularization all seem(ed) to confirm UV-insensitive result

Debate closed in favour of UV-insensitiveness* ... but ...

* In the absence of FI terms

Ghilencea, Groot-Nibbelink, Nilles

# $5D$ calculation from the outset in a toy model

$$\mathcal{S}_{(5)} = \int dz\, d^4x \left( \frac{1}{2}\,\partial_a\widehat{\Phi}\,\partial^a\widehat{\Phi} + \partial_a\widehat{\chi}\,\partial^a\widehat{\chi}^\dagger + \frac{m_\Phi^2}{2}\,\widehat{\Phi}^2 + m_\chi^2\,\widehat{\chi}\widehat{\chi}^\dagger + \frac{\widehat{\lambda}}{4!}\,\widehat{\Phi}^4 + \frac{\widehat{g}}{2}\,\widehat{\Phi}^2\widehat{\chi}\widehat{\chi}^\dagger \right)$$

$$\widehat{\Phi}(x, z + 2\pi R) = \widehat{\Phi}(x, z) \quad ; \quad \widehat{\chi}(x, z + 2\pi R) = e^{2\pi i q}\,\widehat{\chi}(x, z)$$

$q \equiv q' - [q'] \to q \in [0, 1]$

(similar for $\widehat{\Phi}$)

Fourier expansion of $\widehat{\chi}(x, z)$: EFT up to $\Lambda$

$$\widehat{\chi}(x, z) = e^{iq\frac{z}{R}} \left( \sum_n \int \frac{d^4p}{(2\pi)^5 R} \right)' \widehat{\chi}_{n,p}\, e^{i\left(p\cdot x + n\frac{z}{R}\right)}$$

$$\left( \frac{1}{2\pi R} \sum_n \int \frac{d^4p}{(2\pi)^4} \right)' \equiv \frac{1}{2\pi R} \sum_{n=-[R\Lambda]}^{[R\Lambda]} \int^{C_\Lambda^n} \frac{d^4p}{(2\pi)^4}, \quad C_\Lambda^n \equiv \sqrt{\Lambda^2 - \frac{n^2}{R^2}}$$

$$\widehat{\chi}(x, z) = e^{iq\frac{z}{R}} \sum_{n=-[R\Lambda]}^{[R\Lambda]} \frac{\chi_n^\Lambda(x)\, e^{in\frac{z}{R}}}{\sqrt{2\pi R}}; \quad \chi_n^\Lambda(x) \equiv \frac{1}{\sqrt{2\pi R}} \int^{C_\Lambda^n} \frac{d^4p}{(2\pi)^4}\, \widehat{\chi}_{n,p}\, e^{ip\cdot x}$$

# 4D Effective Potential from 5D Effective Potential

$$\mathcal{V}^{(5)}_{1l}(\widehat{\Phi}) = \frac{1}{2}\mathrm{Tr}_5 \log \frac{p^2 + \frac{n^2}{R^2} + m_\phi^2 + \frac{\widehat{\lambda}}{2}\widehat{\Phi}^2}{p^2 + \frac{n^2}{R^2}} + \frac{1}{2}\mathrm{Tr}_5 \log \frac{p^2 + \left(\frac{n}{R} + q\right)^2 + m_\chi^2 + \frac{\widehat{g}}{2}\widehat{\Phi}^2}{p^2 + \frac{n^2}{R^2}}$$

- $p$ & $n$ intertwined: NO hierarchy when including asymptotics

$$\mathrm{Tr}_5 = \left(\frac{1}{2\pi R}\sum_n \int \frac{d^4p}{(2\pi)^4}\right)' = \frac{1}{2\pi R}\sum_{n=-[R\Lambda]}^{[R\Lambda]} \int^{C^n_\Lambda} \frac{d^4p}{(2\pi)^4}$$

Performing $z$ integration $\rightarrow$ effective $V^{(4)}_{1l}(\phi)$ with $\phi = \phi_0$

$$V^{(4)}_{1l}(\phi) = \frac{1}{2}\sum_{n=-[R\Lambda]}^{[R\Lambda]} \int^{C^n_\Lambda} \frac{d^4p}{(2\pi)^4}\left(\log \frac{p^2 + \frac{n^2}{R^2} + m_\phi^2 + \frac{\lambda}{2}\phi^2}{p^2 + \frac{n^2}{R^2}} + \log \frac{p^2 + \left(\frac{n+q}{R}\right)^2 + m_\chi^2 + \frac{g}{2}\phi^2}{p^2 + \frac{n^2}{R^2}}\right)$$

