SAŠO GROZDANOV

THE UNIVERSITY of EDINBURGH

Univerza v Ljubljani

THERMAL SPECTRA IN QFTS AND THEIR RECONSTRUCTIONS

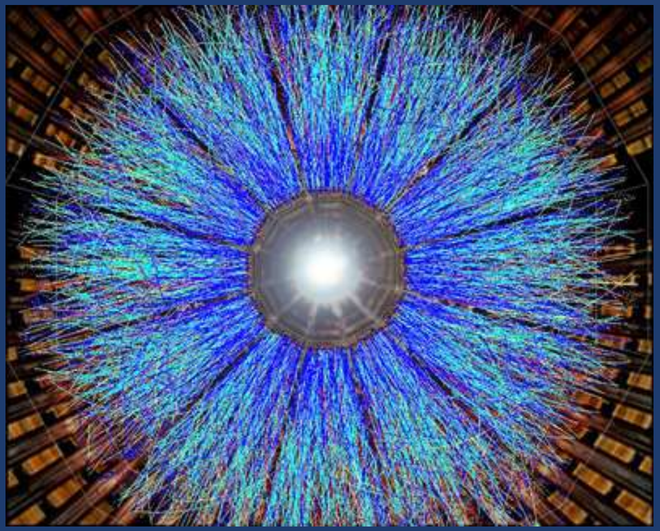
GENOVA, 20.3.2024

OUTLINE

- hydrodynamics, kinetic theory and thermal spectra
- holography and reconstruction of spectra
- pole-skipping and the reconstruction
- summary and future directions

HYDRODYNAMICS, KINETIC THEORY AND THERMAL SPECTRA

THERMAL FIELD THEORY



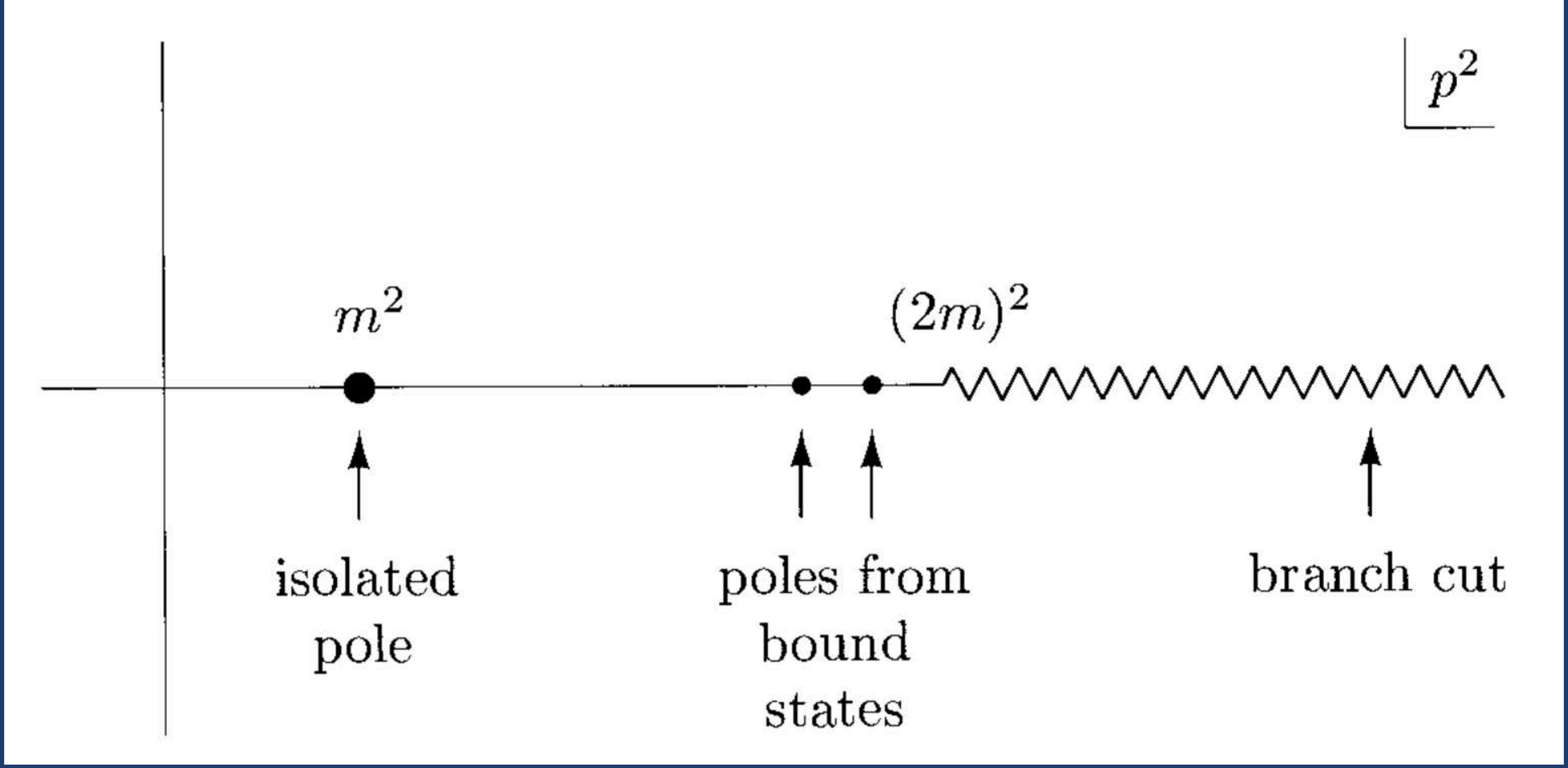
$$\boxed{Z[\beta = 1/T] = \int \mathcal{D}\Phi \, e^{-\beta H} e^{\frac{i}{\hbar}\int d^d x \, \mathcal{L}(\Phi,\lambda)}}$$

expansions of observables:

$$\mathcal{O} = \mathcal{O}_{T,0} + T\,\mathcal{O}_{T,1} + \ldots$$
$$\mathcal{O} = \mathcal{O}_{\lambda,0} + \lambda\,\mathcal{O}_{\lambda,1} + \ldots$$
$$\mathcal{O} = \mathcal{O}_{p,0} + p\,\mathcal{O}_{p,1} + \ldots$$
$$\mathcal{O} = \mathcal{O}_{N,0} + \frac{1}{N}\,\mathcal{O}_{N,1} + \ldots$$
$$\mathcal{O} = \mathcal{O}_{d,0} + \frac{1}{d}\,\mathcal{O}_{d,1} + \ldots$$

SPECTRUM OF A SIMPLE $T = 0$ CORRELATOR

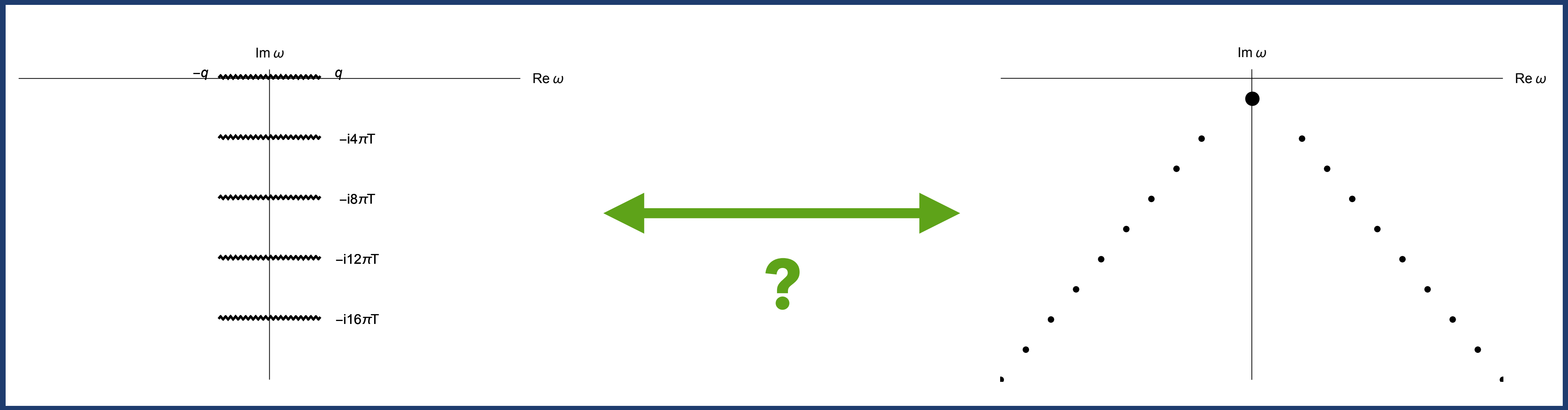
$$\langle \phi(p)\phi(-p)\rangle = \frac{Z(p^2)}{p^2 - m^2 + \Sigma(p^2)}$$



[from Peskin and Schroeder]

ANALYTIC STRUCTURE OF THERMAL CORRELATORS

$$\langle T_{\mu\nu}(-\omega,-q), T_{\rho\sigma}(\omega,q)\rangle_R \sim \frac{b(\omega,q)}{a(\omega,q)}$$



$\lambda \to 0$

[Hartnoll, Kumar, (2005)]

$\lambda \to \infty$

holography

what is the structure of thermal correlators and what is the minimal information necessary to determine them completely?

