Study of the Inflationary Paradigm

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Abstract

In this work we study inflation, a theory proposed as an extension to the standard cosmological scenario to solve some consistency problems. It is characterized by a primordial exponential accelerated expansion. We present the parameters ε and η that guarantee that inflation can last long enough to provide a region of causality and stretching of the curvature of space-time. Furthermore, we present the slow-roll mechanism for the scalar field and its respective approximations for the parameters ε_V and η_V. Finally, we exemplify the theory with a quadratic potential and analyze under which conditions such potential leads to inflation.

Fundamentals

The metric $g_{\mu\nu}$ describing the expanding spacetime and adhering to the Cosmological Principle (CP) is the Friedmann-Lemaître-Robertson-Walker (FLRW) metric, which has the follow line element

$$ds^2 = -c^2dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2/R_0^2} + r^2 d\Omega^2\right], \tag{1}$$

With the temporal part of Einstein's equation $G_{\mu\nu} = 8\pi G T_{\mu\nu}$ we have the connection of General Relativity (GR) with Cosmology, providing the **Friedmann Equation**, in terms of the Hubble parameter defined as $H \equiv \frac{\dot{a}}{a}$,

$$H^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2R_0^2} + \frac{\Lambda c^2}{3}, \tag{2}$$

where k is the term related to curvature, c is the speed of light and Λ is the cosmological constant. In this work, we portray a possible correction to the problems present in standard cosmology, known as inflation. The shrinking Hubble sphere is the **fundamental definition of inflation**

$$\frac{d}{dt}(aH)^{-1} < 0\,. \tag{3}$$

We ask ourselves how long inflation needs to last in order to solve the problems. In particular, they are solved if the entire observable universe is smaller than the comoving Hubble radius at the beginning of inflation. Therefore, we have the condition $(a_0H_0)^{-1} < (a_iH_i)^{-1}$. We need to define a set that quantifies the increasing scale factor. This is characterized by e-folds N:

$$N_{\text{tot}} \equiv \ln(a_e/a_i)\,. \tag{4}$$

We have determined the minimum number of e-folds needed to result in the universe we live in. In this regard, since we know the amount by which the Hubble radius grew during the evolution from the Big Bang through the maximum temperature T_R of the thermal plasma, we have

$$N_{\text{tot}} > 64,47 + \ln\left(\frac{T_R}{10^{15}GeV}\right). \tag{5}$$

This is the famous statement that solving the horizon problem requires about 60 e-folds of inflation.

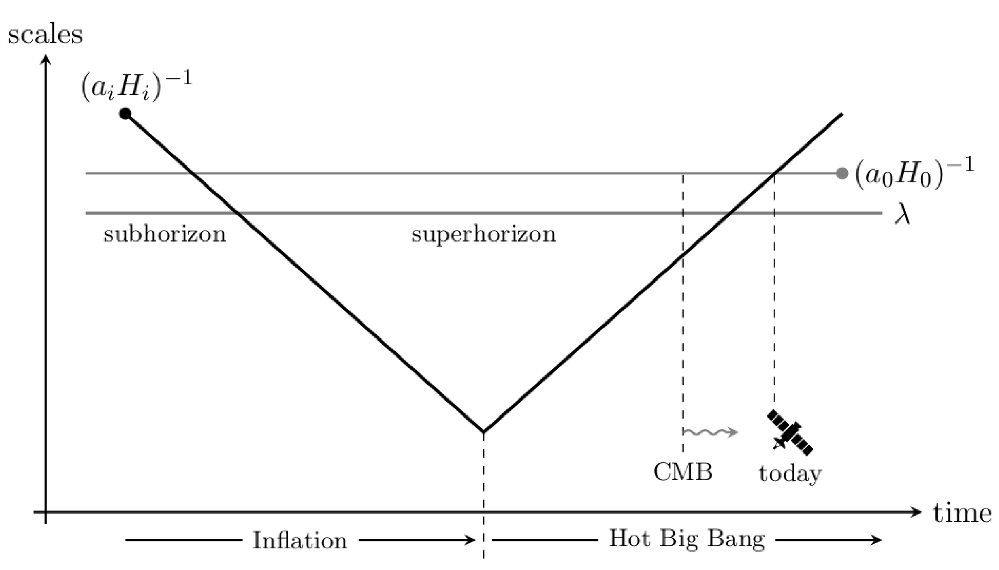

Figure 1: Illustration of how inflation resolves the problem of superhorizon correlations. The Hubble radius is represented by the black line. A representative scale λ is superhorizon when the CMB was created and subhorizon during inflation. Source: Ref. [1].