$\lambda \equiv \frac{\widehat{\lambda}}{2\pi R}$ ; $g \equiv \frac{\widehat{g}}{2\pi R}$ ; $\widehat{\Phi} = \frac{\phi}{\sqrt{2\pi R}}$

$$V^{(4)}_{1l}(\phi) = 2\pi R\,\mathcal{V}^{(5)}_{1l}(\widehat{\Phi})$$

only if we respect the asymptotics

# UV-sensitivity and non-trivial topology

Performing the calculation this way

$$V_{1l}(\phi) = \frac{5m^2 + 3\frac{q^2}{R^2}}{180\pi^2} R\Lambda^3 - \frac{35m^4 + 14m^2\frac{q^2}{R^2} + 3\frac{q^4}{R^4}}{840\pi^2} R\Lambda + \frac{m^5 R}{60\pi} - \sum_{k=1}^{\infty} \frac{e^{-2\pi k m R}(2\pi k m R(2\pi k m R + 3) + 3)\cos(2\pi k q/R)}{64\pi^6 k^5 R^4}$$

New $q$-dependent UV-sensitive terms:

- NOT canceled by SUSY! $\propto \boxed{(q_b^2 - q_f^2)}\, m^2(\phi)\Lambda$
- Topological origin
  1. $= 0$ for $q = 0$ ($q\,\exists$ in multiply connected spacetime)
  2. $q \in [0,1]$: $q$-dependent UV terms $\to 0$ in decompactification limit ("$R \to \infty$")
  3. UV-insensitive terms: $\neq 0$ for $q = 0$ ($\to 0$ for $R \to \infty$)

# Alternatively : Infinite sum & Smooth cut

Typical argument: cut on sum $\rightarrow$ spurious "divergences" ... But ...

$$V_{1l}(\phi) = \frac{1}{2}\sum_{n=-\infty}^{\infty}\int \frac{d^4p}{(2\pi)^4}\log\left(\frac{p^2+m^2+\frac{(n+q)^2}{R^2}}{p^2+\frac{n^2}{R^2}}\right) e^{-\frac{p^2+\frac{n^2}{R^2}}{\Lambda^2}}$$

$\Rightarrow$ **Same result** is found

UV-sensitive terms are **NOT** due to the sharp cut of the sum!
They come from a **careful treatment of $\widehat{p}$ asymptotics**

**So ... why do "Proper time", "Thick brane" and "Pauli-Villars"**

**give UV-insensitive results ?**

# Secret liaison between proper time , thick brane & PV

Thick brane: $\sum_{n=-\infty}^{\infty} \int^{(\Lambda)} \frac{d^4p}{(2\pi)^4} \frac{e^{-\frac{(n+q)^2}{R^2\Lambda^2}}}{p^2+m^2+\left(\frac{n}{R}+q\right)^2}$ — Delgado, von Gersdorff, John, Quiros

Pauli-Villars: $\sum_{n=-\infty}^{\infty} \int \frac{d^4p}{(2\pi)^4} \frac{\Lambda^4}{\Lambda^4+\left(p^2+(\frac{n+q}{R})^2\right)^2} \frac{1}{p^2+m^2+\left(\frac{n+q}{R}\right)^2}$ — Contino, Pilo

Proper Time: — Antoniadis, Quiros

$$V_{1l}^{(4)}(\phi) = -\sum_{n=-\infty}^{\infty} \int \frac{d^4p}{(2\pi)^4} \int_{\frac{1}{\Lambda^2}}^{\infty} \frac{ds}{s} e^{-s\left(p^2+m^2+\left(\frac{n+q}{R}\right)^2\right)}$$

$$= -\sum_{n=-\infty}^{\infty} \int \frac{d^4p}{(2\pi)^4} \Gamma\left(0, \frac{p^2+m^2+\left(\frac{n+q}{R}\right)^2}{\Lambda^2}\right)$$

> Smooth cut function of $\frac{n+q}{R}$: artificial re-absorption of $q$

**Equivalent to introduce a hierarchy between $(p_1, p_2, p_3, p_4)$ and $p_5$**

$\Rightarrow$ **Again : artificial wash-out of UV-sensitive terms**

# First take-home message

$V_{1l}(\phi)$ is UV-sensitive even with SUSY

Due to the non-trivial topology of the spacetime

Both with hard and smooth cutoff

Now ... we're ready for the Cosmological Constant ...