- kinetic theory: solving the collision integral
- perturbative QFT: calculation of Feynman diagrams
- holography: solving differential equations

HYDRODYNAMIC PART OF THE THERMAL SPECTRUM

- low-energy limit of QFTs – a Schwinger-Keldysh effective field theory
  [SG, Polonyi (2013); Crossley, Glorioso, Liu (2015); Haehl, Loganayagam, Rangamani (2015); …]

- conservation laws (equations of motion) of globally conserved operators

$$\boxed{\nabla_\mu T^{\mu\nu} = 0 \qquad \nabla_\mu J^{\mu} = 0 \qquad \ldots \nabla_\mu J^{\mu\nu} = 0}$$

higher-form currents in MHD
[SG, Hofman, Iqbal,
PRD (2017)]

- tensor structures (symmetries, gradient expansions) and transport coefficients (QFT)

$$\boxed{T^{\mu\nu} = \sum_{n=0}^{\infty} \left[\sum_i^N \lambda_i^{(n)} \mathcal{T}^{\mu\nu}_{(n)} \right]} \xrightarrow[u^\mu \sim T \sim e^{-i\omega t + iqz}]{\nabla_\mu T^{\mu\nu} = 0} \boxed{\omega(q) = \sum_{n=1}^{\infty} \alpha_n q^n}$$

$$\boxed{\partial u^\mu \sim \partial T \ll 1} \longrightarrow \boxed{\omega / T \sim q/T \ll 1}$$

equilibrium
temperature

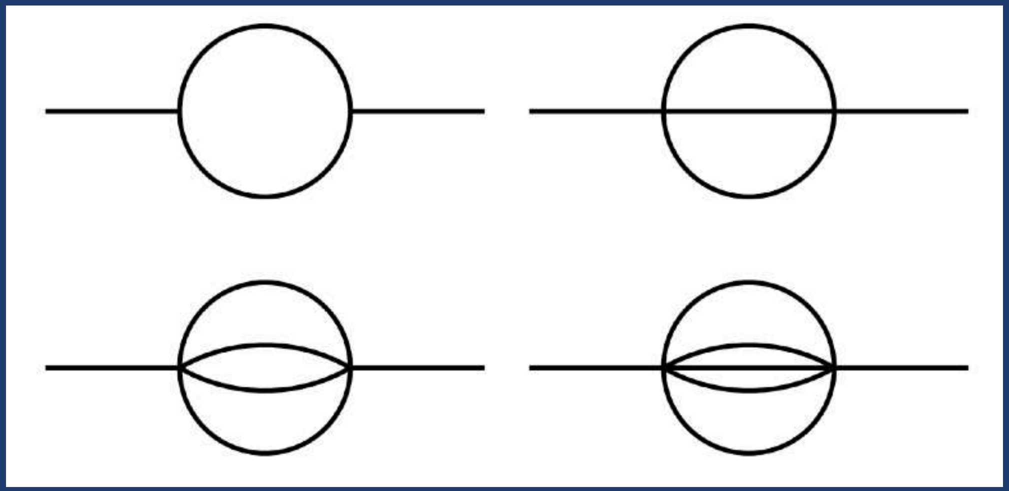
$$q = \sqrt{\mathbf{q}^2}$$

- dispersion relations:

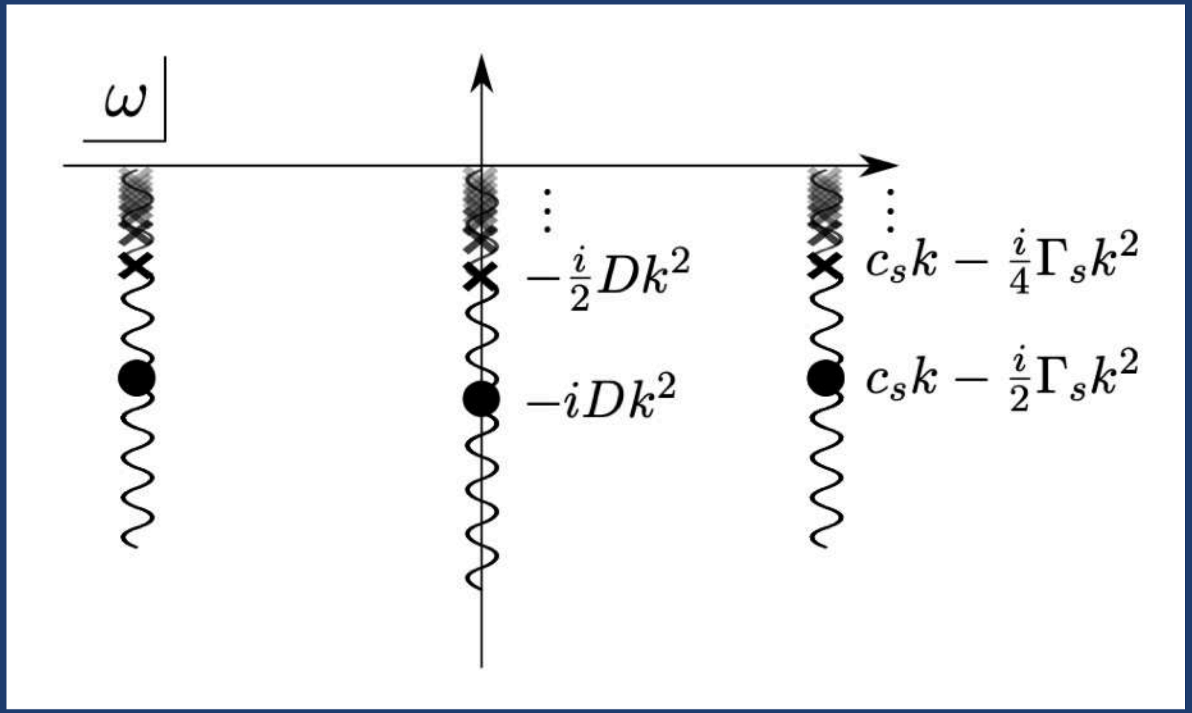
shear diffusion	sound
$\omega = -iDq^2$	$\omega = \pm v_s q - i\Gamma q^2$

HYDRODYNAMIC PART OF THE THERMAL SPECTRUM

- Schwinger-Keldysh effective field theory of diffusion to all loops: long time tails…
  [Chen-Lin, Delacretaz, Hartnoll (2019) ; Delacretaz (2020); SG, Lemut, Soloviev, *to appear*]
- tree-level result (classical hydrodynamics) has one diffusive pole at $\boxed{\omega = -iDq^2}$
- one-loop result has one (renormalised) diffusive pole at $\boxed{\omega = -i(D + \delta D)q^2}$ and one branch point at $\boxed{\omega = -iDq^2/2}$
- *n*-loop result has a branch point at $\boxed{\omega = -iDq^2/(n+1)}$



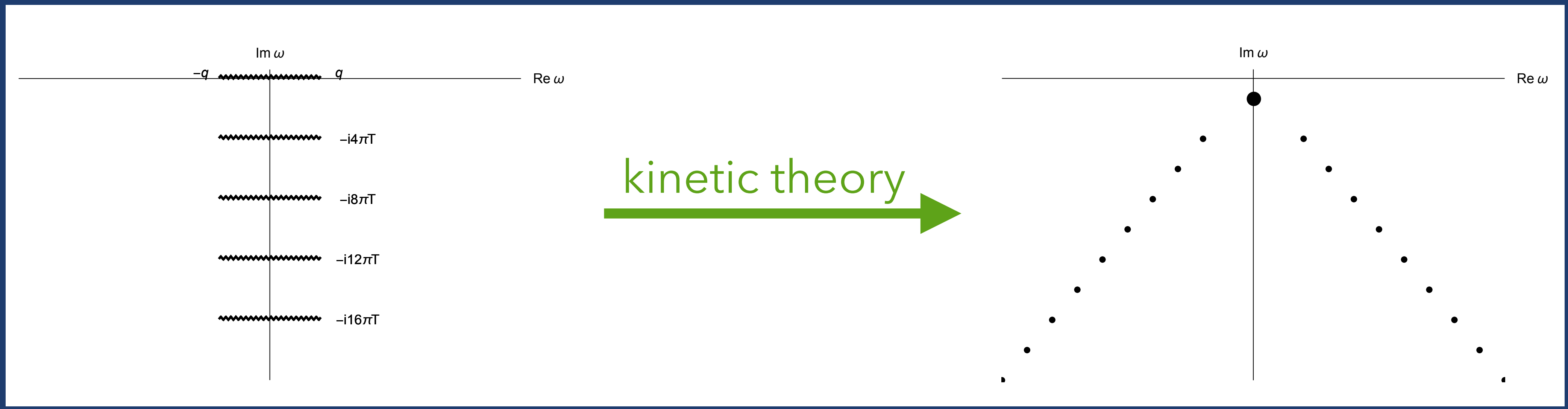
dominant one-, two-, three- and four-loop diagrams responsible for the cut



[plot from Delacretaz (2020)]

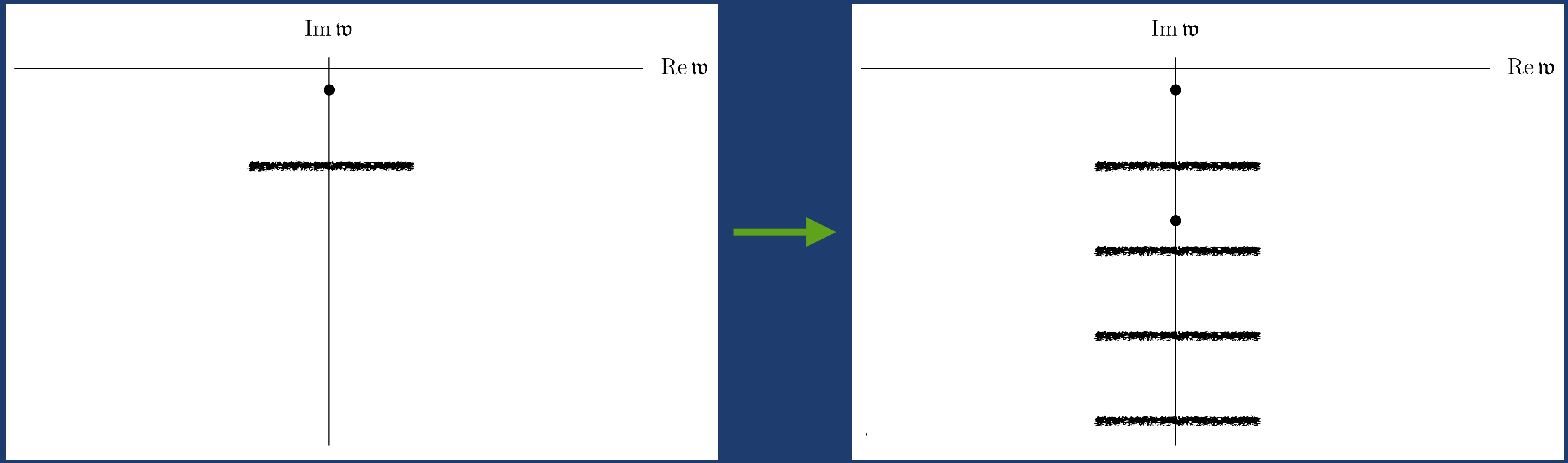
KINETIC THEORY AND THERMAL SPECTRUM

$$\boxed{\langle T_{\mu\nu}(-\omega,-q), T_{\rho\sigma}(\omega,q)\rangle_R \sim \frac{b(\omega,q)}{a(\omega,q)}}$$



$\lambda \to 0$ $\lambda \to \infty$

poles from a cut?



