Weyl cohomology and the conformal anomaly in the presence of torsion

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The conformal anomaly is not a bug

- Knowledge of the trace anomaly (partially) yields $\Gamma[g, \tau]$
  - 👍 No complicated renorm procedures, mostly inde of the spin of fields integrated
  - 👎 Γ_c
- Cohomology yields the most general *model independent* anomaly
  - 👍 greatly generalize previous results
  - 👎 torsion decrees applicability range + mixed anomalies, but . . .

Some terminology

- Trace anomaly consists of: $-\frac{1}{\sqrt{g}}\frac{\delta\Gamma}{\delta\sigma} \subseteq \underbrace{\text{nontrivial}}_{a+b} + \underbrace{\text{trivial}}_{a'}$

- Torsion is just another field

$$A^{\mu}{}_{\nu\rho} = \Gamma^{\mu}{}_{\nu\rho} + K^{\mu}{}_{\nu\rho}(T)$$

A general linear connection, Γ is Levi-Civita, and K contortion. Torsion irreps:

$$T^{\mu}{}_{\nu\rho} = \frac{1}{d-1}\left(\delta^{\mu}{}_{\rho}\boldsymbol{\tau_\nu} - \delta^{\mu}{}_{\nu}\boldsymbol{\tau_\rho}\right) + H^{\mu}{}_{\nu\rho} + \kappa^{\mu}{}_{\nu\rho}$$

Only $\boldsymbol{\tau}$ admits nontrivial Weyl transf:

1. **strongly**: $\tau'_\mu = \tau_\mu + b\,\partial_\mu\sigma$
2. **weakly**: $\tau'_\mu = \tau_\mu$

Cohomological method in a nutshell

Classically: $\delta_\sigma S = 0$. At the quantum level

$$\delta_\sigma \Gamma^{(1)}[g,\tau] = \omega[\sigma; g,\tau]$$

if σ grassmannian[1], $\boldsymbol{\delta_\sigma^2 = 0}$ for 1./2. we get CCs

$$\boxed{\delta_\sigma \omega[\sigma; g,\tau] = 0}$$

and $\exists$ **analogue of de Rham cohom** w.r.t. δ_σ:

- $\omega_{TA} = \delta_\sigma F[g,\tau] \Leftrightarrow$ exact 1-forms
- $\delta_\sigma \omega_{NT} = 0$ but such $F \nexists \Leftrightarrow$ closed but not ex, they $\in H^1_{dR}(M)$

Strategy: basis of 1-forms (or cochains) + consistency cond!

[1] L. Bonora, P. Pasti and M. Bregola, Class. Quant. Grav. 3 (1986), 635

invariant torsion in $2d$: consistency conditions at work

Standard Weyl gen: $\delta_\sigma = 2\int d^2x\, \sigma\, g_{\mu\nu} \frac{\delta}{\delta g_{\mu\nu}}$

♦ 1-cochain basis

$$\omega_1 = \int d^2x \sqrt{g}\sigma J\,, \qquad \omega_2 = \int d^2x \sqrt{g}\sigma\,(\nabla\cdot\tau)\,, \qquad \omega_3 = \int d^2x \sqrt{g}\sigma\,(\tau\cdot\tau)$$

where $J = R/2$. CCs: $\delta_\sigma \sum_i c_i\omega_i = 0$ yield no constraints since

$$\delta_\sigma \omega_i[\sigma; g, \tau] = 0\,, \quad i = 1,2,3\,,$$

and thus

$$\boxed{\omega_\sigma = \int \sqrt{g}\sigma\Big\{c_1 J + c_2\,(\nabla\cdot\tau) + c_3\,(\tau\cdot\tau)\Big\}}$$

♦ What's trivial? It is simple to see

$$\xi_1 = \int d^2x\sqrt{g}J\,, \quad \xi_2 = \int d^2x\sqrt{g}\,(\nabla\cdot\tau)\,, \quad \xi_3 = \int d^2x\sqrt{g}\,(\tau\cdot\tau) \quad \Rightarrow \quad \delta_\sigma\xi_i[g,\tau] = 0$$

so, no trivial anomalies.