# The Dark Dimension

# Swampland conjectures and experimental bounds

Swampland ingredients: Montero, Vafa, Valenzuela

- (A)dS distance conjecture: when $\Lambda_{\rm cc} \to 0$ Lüst, Palti, Vafa

$$\mu_{tow} \sim \left|\frac{\Lambda_{\rm cc}}{M_P^4}\right|^{\alpha} M_P$$

$\Lambda_{\rm cc}$ physical vacuum energy

- Emergent string conjecture: $\mu_{tow} = m_{\rm KK}$ or $\mu_{tow} = M_s$ Lee, Lerche, Weigand
- 1l string calculations: $\rho_4 \sim M_s^4 \quad (\to \rho_4 \sim \mu_{\rm tow}^4)$
- Higuchi bound $\alpha \leq 1/2$ Higuchi

$$\Rightarrow \frac{1}{4} \leq \alpha \leq \frac{1}{2} \Leftarrow \text{Assumed as starting point for DD proposal}$$

Experimental bounds on violations of $\frac{1}{r^2}$ Newton's law : $\mu_{tow} \gtrsim 6.6$ meV

Energy scale associated to $\Lambda_{\rm cc}$: $\Lambda_{\rm cc}^{1/4} \sim 2.31$ meV

$$\Rightarrow \alpha = \frac{1}{4}, \text{ “experimental value”: } \mu_{tow}^{exp} \sim \mathrm{meV} \ (\sim \text{neutrino scale})$$

# The Dark Dimension

In principle $\mu_{tow} = M_s$ possible, but ... "ruled out by experiments":

"we can describe physics above the neutrino scale with EFT", no sign of string excitations at these scales

Only possibility left: EFT decompactification scenario

$$m_{\mathrm{KK}} \sim \mu_{tow}^{exp} \sim \mathrm{meV}$$

This conclusion takes us to EFT: DD takes place in the (deep) EFT realm

Assuming the DD, i.e. $\Lambda_{\mathrm{cc}} \sim m_{\mathrm{KK}}^4$ true prediction of string theory

- EFT reproduces it: ✓
- EFT does not: Attention needs to be paid!
  1. Can we put the pieces together? How? How to frame it?
  2. Is there really a string theory realizing the DD in our Universe?

# Set-up: $(4+1)D$ theory with gravity

Compactification with gravity $\widehat{g}_{MN} = \begin{pmatrix} e^{2\alpha\phi} g_{\mu\nu} - e^{2\beta\phi} A_\mu A_\nu & e^{2\beta\phi} A_\mu \\ e^{2\beta\phi} A_\nu & -e^{2\beta\phi} \end{pmatrix}$

Background configuration $g^0_{\mu\nu} = \eta_{\mu\nu}, A_\mu = 0, \phi = \phi_0$ (hereafter $\phi$)

$$M^2_{i,n}(\phi,\varphi) = m^2 e^{2\alpha\phi} + \frac{(n+q_i)^2}{R^2} e^{6\alpha\phi}$$

$e^{6\alpha\phi} R^{-2} \equiv R_\phi^{-2} \equiv m^2_{\text{KK}}$

$\phi$ radion, $\varphi$ matter field

Fourier expansion $\rightarrow$ cutoff $\Lambda$ (spherical):

$$\hat{p}^2 \leq \Lambda^2 \quad \rightarrow \quad p^2 + \frac{n^2}{R_\phi^2} \leq \Lambda^2\, e^{2\alpha\phi} \quad \left(= m_{\text{KK}}^{1/3} R^{1/3} \Lambda \equiv \Lambda_\phi^2\right)$$

$\Lambda_\phi < \Lambda$ is

- Cutoff for the rescaled momenta
- Cutoff for $4D$ brane fields ($\Lambda_{\text{SM}} = \Lambda_\phi$)

# One-loop vacuum energy

Contribution of a bulk field:

$$\rho_4^{1l} = \frac{5\log\frac{\Lambda^2 e^{2\alpha\phi}}{\mu^2} - 2}{300\pi^2} e^{2\alpha\phi} R\Lambda^5 + \frac{5m^2 + 3\frac{q^2}{R^2}e^{4\alpha\phi}}{180\pi^2} e^{2\alpha\phi} R\Lambda^3 - \frac{35m^4 + 14m^2\frac{q^2}{R^2}e^{4\alpha\phi} + 3\frac{q^4}{R^4}e^{8\alpha\phi}}{840\pi^2} e^{2\alpha\phi} R\Lambda + \frac{m^5}{60\pi} e^{2\alpha\phi} R + \frac{3\log\frac{\Lambda^2 e^{2\alpha\phi}}{\mu^2} + 2}{2880\pi^2 R^4} e^{10\alpha\phi} R\Lambda + R_4 + \mathcal{O}(\Lambda^{-1}) = 2\pi R\, e^{2\alpha\phi} \rho_5^{1l}$$

$$R_4 = -\frac{x^2 \mathrm{Li}_3\left(re^{-x}\right) + 3x\mathrm{Li}_4\left(re^{-x}\right) + 3\mathrm{Li}_5\left(re^{-x}\right) + 6\zeta(5)}{128\pi^6 R^4} e^{12\alpha\phi} + h.c.$$

$$r \equiv e^{2\pi i q} \quad , \quad x \equiv 2\pi e^{-2\alpha\phi} R\sqrt{m^2} \implies R_4 \propto \frac{e^{12\alpha\phi}}{R^4} = m_{KK}^4$$

