SPHERICALLY SYMMETRIC GEOMETRODYNAMICS IN THE JORDAN AND EINSTEIN FRAMES

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Jordan-Einstein Frames

- (Faraoni and Nadeau, Phys. Rev D **75**, 023501 (2007)) Suppose the proton mass is m_p in mass units m_u and, in "natural units", we scale the unit of measurement by a factor λ^{-1} (length)$^{-1}$ $\tilde{m}_u = \lambda^{-1} m_u$

(notice that λ could be a local function $\lambda(x)$). In the new unit the proton mass $\tilde{m}_p = \lambda^{-1} m_p$

- Confronting the measurement of the proton mass in the two mass units

$$\frac{\tilde{m}_p}{\tilde{m}_u} = \frac{\lambda^{-1} m_p}{\lambda^{-1} m_u} = \frac{m_p}{m_u}$$

Jordan-Einstein Frames

- Since $d\tilde{s} = \lambda ds$ and $ds = \left(g_{ij}dx^i dx^j\right)^{\frac{1}{2}}$, (Dicke, Phys. Rev. (1962) **125,** 6 2163-2167) then the covariant metric functions scales as

$$\tilde{g}_{\mu\nu} = \lambda^2 g_{\mu\nu}$$

- Invariance under rescaling of units of measurement implies Weyl (conformal) invariance of the metric tensor

- The starting frame is called "Jordan" frame and the conformal transformed the "Einstein Frame. One observable can be computed in both frames. Its measure, obviously different in the two frames, is related by conformal rescaling according to the observable's dimensions.(e.g. $\tilde{m}_p = \lambda^{-1} m_p$).

- Dicke highlights that free falling particles, in the Jordan frame, mapped into the Einstein frame, do not move on geodesic curves. The equivalence principle does not seem to hold.

Scalar-Tensor Theory

- In general, one starts from a scalar-tensor theory, with GHY-like boundary term, in the Jordan Frame

$$S=\int_M d^n x\sqrt{-g}\left(f(\phi)R-\frac{1}{2}\lambda(\phi)g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi-U(\phi)\right)+2\int_{\partial M}d^{n-1}\sqrt{h}f(\phi)K$$

- and passes to the Einstein Frame with the transformation

$$\tilde{g}_{\mu\nu}=\Big(16\pi G f(\phi)\Big)^{\frac{2}{n-2}}g_{\mu\nu}\ ,$$

- therefore, the action becomes

$$S=\int_M d^n x\sqrt{-\tilde{g}}\left(\frac{1}{16\pi G}\tilde{R}-A(\phi)\tilde{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi-V(\phi)\right)+\frac{1}{8\pi G}\int_{\partial M}d^{n-1}\sqrt{\tilde{h}}\tilde{K}$$

$$A(\phi)=\frac{1}{16\pi G}\left(\frac{\lambda(\phi)}{2f(\phi)}+\frac{n-1}{n-2}\frac{(f'(\phi))^2}{f^2(\phi)}\right),V(\phi)=\frac{U(\phi)}{[16\pi G f(\phi)]^{\frac{n}{n-2}}}$$

- It is assumed that if $\big(g_{\mu\nu}(x),\phi(x)\big)$ is solution of the E.O.M also $\big(\tilde{g}_{\mu\nu}(x,\phi),\phi(x)\big)$ is solution (True?). This reasoning seems to address that the transformation from the Jordan to the Einstein frame look like a canonical transformation in the Hamiltonian theory.

Brans-Dicke Theory

- Brans-Dicke, with GHY boundary term, is a particular case of Scalar Tensor theory ($f(\phi)=\phi$)

$$S=\int_M d^4x\sqrt{-g}\left(\phi\,{}^4R-\frac{\omega}{\phi}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi-U(\phi)\right)+2\int_{\partial M}d^3x\sqrt{h}\phi K \quad .$$

- How to perform canonical analysis of this theory?

Garay and Gracìa-Bellido NPB 400 (1993): the transformations are Hamiltonian canonical.

$$\bar{q}_{ik}=\Phi q_{ik}\,,\quad \phi=-\frac{1}{2\beta}\log\Phi$$

$$\overline{N}^2=\Phi N^2\,,\quad \overline{N}_i=\Phi N_i$$

Deruelle, Sendouda, Youssef PRD 80, (2009). They still claim that the transformations are Hamiltonian canonical

$$\tilde{h}_{ab}=\phi h_{ab},\qquad \tilde{N}^a=N^a,\qquad \tilde{N}=\sqrt{\phi}N,\qquad \tilde{\phi}=\sqrt{\frac{3}{2}}\ln\phi$$

$$\tilde{p}^{ab}=\frac{1}{\phi}p^{ab},\qquad \tilde{\pi}=\sqrt{\frac{2}{3}}(\phi\pi-p)$$

$$\{\tilde{h}_{ab},\tilde{p}^{cd}\}_J=\{h_{ab},p^{cd}\}_J,\ \{\tilde{\phi},\tilde{\pi}\}_J=$$

$$\{\phi,\pi\}_J,\quad \{\tilde{p}^{ab},\tilde{\pi}\}_J=0,\quad \{\tilde{h}_{ab},\tilde{\phi}\}_J=0,\quad \{\tilde{h}_{ab},\tilde{\pi}\}_J=0$$

$$\{\tilde{p}^{ab},\tilde{\phi}\}_J=0$$

Brans-Dicke Theory

- The Hamiltonian Weyl (conformal) transformations from the Jordan to the Einstein frames are

$$\tilde{N} = N(16\pi G\phi)^{\frac{1}{2}}\,;\tilde{N}_i = N_i(16\pi G\phi)\,;\tilde{h}_{ij} = (16\pi G\phi)\,h_{ij}\,;\tilde{\pi} = \frac{\pi}{(16\pi G\phi)^{\frac{1}{2}}}\,;$$

$$\tilde{\pi}^i = \frac{\pi^i}{(16\pi G\phi)},;\tilde{\pi}^{ij} = \frac{\pi^{ij}}{16\pi G\phi}\,;\phi = \phi\,;\tilde{\pi}_\phi = \frac{1}{\phi}(\phi\pi_\phi - \pi_h)$$

- They are not Hamiltonian canonical

$$\{\tilde{N},\tilde{\pi}_\phi\} = \frac{8\pi GN}{\sqrt{16\pi G\phi}} \neq 0, \text{and } \{\tilde{N}_i,\tilde{\pi}_\phi\} = 16\pi GN_i \neq 0$$

