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# Dynamics of Accelerated Waves

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## 1 Introduction

Higher-order wave equations are useful for describing propagation phenomena in non-homogeneous media. For example, when a wave changes from one material medium to another, it experiences an acceleration in propagation at the interface. Therefore, in this work, we derive a fourth-order wave equation that describes accelerated waves. Additionally, we obtain a specific solution that, in the limit of high accelerations and low frequencies, behaves like a wave governed by the second-order wave equation.

### Main goals

1. Deriving fourth-order equations from the equations of motion of one-dimensional coupled oscillators,
2. Obtaining the fourth-order field equation from the continuum limit of the equations of motion of higher order,
3. Propose boundary and initial conditions to obtain a particular solution for the fourth-order field equation,
4. Seek limit situations in which this solution reduces to a particular solution of the wave equation.

## 2 Theoretical Model

Consider the model of one-dimensional coupled oscillators

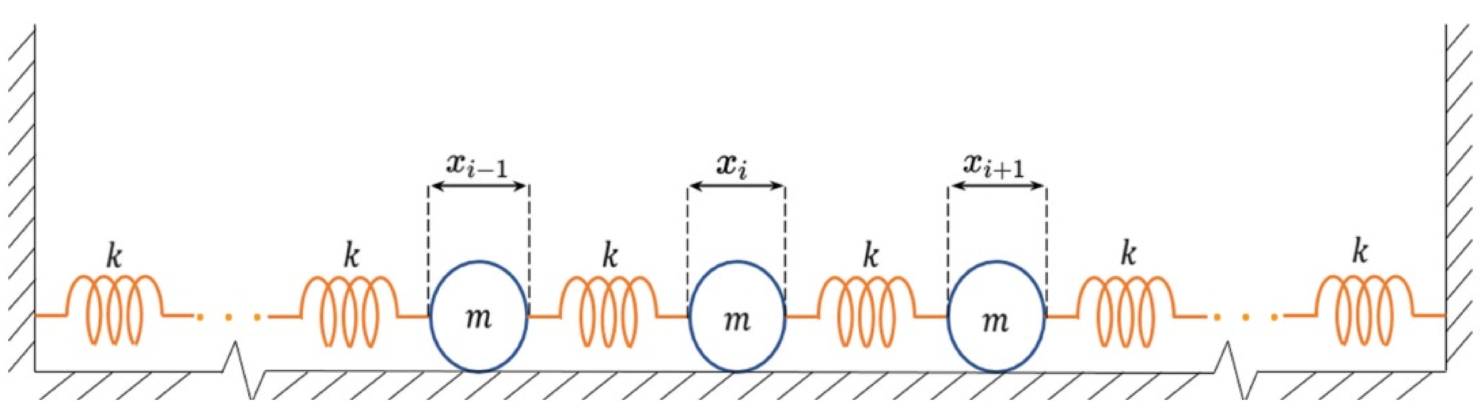


**Figure 1:** One-dimensional coupled oscillators.[2]

The system's equations of motion are given by

$$\ddot{x}_i - \omega^2(x_{i+1} - 2x_i + x_{i-1}) = 0 \quad (i = 1, ...N), \tag{1}$$

In the continuum limit, this system is represented by a wave equation [3]

$$\frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2} - \frac{\partial^2 \psi}{\partial x^2} = 0. \tag{2}$$

It is possible to decouple Eq. (1) and generate equations of the type [4, 5]

$$\ddddot{x}_i + 4\omega^2 \ddot{x}_i - \omega^4(x_{i+2} - 2x_i + x_{i-2}) = 0 \quad (i = 1, ..., N), \tag{3}$$

The continuum limit of this generalization is given by:

$$\frac{1}{a^2}\frac{\partial^4 \Psi}{\partial t^4} + \frac{1}{v^2}\frac{\partial^2 \Psi}{\partial t^2} - \frac{\partial^2 \Psi}{\partial x^2} = 0, \tag{4}$$

Eq. (4) represents a generalization of the wave equation, incorporating a propagation acceleration $a$.

## 3 Discussion

The coupled oscillators system suggests the use of the following boundary conditions: $\psi(0,t) = \psi(L,t) = 0$. Regarding the initial conditions, we consider $\psi(x,0) = \psi_0(x)$ and $\psi_t(x,0) = 0$. This results in the following solution to the wave equation:

$$\psi(x,t) = \sum_{n=0}^{\infty} A_n \sin(k_n x)\cos(\omega_n t); \tag{5}$$

In the broader case, we adopt the same boundary and initial conditions mentioned earlier, along with the conditions $\psi_{tt}(x,0) = \psi_{ttt}(x,0) = 0$. Therefore, the solution of Eq. (4) is:

$$\Psi(x,t) = \sum_{n=0}^{\infty} \bar{A}_n \sin(k_n x)\left[\cos(\Omega_{1n}) - \left(\frac{\Omega_{1n}}{\Omega_{2n}}\right)^2 \cos(\Omega_{2n} t)\right]; \tag{6}$$

$$\bar{A}_n = A_n \left[1 - \left(\frac{\Omega_{1n}}{\Omega_{2n}}\right)^2\right]^{-1},$$

Considering the frequencies $\Omega_{1n} = \omega_0\sqrt{2(1-\sqrt{1-\beta_n^4})}$ and $\Omega_{2n} = \omega_0\sqrt{2(1+\sqrt{1-\beta_n^4})}$, defined by $\omega_0 = a/v$ and $\beta_n = \sqrt{\omega_n/\omega_0}$.

In the limit $a \to \infty$, such that $\omega_0 \gg \omega_n$, this implies $\beta_n \to 0$. Under these conditions, it is possible to show that $\Omega_{1n}/\Omega_{2n} \sim \beta_n^2 \to 0$ and $\Omega_{1n} \sim \omega_n$, which leads to recovering the solution of the wave equation $\Psi(x,t) \sim \psi(x,t)$.

## 4 Conclusions

In summary, Equation (4) describes a wave with acceleration $a$. For very small vibration modes, Equation (6) reduces to Equation (5), but this approximation is not valid for high frequencies. We interpret that the wave undergoes a sudden acceleration $a \to 0$ in the initial moments and subsequently propagates like an unaccelerated wave. This study continues with an approach using singular perturbation theory. [1].

### References

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