As for $V_{1l}$, $q$-terms are absent in the literature

# One-loop vacuum energy

Most "divergent" terms:

- SUSY: $\rho_4^{1l} \sim \left(q_b^2 - q_f^2\right) e^{6\alpha\phi} R^{-1} \Lambda^3 = \left(q_b^2 - q_f^2\right) m_{KK}^2 R \Lambda^3$
- NON-SUSY: $\rho_4^{1l} \sim e^{2\alpha\phi} R \Lambda^5 = m_{KK}^{2/3} \left(R^{\frac{1}{3}} \Lambda\right)^5$

$\rho_4^{1l} \sim m_{KK}^4$ has divergences that do not disappear even in as SUSY theory

Even in the swampland scenario, that requires the light tower limit $\phi \to -\infty$, no term can overthrow these contributions
No light tower regime where $\rho_4^{1l} \sim m_{KK}^4$

# What is the lesson?

In a $(4+1)D$ EFT quantum fluctuations "heavily" dress $\rho_4$

No automatic result $\rho_4 = \Lambda_{\rm cc} \sim m_{KK}^4$ (as often claimed)

To reach $\rho_4 = \Lambda_{\rm cc} \sim m_{KK}^4$ fine-tuning is needed

$\Rightarrow$ even if we believe the "swampland" conjectured

$$\rho_4 = \Lambda_{\rm cc} \sim m_{KK}^4$$

there is an issue of matching between this finite result for $\rho_4$

and the EFT result

unless we resort to this fine tuning

# Summary & Conclusions

- Usual calculations mistreat the asymptotics of the loop momenta
- Careful treatment of loop momenta unveils the presence of UV-sensitive terms of topological origin, previously missed

Our first conclusions

- No solution to the naturalness/hierarchy problem
- No solution to the CC problem
- Fine tuning and renormalization are required
- Is it possible to put pieces together?

To put things together ...

- Can this fine-tuning result from piling up of quantum fluctuations?

# Backup slides

# Computation of the one-loop potential $(i = b, f)$

$$V^i_{1l}(\phi) = \frac{1}{2}\sum_{n=-L}^{L}\int^{\Lambda}\frac{d^4p}{(2\pi)^4}\log\frac{p^2+M^2+(\frac{n}{R}+q_i)^2}{p^2+\frac{n^2}{R^2}}$$

$$= \sum_{n=-L}^{L}\frac{1}{64\pi^2}\left[\Lambda^4\log\frac{\Lambda^2+M^2+\left(\frac{n}{R}+q_i\right)^2}{\Lambda^2+\frac{n^2}{R^2}}+\Lambda^2\left(M^2+\left(\frac{n}{R}+q_i\right)^2-\frac{n^2}{R^2}\right)\right.$$

$$\left.+\left(M^2+\left(\frac{n}{R}+q_i\right)^2\right)^2\log\frac{M^2+\left(\frac{n}{R}+q_i\right)^2}{\Lambda^2+M^2+\left(\frac{n}{R}+q_i\right)^2}-\frac{n^4}{R^4}\log\frac{\frac{n^2}{R^2}}{\Lambda^2+\frac{n^2}{R^2}}\right]\equiv\sum_{n=-L}^{L}F(n). \quad (1)$$

Euler-McLaurin (EML) formula

$$V^i_{1l}(\phi) = \int_{-L}^{L}dx\,F(x)+\frac{F(L)+F(-L)}{2}+\sum_{k=1}^{r}\frac{B_{2k}}{(2k)!}\left(F^{(2k-1)}(L)-F^{(2k-1)}(-L)\right)+R_{2r}, \quad (2)$$

with $r$ is an integer, $B_n$ the Bernoulli numbers, and the rest $R_{2r}$ is

$$R_{2r} = \sum_{k=r+1}^{\infty}\frac{B_{2k}}{(2k)!}\left(F^{(2k-1)}(L)-F^{(2k-1)}(-L)\right)=\frac{(-1)^{2r+1}}{(2r)!}\int_{-L}^{L}dx\,F^{(2r)}(x)B_{2r}(x-[x]), \quad (3)$$

$B_n(x)$ Bernoulli polynomials, $[x]$ integer part of $x$.

- If in (1), (2) and (3) we send $L \to \infty$ while keeping $\Lambda$ fixed, we get for $V^i_{1l}(\phi)$ the usual UV-insensitive (finite) result.