Boltzmann equation – RTA
[Romatschke, (2016)]

BBGKY hierarchy – RTA-like truncations [SG, Soloviev, *to appear*]

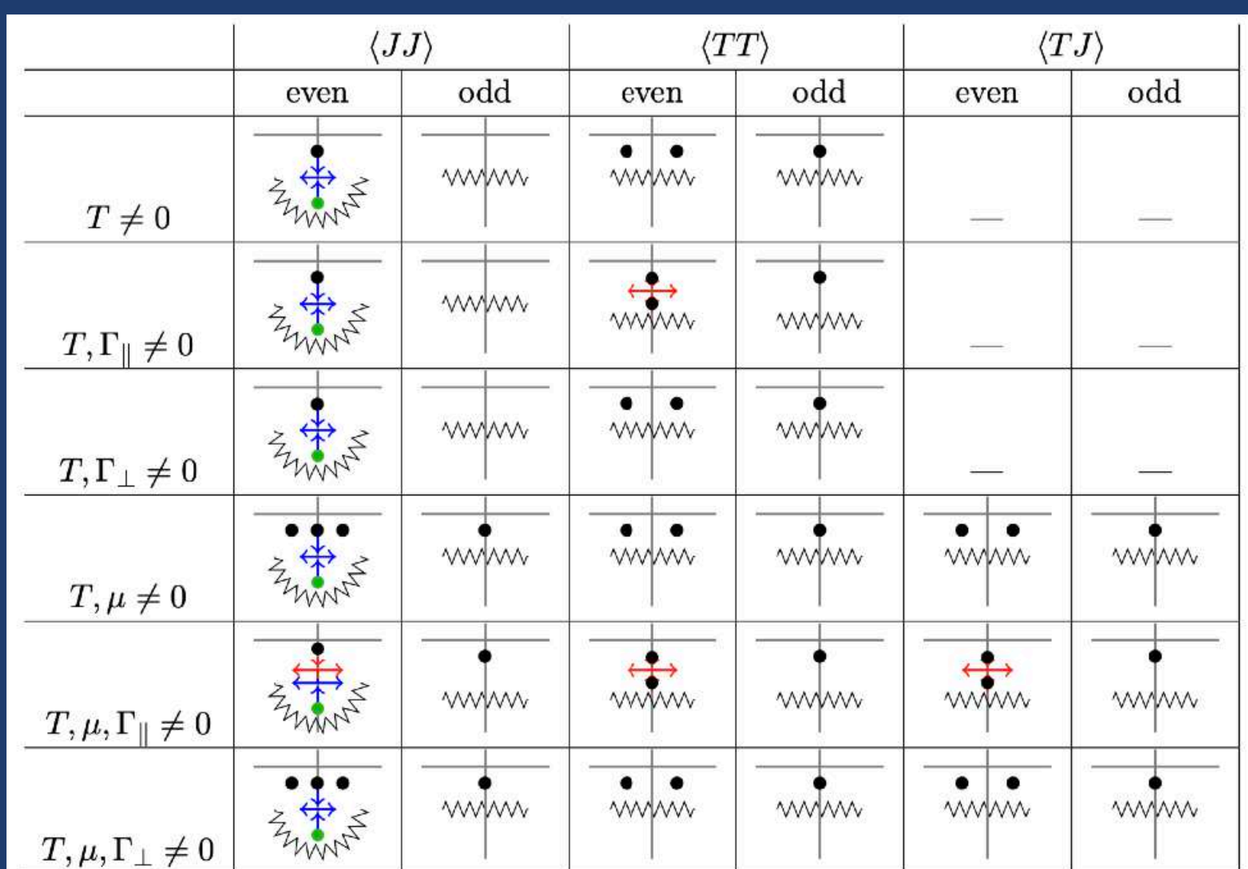
KINETIC THEORY AND THERMAL SPECTRUM

- two-point functions of $T^{\mu\nu}$ and J^{μ} from RTA kinetic theory at finite T, μ, Γ
  [Bajec, SG, Soloviev, *to appear*]

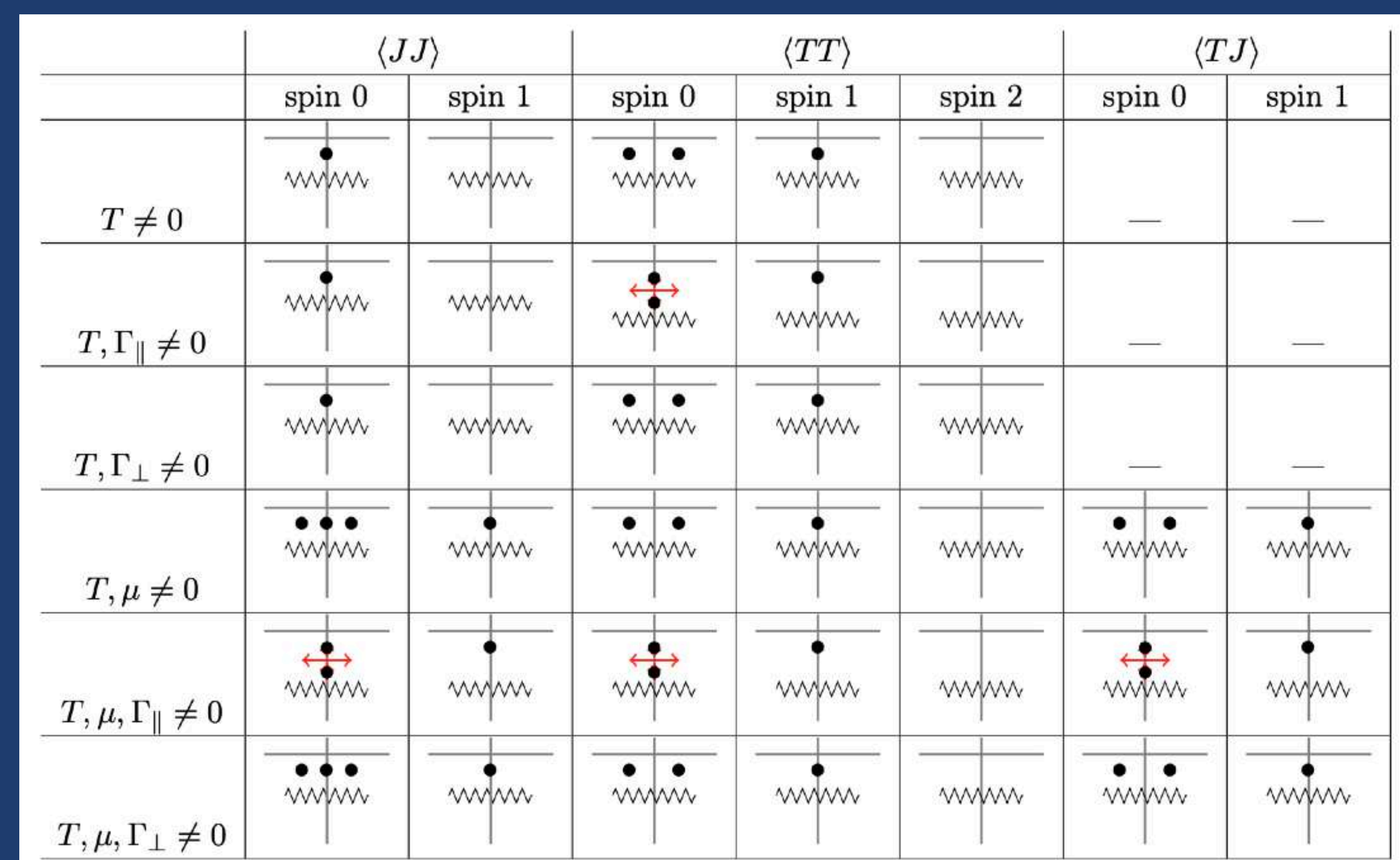
$$\boxed{[p^{\mu}\partial_{\mu} + F^{\mu}\nabla_{p^{\mu}}] = C[f] = \frac{p^{\mu}u_{\mu}}{\tau_R}\left(f - f^{\text{eq}}\right)}$$

$$\boxed{\begin{array}{l}\nabla_{\mu}T^{\mu 0} = 0, \ \nabla_{\mu}T^{\mu i} = -\Gamma T^{0i} \\ \nabla_{\mu}J^{\mu} = 0\end{array}}$$

- analytic expression for all correlators that exhibit hydrodynamics (diffusion and sound), quasihydrodynamics, a branch cut running between $\omega = -i/\tau_R \pm q$, thermoelectric effect
- a 'periodic system' of correlators:



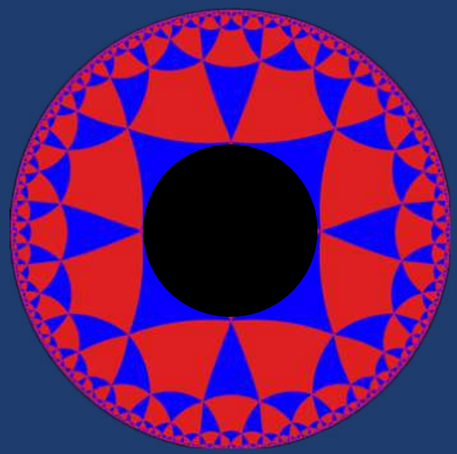
2+1 dimensions



3+1 dimensions

HOLOGRAPHY AND RECONSTRUCTION OF HOLOGRAPHIC SPECTRA

HOLOGRAPHY AND HYDRODYNAMICS

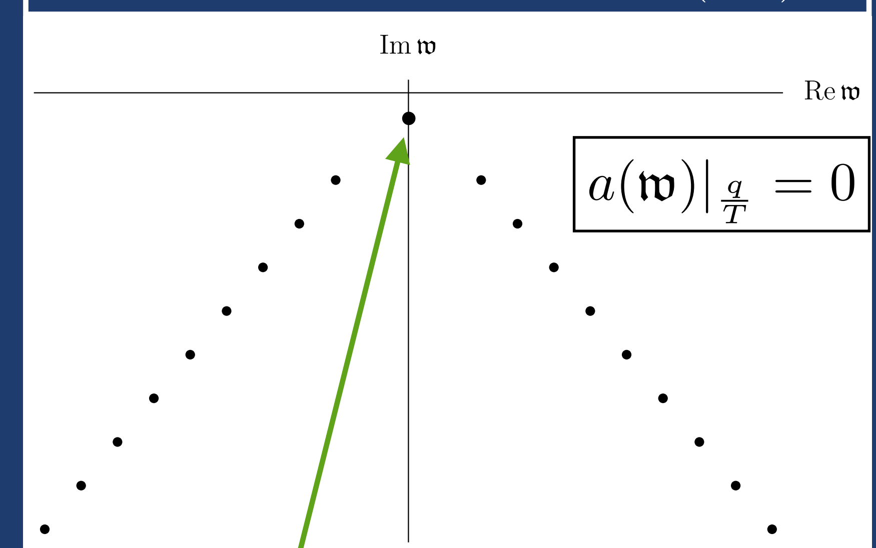


- duality: *theory A* = *theory B*
- a result of string theory (quantum gravity) [Maldacena (1997)]

strongly coupled quantum theory (extremely hard) = *weakly coupled gravity* (much easier)

