Field Equations of General Relativity via Variational Principle

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Field equations of GR

The variational principle is one of the most fundamental principles of physics and most modern theories are formulated in its terms. Although the field equations of General Relativity were first obtained by Einstein through heuristic arguments, Hilbert showed that it is possible to obtain them through the variational principle. The action is defined as being the integral of a Lagrangian density over space,

$$S = \int d^4x\mathcal{L}. \quad (1)$$

In principle, we can define a total action S such that this action represents the sum of the contributions of the gravitational field S_g and the matter fileds S_m,

$$S = S_g + S_m. \quad (2)$$

Let us first consider the action of the gravitational field. The action S_g is given by

$$S_g = \frac{c^3}{16\pi G}\int d^4x R\sqrt{-g}. \quad (3)$$

Henceforth, we consider variation δ with respect to the gravitational field, $g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$. The Lagrangian density of the gravitational field is recognized as

$$\mathcal{L}_g = \kappa\sqrt{-g}R; \ \kappa = \frac{c^3}{16\pi G}. \quad (4)$$

By taking its variation with respect to the metric:

$$\delta\mathcal{L}_g = \kappa\left[\sqrt{-g}R_{\mu\nu}\left(\delta g^{\mu\nu}\right) + R\left(\delta\sqrt{-g}\right) + \sqrt{-g}g^{\mu\nu}\left(\delta R_{\mu\nu}\right)\right]. \quad (5)$$

We can show that the last term in the above equation is a surface term and can therefore be neglected, as it does not contribute to the field equations.

Next, we consider the variation for the Christoffel symbol $\delta\Gamma^{\lambda}{}_{\mu\nu}$. We use the reciprocity property of the metric tensor to write: $g^{\lambda\sigma}g_{\sigma\rho} = \delta^{\lambda}{}_{\rho} \Longrightarrow \delta g^{\lambda\sigma} = -g^{\lambda\kappa}g^{\sigma\rho}\delta g_{\kappa\rho}$. Moreover, the object $\delta g_{\mu\sigma}$ is a genuine tensor so that we are authorized to calculate its covariant derivative: $\nabla_{\nu}\left(\delta g_{\mu\sigma}\right)$. The expression for this quantity is the same as:

$$\partial_{\nu}\left(\delta g_{\mu\sigma}\right) = \nabla_{\nu}\left(\delta g_{\mu\sigma}\right) + \Gamma^{\rho}{}_{\mu\nu}\left(\delta g_{\rho\sigma}\right) + \Gamma^{\rho}{}_{\sigma\nu}\left(\delta g_{\rho\mu}\right). \quad (6)$$

Analogous expressions are obtained for $\partial_{\mu}\left(\delta g_{\nu\sigma}\right)$ and $\partial_{\sigma}\left(\delta g_{\mu\sigma}\right)$. Using these results on the expression for the variation of the Christoffel symbol, we can show that the terms involving the connections cancel each other out and so we get

$$\delta\Gamma^{\lambda}{}_{\mu\nu} = \frac{1}{2}g^{\lambda\sigma}\left[\nabla_{\nu}\left(\delta g_{\mu\sigma}\right) + \nabla_{\mu}\left(\delta g_{\nu\sigma}\right) - \nabla_{\sigma}\left(\delta g_{\mu\nu}\right)\right]. \quad (7)$$

Thus, even though the connection $\Gamma^{\lambda}{}_{\mu\nu}$ is not a tensor, $\delta\Gamma^{\lambda}{}_{\mu\nu}$ is a tensor. Therefore, its covariant derivative can be computed:

$$\nabla_{\kappa}\left(\delta\Gamma^{\lambda}{}_{\mu\nu}\right) = \partial_{\kappa}\left(\delta\Gamma^{\lambda}{}_{\mu\nu}\right) - \Gamma^{\rho}{}_{\mu\kappa}\left(\delta\Gamma^{\lambda}{}_{\rho\nu}\right) - \Gamma^{\rho}{}_{\nu\kappa}\left(\delta\Gamma^{\lambda}{}_{\rho\mu}\right) + \Gamma^{\lambda}{}_{\rho\kappa}\left(\delta\Gamma^{\rho}{}_{\mu\nu}\right). \quad (8)$$

Also, by writing the covariant derivative $\nabla_{\nu}\left(\delta\Gamma^{\lambda}{}_{\mu\kappa}\right)$, by contracting the indices ν and λ in both covariant derivatives and by taking the difference between them, we get:

$$\nabla_{\lambda}\left(\delta\Gamma^{\lambda}{}_{\mu\kappa}\right) - \nabla_{\kappa}\left(\delta\Gamma^{\lambda}{}_{\mu\lambda}\right) = \partial_{\lambda}\left(\delta\Gamma^{\lambda}{}_{\mu\kappa}\right) - \partial_{\kappa}\left(\delta\Gamma^{\lambda}{}_{\mu\lambda}\right) - \Gamma^{\rho}{}_{\mu\lambda}\left(\delta\Gamma^{\lambda}{}_{\rho\kappa}\right) + \Gamma^{\lambda}{}_{\rho\lambda}\left(\delta\Gamma^{\rho}{}_{\mu\kappa}\right) + \Gamma^{\rho}{}_{\mu\kappa}\left(\delta\Gamma^{\lambda}{}_{\rho\lambda}\right) - \Gamma^{\lambda}{}_{\rho\kappa}\left(\delta\Gamma^{\rho}{}_{\mu\lambda}\right).$$