invariant torsion in $2d$: anomalous actions

Finite conformal transformations

$$\sqrt{g}\,(\nabla\cdot\tau)=\sqrt{g'}\,(\nabla'\cdot\tau)\ ,\qquad \sqrt{g}\,(\tau\cdot\tau)=\sqrt{g'}\,(\tau\cdot\tau)\ ,\qquad \sqrt{g}\,J=\sqrt{g'}\,J'-\Delta_2'\sigma$$

Nonlocal action

$$\Gamma_{NL}[g,\tau]=-\int d^2x\sqrt{g}\Big\{\Big(\frac{c_1}{2}J+c_2\,(\nabla\cdot\tau)+c_3\,(\tau\cdot\tau)\Big)\frac{1}{\Delta_2}J\Big\}+\Gamma_c[g,\tau]$$

Localized action

$$\Gamma_{loc}[g,\varphi,\psi]=\sum_{i=1}^{2}\int d^2x\sqrt{g}\Big\{\frac{1}{2}\varphi_i\Delta_2\varphi_i+\alpha_i\varphi_iJ+\beta_i\varphi_i\mathcal{T}_i-\frac{1}{2}\psi_i\Delta_2\psi_i-\beta_i\psi_i\mathcal{T}_i\Big\}$$

$$\mathcal{T}_1=\nabla\cdot\tau\,,\qquad \mathcal{T}_2=\tau\cdot\tau$$

where

$$\alpha_1=\alpha_2=\sqrt{\frac{c_1}{2}},\qquad \beta_1=\sqrt{\frac{2}{c_1}}\,c_2\,,\qquad \beta_2=\sqrt{\frac{2}{c_1}}\,c_3$$

invariant torsion in $2d$ in a toy model: Wald entropy on Rindler spaces

Wald entropy for nonlocal actions[2] yields

$$\frac{1}{2\pi} S_{\varphi_{1,2} J} = - \left(\frac{\partial L_{loc}}{\partial R_{\alpha\beta\mu\nu}} \right) \epsilon_{\alpha\beta} \epsilon_{\mu\nu} \Big|_{x=x_h} = \left\{ c_1 \frac{1}{\Delta_2} J + c_2 \frac{1}{\Delta_2} \nabla \cdot \tau + c_3 \frac{1}{\Delta_2} \tau \cdot \tau \right\} \Big|_{x=x_h}$$

Near horizon $ds^2 = dr^2 + \left(\frac{2\pi}{\beta_H}\right)^2 r^2 d\eta^2 \sim$ Rindler. In an out-of-eq approach an using $\theta = \beta^{-1}\eta$ becomes a cone

$$ds^2 = dr^2 + \alpha^2 r^2 d\theta^2 \,, \qquad \alpha = \frac{\beta}{\beta_H}$$

Laplacian's Green function[3]

$$G(\vec{r}, \vec{r}_1) = \frac{2(\alpha - 1)}{\alpha} \ln |\vec{r} - \vec{r}_1|$$

and we choose

$$\tau_r \sim \frac{1}{r^{1+\epsilon}}$$

[2] R. C. Myers, Phys. Rev. D 50 (1994)
[3] S. N. Solodukhin, Phys. Rev. D 51 (1995)

invariant torsion in $2d$ in a toy model: Wald entropy on Rindler spaces

Wald's formula yiels

$$\frac{1}{\Delta_2}\tau\cdot\tau\Big|_{r_1=0} = \frac{2(\alpha-1)^3}{\alpha}\int_0^{2\pi}\int_a^{\infty} r\alpha d\theta dr\,\frac{\ln r}{r^{2(1+\epsilon)}} = (\alpha-1)^3\,\frac{\pi\epsilon^{-2}(1+2\epsilon\ln a)}{a^{2\epsilon}}$$

and

$$\frac{1}{\Delta_2}\nabla\cdot\tau\Big|_{r_1=0} = \frac{2(\alpha-1)^2(1+\epsilon)}{\alpha}\int_0^{2\pi}\int_a^{\infty} r\alpha d\theta dr\,\frac{\ln r}{r^{2(1+\epsilon)}} = -(\alpha-1)^2\,\frac{4\pi\epsilon^{-2}(1+\epsilon)(1+\epsilon\ln a)}{a^{\epsilon}}$$