- perturbations of black holes (quasinormal modes) give spectra of QFT operators for $\mathfrak{w} \equiv \frac{\omega}{2\pi T} \in \mathbb{C}$
- large-N QFT calculations –> ODEs
- invaluable explicit (toy) models: the $\mathcal{N}=4$ supersymmetric Yang-Mills theory [SG, Kovtun, Starinets, Tadić, JHEP (2019)]

$$\langle T_{\mu\nu}(-\omega,-q), T_{\rho\sigma}(\omega,q)\rangle_R \sim \frac{b(\omega,q)}{a(\omega,q)}$$



sound:

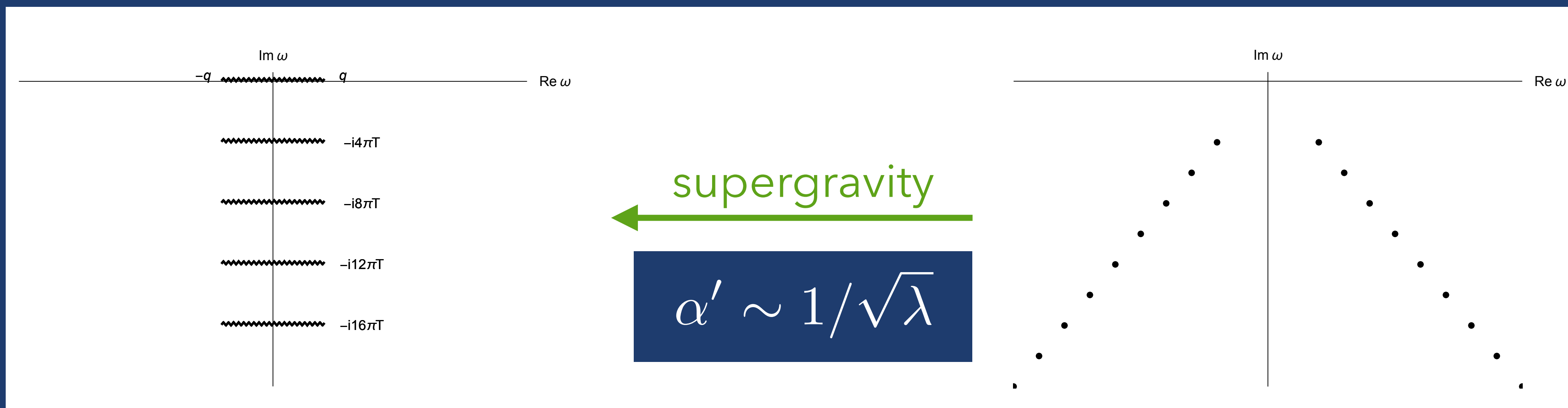
$$\omega = \pm\frac{1}{\sqrt{3}}q - \frac{i}{6\pi T}q^2 \pm \frac{3-2\ln 2}{24\sqrt{3}\pi^2T^2}q^3 - \frac{i\left(\pi^2-24+24\ln 2-12\ln^2 2\right)}{864\pi^3T^3}q^4 \pm \cdots$$

shear diffusion:

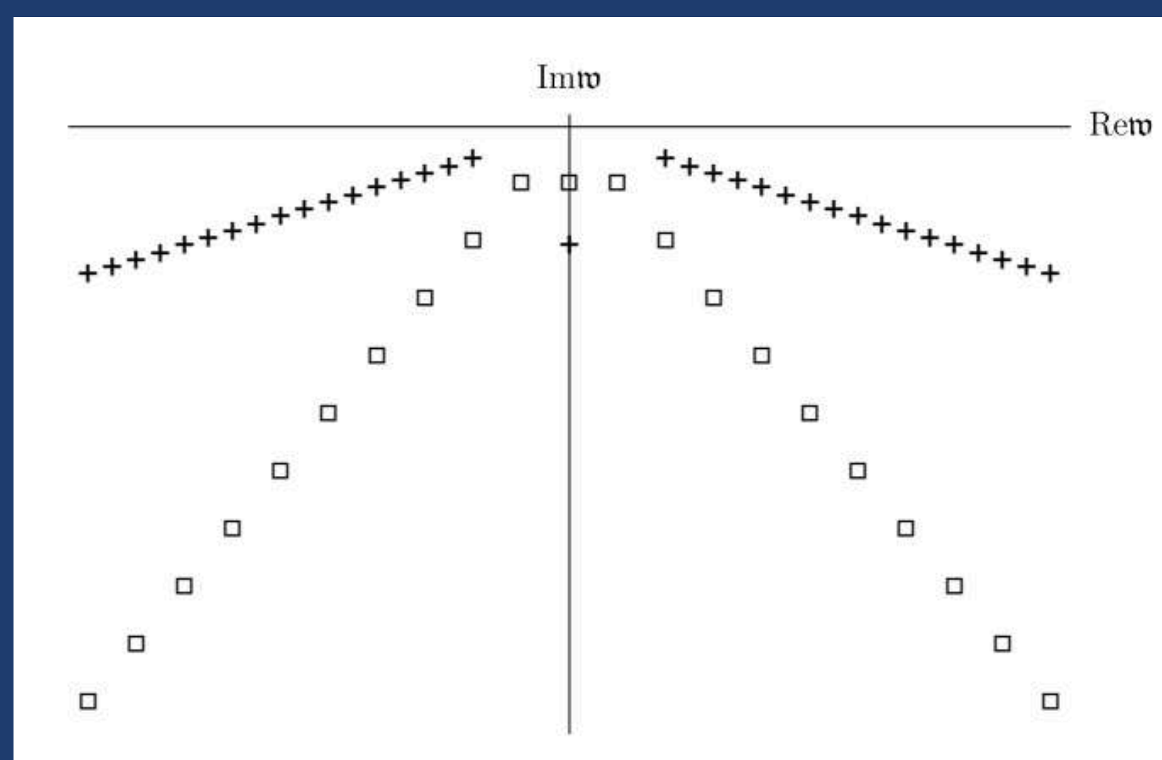
$$\omega = -\frac{i}{4\pi T}q^2 - \frac{i(1-\ln 2)}{32\pi^3T^3}q^4 - \frac{i(24\ln^2 2-\pi^2)}{96\,(2\pi T)^5}q^6 - \frac{i\left[2\pi^2(\ln 32-1)-21\zeta(3)-24\ln 2(1+\ln 2(\ln 32-3))\right]}{384\,(2\pi T)^7}q^8 + \cdots$$

HOLOGRAPHY AND THERMAL SPECTRUM

$$\langle T_{\mu\nu}(-\omega,-q), T_{\rho\sigma}(\omega,q)\rangle_R \sim \frac{b(\omega,q)}{a(\omega,q)} = \frac{B(\omega,q)}{\prod_{i=0}^{\infty}(\omega-\omega_i(q))}$$



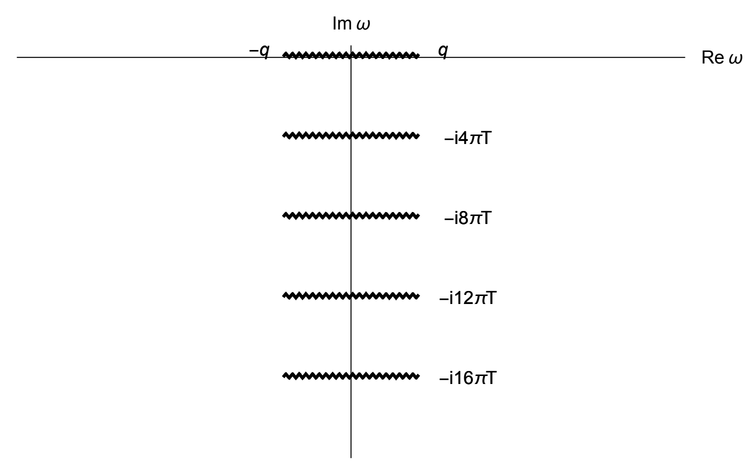
$\lambda \to 0$ cut from poles $\lambda \to \infty$



[SG, Starinets …, several papers]

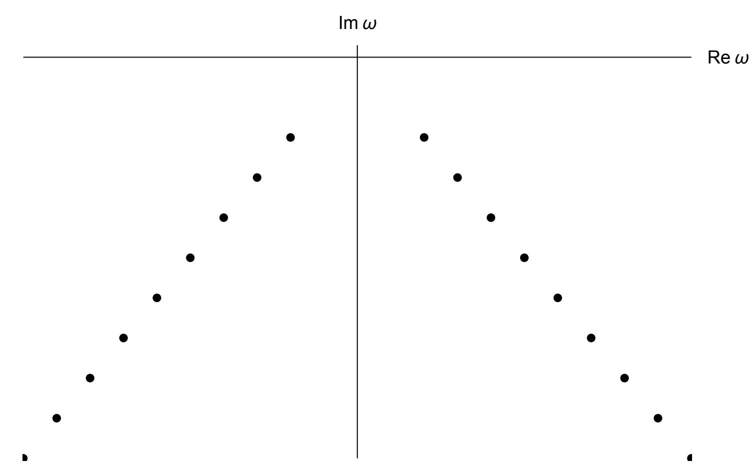
HOLOGRAPHY AND THERMAL SPECTRUM

$$\boxed{\langle T_{\mu\nu}(-\omega,-q), T_{\rho\sigma}(\omega,q)\rangle_R \sim \frac{b(\omega,q)}{a(\omega,q)} = \frac{B(\omega,q)}{\prod_{i=0}^{\infty}(\omega-\omega_i(q))}}$$