- To properly take into account the asymptotics of the loop momenta $p^{(5)} = (p_1, p_2, p_3, p_4, n/R)$, we include them in (1) keeping

$$\frac{L}{R\Lambda} \quad \text{finite} \quad \text{when} \quad L, \Lambda \to \infty \,. \tag{4}$$

- From the physical meaning of the UV cuts: only values of $M$ and $q_i$ that fulfill the conditions

$$M^2, q_i^2 \ll \Lambda^2, L^2/R^2 \,. \tag{5}$$

- The conditions (4) and (5) are easily implemented in our calculations if we write ($\xi$ dimensionless finite number).

$$L = \xi R\Lambda \,, \tag{6}$$

and expand each term in (2) for $M^2/\Lambda^2, q_i^2/\Lambda^2 \ll 1$. We get

$$V_{1l}(\phi) = \frac{2M^2 \tan^{-1}\xi + \xi\left(\xi^2 \log\frac{\xi^2}{\xi^2+1} + 1\right)\left(M^2 + 3q_i^2\right)}{48\pi^2} R\Lambda^3$$
$$+ \frac{\xi^2\left(M^2+3q_i^2\right) + \xi^2\left(\xi^2+1\right)\left(M^2+3q_i^2\right)\log\frac{\xi^2}{\xi^2+1} + M^2 + q_i^2}{32\pi^2\left(\xi^2+1\right)}\Lambda^2$$
$$+ \frac{\xi\, M^2\left(6q_i^2R^2+1\right)\left(\xi^2+1\right) + \xi\, q_i^2\left(q_i^2R^2+1\right)\left(3\xi^2+5\right)}{96\pi^2\left(\xi^2+1\right)^2}\frac{\Lambda}{R}$$
$$+ \frac{\xi \log\frac{\xi^2}{\xi^2+1}\left(3R^2\left(M^2+q_i^2\right)^2 + M^2 + 3q_i^2\right) - 3M^4R^2\tan^{-1}\xi}{96\pi^2}\frac{\Lambda}{R}$$
$$+ \frac{3\left(\xi^2+1\right)^2 M^4 + 6\left(\xi^4+4\xi^2+3\right)M^2q_i^2 + \left(3\xi^4+6\xi^2+11\right)q_i^4}{192\pi^2\left(\xi^2+1\right)^3}$$
$$+ \frac{16\pi M^5 R + 15\log\frac{\xi^2}{\xi^2+1}\left(M^2+q_i^2\right)^2}{960\pi^2} + R_2 + \mathcal{O}\left(\Lambda^{-1}\right). \quad (7)$$

To compare (7) with the usual calculations, we take limit $\xi \to \infty$, with $\Lambda$ kept finite

$$V_{1l}^i(\phi) \sim \frac{R\Lambda^3 M^2}{48\pi} - \frac{R\Lambda M^4}{64\pi} + \frac{RM^5}{60\pi} + \widetilde{R}_2 + \mathcal{O}\left(\xi^{-1}\right). \quad (8)$$

with

$$\widetilde{R}_2 \equiv \lim_{\xi\to\infty} R_2 = \frac{3\zeta(5)}{64\pi^6 R^4} - \frac{1}{128\pi^6 R^4}\left[x^2\mathrm{Li}_3\left(r_i e^{-x}\right) + 3x\mathrm{Li}_4\left(r_i e^{-x}\right) + 3\mathrm{Li}_5\left(r_i e^{-x}\right) + h.c.\right].$$

# Vacuum energy calculation

Relation between the cutoff $\Lambda$ of the $(4+1)$D theory and the 4D cutoff $\Lambda_{\rm SM}$ of the Standard Model. $(4+1)$D theory, with compact space dimension in the shape of a circle of radius $R$, defined by

$$\mathcal{S} = \mathcal{S}_{\rm grav} + \mathcal{S}_{\rm mat} \tag{9}$$

$$\mathcal{S}_{\rm grav} = \frac{1}{2\hat{\kappa}^2} \int d^4x dz \sqrt{\hat{g}} \left(\hat{\mathcal{R}} - 2\hat{\Lambda}_{cc}\right) \tag{10}$$

is the $(4+1)$D Einstein-Hilbert action and as an example for the matter action we take

$$\mathcal{S}_{\rm mat} = \int d^4x dz \; \sqrt{\hat{g}} \left(\hat{g}^{MN} \partial_M \hat{\Phi}^* \partial_N \hat{\Phi} - m^2 |\hat{\Phi}|^2\right) , \tag{11}$$

with $\hat{\Phi}$ a $(4+1)$D scalar field that obeys the boundary condition $\hat{\Phi}(x, z + 2\pi R) = \hat{\Phi}(x, z)$. We indicate with $x$ the 4D coordinates and with $z$ the coordinate along the compact dimension. Using the signature $(+, -, -, -, -)$, the $(4+1)$D metric is parametrized as

$$\hat{g}_{MN} = \begin{pmatrix} e^{2\alpha\phi} g_{\mu\nu} - e^{2\beta\phi} A_\mu A_\nu & e^{2\beta\phi} A_\mu \\ e^{2\beta\phi} A_\nu & -e^{2\beta\phi} \end{pmatrix} \tag{12}$$

