PPGF UNIFAL-MG/UFSJ International Conference on Multimessenger Astrophysics

GRAVITATIONAL WAVES IN A PRIMORDIAL UNIVERSE

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Introduction

Nowadays it's possible to detect gravitational waves of black holes and neutron stars coalescences [1]. But these detections were made in a region not very different, where there is practically no curvature of spacetime. So when we want to study gravitational waves from the primordial universe, we should use the Friedmann-Lemaître-Robertson-Walker (FLRW) metric [4], which is the standard model for an expanding universe. In this way, the background metric is no longer Minkowski, that is, flat.

Shortwave approximation

For a general metric, was made a perturbation $h_{\mu\nu}$ on the background $g^{(\mathbf{B})}_{\mu\nu}$

$$g_{\mu\nu} = g^{(\mathbf{B})}_{\mu\nu} + h_{\mu\nu}, \tag{1}$$

and the raising and lowering of indices was made with respect to the background metric $g^{(\mathbf{B})}$. As an analogy, one can think these perturbation as a roughness in a surface of an orange, that does not change its geometry in "large scale".

In order to use this new metric in the Einstein equations, which is given by [5]:

$$R_{\mu\nu} = \chi\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right), \tag{2}$$

first one need to find the inverse metric in second order:

$$g^{\mu\nu} = g^{(\mathbf{B})\mu\nu} - h^{\mu\nu} + h^{\mu\alpha}h^{\nu}_{\alpha} - h^{\mu\alpha}h^{\beta}_{\alpha}h^{\nu}_{\beta} + O(h^3) \tag{3}$$

In this manner, one can find the Ricci tensor:

$$R_{\mu\nu} = R^{\mathbf{B}}_{\mu\nu} + R^{(1)}_{\mu\nu} + R^{(2)}_{\mu\nu} + ... \tag{4}$$

Since it is a metrics variation, the Ricci tensor also has variation, described by Eq (4). But to discriminate each one of its terms, the variation of Ricci tensor:

$$\delta R_{\mu\nu} = \partial_\mu \delta\Gamma_{\mu\nu}^{\ \ \alpha} - \partial_\alpha \delta\Gamma_{\mu\nu}^{\ \ \alpha} + \Gamma_{\mu\lambda}^{\ \ \alpha}\delta\Gamma_{\alpha\nu}^{\ \ \lambda} + \Gamma_{\alpha\nu}^{\ \ \lambda}\delta\Gamma_{\mu\lambda}^{\ \ \alpha} - \Gamma_{\alpha\lambda}^{\ \ \alpha}\delta\Gamma_{\mu\nu}^{\ \ \lambda} - \Gamma_{\mu\nu}^{\ \ \lambda}\delta\Gamma_{\alpha\lambda}^{\ \ \alpha} \tag{5}$$

Applying the Eq. (3) in Eq. (5)[3]:

$$\delta R_{\mu\nu} = \frac{1}{2}\left[\nabla_\nu\nabla_\mu h + \nabla_\alpha\nabla_\nu h_\mu^{\ \alpha} - \nabla_\alpha\nabla_\nu h^\alpha_{\ \nu} + \Box h_{\mu\nu}\right] + O(h^2) \tag{6}$$

where the first term that has order h refers to $R^{(1)}_{\mu\nu}$. So, the first-order approximation and the propagation of vacuum space waves will be considered, such as:

$$R^{(1)}_{\mu\nu} = \frac{1}{2}\left[\nabla_\nu\nabla_\mu h + \nabla_\alpha\nabla_\nu h_\mu^{\ \alpha} - \nabla_\alpha\nabla_\nu h^\alpha_{\ \nu} + \Box h_{\mu\nu}\right] = 0 \tag{7}$$

In constructing the linearized theory, the reverse trace of $h_{\mu\nu}$ was employed, enabling its application here:

$$h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2}g^{(\mathbf{B})}_{\mu\nu}\bar{h} \tag{8}$$

Finally, it was identified the propriety of second order covariant derivative tensor:

$$\nabla_\alpha\nabla_\beta S^{\mu\nu} = \nabla_\beta\nabla_\alpha S^{\mu\nu} + R_{\alpha\rho\beta}^{\ \ \ \ \mu}S^{\rho\nu} + R_{\alpha\rho\beta}^{\ \ \ \ \nu}S^{\mu\rho} \tag{9}$$