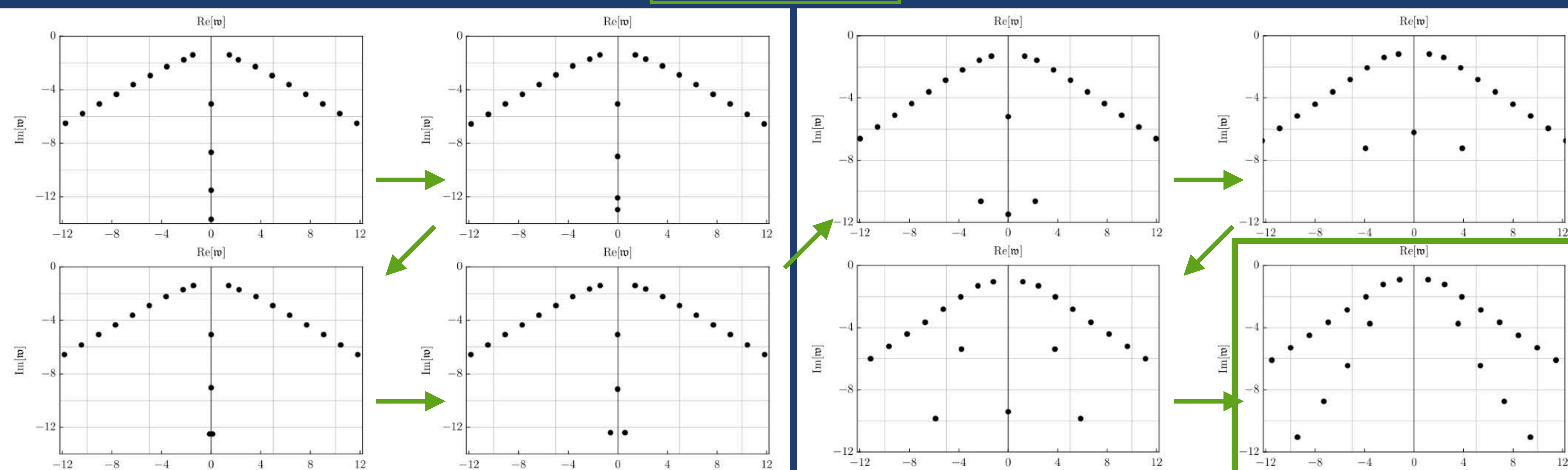


supergravity

$$\alpha' \sim 1/\sqrt{\lambda}$$



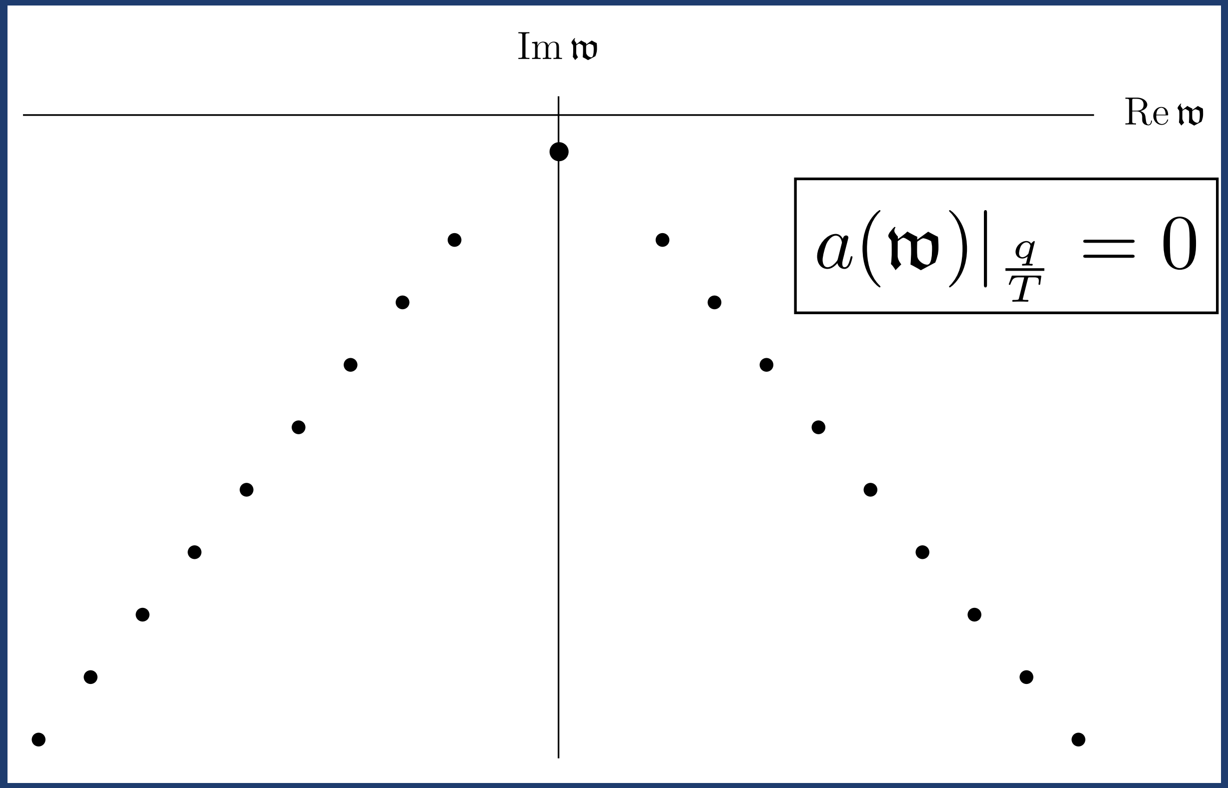
$\lambda \to 0$ cut from poles $\lambda \to \infty$



[SG, Starinets …, several papers]

HOLOGRAPHY AND THERMAL SPECTRUM

$$\boxed{\langle T_{\mu\nu}(-\omega,-q), T_{\rho\sigma}(\omega,q)\rangle_R \sim \frac{b(\omega,q)}{a(\omega,q)} = \frac{B(\omega,q)}{\prod_{i=0}^{\infty}(\omega-\omega_i(q))}}$$



meromorphic momentum space correlator

quantum field theory

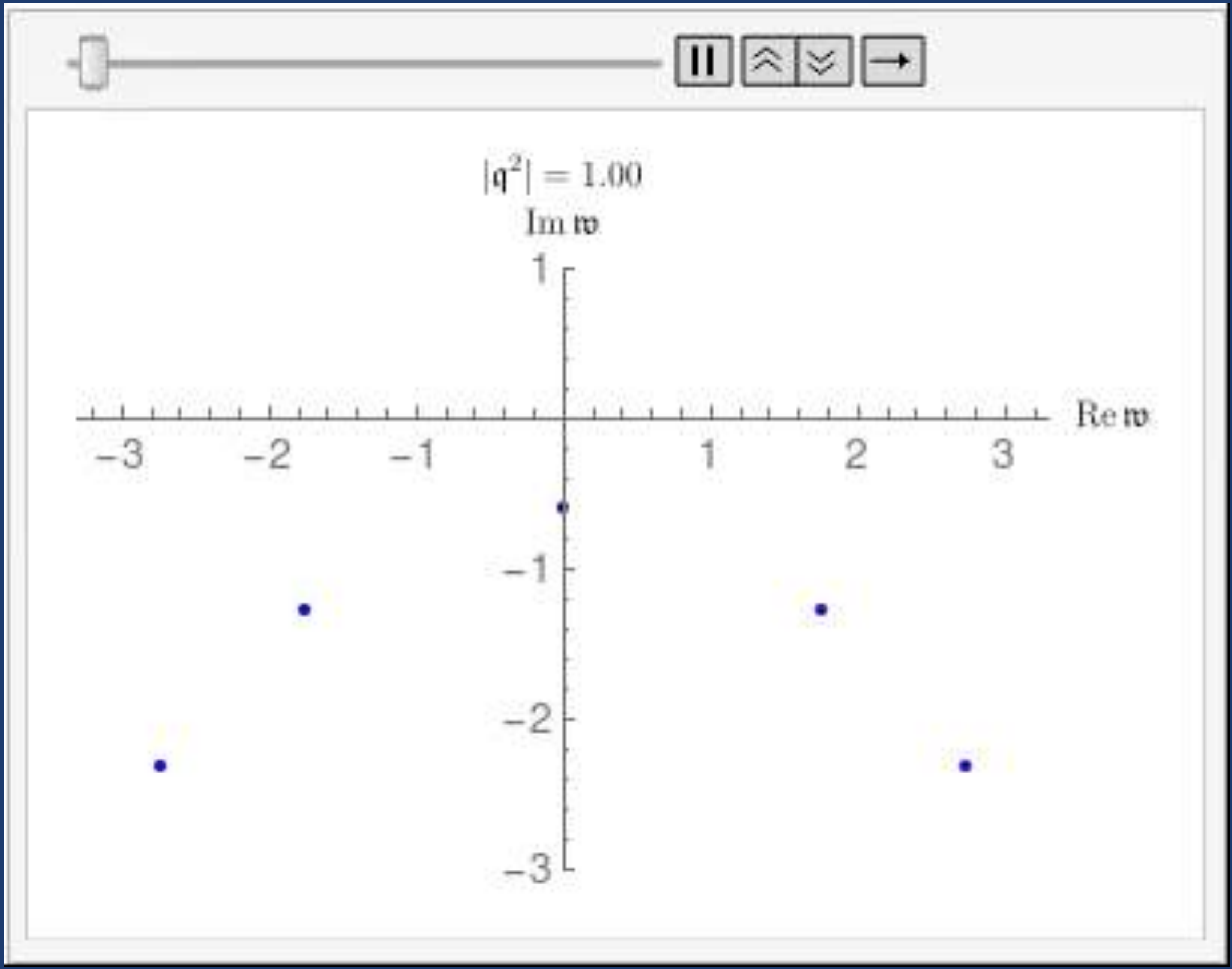
spectra of linear non-Hermitian operators