Notice

- log div for $a\to 0$, but for $\alpha=1$ we get, as expected, zero
- $\alpha=1$ choose the Rindler temperature $\sim$ Minkowski vacuum

Weyl generator

$$\delta_\sigma = \delta_\sigma^g + \delta_\sigma^\tau = 2\int d^dx\,\sigma\, g_{\mu\nu}\frac{\delta}{\delta g_{\mu\nu}} + b\int d^dx\,\partial_\mu\sigma\,\frac{\delta}{\delta\tau_\mu}$$

Nöether id for classical Weyl inv of $S[g,\tau]$ is

$$T^\mu{}_\mu = -b\nabla_\mu\mathcal{D}^\mu$$

$\mathcal{D}^\mu$ (virial) current coupled to τ_μ

$$\frac{\delta}{\sqrt{g}\delta\tau_\mu}S[g,\tau] = \mathcal{D}^\mu$$

In flat space limit: *scale anomaly.*

affinely transforming torsion in $2d$: consistent anomaly

◆ 1-cochain basis

$$\omega_1 = \int d^2x\sqrt{g}\sigma\tilde{J}, \qquad \omega_2 = \int d^2x\sqrt{g}\sigma\left(\nabla\cdot\tau\right), \qquad \omega_3 = \int d^2x\sqrt{g}\sigma\left(\tau\cdot\tau\right)$$

$\tilde{J} = J + \frac{1}{b}\nabla\cdot\tau$ transf homogen with weight -2. CCs yield

$$-2bc_3\int d^2x\sqrt{g}\sigma\left(\nabla_\mu\sigma\right)\tau^\mu = 0 \quad \Rightarrow \quad c_3 = 0$$

Thus

$$\boxed{\omega_\sigma = \int\sqrt{g}\sigma\Big\{c_1\tilde{J} + c_2\left(\nabla\cdot\tau\right)\Big\}}$$

◆ What's trivial? 0-cochains basis

$$\xi_1 = \int d^2x\sqrt{g}\tilde{J}, \qquad \xi_2 = \int d^2x\sqrt{g}\left(\nabla\cdot\tau\right), \qquad \xi_3 = \int d^2x\sqrt{g}\left(\tau\cdot\tau\right)$$

Only ω_2: $\boxed{\omega_2 = -\frac{1}{2b}\delta_\sigma\xi_3}$

We have

$$\sqrt{g'}\,\tilde{J}' = \sqrt{g}\,\tilde{J}\,, \qquad \sqrt{g'}\,(\nabla'\cdot\tau') = \sqrt{g}\,(\nabla\cdot\tau + b\Delta_2\sigma)$$

$\boldsymbol{\nabla}\cdot\boldsymbol{\tau}$: *mixed* $(a + a')$-anomaly $\rightarrow$ integrate *simultaneously*

$$\Gamma_{WZ}[\sigma; g, \tau] = \int d^2x\sqrt{g}\sigma\left\{c_1\tilde{J} + \mathfrak{C}_2\,(\nabla\cdot\tau) + b\frac{\mathfrak{C}_2}{2}\Delta_2\sigma\right\} - \frac{\mathfrak{C}_3}{2b}\int d^2x\sqrt{g}\,(\tau\cdot\tau)$$

with $\quad \mathfrak{C}_2 + \mathfrak{C}_3 = c_2$

But "p_1 transforms as p_2" equivalence rel: *Weyl classes*.

Ex1: $\tilde{J} \in [J]_{dR}$ but different Weyl class.

Ex2: $a' \in [0]_{dR}$: $\neq$ representative may belong to $\neq$ Weyl classes, integ accordingly!