Now, by taking the variation of the Ricci tensor, $\delta R_{\mu\kappa}$, we obtain an expression that is exactly the same as the expression in the last equation, in such a way that

$$\delta R_{\mu\kappa} = \nabla_{\lambda}\left(\delta\Gamma^{\lambda}{}_{\mu\kappa}\right) - \nabla_{\kappa}\left(\delta\Gamma^{\lambda}{}_{\mu\lambda}\right). \quad (9)$$

This equation is known as the Palatini identity.

From the Palatini identity, we can then write the last term of the Eq. (5) in the form

$$\sqrt{-g}g^{\mu\kappa}\delta R_{\mu\kappa} = \sqrt{-g}\left[\nabla_{\lambda}\left(g^{\mu\kappa}\delta\Gamma^{\lambda}{}_{\mu\kappa}\right) - \nabla_{\kappa}\left(g^{\mu\kappa}\delta\Gamma^{\lambda}{}_{\mu\lambda}\right)\right], \quad (10)$$

where we used $\nabla_{\lambda}g^{\mu\kappa} = 0$. We can also write that the covariant derivative of a contravariant vector with its contracted indices is given by

$$\nabla_{\mu}V^{\mu} = \frac{1}{\sqrt{-g}}\partial_{\mu}\left(\sqrt{-g}V^{\mu}\right). \quad (11)$$

Plugging this result in the Eq. (10),

$$\int_{\Omega} d^4x\sqrt{-g}g^{\mu\kappa}\delta R_{\mu\kappa} = \int_{\Omega} d^4x\partial_{\lambda}W^{\lambda}, \quad (12)$$

where

$$W^{\lambda} = \sqrt{-g}g^{\mu\kappa}\delta\Gamma^{\lambda}{}_{\mu\kappa} - \sqrt{-g}g^{\mu\lambda}\delta\Gamma^{\kappa}{}_{\mu\kappa}. \quad (13)$$

Eq. (12) displays the integral sign inherited from the definition of S_g.

By Gauss's theorem,

$$\int_{\Omega} d^4x\partial_{\mu}W^{\mu} = \int_{\partial\Omega} W^{\mu}dS_{\mu}. \quad (14)$$

Eqs. (12) and (14) combine into:

$$\int_{\Omega} d^4x\sqrt{-g}g^{\mu\kappa}\delta R_{\mu\kappa} = \int_{\partial\Omega} W^{\mu}dS_{\mu}. \quad (15)$$

Notice that we are evaluating the left-hand side over a spacetime region Ω of boundary $\partial\Omega$ on which $\delta g_{\mu\nu} = 0$ (see Fig. 1).

Hence, the boundary $\partial\Omega$:

$$\delta\Gamma^{\lambda}{}_{\mu\kappa}\left(\delta g_{\mu\nu}\right)\big|_{\partial\Omega} = 0 \Rightarrow W^{\mu}\left(\delta\Gamma^{\lambda}{}_{\mu\kappa}\right)\big|_{\partial\Omega} = 0. \quad (16)$$

Thus the total divergence does not contribute to the field equations, that is, the right-hand side of Eq. (15) vanishes:

$$\int d^4x\sqrt{-g}g^{\mu\kappa}\delta R_{\mu\kappa} = 0. \quad (17)$$

This eliminates the last term in Eq. (5) under the integral sign requires by Eq. (3).