- The Dirac's constraint analysis of the Hamiltonian theory has to be done, independently, in the Jordan and Einstein frames. We have studied the Hamiltonian constrained theory in Jordan and Einstein frames for both cases $\omega \neq -\frac{3}{2}$ and , $\omega = -\frac{3}{2}$. In the case $\omega = -\frac{3}{2}$ the theory has an extra Weyl(conformal) symmetry with an associated primary first class constraint C_ϕ

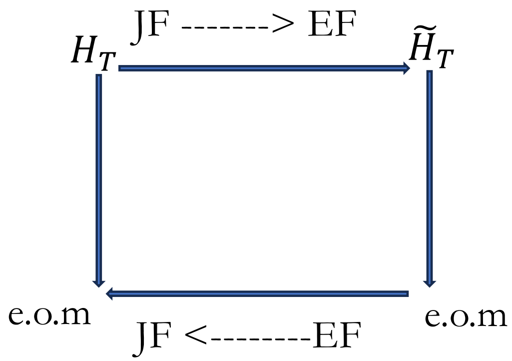
Hamiltonian Analysis of BD for $\omega \neq -\frac{3}{2}$	
in Jordan Frame	**in Einstein Frame**
constraints $\pi \approx 0; \pi^i \approx 0; \mathcal{H} \approx 0; \mathcal{H}_i \approx 0;$	*constraints* $\widetilde{\pi} \approx 0; \widetilde{\pi}_i \approx 0; \widetilde{\mathcal{H}} \approx 0; \widetilde{\mathcal{H}}_i \approx 0;$
constraint algebra $\{\pi, \pi_i\} = 0; \{\pi, \mathcal{H}\} = 0; \{\pi, \mathcal{H}_i\} = 0; \{\pi_i, \mathcal{H}\} = 0;$ $\{\pi_i, \mathcal{H}_j\} = 0;\ \{\mathcal{H}(x), \mathcal{H}_i(x')\} = -\mathcal{H}(x')\partial_i'\delta(x, x');$ $\{\mathcal{H}_i(x), \mathcal{H}_j(x')\} = \mathcal{H}_i(x')\partial_j\delta(x, x') - \mathcal{H}_j(x)\partial_i{}'\delta(x, x');$ $\{\mathcal{H}(x), \mathcal{H}(x')\} = \mathcal{H}^i(x)\partial_i\delta(x, x') - \mathcal{H}^i(x')\partial_i'\delta(x, x');$	*constraint algebra* $\{\widetilde{\pi}, \widetilde{\pi}_i\} = 0; \{\widetilde{\pi}, \widetilde{\mathcal{H}}\} = 0; \{\widetilde{\pi}, \widetilde{\mathcal{H}}_i\} = 0; \{\widetilde{\pi}_i, \widetilde{\mathcal{H}}\} = 0;$ $\{\widetilde{\pi}_i, \widetilde{\mathcal{H}}_j\} = 0;\ \left\{\widetilde{\mathcal{H}}(x), \widetilde{\mathcal{H}}_i(x')\right\} = -\widetilde{\mathcal{H}}(x')\partial_i'\delta(x, x');$ $\{\widetilde{\mathcal{H}}_i(x), \widetilde{\mathcal{H}}_j(x')\} = \widetilde{\mathcal{H}}_i(x')\partial_j\delta(x, x') - \widetilde{\mathcal{H}}_i(x)\partial_i{}'\delta(x, x');$ $\{\widetilde{\mathcal{H}}(x), \widetilde{\mathcal{H}}(x')\} = \widetilde{\mathcal{H}}^i(x)\partial_i\delta(x, x') - \widetilde{\mathcal{H}}^i(x')\partial_i'\delta(x, x');$

Hamiltonian Analysis of BD for $\omega = -\frac{3}{2}$	
in Jordan Frame	**in Einstein Frame**
constraints $\pi_N \approx 0; \pi^i \approx 0; C_\phi \approx 0; \mathcal{H}^{(-3/2)} \approx 0; \mathcal{H}_i^{(-3/2)} \approx 0;$	*constraints* $\widetilde{\pi}_N \approx 0; \widetilde{\pi}_i \approx 0; \widetilde{C}_\phi = -\widetilde{\phi}\widetilde{\pi}_\phi \approx 0; \widetilde{\mathcal{H}}^{(-3/2)} \approx 0; \widetilde{\mathcal{H}}_i^{(-3/2)} \approx 0;$
constraint algebra $\{\pi_N, \pi_i\} = \{\pi_N, \mathcal{H}^{(-3/2)}\} = \{\pi_N, \mathcal{H}_i^{(-3/2)}\} = 0;$ $\{\pi_i, \mathcal{H}^{(-3/2)}\} = \{\pi_i, \mathcal{H}_j^{(-3/2)}\} = 0;$ $\left\{C_\phi(x), \mathcal{H}_i^{(-3/2)}(x')\right\} = -\partial'_i\delta(x,x')C_\phi(x');$ $\left\{C_\phi(x), \mathcal{H}^{(-3/2)}(x')\right\} = \frac{1}{2}\mathcal{H}^{(-3/2)}(x)\delta(x,x');$ $\left\{\mathcal{H}^{(-3/2)}(x), \mathcal{H}_i^{(-3/2)}(x')\right\} = -\mathcal{H}^{(-3/2)}(x')\partial'_i\delta(x,x');$ $\left\{\mathcal{H}_i^{(-3/2)}(x), \mathcal{H}_j^{(-3/2)}(x')\right\} = \mathcal{H}_i^{(-3/2)}(x')\partial_j\delta(x,x')$ $-\mathcal{H}_j^{(-3/2)}(x)\partial_i{}'\delta(x,x');$ $\{\mathcal{H}^{(-3/2)}(x), \mathcal{H}^{(-3/2)}(x')\} =$ $\mathcal{H}_i^{(-3/2)}(x)\partial^i\delta(x,x') - \mathcal{H}_i^{(-3/2)}(x')\partial'^i\delta(x,x') +$ $\left[D^i(\log\phi(x))\right] C_\phi(x)\partial_i\delta(x,x')$ $-\left[D^i(\log\phi(x'))\right] C_\phi(x')\partial'_i\delta(x,x');$	*constraint algebra* $\{\widetilde{\pi}_N, \widetilde{\pi}_i\} = \{\widetilde{\pi}_N, \widetilde{\mathcal{H}}^{(-3/2)}\} = 0; \{\widetilde{\pi}_N, \widetilde{\mathcal{H}}_i^{(-3/2)}\} = 0;$ $\{\widetilde{\pi}_i, \widetilde{\mathcal{H}}^{(-3/2)}\} = \{\widetilde{\pi}_i, \widetilde{\mathcal{H}}_j^{(-3/2)}\} = 0;$ $\left\{\widetilde{C}_\phi(x), \widetilde{\mathcal{H}}_i^{(-3/2)}(x')\right\} = 0;$ $\left\{\widetilde{C}_\phi(x), \widetilde{\mathcal{H}}^{(-3/2)}(x')\right\} = 0;$ $\left\{\widetilde{\mathcal{H}}^{(-3/2)}(x), \widetilde{\mathcal{H}}_i^{(-3/2)}(x')\right\} = -\widetilde{\mathcal{H}}^{(-3/2)}(x')\partial'_i\delta(x,x');$ $\{\widetilde{\mathcal{H}}_i^{(-3/2)}(x), \widetilde{\mathcal{H}}_j^{(-3/2)}(x')\} = \widetilde{\mathcal{H}}_i^{(-3/2)}(x')\partial_j\delta(x,x')$ $-\widetilde{\mathcal{H}}_i^{(-3/2)}(x)\partial_i{}'\delta(x,x');$ $\{\widetilde{\mathcal{H}}^{(-3/2)}(x), \widetilde{\mathcal{H}}^{(-3/2)}(x')\} =$ $\widetilde{\mathcal{H}}_i^{(-3/2)}(x)\partial^i\delta(x,x') - \widetilde{\mathcal{H}}_i^{(-3/2)}(x')\partial'_i\delta(x,x');$