Localized action also displays a seemgly *unphysical* dependence

$$\Gamma_{NL}[g,\varphi,\psi] = \int d^2x\sqrt{g}\left\{\frac{1}{2}\varphi\Delta_2\varphi + \alpha\varphi\tilde{J} + \beta\varphi\left(\nabla\cdot\tau\right) - \frac{1}{2}\psi\Delta_2\psi - \gamma\psi\tilde{J}\right\} - \frac{\mathfrak{C}_2 - c_2}{2}\int d^2x\sqrt{g}\left(\tau\cdot\tau\right)$$

with

$$\beta = \sqrt{\mathfrak{C}_2}\,, \qquad \alpha = \frac{c_1}{\sqrt{\mathfrak{C}_2}}\,, \qquad \gamma^2 = \frac{c_1^2}{\mathfrak{C}_2}$$

However, it does **not** appear in the Wald entropy

$$\frac{1}{2\pi}S_{\varphi\tilde{J}+\psi\tilde{J}} = -\left(\frac{\partial L}{\partial R_{\alpha\beta\mu\nu}}\right)\epsilon_{\alpha\beta}\epsilon_{\mu\nu}\Big|_{x=x_h} = c_1\frac{1}{\Delta_2}\left(\nabla\cdot\tau\right)\Big|_{x=x_h}$$

Conclusions and outlook

We got

- most general and τ-dependent anomaly, mixed anomalies
- even if it is less powerful some observables are well-defined in $2d$

Some future directions

- Including torsion irreps straightforward but involved
- Feynman propagator on Rindler e.g. from Lorentz-boost eigenfunctions
- Full-fledged application to bh thermo

$$\ln \mathcal{Z}(\beta) \approx -S_{cl}[g,\tau] - \Gamma_{loc}[g,\varphi_i,\psi_i]$$

Thanks for listening!

Thank you for your attention, questions are welcome!

$d = 2$ and Polyakov action, a simple but paradigmatic example

No b- and a'-anomaly

$$\langle T^\mu{}_\mu \rangle = a E_2 = a R \,.$$

A conf scalar with 2 derivatives has $a = \frac{1}{24\pi}$.

From $g'_{\mu\nu} = e^{2\sigma} g_{\mu\nu}$ and the chain rule

$$2 g'_{\mu\nu} \frac{\delta}{\delta g'_{\mu\nu}} \Gamma[g'_{\mu\nu}] = \frac{\delta}{\delta\sigma} \Gamma[e^{2\sigma} g_{\mu\nu}] = \sqrt{g'} \langle T^\mu{}_\mu \rangle \equiv a \sqrt{g'} R' \,.$$

In $d = 2$

$$\sqrt{g'} R' = \sqrt{g} (R - 2\Box\sigma) \quad \Rightarrow \quad \Gamma[\sigma, g_{\mu\nu}] = a \int d^2x \sqrt{g} (R\sigma - \sigma\Box\sigma)$$

by using $\sigma = \frac{1}{2} \frac{1}{\Box} R$ we get the celebrated **Polyakov non-local action**

$$\Gamma_{NL}[g] = \frac{a}{4} \int d^2x \sqrt{g} R \frac{1}{\Box} R \,.$$

Localizing Polyakov action

Polyakov non-local action

$$\Gamma_{NL}[g] = \frac{a}{4} \int d^2x \sqrt{g} R \frac{1}{\Box} R$$

It is easy to localize. The local action

$$\Gamma_{loc}[g, \varphi] = -\frac{a}{2} \int d^2x \sqrt{g} \{ \frac{1}{2} \varphi \Box \varphi - R\varphi \}$$

yields $\Gamma_{NL}[g]$ one we go on-shell using

$$\Box \varphi = R$$

Thus the strategy is to write our non local action "completing the square" so to write them in a form that resemble Polyakov.

Schouten and its trace are defined as

$$K_{\mu\nu} = \frac{1}{d-2}\left(R_{\mu\nu} - \frac{1}{2(d-1)}Rg_{\mu\nu}\right) \qquad \mathcal{J} = g^{\mu\nu}K_{\mu\nu} = \frac{1}{2(d-1)}R\,,$$

Weyl

$$W_{\mu\nu\lambda\theta} = R_{\mu\nu\lambda\theta} - g_{\mu\lambda}K_{\nu\theta} + g_{\nu\lambda}K_{\mu\theta} - g_{\nu\theta}K_{\mu\lambda} + g_{\mu\theta}K_{\nu\lambda}$$

Cotton and Bach

$$C_{\mu\nu\lambda} = \nabla_\lambda K_{\mu\nu} - \nabla_\nu K_{\mu\lambda}$$
$$B_{\mu\nu} = \nabla^\lambda C_{\mu\nu\lambda} + K^{\lambda\theta}W_{\lambda\mu\theta\nu}$$