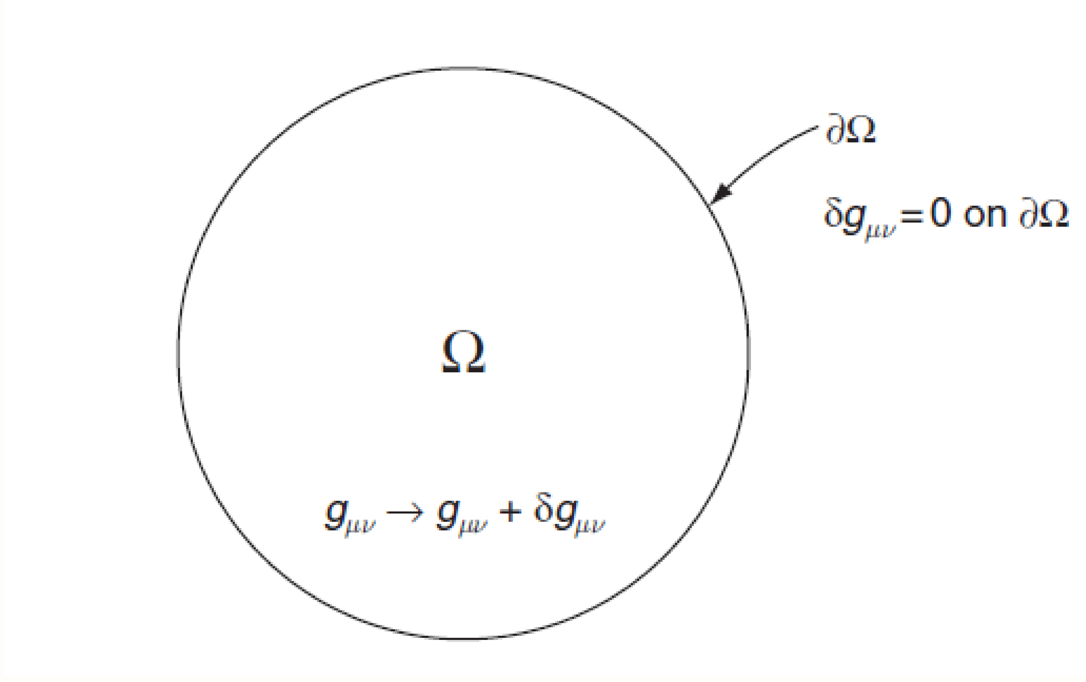

Figure 1: A spacetime Ω with boundary $\partial\Omega$. The variation of the metric tensor vanishes on the boundary.

Let us now look at the term $\delta\left(\sqrt{-g}\right)$, which appears in the second term in Eq. (5). The derivative of the determinant of the metric $g_{\mu\nu}$ is given by

$$\partial_{\rho}g = gg^{\mu\nu}\partial_{\rho}g_{\mu\nu}. \quad (18)$$

We can also write that

$$\partial_{\rho}\sqrt{-g} = \frac{1}{2}\sqrt{-g}g^{\mu\nu}\partial_{\rho}g_{\mu\nu} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\left(\delta g^{\mu\nu}\right). \quad (19)$$

This reduces the action integral to:

$$\delta S_g = \frac{c^3}{16\pi G}\int_{\Omega} d^4x\left[\sqrt{-g}\delta g^{\mu\nu}\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right)\right]. \quad (20)$$

The principle of least action prescribes that δS_g must be zero for an arbitrary $\delta g^{\mu\nu}$ in any hypervolume. For this, the term in square brackets must be zero, since the term $\sqrt{-g}\delta g^{\mu\nu}$ cannot be zero within Ω. In this way, we obtain:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0, \quad (21)$$

which are the Einstein's field equations in vacuum.

In order to model the interaction of matter and energy with the gravitational field, we introduce the energy-momentum tensor definition. Accordingly, a variation $\delta g_{\mu\nu}$ in the gravitational field will produce a variation in the action of matter S_m, Eq. (2), postulated as:

$$\delta S_m = -\frac{1}{2c}\int_{\Omega} d^4x\sqrt{-g}\delta g^{\mu\nu}T_{\mu\nu}\left(x\right). \quad (22)$$

Moreover, Eq. (2) under the variation of $g_{\mu\nu}$ reads:

$$\begin{aligned}\delta S &= \delta S_g + \delta S_m \\ &= \frac{c^3}{16\pi G}\int_{\Omega} d^4x\sqrt{-g}\delta g^{\mu\nu}\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right) \\ &\quad - \frac{1}{2c}\int_{\Omega} d^4x\sqrt{-g}\delta g^{\mu\nu}T_{\mu\nu}\left(x\right). \quad (23)\end{aligned}$$

Taking $\delta S = 0$ and factorizing the integrals, yields:

$$\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right) = \frac{16\pi G}{c^3}\frac{1}{2c}T_{\mu\nu}\left(x\right). \quad (24)$$

By rearranging the constants, we obtain

$$\boxed{R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4}T_{\mu\nu}.} \quad (25)$$

These are the famous Einstein's field equations in the presence of a source (matter and energy).

We can also obtain an expression of $T_{\mu\nu}$ in terms of the matter Lagrangian density from Eq. (22). In fact, from the definitions of the action of matter

$$S_m = \int d^4x\sqrt{-g}\mathcal{L}_m, \quad (26)$$

we compute its variation

$$\delta S_m = \int d^4x\delta\left(\sqrt{-g}\mathcal{L}_m\right), \quad (27)$$

and compare with Eq. (22) to write:

$$\int d^4x\delta\left(\sqrt{-g}\mathcal{L}_m\right) = -\frac{1}{2c}\int d^4x\sqrt{-g}\delta g^{\mu\nu}T_{\mu\nu}, \quad (28)$$

i.e.,

$$\delta\left(\sqrt{-g}\mathcal{L}_m\right) = \frac{\sqrt{-g}}{(-2c)}\delta g^{\mu\nu}T_{\mu\nu}. \quad (29)$$

Isolating $T_{\mu\nu}$, we get:

$$T_{\mu\nu} = \frac{-2c}{\sqrt{-g}}\frac{\delta\left(\sqrt{-g}\mathcal{L}_m\right)}{\delta g^{\mu\nu}}, \quad (30)$$

which provides a definition for the energy-momentum tensor in terms of the variation of the matter Lagrangian density with respect to the gravitational field metric.

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