FLAT FLRW Brans-Dicke theory

$$ds^2 = -N^2(t)dt^2 + a^2(t)dx^3$$

$$\mathcal{L}_{FLRW} = -\frac{6a\dot{a}^2}{N}\phi - \frac{6a^2\dot{a}}{N}\dot{\phi} + \frac{\omega a^3}{N\phi}(\dot{\phi})^2 - Na^3U(\phi)$$



$$\dot{N} \approx \lambda_N\,, \tag{1}$$

$$\dot{\pi}_N = -H \approx 0\,, \tag{2}$$

$$\dot{a} \approx -\frac{N}{2a(2\omega+3)}\left(\frac{\omega\pi_a}{3\phi} + \frac{\pi_\phi}{a}\right), \tag{3}$$

$$\dot{\pi}_a \approx -\frac{N}{2a^2(2\omega+3)}\left(\frac{\omega\pi_a^2}{6\phi} + \frac{2\pi_a\pi_\phi}{a} - \frac{3\phi\pi_\phi^2}{a^2}\right) - 3Na^2U(\phi)\,, \tag{4}$$

$$\dot{\phi} \approx \frac{N}{2a^2(2\omega+3)}\left(-\pi_a + \frac{2\phi\pi_\phi}{a}\right), \tag{5}$$

$$\dot{\pi}_\phi \approx -\frac{N}{2a(2\omega+3)}\left(\frac{\omega\pi_a^2}{6\phi^2} + \frac{\pi_\phi^2}{a^2}\right) - Na^3\frac{dU}{d\phi} \tag{6}$$

$$\dot{N} \approx \frac{\widetilde{\lambda}_N}{(16\pi G\phi)^{\frac{1}{2}}} - \frac{N^2}{2a^2(2\omega+3)}\left(\frac{\pi_\phi}{a} - \frac{\pi_a}{2\phi}\right), \tag{1}$$

$$\dot{\pi}_N \approx -H + \frac{N\pi_N}{2a^2(2\omega+3)}\left(\frac{\pi_\phi}{a} - \frac{\pi_a}{2\phi}\right), \tag{2}$$

$$\dot{a} \approx -\frac{N}{2a(2\omega+3)}\left(\frac{\omega\pi_a}{3\phi} + \frac{\pi_\phi}{a}\right), \tag{3}$$

$$\dot{\pi}_a \approx -\frac{N}{2a^2(2\omega+3)}\left(\frac{\omega\pi_a^2}{6\phi} + \frac{2\pi_a\pi_\phi}{a} - \frac{3\phi\pi_\phi^2}{a^2}\right) - 3Na^2U(\phi)\,, \tag{4}$$

$$\dot{\phi} \approx \frac{N}{2a^2(2\omega+3)}\left(-\pi_a + \frac{2\phi\pi_\phi}{a}\right), \tag{5}$$

$$\dot{\pi}_\phi \approx -\frac{N}{2a(2\omega+3)}\left(\frac{\omega\pi_a^2}{6\phi^2} + \frac{\pi_\phi^2}{a^2}\right) - Na^3\frac{dU}{d\phi} + \frac{H}{2\phi}\,. \tag{6}$$

CANONICAL EQUIVALENCE OF JF AND EJ VIA GAUGE FIXING

- We perform a gauge-fixing in the Jordan Frame

$N = c_0(x); N^i = c^i(x)$ we implement this gauge choice as secondary constraints $N - c_0(x) \approx 0; N^i - c^i(x) \approx 0$

- Previous gauge fixing implies a gauge fixing in the Einstein frame

$$\widetilde{N} = c_0(x)(16\pi G\phi)^{\frac{1}{2}}; \widetilde{N}^i = c^i(x)(16\pi G\phi)^{\frac{1}{2}} \longrightarrow \widetilde{N} - c_0(x)(16\pi G\phi)^{\frac{1}{2}} \approx 0; \widetilde{N}^i - c^i(x)(16\pi G\phi)^{\frac{1}{2}} \approx 0$$

- The primary first class constraints become second class both in the Jordan and Einstein frames

$$\{N(x) - c_0, \pi_N(x')\} \approx \delta^{(3)}(x - x') \qquad \{\widetilde{N}(x) - c_0\,(16\pi G\phi)^{\frac{1}{2}}, \widetilde{\pi}_N(x')\} \approx \delta^{(3)}(x - x'),$$

$$\{N_i(x) - c_i, \pi^j(x')\} = \delta_i^j\,\delta^{(3)}(x - x') \qquad \{\widetilde{N}_i(x) - c_i\,(16\pi G\phi(x)), \widetilde{\pi}^j(x')\} \approx \delta_i^j\delta^{(3)}(x - x')$$

CANONICAL EQUIVALENCE OF JF AND EJ VIA GAUGE FIXING

- Following Dirac, we define Dirac's brackets and substitute them to the Poisson brackets

$$\{\,,\,\}_{DB} \equiv \{\,,\,\} - \{\,,\varphi_\alpha\}C^{-1}_{\alpha\beta}\{\varphi_\beta,\,\} \;\; C_{\alpha\beta} \equiv \{\varphi_\alpha, \varphi_\beta\} \text{ being } \varphi_\alpha, \varphi_\beta \text{ second class contraints}$$

- Dirac's algorithm implies that we calculate the equations of motion using Dirac's brackets.