Properties Cotton

$$C_{\mu\nu\lambda} = -C_{\mu\lambda\nu} \qquad C_{\mu\nu\lambda} + C_{\nu\lambda\mu} + C_{\lambda\mu\nu} = 0 \qquad C^{\mu}{}_{\mu\lambda} = 0 \qquad \nabla^\mu C_{\mu\nu\lambda} = 0$$

Modified conformal tensors

Consider

$$\tilde{J} = J + c\,\nabla\cdot\tau + c_1\,\tau\cdot\tau\,.$$

To ensure it transforms homogeneously, i.e., $\tilde{J} = \tilde{J}' - 2\sigma\tilde{J}$, the coefficients must satisfy

$$c = \frac{1}{b}\,,\qquad c_1 = -\frac{d-2}{2b^2}\,.$$

So $\sqrt{g}\,\tilde{J}$ is inv in $d=2$ and $\sqrt{g}\,\tilde{J}^2$ in $d=4$. Using conformal tensors

$$\tilde{J}^2 = J^2 - \frac{d-2}{b^2}J\,(\tau\cdot\tau) + \frac{2}{b}J\,(\nabla\cdot\tau) + \frac{1}{b^2}\,(\nabla\cdot\tau)^2 + \frac{(d-2)^2}{4b^4}\,(\tau\cdot\tau)^2 - \frac{d-2}{b^3}\,(\nabla\cdot\tau)\,(\tau\cdot\tau)\,,$$

Similarly, requiring

$$\tilde{K}^{\mu\nu} = K^{\mu\nu} + b_1\,(\nabla_\mu\tau_\nu + \nabla_\nu\tau_\mu) + b_2\,\tau_\mu\tau_\nu + b_3\,g^{\mu\nu}\tau\cdot\tau + b_4\,g^{\mu\nu}\nabla\cdot\tau$$

transforms $\tilde{K}^{\mu\nu} = \tilde{K}'^{\mu\nu} - 4\sigma\tilde{K}^{\mu\nu}$, we get

$$b_1 = \frac{1}{2b}\,,\qquad b_2 = \frac{1}{b^2}\,,\qquad b_3 = -\frac{1}{2b^2}\,,\qquad b_4 = 0\,.$$

Naturally

$$\tilde{K}^\mu{}_\mu = \tilde{J}\,,$$

and, for the same choice of constants, that $\tilde{K}'_{\mu\nu} = \tilde{K}_{\mu\nu}$.

Modified conformal tensors

$\sqrt{g}\tilde{K}^2_{\mu\nu}$ inv in $d=4$. In terms of conf tensor

$$\tilde{K}^2_{\mu\nu} = K^2_{\mu\nu} - \frac{1}{b^2} J\,(\tau\cdot\tau) + \frac{2}{b^2} K_{\mu\nu}\,(\tau^\mu\;\tau^\nu) + \frac{2}{b} K_{\mu\nu}\,(\nabla^\mu\;\tau^\nu) + \frac{d}{4b^4}\,(\tau\cdot\tau)^2$$
$$+ \frac{2}{b^3}\,(\tau^\mu\tau^\nu)\,\nabla_\mu\tau_\nu - \frac{1}{b^3}\,(\nabla\cdot\tau)\,(\tau\cdot\tau) + \frac{1}{2b^2}\,(\nabla_\mu\tau_\nu\nabla^\nu\tau^\mu) + \frac{1}{2b^2}\,(\nabla_\mu\tau_\nu)^2\ .$$

Homothetic curvature

$$\Omega_{\mu\nu} = (\nabla_\mu\tau_\nu - \nabla_\nu\tau_\mu) = (\partial_\mu\tau_\nu - \partial_\nu\tau_\mu)\ .$$

so $\sqrt{g}\Omega^2_{\mu\nu}$ inv in $d=4$.