Physics of Inflation

A key feature of inflation is that all physical quantities are varying slowly, despite the rapid expansion of spacetime. In this sense, we write the time derivative of the comoving Hubble radius as

$$\frac{d}{dt}(aH)^{-1} = -\frac{1}{a}(1-\varepsilon)\,, \tag{6}$$

where we introduce the **first Hubble slow-roll parameter**

$$\varepsilon \equiv -\frac{d\ln H}{dN} = -\frac{\dot{H}}{H^2}\,. \tag{7}$$

Here $dN \equiv d\ln a = Hdt$. **The near scale-invariance of the observed fluctuations requires that** $\varepsilon \ll 1$. We want inflation to generally achieve 60 e-folds. This requires ε to remain small for a sufficiently large number of Hubble times. This condition is measured by a **second Hubble slow-roll parameter**.

$$\eta \equiv \frac{d\ln\varepsilon}{dN} = \frac{\dot{\varepsilon}}{H\varepsilon}\,. \tag{8}$$

For $|\eta| < 1$, the fractional change of ε per e-fold is small, and inflation persists.

Scalar Field Dynamics

The simplest models of inflation implement time-dependent dynamics during inflation in terms of the evolution of a scalar field, $\phi(t,x)$, called the **inflaton**, as shown in Fig. 2. If the energy associated with the scalar field $\phi(t,x)$ dominates the universe, then it drives the evolution of the FLRW background. We want to determine under what conditions this can lead to an inflationary expansion.

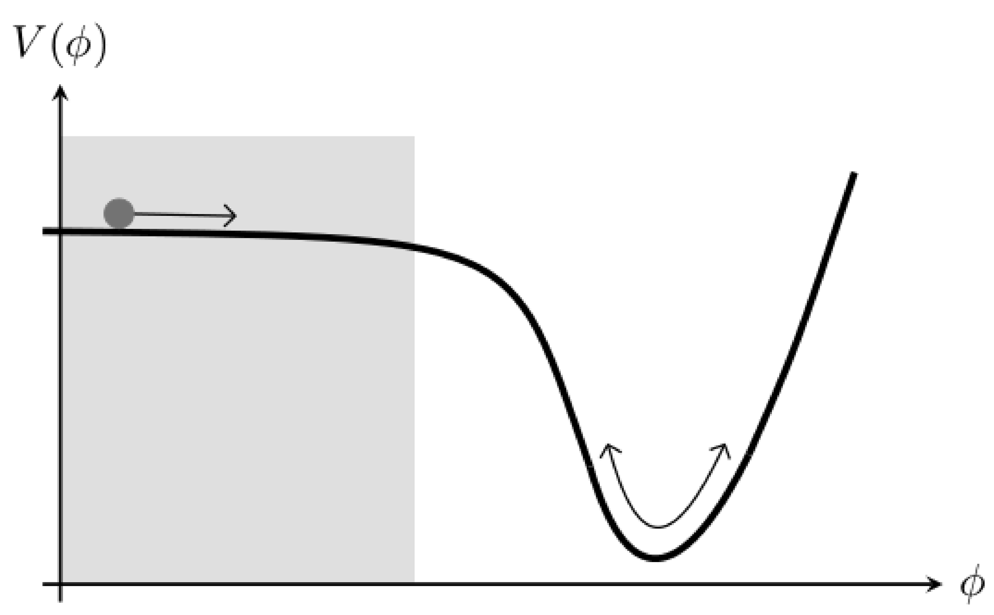

Figure 2: Example of a slow-roll potential. The inflaton scalar field is represented by the ball. Inflation occurs in the shaded part of the potential; in the potential well, the universe undergoes reheating, where particles of the standard model are formed. Source: Ref. [1].

The action of the scalar field involves a kinetic part and a potential part, as described below:

$$S = \int d^4x\sqrt{-g}\left[-\frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - V(\phi)\right], \tag{9}$$

where $g \equiv \det(g_{\mu\nu})$. When we vary the action S with respect to the field ϕ, we obtain the general Klein-Gordon equation

$$\Box\phi = -\frac{\partial V}{\partial\phi}\,. \tag{10}$$

Specifying this for the FLRW metric, we obtain the equation of motion:

$$\ddot{\phi} + 3H\dot{\phi} = -\frac{\partial V}{\partial\phi}\,. \tag{11}$$

The expansion of space-time generates Hubble friction $3H\dot{\phi}$. Given Eq. (9), it is natural to assume the Hamiltonian, or the energy density, as $\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi)$. With the continuity equation

$$\dot{\rho_\phi} = -3H\left(\rho_\phi + P_\phi\right), \tag{12}$$

and with the result $\dot{\rho_\phi} = 3H\dot{\phi}^2$, we infer the pressure

$$P_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi)\,. \tag{13}$$

This pressure will determine the acceleration of expansion linked to the **Raychaudhuri Equation**, which is obtained through the spatial part of Einstein Field Equation. There is a proportionality $\ddot{a} \propto -\left(\rho_\phi + 3P_\phi\right)$. However, if the kinetic energy of inflation is much less than the potential energy, then $P_\phi \approx -\rho_\phi$. **That is, the inflationary potential temporarily acts as a cosmological constant, promoting a period of exponential expansion**.