- We resolve *strongly* the second class constraints both in the JF and EF

- On this reduced phase space, the transformations from JF to EF are Hamiltonian canonical transformations.

CANONICAL EQUIVALENCE AND PHYSICAL EQUIVALENCE

- Harmonic Oscillator (Goldstein)

$$H = \frac{p^2}{2m} + \frac{m\omega^2}{2} q^2$$

- Canonical transformations (not symmetry of the system…)

$$q = \sqrt{\frac{2P}{m\omega}} sinQ \, , p = \sqrt{2m\omega P} cosQ$$

- Therefore the Hamiltonian becomes

$$H = \omega P$$

- and then,

$$P = \frac{E}{\omega}, \;\; \dot{Q} = \frac{\partial H}{\partial P} = \omega, \;\; Q = \omega t + \alpha, \;\; q(t) = \sqrt{\frac{2E}{m\omega^2}} sin(\omega t + \alpha)$$

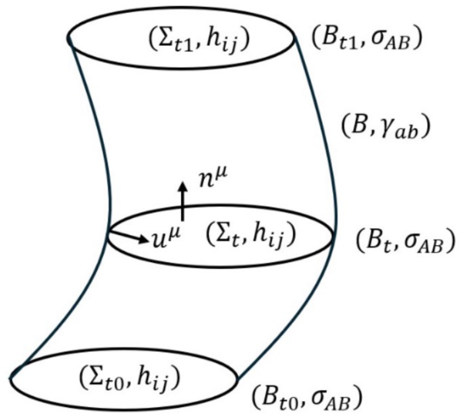
- Notice that the harmonic oscillator is mapped into a free particle

GEOMETRODYNAMICS IN SPHERICAL SYMMETRY IN THE JF AND EF

- Lorentian Manifold (M,g) where $M = \mathbb{R} \times \Sigma_t$ and $\Sigma_t = \mathbb{R}\times S^2$ $(-\infty < r < +\infty)$. The ADM metric is

$$
\begin{aligned}
g_{\mu\nu}dx^\mu dx^\nu &= -N^2 dt^2 + \Lambda^2 \left(dr + N^r dt\right)^2 + R^2 d\Omega^2 \\
&= -\left(N^2 - \Lambda^2 (N^r)^2\right) dt^2 + 2\Lambda^2 N^r dt dr + \Lambda^2 dr^2 + R^2 d\Omega^2
\end{aligned}
$$

- The foliation $M = \mathbb{R} \times \Sigma_t$ can be visualized as



- The boundary $\partial M = \Sigma_{t_0} \cup \Sigma_{t_1} \cup B$, where Σ_t is space-like and B is time-like.

GEOMETRODYNAMICS IN SPHERICAL SYMMETRY IN THE JF AND EF

- The corresponding E-H action with all the boundary terms

$$S = \frac{1}{16\pi G}\int_M d^4x\sqrt{-g}\,{}^{(4)}R + \frac{1}{8\pi G}\int_{\Sigma_{t_1}} d^3x\sqrt{h}K - \frac{1}{8\pi G}\int_{\Sigma_{t_0}} d^3x\sqrt{h}K$$
$$-\frac{1}{8\pi G}\int_B d^3x\sqrt{-\gamma}\,\Theta - \frac{1}{8\pi G}\int_{B_{t_0}} d^2x\sqrt{\sigma}\,\mathrm{arcsinh}(n^\mu u_\mu)$$
$$+\frac{1}{8\pi G}\int_{B_{t_1}} d^2x\sqrt{\sigma}\,\mathrm{arcsinh}(n^\mu u_\mu).$$

- The 3+1 decomposition on the trace ${}^{(4)}R$ of the Ricci tensor is

$${}^{(4)}R = {}^{(3)}R + K_{ij}K^{ij} - K^2 + 2\nabla_\mu\left(-n^\mu K - a^\mu\right)$$

- Implementing this 3+1 decomposition in the previous action, we get

$$S = \frac{1}{16\pi G}\int dt\int_{\Sigma_t} d^3x\left(\sqrt{h}N({}^{(3)}R + K_{ij}K^{ij} - K^2)\right)$$
$$-\frac{1}{8\pi G}\int_B d^3x\sqrt{-\gamma}\,\left(\Theta + {}^{(3)}r_i n^i K - N\,{}^{(3)}r_i h^{ij}\nabla_j n^0\right)$$
$$-\frac{1}{8\pi G}\int_{B_{t_0}} d^2x\sqrt{\sigma}\,\mathrm{arcsinh}(n^\mu u_\mu) + \frac{1}{8\pi G}\int_{B_{t_1}} d^2x\sqrt{\sigma}\,\mathrm{arcsinh}(n^\mu u_\mu)$$

GEOMETRODYNAMICS IN SPHERICAL SYMMETRY IN THE JF AND EF

- Previous action is finite for compact geometries. For non-compact geometries, we define a physical action substracting a background geometry g_0, an asymptotically flat static solution of the equations of motion

$$S^{\mathrm{PHYS}}(g) \equiv S(g) - S(g_0)$$

- Then, we get always a bulk and boundary in the action

$$S^{\mathrm{PHYS}} = \frac{1}{16\pi G}\int_{t_0}^{t_1} dt \int_{\Sigma_t} d^3x N\sqrt{h}\left({}^{(3)}R + K_{ij}K^{ij} - K^2\right) - \frac{1}{8\pi G}\int_{t_0}^{t_1} dt \int_{B_t} d^2x N\sqrt{\sigma}\left({}^{(2)}k - {}^{(2)}k_0\right) .$$