Modified conformal tensors

"Boundary" term $\widetilde{\Box}\tilde{J}$ (in $d=4$). Ansatz for $\widetilde{\Box}$

$$\widetilde{\Box} = \Box + g_1 \tau^\mu \nabla_\mu + g_2 \left(\nabla_\mu \tau^\mu\right) + g_3 \left(\tau^\mu \tau_\mu\right) \, .$$

By imposing $\widetilde{\Box}\tilde{J} = (\widetilde{\Box}\tilde{J})' - 4\sigma(\widetilde{\Box}\tilde{J})'$, we get

$$g_1 = \frac{d-6}{b} \, , \qquad g_2 = \frac{2}{b} \, , \qquad g_3 = -\frac{2(d-4)}{b^2} \, .$$

In terms of the conf tensors

$$\begin{aligned}
\widetilde{\Box}\tilde{J} = {} & \Box J + \frac{1}{b}\Box \nabla\cdot\tau - \frac{d-2}{b^2}\tau^\mu \Box \tau_\mu - \frac{d-2}{b^2}\left(\nabla_\mu \tau_\nu\right)^2 - \frac{d-6}{b}\tau\cdot(\nabla J) + \frac{2}{b} J\left(\nabla\cdot\tau\right) \\
& - \frac{d-6}{b^2}\tau^\mu \nabla_\mu \left(\nabla\cdot\tau\right) + \frac{(d-6)(d-2)}{b^3}\tau^\mu\tau^\nu\left(\nabla_\mu\tau_\nu\right) + \frac{10-3d}{b^3}\tau\cdot\tau\left(\nabla\cdot\tau\right) \\
& - \frac{2(d-4)}{b^2} J\left(\tau\cdot\tau\right) + \frac{2}{b^2}\left(\nabla\cdot\tau\right)^2 + \frac{(d-4)(d-2)}{b^4}\left(\tau\cdot\tau\right)^2 \, ,
\end{aligned}$$

$\sqrt{g}\,\widetilde{\Box}\tilde{J}$ Weyl-inv BT in $d4$.
The combinations are very useful the cohomological analysis.

Diff invariance is not broken

Nöther id Diff:

$$\nabla_\mu T^\mu{}_\nu = \mathcal{D}^\mu \Omega_{\mu\nu} + \tau_\nu \nabla_\mu \mathcal{D}^\mu$$

as expected it holds also for $\langle T^\mu{}_\nu \rangle$ and $\langle \mathcal{D}^\mu \rangle$ (therefore when computed using our anomalous actions).

2-cochains basis for CCs

For $d = 4$, $\delta_\sigma \omega_i[\sigma; g, \tau]$ can be expressed using

$$\int d^4x\sqrt{g}\,(\sigma\nabla_\mu\sigma)\,v_i^\mu\,, \qquad \int d^4x\sqrt{g}\,(\nabla_\mu\sigma\Box\sigma)\,w_j^\mu\,,$$

where v_i^μ and w_j^μ counts, respectively, as 3 and 1 derivatives, and are constructed from the curvatures, τ and covariant derivatives.

Ex: using this basis we get (affine case)

$$\sum_{j,n}\int d^4x\sqrt{g}\left\{h_j(f_i, c_k)\,(\nabla_\mu\sigma\Box\sigma)\,w_j^\mu + l_n(f_i, c_k)\,(\sigma\nabla_\mu\sigma)\,v_n^\mu\right\} = 0 \implies h_j = 0\,,\; l_n = 0$$

leading to

$$f_1 = \frac{f_6}{2b^3}\,, \qquad f_5 = \frac{f_6}{b}\,, \qquad f_7 = 0\,, \qquad f_8 = \frac{f_{10}}{2} - \frac{f_6}{b^2}\,,$$

$$f_9 = \frac{f_6}{2b} - f_{11}\,, \qquad f_{12} = f_{11}\,, \qquad f_{13} = -f_{11}\,, \qquad c_2 = -\frac{bf_6}{2} - c_1\,,$$

with $f_2\,, f_3\,, f_4\,, f_{14}$ arbitrary.