$A_\mu$ is the graviphoton and $\phi$ the radion field. Considering only zero modes for $\hat{g}_{MN}$, i.e. $g_{\mu\nu}(x)$, $A_\mu(x)$ and $\phi(x)$ only depend on $x$. Integrating over $z$, for the 4D gravitational action $\mathcal{S}^{(4)}_{\rm grav}$ we get

$$\mathcal{S}^{(4)}_{\rm grav} = \frac{1}{2\kappa^2} \int {\rm d}^4x \sqrt{-g} \left[\mathcal{R} - 2e^{2\alpha\phi}\hat{\Lambda}_{cc} + 2\alpha\Box\phi + \frac{(\partial\phi)^2}{2} - \frac{e^{-6\alpha\phi}}{4} F^2\right] , \tag{13}$$

where the 4D constant $\kappa = M_P^2$ is related to the $(4+1)$D $\hat{\kappa} = \hat{M}_P^3$ through the relation $\kappa^2 = \hat{\kappa}^2/(2\pi R)$.

The fields $\phi$ and $A_\mu$ in the above equation are dimensionless (dimensionful fields are obtained through the redefinition $\phi \to \phi/(\sqrt{2}\kappa)$, $A_\mu \to A_\mu/(\sqrt{2}\kappa)$), and we used $2\alpha + \beta = 0$. The canonical kinetic term in (13) for the radion field is obtained taking $\alpha = 1/\sqrt{12}$.
Considering the Fourier decomposition of $\hat{\Phi}(x, z)$, for the $4D$ matter action (11) we have

$$\mathcal{S}^{(4)}_{\rm mat} = \int d^4x \sqrt{-g} \sum_n \left[ |D\varphi_n|^2 - \left( e^{\sqrt{\frac{2}{3}}\frac{\phi}{M_P}} m^2 + e^{\sqrt{6}\frac{\phi}{M_P}} \frac{n^2}{R^2} \right) |\varphi_n|^2 \right] , \qquad (14)$$

where $D_\mu \equiv \partial_\mu - i\,(n/R)\,A_\mu$, and $\varphi_n(x)$ are the KK modes of $\hat{\Phi}(x, z)$. Taking a constant background radion field $\phi$, and the trivial background for $A_\mu$, the metric (12) becomes

$$\hat{g}^0_{MN} = \begin{pmatrix} e^{\sqrt{\frac{2}{3}}\frac{\phi}{M_P}} \eta_{\mu\nu} & 0 \\ 0 & -e^{-2\sqrt{\frac{2}{3}}\frac{\phi}{M_P}} \end{pmatrix} . \qquad (15)$$

From (14) we define the $\phi$-dependent radius $R_\phi \equiv R\, e^{-\sqrt{\frac{3}{2}}\frac{\phi}{M_P}}$. With such a definition, we immediately see that, when computing radiative corrections, the $(4+1)$D momentum $\hat{p} \equiv (p, n/R)$ is cut as

$$\hat{p}^2 = e^{-\sqrt{\frac{2}{3}}\frac{\phi}{M_P}} \left( p^2 + \frac{n^2}{R^2_\phi} \right) \leq \Lambda^2 . \qquad (16)$$

This latter equation is conveniently rewritten as

$$p^2 + \frac{n^2}{R_\phi^2} \leq \Lambda_\phi^2, \tag{17}$$

where we defined $\Lambda_\phi \equiv \Lambda\, e^{\frac{1}{\sqrt{6}}\frac{\phi}{M_P}}$. In terms of the dimensionless $\phi$ of (12) and (13), and before using $\alpha = 1/\sqrt{12}$, it is $\Lambda_\phi = e^{\alpha\phi}\Lambda = m_{\rm KK}^{1/3} R^{1/3}\Lambda$.

Since $p^2$ in (17) is the modulus of the four-momentum on the brane, this equation tells us that $\Lambda_\phi$ is the cutoff $\Lambda_{\rm SM}$ of the SM (or more generally of the BSM model that lives on the 3-brane, where fields have $n = 0$). Therefore:

$$\Lambda_{\rm SM} = \Lambda_\phi = \Lambda\, e^{\frac{1}{\sqrt{6}}\frac{\phi}{M_P}}. \tag{18}$$

Finally, as the DD scenario is realized for negative values of $\phi$, from (18) we see that $\Lambda_{\rm SM} \leq \Lambda$, i.e. the SM cutoff is lower than the cutoff of the $(4+1)$-dimensional EFT that implements the DD scenario.