- The corresponding total Hamiltonian density , $\mathcal{H}_{ADM} \equiv \pi^{ij}\dot{h}_{ij} - \mathcal{L}_P$ therefore the total hamiltonian is

$$H_{ADM} = \int_{\Sigma_t} d^3x \left(N\mathcal{H} + N_i\mathcal{H}^i\right) + 2\int_{B_t} d^2x\sqrt{\sigma}\left(N_i\frac{\pi^{ij}}{\sqrt{h}}r_j - N_i\frac{\pi^{ij}}{\sqrt{h}}r_j\Big|_0\right) - \frac{1}{8\pi G}\int_{B_t} d^2x\sqrt{\sigma}\left({}^{(2)}k - {}^{(2)}k_0\right) ,$$

$$\mathcal{H} \equiv \sqrt{h}\left[\frac{1}{16\pi G}{}^{(3)}R - \frac{16\pi G}{h}\left(\pi_{ij}\pi^{ij} - \frac{\pi_h^2}{2}\right)\right]$$

$$\mathcal{H}^i \equiv -2\sqrt{h}\nabla_j\left(\frac{\pi^{ij}}{\sqrt{h}}\right)$$

SPHERICAL SYMMETRY IN THE JORDAN FRAME

- The Lagrangian density $\mathcal{L}_{JF}$ in the JF (from now on G=1)

$$\mathcal{L}_{JF} = \frac{1}{16\pi}\left[\left(1-\frac{\phi^2}{6}\right){}^{(4)}R - g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right]$$

- The action in the JF with the boundary terms is

$$\begin{aligned}
S_{JF} = & \frac{1}{16\pi}\int_M d^4x\sqrt{-g}\left[\left(1-\frac{\phi^2}{6}\right){}^{(4)}R - g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right] \\
& + \frac{1}{8\pi}\int_{\Sigma_{t_1}} d^3x\sqrt{h}\left(1-\frac{\phi^2}{6}\right)K - \frac{1}{8\pi}\int_{\Sigma_{t_0}} d^3x\sqrt{h}\left(1-\frac{\phi^2}{6}\right)K \\
& - \frac{1}{8\pi}\int_B d^3x\sqrt{-\gamma}\left(1-\frac{\phi^2}{6}\right)\Theta - \frac{1}{8\pi}\int_{B_{t_0}} d^2x\sqrt{\sigma}\left(1-\frac{\phi^2}{6}\right)\operatorname{arcsinh}(n^\mu u_\mu) \\
& + \frac{1}{8\pi}\int_{B_{t_1}} d^2x\sqrt{\sigma}\left(1-\frac{\phi^2}{6}\right)\operatorname{arcsinh}(n^\mu u_\mu).
\end{aligned}$$

- The bulk term of the physical action results to be

$$\begin{aligned}
S_{JF}^{\mathrm{PHYS}} = & \int dt\int_{-\infty}^{\infty} dr\left\{\left(1-\frac{\phi^2}{6}\right)\left[-\frac{1}{N}\left(\frac{\Lambda}{2}\left(-\dot{R}+R'N^r\right)^2 + R\left(-\dot{\Lambda}+(\Lambda N^r)'\right)\left(-\dot{R}+R'N^r\right)\right)\right.\right. \\
& \left. + N\left(-\frac{RR'}{\Lambda}+\frac{RR'\Lambda'}{\Lambda^2}-\frac{R'^2}{2\Lambda}+\frac{\Lambda}{2}\right)\right] + \frac{1}{4}\left(-\frac{NR^2\phi'^2}{\Lambda}+\frac{\Lambda R^2}{N}(\dot{\phi}-N^r\phi')^2\right) \\
& - \frac{1}{6}\phi\frac{R}{N}\left[R\left(-\dot{\Lambda}+(\Lambda N^r)'\right)+2\Lambda\left(-\dot{R}+R'N^r\right)\right]\left(\dot{\phi}-N^r\phi'\right) \\
& \left. + \frac{1}{6}N\Lambda R^2\left(\frac{\phi'^2}{\Lambda^2}+\frac{\phi\phi''}{\Lambda^2}-\frac{\Lambda'\phi\phi'}{\Lambda^3}+\frac{2\phi R'\phi'}{R\Lambda^2}\right)\right\}
\end{aligned}$$

SPHERICAL SYMMETRY IN THE JORDAN FRAME

- Momenta in the Jordan frame

$$\begin{aligned}
\pi_\Lambda &\equiv \frac{\partial \mathcal{L}_{JF}}{\partial \dot{\Lambda}} = -\frac{1}{N}\left(1-\frac{\phi^2}{6}\right) R\left(\dot{R} - R' N^r\right) + \frac{R^2\phi}{6N}(-N^r\phi' + \dot{\phi})\,, \\
\pi_R &\equiv \frac{\partial \mathcal{L}_{JF}}{\partial \dot{R}} = \frac{1}{N}\left(1-\frac{\phi^2}{6}\right)\left[R\left(-\dot{\Lambda} + (\Lambda N^r)'\right) + \Lambda\left(-\dot{R} + R' N^r\right)\right] + \frac{\phi \Lambda R}{3N}\left(\dot{\phi} - N^r \phi'\right) \\
\pi_\phi &\equiv \frac{\partial \mathcal{L}_{JF}}{\partial \dot{\phi}} = -\frac{\phi R}{6N}\left[R\left(-\dot{\Lambda} + (\Lambda N^r)'\right) + 2\Lambda\left(-\dot{R} + R' N^r\right)\right] + \frac{\Lambda R^2}{2N}\left(\dot{\phi} - N^r \phi'\right)
\end{aligned}$$

- Hamiltonian density function

$$\mathcal{H}_{JF} = N^r H_r + N H$$

$$H_r = \pi_R R' - \pi'_\Lambda \Lambda + \pi_\phi \phi'$$

$$\begin{aligned}
H = \; & \frac{\phi^2 \pi_R^2}{36\Lambda}\left(1-\frac{\phi^2}{6}\right)^{-1} + \frac{\Lambda \pi_\Lambda^2}{2R^2}\left(1+\frac{\phi^2}{18}\right)\left(1-\frac{\phi^2}{6}\right)^{-1} + \frac{\pi_\phi^2}{R^2\Lambda}\left(1-\frac{\phi^2}{6}\right) \\
& -\frac{\pi_R \pi_\Lambda}{R}\left(1-\frac{\phi^2}{18}\right)\left(1-\frac{\phi^2}{6}\right)^{-1} + \frac{\phi \pi_R \pi_\phi}{3R\Lambda} + \frac{\phi \pi_\Lambda \pi_\phi}{3R^2} \\
& + \left(1-\frac{\phi^2}{6}\right)\left(-\frac{\Lambda}{2} + \frac{R'^2}{2\Lambda} - \frac{RR'\Lambda'}{\Lambda^2} + \frac{RR''}{\Lambda}\right) + \frac{R^2\phi'^2}{12\Lambda} + \frac{R^2\phi\Lambda'\phi'}{6\Lambda^2} - \frac{R^2\phi\phi''}{6\Lambda} - \frac{R\phi R'\phi'}{3\Lambda} \qquad (1)
\end{aligned}$$