Applying the Eq. (9) and Eq. (8) in Eq. (7):

$$\Box\bar{h}_{\mu\nu} - g^{(\mathbf{B})}_{\mu\nu}\nabla_\alpha\nabla_\beta\bar{h}^{\beta\alpha} - \nabla_\nu\nabla_\lambda\bar{h}_\mu^{\ \lambda}$$
$$\nabla_\mu\nabla_\lambda\bar{h}^\lambda_{\ \nu} + 2R_{\mu\lambda\nu\gamma}\bar{h}^{\gamma\lambda} - R_{\nu\gamma}\bar{h}_\mu^{\ \gamma} - R_{\mu\gamma}\bar{h}^\gamma_{\ \nu} = 0 \tag{10}$$

In a vacuum propagation scenario, the last two terms of Eq. (8) have vanished. Once again, retrieval from the linearized theory occurs through an appropriate choice of a quadrifunction, leading to the application of the Lorenz gauge, thus:

$$\nabla_\lambda\bar{h}^{\lambda\alpha} = 0 \tag{11}$$

The equation to describe the gravitational waves propagation in a curved background is:

$$\Box\bar{h}_{\mu\nu} + 2R^{(\mathbf{B})}_{\mu\alpha\nu\beta}\bar{h}^{\alpha\beta} = 0 \tag{12}$$

But we can take a first approximation, and the curvature tensor drops out[2]:

$$\Box\bar{h}_{\mu\nu} = 0 \tag{13}$$

FLRW metric

In an expanding universe, the metric cannot be stationary, so the terms related to spatial coordinates gain the expansion factor $a(t)$, and for this model, we use the FLRW metric:"

$$ds^2 = -c^2dt^2 + a(t)^2[dr^2 + S_\kappa(r)^2 d\Omega^2] \tag{14}$$

Let us consider a flat universe, that is, $\kappa = 0$, so the curvature tensors and connections will be:

$$\Gamma_{ij}^{\ \ 0} = a^2 H\delta_{ij} \tag{15}$$
$$\Gamma_{i0}^{\ \ j} = H\delta_i^j \tag{16}$$
$$R_{010}^{\ \ \ \ 1} = R_{020}^{\ \ \ \ 2} = R_{030}^{\ \ \ \ 3} = \dot{H} + H^2 \tag{17}$$
$$R_{101}^{\ \ \ \ 0} = R_{202}^{\ \ \ \ 0} = R_{303}^{\ \ \ \ 0} = -a^2\left(\dot{H} - H^2\right) \tag{18}$$
$$R_{131}^{\ \ \ \ 3} = R_{232}^{\ \ \ \ 3} = R_{121}^{\ \ \ \ 2} = -a^2H^2 \tag{19}$$

Now we apply the Lorenz gauge, Eq. (11) and the TT gauge, put in the Eq. (13):

$$\ddot{h}^{ij} - \frac{1}{a^2}\nabla^2 h^{ij} - 3H\dot{h}^{ij} = 0. \tag{20}$$

The solution of plane waves by a Fourier transformation is[5]:

$$h_{ij} = e_{ij}\mathscr{D}_q(t)e^{i\mathbf{q}\cdot\mathbf{x}} \tag{21}$$

So:

$$\ddot{\mathscr{D}}_q + 3H\dot{\mathscr{D}}_q + \frac{q^2}{a^2}\mathscr{D}_q = 0 \tag{22}$$

where the $\mathscr{D}_q$ depends on the era of the universe. This is a damped harmonic oscillator of gravitational waves, with the expansion factor defines the frequency of the wave and the damping.

For another hand, if we use the Eq. (12), considering the curvature tensor:

$$\ddot{h}_{ij} - \frac{1}{a^2}\nabla^2 h_{ij} + 3H\dot{h}_{ij} + 2\dot{a}^2 h_{ij} = 0. \tag{23}$$

and the spatial solution is:

$$\ddot{\mathscr{D}}_q + 3H\dot{\mathscr{D}}_q + \left(\frac{q^2}{a^2} + 2\dot{a}^2\right)\mathscr{D}_q = 0 \tag{24}$$

It is important to note that the curvature tensor changes the frequency of gravitational waves. This suggests that during extreme times, we may consider the shortwave formalism. However, if we neglect the curvature tensor and focus solely on the past, we can calculate the gravitational waves.

Conclusion

The shortwave formalism can describe the propagation in a curved background and can be used to explore distant eras of the universe using the FLRW metric. The expansion factor is responsible for the frequency and damping of the gravitational wave, and the approximation of the curvature tensor in Eq. (13) allows us to study the earlier universe because the curvature tensor drops out. However, if we want to study the future of the universe, this tensor needs to be considered.

References

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Acknowledgements

RPRO thanks FAPEMIG for full financial support.
CAMM thanks FAPEMIG grants APQ-00544-23 and APQ-05218-23 for partial financial support