torsion dep basis

$$\omega_5 = \int d^4x\sqrt{g}\sigma\,(\tau\cdot\tau)^2\,,\qquad \omega_6 = \int d^4x\sqrt{g}\sigma\tilde{J}^2\,,\qquad \omega_7 = \int d^4x\sqrt{g}\sigma\tilde{K}^2_{\mu\nu}\,,$$

$$\omega_8 = \int d^4x\sqrt{g}\sigma\Omega^2_{\mu\nu}\,,\qquad \omega_9 = \int d^4x\sqrt{g}\sigma\tilde{J}\,(\tau\cdot\tau)\,,\qquad \omega_{10} = \int d^4x\sqrt{g}\sigma\left(\nabla\tilde{J}\right)\cdot\tau\,,$$

$$\omega_{11} = \int d^4x\sqrt{g}\sigma\tilde{K}^{\mu\nu}\tau_\mu\tau_\nu\,,\qquad \omega_{12} = \int d^4x\sqrt{g}\sigma\,\tau\cdot\tau\,(\nabla\cdot\tau)\,,\qquad \omega_{13} = \int d^4x\sqrt{g}\sigma\,(\nabla\cdot\tau)^2\,,$$

$$\omega_{14} = \int d^4x\sqrt{g}\sigma\,(\tau^\mu\tau^\nu)\,\nabla_\mu\tau_\nu\,,\qquad \omega_{15} = \int d^4x\sqrt{g}\sigma\,\tau_\mu\Box\tau^\mu\,,\qquad \omega_{16} = \int d^4x\sqrt{g}\sigma\,(\nabla_\mu\tau_\nu)^2\,,$$

$$\omega_{17} = \int d^4x\sqrt{g}\sigma\,\tau^\mu\nabla_\mu\nabla\cdot\tau\,,\qquad \omega_{18} = \int d^4x\sqrt{g}\sigma\widetilde{\Box}\tilde{J}\,,$$

Plus usual metric dep basis

$$\omega_1 = \int d^4x\sqrt{g}\sigma W^{\alpha\beta\rho\gamma}W_{\alpha\beta\rho\gamma}\,,\qquad \omega_2 = \int d^4x\sqrt{g}\sigma K^2_{\mu\nu}\,,$$

$$\omega_3 = \int d^4x\sqrt{g}\sigma J^2\,,\qquad \omega_4 = \int d^4x\sqrt{g}\sigma\Box J\,.$$

since $E_4 = 8(J^2 - K^2_{\mu\nu}) + W^2_{\alpha\beta\rho\gamma} \rightarrow$ only E_4 and W^2.

Similar (but much involved) analysis

$$\omega_\sigma = \int \sqrt{g}\sigma\Big\{ f_2\tilde{J}^2 + f_3\tilde{K}^2_{\mu\nu} + f_4\Omega^2_{\mu\nu} + f_{14}\tilde{\Box}\tilde{J} + f_6\nabla_\mu\big(\tilde{J}\tau^\mu\big) + \frac{f_{10}}{2}\nabla_\mu\left(\tau^\mu\,\tau\cdot\tau\right) + f_{11}\Big(\frac{1}{2}\Box\tau^2 - \nabla_\mu(\tau^\mu\,\nabla\cdot\tau)\Big)\Big\}$$

$\tilde{\Box}\tilde{J}$: $(b + a')$-anomaly.

Local action

$$\Gamma_{int}[g,\varphi_i,\psi_i] = \sum_{i=1}^{5}\int d^4x\sqrt{g}\Big\{\frac{1}{2}\varphi_i\Delta_4\varphi_i - \alpha_i\varphi_i Q_4 - \beta_i\varphi_i\mathcal{T}_i - \frac{1}{2}\psi_i\Delta_4\psi_i - \beta_i\psi_i\mathcal{T}_i\Big\}$$

$$\mathcal{T}_1 = W^2\,, \qquad \mathcal{T}_2 = \tilde{J}^2\,, \qquad \mathcal{T}_3 = \tilde{K}^2_{\mu\nu} \qquad \mathcal{T}_4 = \Omega^2_{\mu\nu}\,, \qquad \mathcal{T}_5 = \tilde{\Box}\tilde{J}$$

but with

$$\alpha_{i=1,\dots,5} = \sqrt{-\frac{a_1}{5}}\,, \qquad \beta_1 = -b_1\sqrt{-\frac{5}{a_1}}\,\dots\,, \quad \beta_5 = -\mathfrak{c}_{14}\sqrt{-\frac{5}{a_1}}$$