SPHERICAL SYMMETRY IN THE EINSTEIN FRAME

- The bulk physical action, in spherical symmetry, in the Einstein Frame

$$\begin{aligned} S_{EF}^{\mathrm{PHYS}} &= \int dt \int_{-\infty}^{\infty} dr \Bigg[-\frac{1}{\widetilde{N}} \Bigg(\widetilde{R}\big(-\dot{\widetilde{\Lambda}} + (\widetilde{\Lambda}\widetilde{N}^r)'\big)\big(-\dot{\widetilde{R}} + \widetilde{R}'\widetilde{N}^r\big) + \frac{\widetilde{\Lambda}}{2}\big(-\dot{\widetilde{R}} + \widetilde{R}'\widetilde{N}^r\big)^2 \Bigg) \\ &\quad + \widetilde{N}\left(-\frac{\widetilde{R}\widetilde{R}''}{\widetilde{\Lambda}} + \frac{\widetilde{R}\widetilde{R}'\widetilde{\Lambda}'}{\widetilde{\Lambda}^2} - \frac{\widetilde{R}'^2}{2\widetilde{\Lambda}} + \frac{\widetilde{\Lambda}}{2} \right) + \frac{1}{4}\Big(\frac{\widetilde{\Lambda}\widetilde{R}^2}{\widetilde{N}} \big(\dot{\widetilde{\phi}} - \widetilde{N}^r\widetilde{\phi}'\big)^2 - \frac{\widetilde{N}\widetilde{R}^2\widetilde{\phi}'^2}{\widetilde{\Lambda}} \Big) \Bigg] \end{aligned}$$

- The canonical momenta are

$$\begin{aligned} \widetilde{\pi}_\Lambda &\equiv \frac{\partial \mathcal{L}_{EF}}{\partial \dot{\widetilde{\Lambda}}} = -\frac{\widetilde{R}\big(\dot{\widetilde{R}} - \widetilde{R}'\widetilde{N}^r\big)}{\widetilde{N}}, \\ \widetilde{\pi}_R &\equiv \frac{\partial \mathcal{L}_{EF}}{\partial \dot{\widetilde{R}}} = -\frac{\widetilde{\Lambda}\big(\dot{\widetilde{R}} - \widetilde{R}'\widetilde{N}^r\big) + \widetilde{R}\big(\dot{\widetilde{\Lambda}} - (\widetilde{\Lambda}\widetilde{N}^r)'\big)}{\widetilde{N}}, \\ \widetilde{\pi}_\phi &\equiv \frac{\partial \mathcal{L}_{EF}}{\partial \dot{\widetilde{\phi}}} = \frac{\widetilde{\Lambda}\widetilde{R}^2}{2\widetilde{N}}\big(\dot{\widetilde{\phi}} - \widetilde{N}^r\widetilde{\phi}'\big). \end{aligned}$$

- The Hamiltonian density is then

$$\begin{aligned} \widetilde{H} = \quad & \widetilde{N}\Big(-\frac{\widetilde{\pi}_R\widetilde{\pi}_\Lambda}{\widetilde{R}} + \frac{\widetilde{\Lambda}\widetilde{\pi}_\Lambda^2}{2\widetilde{R}^2} + \frac{\widetilde{R}\widetilde{R}''}{\widetilde{\Lambda}} - \frac{\widetilde{R}\widetilde{R}'\widetilde{\Lambda}'}{\widetilde{\Lambda}^2} + \frac{\widetilde{R}'^2}{2\widetilde{\Lambda}} - \frac{\widetilde{\Lambda}}{2} + \frac{\widetilde{\pi}_\phi^2}{\widetilde{\Lambda}\widetilde{R}^2} + \frac{\widetilde{R}^2\widetilde{\phi}'^2}{4\widetilde{\Lambda}} \Big) \\ & + \widetilde{N}^r\left(\widetilde{\pi}_R\widetilde{R}' - \widetilde{\Lambda}\widetilde{\pi}_\Lambda' + \widetilde{\pi}_\phi\widetilde{\phi}' \right) \end{aligned}$$

CANONICAL TRANSFORMATIONS

- Transformations from the JF and EF

$$\widetilde{g}_{\mu\nu} = \left(1 - \frac{\phi^2}{6}\right) g_{\mu\nu}$$

$$\widetilde{N} = \left(1 - \frac{\phi^2}{6}\right)^{\frac{1}{2}} N\,,\ \widetilde{N}^r = N^r\,,\ \widetilde{\Lambda} = \left(1 - \frac{\phi^2}{6}\right)^{\frac{1}{2}} \Lambda\,,\ \widetilde{R} = \left(1 - \frac{\phi^2}{6}\right)^{\frac{1}{2}} R$$

$$\widetilde{\pi}_\Lambda = \left(1 - \frac{\phi^2}{6}\right)^{-\frac{1}{2}} \pi_\Lambda\,,\ \widetilde{\pi}_R = \left(1 - \frac{\phi^2}{6}\right)^{-\frac{1}{2}} \pi_R\,,$$

$$\widetilde{\pi}_\phi = \left(1 - \frac{\phi^2}{6}\right) \pi_\phi + \frac{1}{6} R\,\phi\,\pi_R + \frac{1}{6}\Lambda\,\phi\,\pi_\Lambda\,.$$

- Poisson Brackets among canonical variables

$$\left\{\widetilde{N}, \widetilde{\pi}_N\right\} = 1\,, \left\{\widetilde{\Lambda}, \widetilde{\pi}_\Lambda\right\} = 1\,, \left\{\widetilde{R}, \widetilde{\pi}_R\ \right\} = 1\ \left\{\widetilde{\phi}, \widetilde{\pi}_\phi\right\} = 1$$

$$\{\widetilde{\pi}_\Lambda, \widetilde{\pi}_\phi\} = 0\,, \{\widetilde{\pi}_R, \widetilde{\pi}_\phi\} = 0$$