Let us note that here we considered a spherical cutoff. Naturally, we can make a different choice, taking for instance a cylindrical cutoff

$$p^2 \leq \Lambda_\phi^2 \qquad \text{and} \qquad \frac{n^2}{R_\phi^2} \leq \Lambda_\phi^2.$$

This choice, that is closer to what is typically done when using the species scale $\Lambda_{\rm sp}$ as the, does not change the above considerations.

# Criticisms

Anchordoqui, Antoniadis, Lüst, Lüst

- We reportedly question the swampland relation $\Lambda_{cc} \sim m_{KK}^4$
- We reportedly claim for $\alpha$ the values $\alpha = 1/2$, $\alpha = 3/2$ for the SUSY and non-SUSY case respectively
- Cutoff dependence of the result, nonsensical to extract relationship between vacuum energy and $m_{KK}$ without fixing the cutoff
- Quantum Gravity dictates UV-IR mixing of the cutoff
- With general $\Lambda$: non-SUSY case requires a too low cutoff not to violate Higuchi bound
- $\Lambda = \Lambda_{sp}$: non-SUSY violates Higuchi
- $\Lambda = m_{KK}$ : DD relation is obtained, correct cutoff
- $T \neq 0$ and Casimir energy: field theory examples with finite result
  $T \neq 0$: $T^4$;     $\mathcal{E}_C : m_{KK}^4$

# Replacement

Criticisms are based on the replacement $\rho_4^{1l} \to \Lambda_{\rm cc}$ in our results for $\rho_4^{1l}$

$$\rho_4^{1l} \sim m_{\rm KK}^2 R \Lambda^3 \quad \text{and} \quad \rho_4^{1l} \sim m_{\rm KK}^{2/3} R^{5/3} \Lambda^5$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$\Lambda_{cc} \sim m_{\rm KK}^2 R \Lambda^3 \quad \text{and} \quad \Lambda_{cc} \sim m_{\rm KK}^{2/3} R^{5/3} \Lambda^5$$

Authors take the result of the one-loop calculation to directly coincide with the physical vacuum energy

- Opposite to what we do
- Not in itself a problem: theoretically legitimate in principle

**We must explore the consequences of the $\rho_4^{1l} \to \Lambda_{\rm cc}$ replacement to determine its viability**

# Fatal flaw

$$\Lambda_{cc} \sim m_{\rm KK}^2 R \Lambda^3 \qquad \text{and} \qquad \Lambda_{cc} \sim m_{\rm KK}^{2/3} R^{5/3} \Lambda^5$$

Most important consequence

The replacement $\rho_4^{1/l} \to \Lambda_{\rm cc}$ fully determines the cutoff $\Lambda$

$\Lambda_{\rm cc} \sim m_{\rm KK}^4$ by definition $\to$ replacement fixes

$$R\Lambda^3 \sim m_{\rm KK}^2$$

This implies:

$$\Lambda_{\rm SM} = \Lambda_\phi \sim m_{\rm KK} \sim \text{meV}$$

**Absurd requirement! $\rho_4^{1/l} \to \Lambda_{\rm cc}$ is unacceptable**

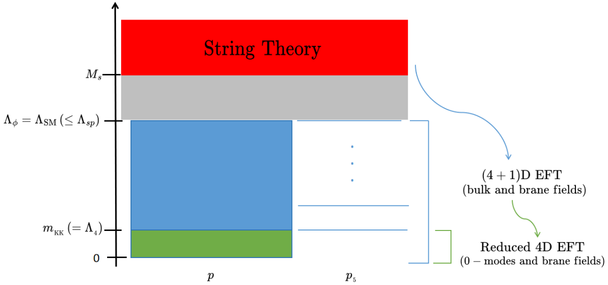
# Inconsistency of the criticisms

This means that

- $\Lambda_{cc} \sim m_{\rm KK}^2 R \Lambda^3$ and $\Lambda_{cc} \sim m_{\rm KK}^{2/3} R^{5/3} \Lambda^5$ do not exist
- Cannot be used to derive any relation and draw any conclusion

On top of that:

$\Lambda_\phi \sim m_{\rm KK}$ leaves no space for the $(4+1)D$ theory of the DD scenario



$$\cancel{\rho_4^{1/} \to \Lambda_{\mathrm{cc}}}$$

We could stop here ... but ...

Let's follow ALL arguments anyway ... Further inconsistencies ...