quasinormal mode spectrum of black holes

zeros of (algebraic) equations

HYDRODYNAMICS AND COMPLEX SPECTRAL CURVES

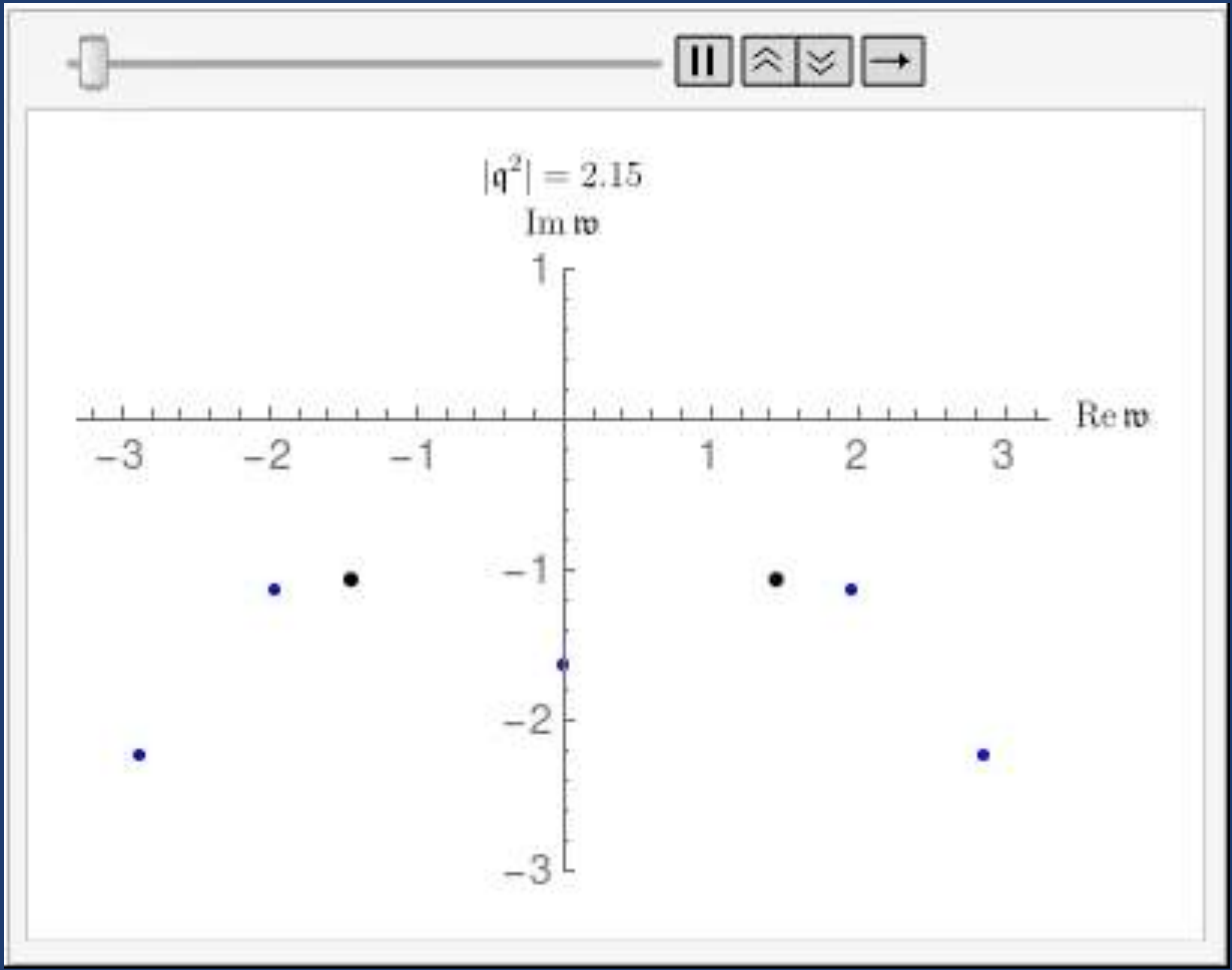
- radius of convergence of $\mathfrak{w}(\mathfrak{q}) = \sum_{i=1}^{\infty} c_n \mathfrak{q}^n\,,\ \ |\mathfrak{q}| < \mathfrak{q}_*$, is set by the lowest momentum at which the hydro pole collides (level-crossing): $\boxed{\mathfrak{q}_* = \min\left[|\mathfrak{q}_{\text{collision}}|\right]}$



$\boxed{\mathfrak{q}^2 = \left|\mathfrak{q}^2\right| e^{i\theta}}$

HYDRODYNAMICS AND COMPLEX SPECTRAL CURVES

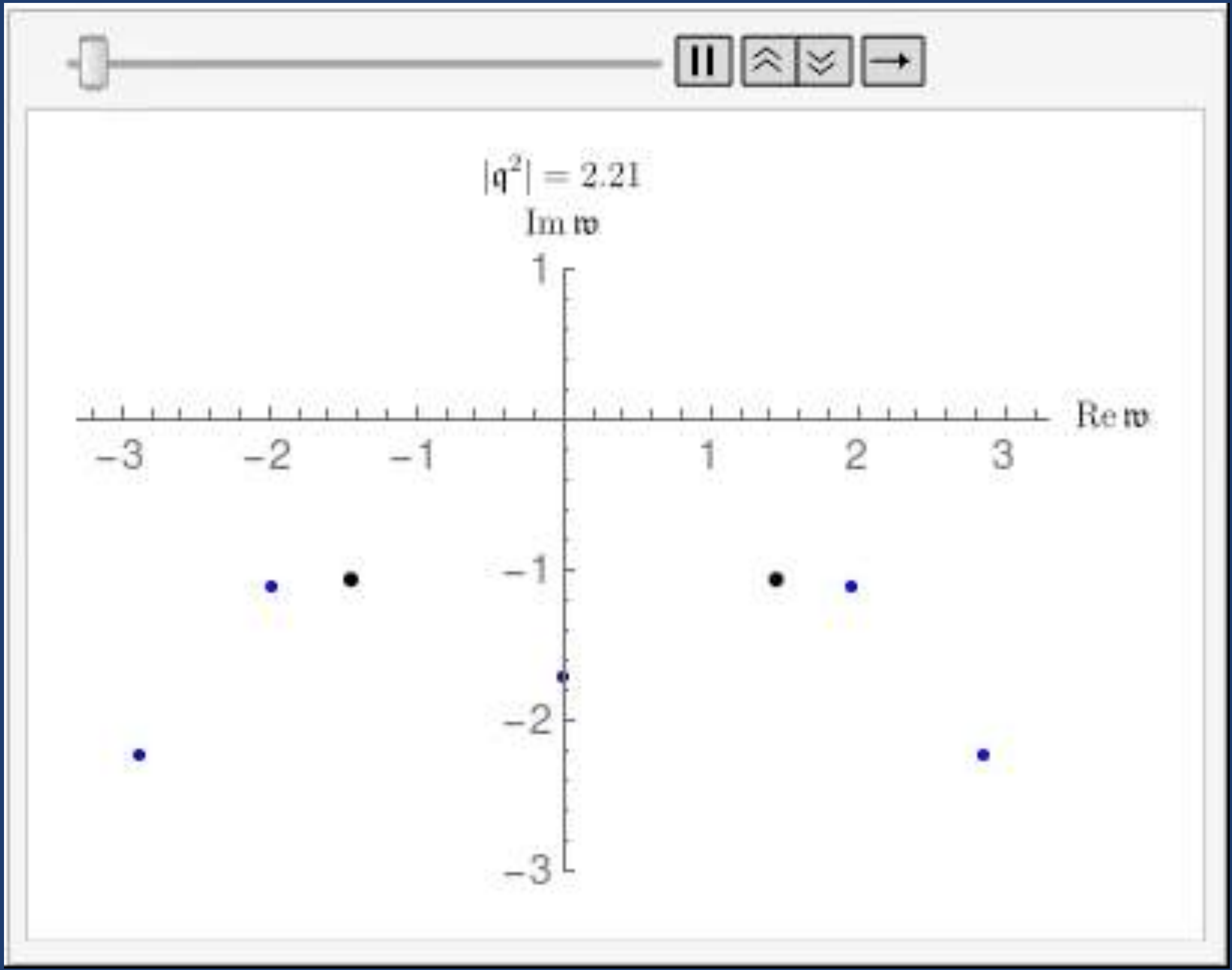
- radius of convergence of $\mathfrak{w}(\mathfrak{q}) = \sum_{i=1}^{\infty} c_n \mathfrak{q}^n$, $|\mathfrak{q}| < \mathfrak{q}_*$, is set by the lowest momentum at which the hydro pole collides (level-crossing): $\boxed{\mathfrak{q}_* = \min\left[|\mathfrak{q}_{\text{collision}}|\right]}$