- The transformations from the Jordan to the Einstein frame are not Hamiltonian canonical

$$\left\{\widetilde{N}, \widetilde{\pi}_\phi\right\} = -N\left(1 - \frac{\phi^2}{6}\right)^{\frac{1}{2}} \frac{\phi}{6}\,,$$

$$\left\{\widetilde{N}_r, \widetilde{\pi}_\phi\right\} = -N_r\left(1 - \frac{\phi^2}{6}\right) \frac{\phi}{3}\,.$$

Gauge fixing implemented as secondary constraints, and intoduction of Dirac's barckets to make the transformation from JF to EF Hamitonian canonical

$$N = c(r)\, N_r = c_r(r)$$

$$\{\cdot,\cdot\}_{DB} \equiv \{\cdot,\cdot\} - \{\cdot,\chi_\alpha\} C^{-1}_{\alpha\beta} \{\chi_\beta,\cdot\}$$

SOME USEFUL REMARKS

- There exist some articles, in the litterature, which do not treat the integration by parts coming from $^{(4)}R$ and use boundary terms, in the action, different from those we introduced above

- In particular, in the Jordan Frame, the 3+1 decomposition of $^{(4)}R$ has a term

$$2\nabla_\mu\left(-n^\mu K - a^\mu\right)$$

that, in scalar tensor theories, generates quantities, by integration by parts, important for the equations of motion. If we discard the above divergence term, we checked, we obtain momenta as

$$\mathcal{L}_{JF} = \frac{1}{16\pi}\left[\left(1-\frac{\phi^2}{6}\right)^{(4)}R - g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right]$$

$$\begin{aligned}
\pi_\Lambda &= -\left(1-\phi^2\right)\frac{R\left(\dot{R}-R'N^r\right)}{N}\,,\\
\pi_R &= -\left(1-\phi^2\right)\frac{\Lambda\left(\dot{R}-R'N^r\right)+R\left(\dot{\Lambda}-(\Lambda N^r)\right)}{N}\,,
\end{aligned}$$

- As usual, we can express the Hamiltonian canonical variables in the Eisntein frame as function of the Hamiltonian canonical variables in the Jordan frame. If we employs the previous definition of the momenta we get the wrong commutation relation.

JANIS SPHERICAL SYMMETRIC SOLUTION

- Consider the following Fisher, Janis, Newman and Winicour (FJNW) solution (in the Einstein frame) (b and γ are related to the mass m of the compact object)

$$d\widetilde{s}^2 = -\left(1-\frac{b}{r}\right)^{\gamma} dt^2 + \left(1-\frac{b}{r}\right)^{-\gamma} dr^2 + r^2\left(1-\frac{b}{r}\right)^{1-\gamma} d\Omega^2 \qquad b = 2\sqrt{m^2+\frac{q^2}{2}}\,,\ \gamma = \frac{2m}{b}\,;$$

$$\widetilde{\phi} = \sqrt{\frac{1-\gamma^2}{2}}\ln\left(1-\frac{b}{r}\right).$$

- In general, for $\gamma \neq 1$, it has two singularities $r = 0$ and $r = b$. $r = b$ is a singularity without a horizon, in fact checking the curvature invariants

$$\widetilde{g}^{\mu\nu}\widetilde{R}_{\mu\nu} = \frac{1-\gamma^2}{2(b-r)^2}\left(1-\frac{b}{r}\right)^{\gamma}\left(\frac{b}{r}\right)^2 \xrightarrow{\gamma=1} 0\,,$$

$$\widetilde{R}^{\mu\nu}\widetilde{R}_{\mu\nu} = \frac{(1-\gamma^2)^2}{4(b-r)^4}\left(1-\frac{b}{r}\right)^{2\gamma}\left(\frac{b}{r}\right)^4 \xrightarrow{\gamma=1} 0\,,$$

$$\widetilde{R}^{\mu\nu\rho\sigma}\widetilde{R}_{\mu\nu\rho\sigma} = \left[12\gamma^2 + \frac{(1+\gamma)^2(3+2\gamma+7\gamma^2)}{4}\left(\frac{b}{r}\right)^2 - 4\gamma(1+\gamma)(1+2\gamma)\frac{b}{r}\right]$$

$$\frac{1}{(b-r)^4}\left(1-\frac{b}{r}\right)^{2\gamma}\left(\frac{b}{r}\right)^2 \xrightarrow{\gamma=1} \frac{12b^2}{r^6} = \frac{48m^2}{r^6}\,.$$

- FJNW is solution of the Equations of motion in the Einstein frame

$$\widetilde{N} = \left(1-\frac{b}{r}\right)^{\gamma/2}\,,\quad \widetilde{N}^r = 0\,,$$

$$\widetilde{\Lambda} = \left(1-\frac{b}{r}\right)^{-\gamma/2}\,,\quad \widetilde{R} = r\left(1-\frac{b}{r}\right)^{(1-\gamma)/2}$$

$$\widetilde{\phi} = \sqrt{\frac{1-\gamma^2}{2}}\ln\left(1-\frac{b}{r}\right).$$

BBMB BLACK HOLES IN THE JORDAN FRAME

- If one pass from the EF to the JF $ds^2 = \frac{1}{1-\phi^2/6} d\tilde{s}^2$ in the case $\gamma = \frac{1}{2}$, we get

$$N(r) = \frac{1}{2}\left(1 + \sqrt{1 - \frac{b}{r}}\right), \quad N^r(r) = 0,$$

$$\Lambda(r) = \frac{1}{2}\left(1 + \frac{1}{\sqrt{1 - \frac{b}{r}}}\right), \quad R(r) = \frac{r}{2}\left(1 + \sqrt{1 - \frac{b}{r}}\right)$$

$$\phi = \sqrt{6}\tanh\left[\frac{1}{4}\ln\left(1 - \frac{b}{r}\right)\right] = -\sqrt{6}\frac{1 - \sqrt{1 - b/r}}{1 + \sqrt{1 - b/r}}$$

- If we pose

$$\sqrt{1 - \frac{b}{r}} = 1 - \frac{b}{2\rho}$$

- The metric in the Jordan frame is the Bocharova-Bronnikov-Melnikov-Bekenstein (BBMB) Black Hole metric

$$ds^2 = -\left(1 - \frac{b}{4\rho}\right)^2 dt^2 + \left(1 - \frac{b}{4\rho}\right)^{-2} d\rho^2 + \rho^2 d\Omega^2$$

$$\phi = -\sqrt{6}\frac{\frac{b}{4}}{\rho - \frac{b}{4}}\,.$$

BBMB BLACK HOLES IN THE JORDAN FRAME

- The . $r = b$ naked singularity in the Einstein frame corresponds here to the Balck Hole horizon $\rho = \rho$