1-cochains basis: inv torsion

torsion dep basis

$$\omega_5 = \int d^4x\sqrt{g}\sigma\,(\tau\cdot\tau)^2\ , \qquad \omega_6 = \int d^4x\sqrt{g}\sigma J\,(\nabla\cdot\tau)\ , \qquad \omega_7 = \int d^4x\sqrt{g}\sigma K^{\mu\nu}\,(\nabla_\mu\tau_\nu)$$

$$\omega_8 = \int d^4x\sqrt{g}\sigma\Omega^2_{\mu\nu}\,, \qquad \omega_9 = \int d^4x\sqrt{g}\sigma J\,(\tau\cdot\tau)\ , \qquad \omega_{10} = \int d^4x\sqrt{g}\sigma\,(\nabla J)\cdot\tau\,,$$

$$\omega_{11} = \int d^4x\sqrt{g}\sigma K^{\mu\nu}\tau_\mu\tau_\nu\,, \qquad \omega_{12} = \int d^4x\sqrt{g}\sigma\,\tau\cdot\tau\,(\nabla\cdot\tau)\ , \qquad \omega_{13} = \int d^4x\sqrt{g}\sigma\,(\nabla\cdot\tau)^2\ ,$$

$$\omega_{14} = \int d^4x\sqrt{g}\sigma\,(\tau^\mu\tau^\nu)\,\nabla_\mu\tau_\nu \qquad \omega_{15} = \int d^4x\sqrt{g}\sigma\,\tau_\mu\Box\tau^\mu\,,\,, \qquad \omega_{16} = \int d^4x\sqrt{g}\sigma\ (\nabla_\mu\tau_\nu)^2\,,$$

$$\omega_{17} = \int d^4x\sqrt{g}\sigma\,\tau^\mu\nabla_\mu\nabla\cdot\tau\,, \qquad \omega_{18} = \int d^4x\sqrt{g}\sigma\Box\nabla\cdot\tau\,.$$

Plus usual metric dep basis.

invariant torsion in $4d$: the Ψ-anomaly

Now

$$\omega^{\tau}_{\sigma} = \int \sqrt{g}\sigma\Big\{ f_1\,(\tau\cdot\tau)^2 + f_4\Omega^2_{\mu\nu} + f_{12}\overbrace{\left(\tau^{\mu}\,\nabla_{\nu}\nabla_{\mu}\tau^{\nu} + (\nabla_{\mu}\tau_{\nu})^2\right)}^{\Psi} + \frac{f_{11}}{2}\Box\tau^2_{\mu}$$
$$+ \frac{f_{10}}{2}\nabla_{\mu}\,(\tau^{\mu}\,\tau\cdot\tau) + f_{13}\nabla_{\mu}\,(\tau^{\mu}\,\nabla\cdot\tau) + f_2(\frac{1}{2}\Box\nabla\cdot\tau + \nabla_{\mu}\,(\tau^{\mu}\,J))\Big\}$$

where

$$\sqrt{g'}\Psi' = \sqrt{g}(\Psi + O(g,\tau))$$

Weyl inv F can be integrated as

$$\int d^4x\sqrt{g}F(g,\tau)\frac{1}{O}\Psi\,, \qquad \int d^4x\sqrt{g}F(g,\tau)\frac{1}{\Delta_4}Q_4$$

Wess-Zumino action

$$\Gamma_{WZ} = \int d^4x\sqrt{g'}\sigma\Big\{f_1\,(\tau\cdot\tau)^2 + f_4\Omega^2_{\mu\nu} + f_5\Psi(g',\tau) - \frac{f_5}{2}O(g',\tau)\sigma\Big\}$$

Nonlocal action

$$\Gamma_{NL} = \int d^4x\sqrt{g'}\Big\{(f_1\,(\tau\cdot\tau)^2 + f_4\Omega^2_{\mu\nu} + b_1 W^2 + \frac{1}{2}Q_4^\tau)\frac{1}{\Delta_4^\tau}Q_4^\tau\Big\} + \Gamma_L$$

where

$$Q_4^\tau = a_1 Q_4 + f_5\Psi(g,\tau)\,, \qquad \sqrt{g}Q_4^\tau = (\sqrt{g'}Q'^\tau_4 + \Delta'^\tau_4)$$

and

$$\Delta_4^\tau \equiv a_1\Delta_4 - f_5 O(g,\tau)$$

is self adjoint and conformally cov by construction.