$\boxed{\mathfrak{q}^2 = |\mathfrak{q}^2|\, e^{i\theta}}$

HYDRODYNAMICS AND COMPLEX SPECTRAL CURVES

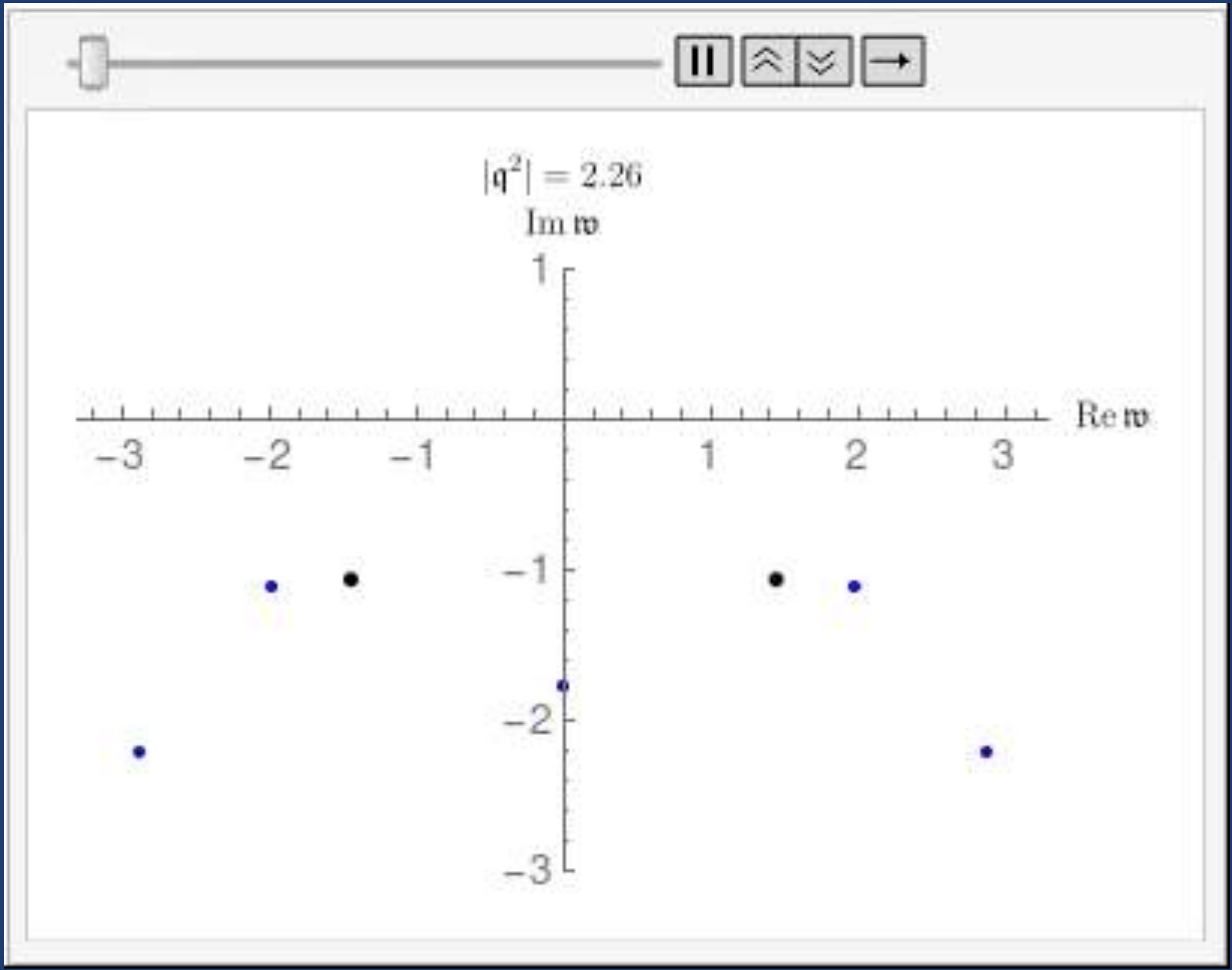
- radius of convergence of $\mathfrak{w}(\mathfrak{q}) = \sum_{i=1}^{\infty} c_n \mathfrak{q}^n$, $|\mathfrak{q}| < \mathfrak{q}_*$, is set by the lowest momentum at which the hydro pole collides (level-crossing): $\boxed{\mathfrak{q}_* = \min\left[|\mathfrak{q}_{\text{collision}}|\right]}$



$\boxed{\mathfrak{q}^2 = |\mathfrak{q}^2|\, e^{i\theta}}$

HYDRODYNAMICS AND COMPLEX SPECTRAL CURVES

- radius of convergence of $\mathfrak{w}(\mathfrak{q}) = \sum_{i=1}^{\infty} c_n \mathfrak{q}^n$, $|\mathfrak{q}| < \mathfrak{q}_*$, is set by the lowest momentum at which the hydro pole collides (level-crossing): $\boxed{\mathfrak{q}_* = \min\left[|\mathfrak{q}_{\text{collision}}|\right]}$



$\boxed{\mathfrak{q}^2 = |\mathfrak{q}^2|\, e^{i\theta}}$

HYDRODYNAMICS AND COMPLEX SPECTRAL CURVES

- hydrodynamic series are convergent Puiseux series (shear *p=1*, sound *p=2*)
  [SG, Kovtun, Starinets, Tadić, PRL (2019); … ; see also Withers; JHEP (2018); Heller, et.al. (2020, …)]

$$\mathfrak{w}_{\text{shear}} = -i \sum_{n=1}^{\infty} c_n \left(\mathfrak{q}^2\right)^n = -i\mathfrak{D}\mathfrak{q}^2 + \ldots$$

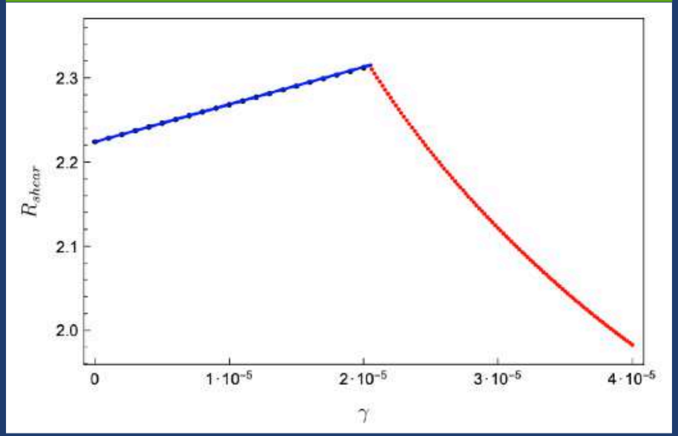
$$\mathfrak{w}_{\text{sound}} = -i \sum_{n=1}^{\infty} a_n e^{\pm \frac{i\pi n}{2}} \left(\mathfrak{q}^2\right)^{n/2} = \pm v_s \mathfrak{q} - \frac{i}{2}\mathfrak{G}\mathfrak{q}^2 + \ldots$$

- dispersion relations are holomorphic in a disk

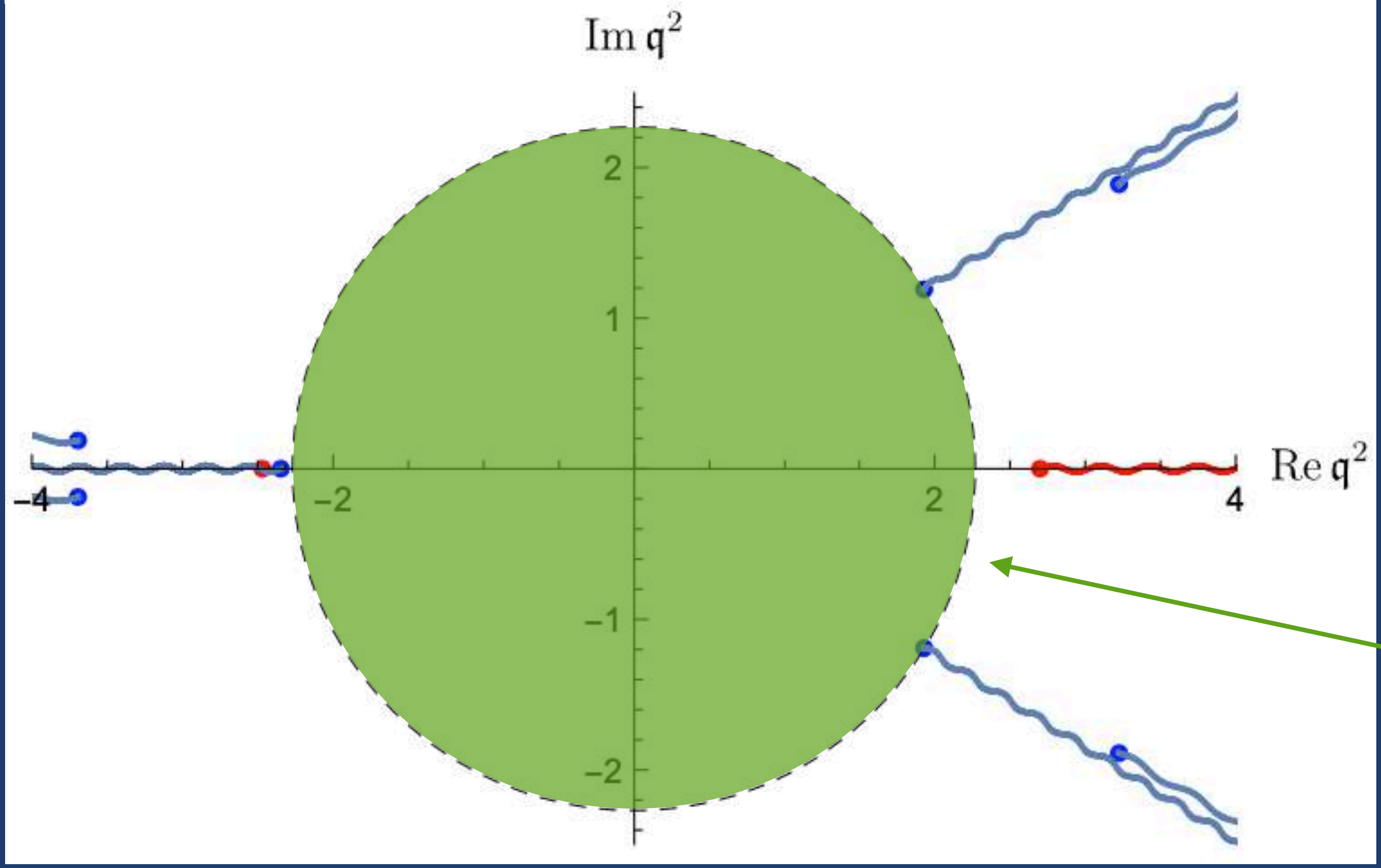
$$R_{\text{shear}}(\lambda) = 2.22 \left(1 + 674.15\, \lambda^{-3/2} + \cdots\right)$$

$$R_{\text{sound}}(\lambda) = 2 \left(1 + 481.68\, \lambda^{-3/2} + \cdots\right)$$

N=4 SYM radius convergence
[SG, Starinets, Tadić, JHEP (2021)]