Slow-roll Inflation

The dynamics during inflation is determined by a combination of Eq. (2) and Eq. (11).

A slowly rolling field

Hereafter, we determine the parameter ε for a slowly rolling field. To do this, we find the Hubble rate by taking the temporal derivative of Eq. (3) with $\kappa = 0$ and $\Lambda = 0$, substituting ρ_ϕ and Eq. (11). Thus, we have

$$\dot{H} = -\frac{1}{2}\frac{\dot{\phi}^2}{M_{Pl}^2}\,. \tag{14}$$

Inflation occurs if the kinetic energy density has a small contribution to the total energy density. This situation is known as slow-roll inflation and has the condition $\varepsilon \ll 1$. The slow-roll behavior persists if the acceleration of the scalar field is also small. It is useful to define a dimensionless acceleration per Hubble time

$$\delta \equiv -\frac{\ddot{\phi}}{H\dot{\phi}}\,. \tag{15}$$

When δ is small, the friction term in Eq. (11) dominates and the velocity of the inflaton is determined by the slope of the potential. Furthermore, as long as δ is small, the kinetic energy of the inflaton remains subdominant and inflationary expansion continues.

Slow-roll approximation

We will use conditions to simplify the equations of motion. This is called the slow-roll approximation. First, we note that the condition $\varepsilon \ll 1$ implies $\dot{\phi}^2 \ll V$, hence $H^2 \approx \frac{V}{3M_{Pl}^2}$. In this approximation, the Hubble expansion rate is entirely determined by the potential. Next, we see that the condition $|\delta| \ll 1$ implies $3H\dot{\phi} \approx -\frac{dV}{d\phi}$. With $\varepsilon = \frac{1}{2}\frac{\dot{\phi}^2}{M_{Pl}^2H^2}$, we have the approximation

$$\varepsilon_V \approx \frac{M_{Pl}^2}{2}\left(\frac{V'}{V}\right)^2. \tag{16}$$

In order to study the parameter δ in the slow-roll approximation, we take the time derivative of ε and obtain

$$\eta_V \equiv \delta + \varepsilon \approx M_{Pl}^2\frac{V''}{V}\,. \tag{17}$$

A successful inflation occurs when these parameters are much smaller than unity, i.e., $\varepsilon_V, \eta_V \ll 1$. The total number of e-folds of accelerated expansion is

$$N_{\text{tot}} \equiv \int_{a_i}^{a_e} d\ln a = \int_{\phi_i}^{\phi_e}\frac{H}{\dot{\phi}}d\phi\,. \tag{18}$$

In the slow-roll regime, we can use Eq. (16) to write the integral over the field space as

$$N_{\text{tot}} \approx \int_{\phi_i}^{\phi_e}\frac{1}{\sqrt{2\varepsilon_V}}\frac{|d\phi|}{M_{Pl}}\,. \tag{19}$$

As we saw in the section "Fundamentals", a solution to the horizon problem requires $N_{\text{tot}} \gtrsim 60$, which provides an important constraint on successful models of inflation.

Study case: quadratic inflation

For example, let us analyze arguably the simplest inflationary model: single-field inflation driven by a mass term

$$V(\phi) = \frac{1}{2}m^2\phi^2\,. \tag{20}$$

This model is ruled out by observations of the CMB, but still provides a useful example to illustrate the mechanism of slow-roll inflation. Given Eq. (20), the parameters ε_V, η_V are

$$\varepsilon_V = \eta_V = 2\left(\frac{M_{Pl}}{\phi}\right)^2. \tag{21}$$

To satisfy the conditions of slow-roll, we need to consider super-Planckian values $\phi_e \equiv \sqrt{2}M_{Pl}$. As the field moves from $\phi_i \to \phi_e$, the number of e-folds of inflationary expansion is

$$N_{\text{tot}} = \frac{\phi_i^2}{4M_{Pl}^2} - \frac{1}{2}\,. \tag{22}$$

To achieve $N_{\text{tot}} > 60$, the initial field value must satisfy

$$\phi_i > 2\sqrt{60}M_{Pl} \sim 15M_{Pl}\,. \tag{23}$$

We note that the total field excursion is super-Planckian, $\Delta\phi = \phi_i - \phi_e \gg M_{Pl}$.

Conclusions

In this study we address the physics of inflation and obtain some results that provide the path to the equation of motion of the inflaton scalar field ϕ, as well as the conditions that cause the inflationary potential to temporarily act as a cosmological constant. In slow-roll inflation, we saw an expression for the total number N of e-folds that can resolve the horizon problem. Finally, we study quadratic inflation and, with the parameter ε_V, we obtain the condition (relating the scalar field ϕ and the Planck mass) that leads to inflation via a quadratic potential.

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References

[1] Baumann, D. *Cosmology. Cambridge University Press*, 2022.

[2] Ryder, L. *Introduction to General Relativity. Cambridge University Press*, 2009.