# $\alpha = 1/2$ & $\alpha = 3/2$

From $m_{\rm KK} \sim \left|\frac{\Lambda_{\rm cc}}{M_P^4}\right|^{\alpha} M_P$, rewritten as $m_{\rm KK} \sim M_P^{1-4\alpha}\Lambda_{\rm cc}^{\alpha}$

- Values of $\alpha$ can only be deduced from the physical vacuum energy
- To conclude $\alpha = 1/2$ and $\alpha = 3/2$ coefficient must be given by correct $M_P$ power

Rewriting the relations for convenience

$$m_{\rm KK} \sim (R\Lambda^3)^{-1/2}\Lambda_{cc}^{1/2} \ ; \ m_{\rm KK} \sim (R\Lambda^3)^{-5/2}\Lambda_{cc}^{3/2}$$

- $\alpha = 1/2$ and $\alpha = 3/2$ require $R\Lambda^3 \sim M_P^2$, in sharp contrast to $R\Lambda^3 \sim m_{\rm KK}^2$

**Even closing an eye on fatal flaw, our results do not amount to $\alpha = 1/2$ and $\alpha = 3/2$**

# Higuchi bound

For a spin 2 massive field in $4D$ dS:

$$m^2 \geq \frac{2}{3}\frac{\Lambda_{cc}}{M_P^2}$$

- Relation between physical parameters

Comparing: $\Lambda_{cc} \sim m_{\text{KK}}^2 R\Lambda^3 \quad \Lambda_{cc} \sim m_{\text{KK}}^{2/3} R^{5/3}\Lambda^5$

$$\downarrow \qquad \downarrow$$

$$R\Lambda^3 \lesssim M_P^2 \qquad R\Lambda^3 \lesssim 10^{-48} M_P^2$$

- First one: not too strong constraint ($\Lambda \leq \hat{M}_P$, no constraint at all)
- Second one: too low cutoff

Bound 2 can be rewritten as $\Lambda_\phi \lesssim 10^5 m_{\text{KK}} \sim 10^2$ eV

Too low ... but later on ... contradiction

# Higuchi bound and "natural" choices for $R$ and $\Lambda$

- Authors note our results still depend on $R$ and $\Lambda$
- Claim: "natural" choices for $R$ and $\Lambda$, eventually consistent with DD

"Natural" choice for $R$: $R = m_{\rm KK}^{-1}$

- Authors miss radion dependence. $R_\phi = Re^{-3\alpha\phi} = m_{\rm KK}^{-1}$ correct relation

"Natural" choices for $\Lambda$: UV-IR mixing

- (1): $\Lambda = \Lambda_{\rm sp} \sim m_{\rm KK}^{1/3} M_P^{2/3}$, Higuchi explicitly violated

- $\Lambda_{\rm sp}$ cut on $p$: $\Lambda_\phi = \Lambda_{\rm sp}(\Lambda = \hat{M}_P)$ correct identification

- (2): $\Lambda = m_{\rm KK}$, everything ok, "correct choice"

- $\Lambda_\phi = m_{\rm KK}$ and not $\Lambda = m_{\rm KK}$ is what needed to have $m_{\rm KK}^4$
- Absurd again: $\Lambda_{\rm SM} \lesssim$ meV and no space for $(4+1)D$ theory

# Finite temperature

Profound difference between the sums in finite $T$ and KK

$$\rho^{1l}\ ,\ F^{1l} \sim \frac{1}{2} \sum_n \int d^d p\, \log(p^2 + m^2 + f_n)\,.$$

KK theories ($\rho^{1l}$)

- $n$ and $p$ intertwined, components of $\hat{p}$
- $p$ and $n$ cut together: no hierarchy when including asymptotics

Finite temperature ($F^{1l}$)

Finite for SUSY theory

- $n$ and $p$ not intertwined
- $\int d^3 p$: trace over quantum fluctuations
- $\sum_n$: statistical average (mixed states)
- Infinite sum: ergodicity! MUST DO: no $q$ dependent "divergences"

$$F^{1l}_T = \frac{T}{2} \sum_{n=-\infty}^{\infty} \int d^3 p\, \log(p^2 + m^2 + f_n) - \frac{1}{2} \int d^4 p\, \log(p^2 + m^2) \sim T^4 = \text{finite}\,.$$

# Casimir energy

By definition

$$\mathcal{E}_C = \rho_R - \rho_\infty$$

$$\mathcal{E}_C^{1l} = \frac{1}{2}\sum_{n=-\infty}^{\infty}\int d^4p\,\log(p^2+m^2+f_n) - \frac{1}{2}\int d^4p\,\log(p^2+m^2).$$

- Infinite sum in $\rho_R$ (literature): $\sim$ finite $T$

- $\rho_R$ and $\rho_\infty$ have the same divergences
- $\mathcal{E}_C \sim m_{\text{KK}}^4$

-No hierarchy when including asymptotics in $\rho_R$ (us): $q$-divergences

- $\rho_R$ and $\rho_\infty$ do not have same divergences when non-trivial boundary charges are present
- $\rho_R - \rho_\infty$ subtraction not sufficient