- If we compute the curvature invariants, there exists one singularity for $\rho = 0$

$$
\begin{aligned}
g^{\mu\nu} R_{\mu\nu} &= 0\,, \\
R^{\mu\nu} R_{\mu\nu} &= \frac{b^4}{64\rho^8}\,, \\
R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} &= \frac{b^2(7b^2 - 48b\rho + 96\rho^2)}{32\rho^8}
\end{aligned}
$$

- The transformation from the Joradn to the Einstein frame is, as immediately seen, singular for $\phi = -\sqrt{6}$, which correspond to the naked singularity $r = b$ in the Einstein frame and the horizon event $\rho = 0$ in the Jordan frame.

CONCLUSIONS

- We have introduced and discussed the Jordan and Einstein frames in the Hamiltonian formalisms
- We have proved that the Weyl (conformal) Hamiltonian transformations from the Jordan to the Einstein frames is Hamiltonian canonical, provided we perform a gauge fixing on the lapse and shifts functions, implement them as secondary constraints, and substitute the relative Dirac's Brackets to the Poisson Brackets.
- We have introduced the ADM analysis in the case of Spherical Symmetry and the relative boundary terms in the action functional. We have stressed the importance of the right boundary terms in deriving the Equations of motions in the Hamiltonian formalism.
- We have shown an example of Spherical Symmetric solutions. The Janis solutition, via a Weyl (conformal) transformation with a singularity, can be mapped to BBMB Black Hole. The naked singularity in mapped into the Black Hole horizon.
- Jordan and Einstein frames, via gauge fixing in the Hamiltonian formalism, are mathematically equivalent via a Hamiltonian canonical transformation. The Weyl(conformal) Hamiltonian transformation maps solutions of the equations of motion in the Jordan Frame into solutions of the equations of motion in the Einstein frame. The Physical equivalence is, in our opinion, still an open problem.

CANONICAL EQUIVALENCE AND PHYSICAL EQUIVALENCE

- JF is canonical equivalent, via gauge-fixing of Lapse N and shifts N_i , to EF (structure of light cone preserved by JF-EF transformations).

- JF is canonical equivalent to Anti-Gravity frame (light cone structure modified by JF- Anti-Gravity transformations).

- JF cannot be equivalent to two physically inequivalent frames. Therefore, Hamiltonioan canonical transformations represent, in our opinion, a mathematical equivalence. These transformations map solutions of e.o.m into solutions of e.o.m.

CONCLUSIONS

- The transformations from the Jordan to the Einstein frames, in the extended phase space, are not Hamiltonian canonical transformations.

- Gauge-fixing the Lapse N and the Shifts N_i and implementing the Dirac's Brackets, Hamiltonian canonical transformations do exist from JF to EF.

- This very fact does not mean, necessarily, that the two frames are "physically" equivalent.

- The equivalence of the physical observables in JF and EF remains still to be studied.

CANONICAL EQUIVALENCE OF JF AND EJ VIA GAUGE FIXING

- We have performed the following gauge fixing in the Jordan Frame and in the Einstein Frame

$$\text{Jordan Frame}\, N \approx c\ , N_i \approx c_i\ \mapsto\ \text{Einstein Frame}\, \widetilde{N} - c(16\pi G\phi)^{\frac{1}{2}} \approx 0\ , \widetilde{N}_i - c_i(16\pi G\phi) \approx 0$$

- The secondary first class constraints $\pi \approx 0$ and $\pi_i \approx 0$ become second class constraints

- It is possible to define Dirac's brackets and solve the second class constraints

$$\{\,,\,\}_{DB} \equiv \{\,,\,\} - \{\,,\varphi_\alpha\}C^{-1}_{\alpha\beta}\{\varphi_\beta,\,\}\ \ C_{\alpha\beta} \equiv \{\varphi_\alpha, \varphi_\beta\}\ \text{being}\ \varphi_\alpha, \varphi_\beta\ \text{second class contraints}$$

- The transformations from the Jordan to the Einstein frames result to be Hamiltonian canonical transformations. Remember: now the phase space is a reduced one, where we have gauge-fixed the lapse function N and the shift functions N_i .

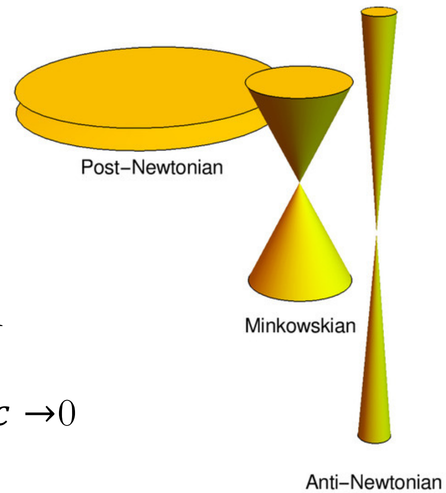
- Does it mean that the two frames are physically equivalent?

ANTI-GRAVITY TRANSFORMATIONS (Canonical Transformations)

- There exist Hamiltonian Canonical Transformations on the extended phase space: The Anti-Gravity transformations

$$\widetilde{N}^* = N \; ; \; \widetilde{\pi}_{N^*} = \pi_N \; ; \widetilde{N_i^*} = N_i \; ; \; \widetilde{\pi}^{*i} = \pi^i \; ; \; \widetilde{h}^*_{ij} = (16\pi G\phi) h_{ij} \; ;$$

$$\widetilde{\pi}^{*ij} = \frac{\pi^{ij}}{(16\pi G\phi)^{\frac{1}{2}}} \; ; \widetilde{\phi}^* = \phi \; ; \; \widetilde{\pi}^*_\phi = \frac{1}{\phi}\left(\phi\pi_\phi - \pi_h\right) ;$$



Carrolian Gravity, G→ ∞ , c →0

M. Niedermaier 2019

- In two dimensions, they look like

$$ds^2 = -dt^2 + \lambda^2 dx^2 ; \lambda > 1$$

- Since this theory is canonically equivalent to B-D theory, the constraint algebra of secondary first class constraints $(\mathcal{H}, \mathcal{H}_i)$ is like B-D theory's one.