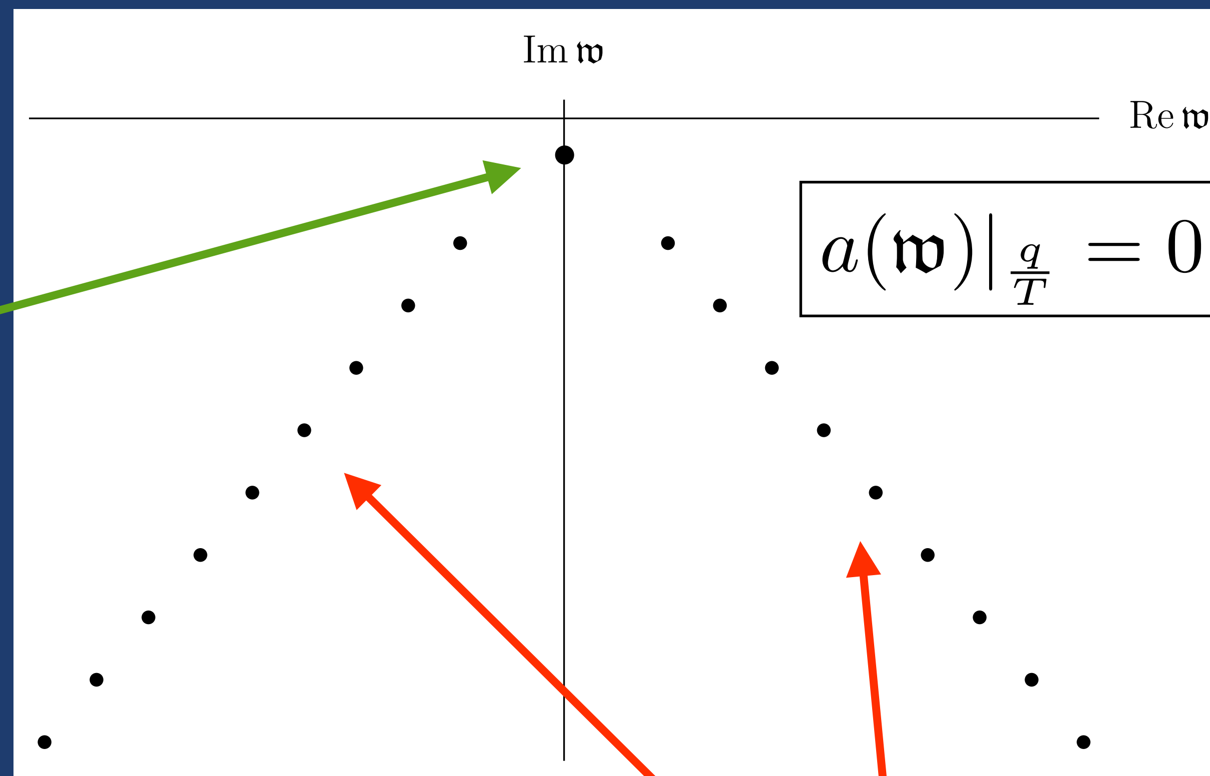
$\omega(q^2)$



holomorphic disk

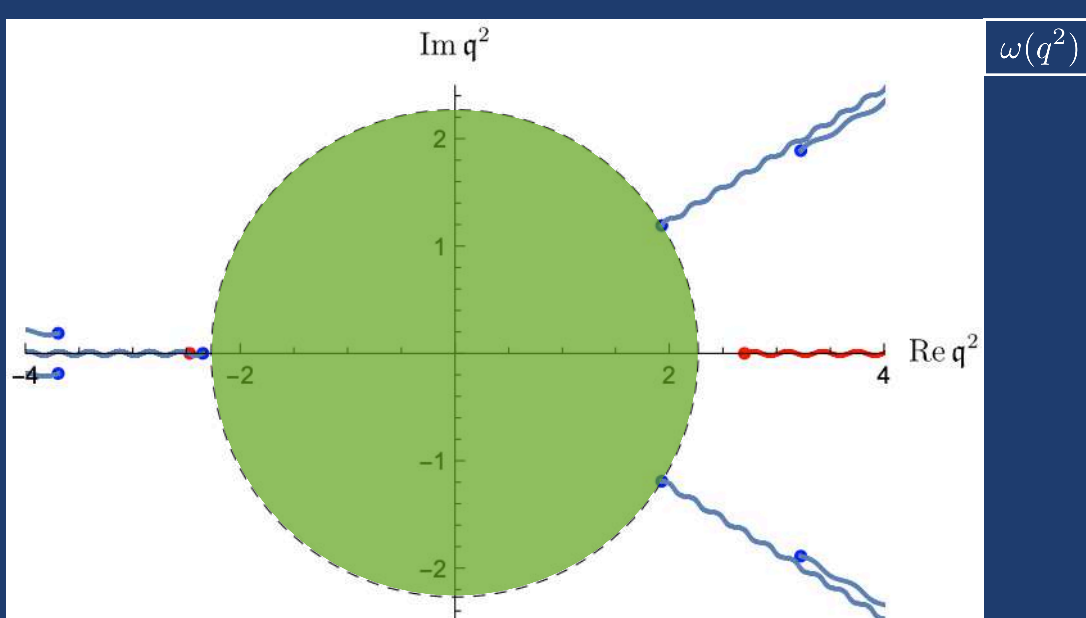
RECONSTRUCTION OF SPECTRA

$$\langle T_{\mu\nu}(-\omega,-q), T_{\rho\sigma}(\omega,q)\rangle_R \sim \frac{b(\omega,q)}{a(\omega,q)} = \frac{B(\omega,q)}{\prod_{i=0}^{\infty}(\omega-\omega_i(q))}$$



hydrodynamics

gapped modes with $\omega(q=0)\neq 0$



[SG, Lemut, JHEP (2023)]

[see also Withers, JHEP (2019)]

PUISEUX AND DARBOUX THEOREMS

- *Puiseux theorem*

  Around a critical point of order *p*, we expect *p* branches of solutions

$$\boxed{f(x_*=0,y_*=0)=0,\ \partial_y f(0,0)=0,\ \ \ldots,\ \partial_y^p f(0,0)\neq 0}$$

$$\boxed{y=Y_j(x)=\sum_{k\geq k_0}^{\infty} a_k x^{k/m_j},\quad j=1,\ldots,p}$$

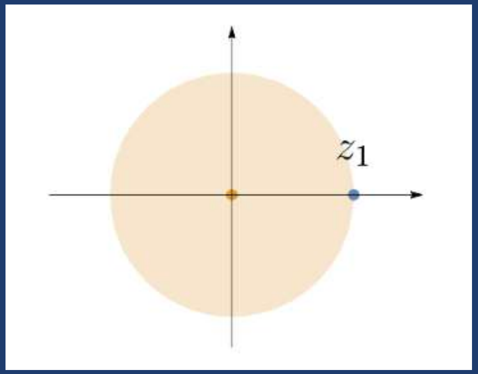
If some $m_j>1$, we necessarily have a family of m_j solutions

$$\boxed{y=Y_l(x)=\sum_{k\geq k_0}^{\infty} a_k\left(e^{\frac{2\pi i l}{m_j}}\right)^k x^{k/m_j},\quad l=0,1,\ldots,m_j-1}$$

- recall: sound

$$\boxed{\mathfrak{w}_{\text{sound}}=-i\sum_{n=1}^{\infty} a_n e^{\pm\frac{i\pi n}{2}}\left(\mathfrak{q}^2\right)^{n/2}=\pm v_s\mathfrak{q}-\frac{i}{2}\mathfrak{G}\mathfrak{q}^2+\ldots}$$

PUISEUX AND DARBOUX THEOREMS



- *Darboux theorem*

  Consider a power series

$$\boxed{f(z) = \sum_{n=0}^{\infty} a_n z^n}$$

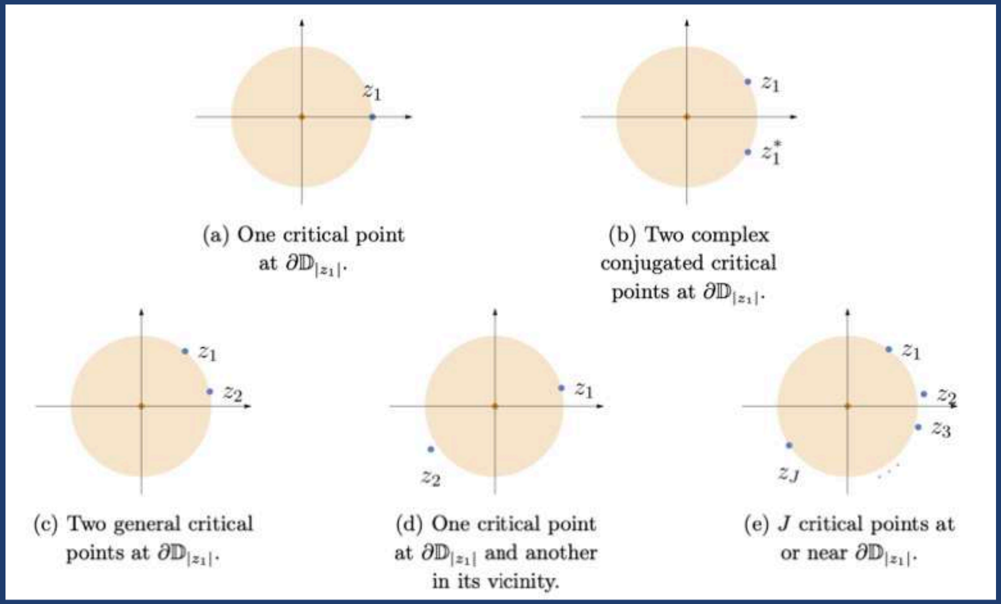
that converges up to a critical point of order $\nu\,[= -1/p]$, which can be computed

$$\boxed{f(z) \sim (z - z_1)^{-\nu\,[=1/2]}\, r(z) + q(z)}$$

$$\boxed{\nu = \lim_{n\to\infty}\left[z_1 (n+1)\frac{a_{n+1}}{a_n} - n\right]}$$

as well as all coefficients in the expansion and subleading (non-singular) terms

$$\boxed{\begin{aligned} r(z) &= \sum_{m=0}^{\infty} r_m\,(z - z_1)^m \\ r_m &= \lim_{n\to\infty}\left[\frac{(-1)^{m-\nu} n!\, z_1^{n-m+\nu} a_n}{(\nu - m)_n} - \sum_{k=0}^{m-1}\frac{(-1)^{m-k}(\nu - k)_n r_k}{(\nu - m)_n z_1^{m-k}}\right]\end{aligned}}$$



(a) One critical point at $\partial\mathbb{D}_{|z_1|}$.

(b) Two complex conjugated critical points at $\partial\mathbb{D}_{|z_1|}$.

(c) Two general critical points at $\partial\mathbb{D}_{|z_1|}$.

(d) One critical point at $\partial\mathbb{D}_{|z_1|}$ and another in its vicinity.

(e) J critical points at or near $\partial\mathbb{D}_{|z_1|}$.

- Problem: need the location of the critical point or the exponent… this is resolved by Hunter and Guerrieri (1980), which we generalise and apply to finite number of coefficients

RECONSTRUCTION OF '*ALL*' UV MODES

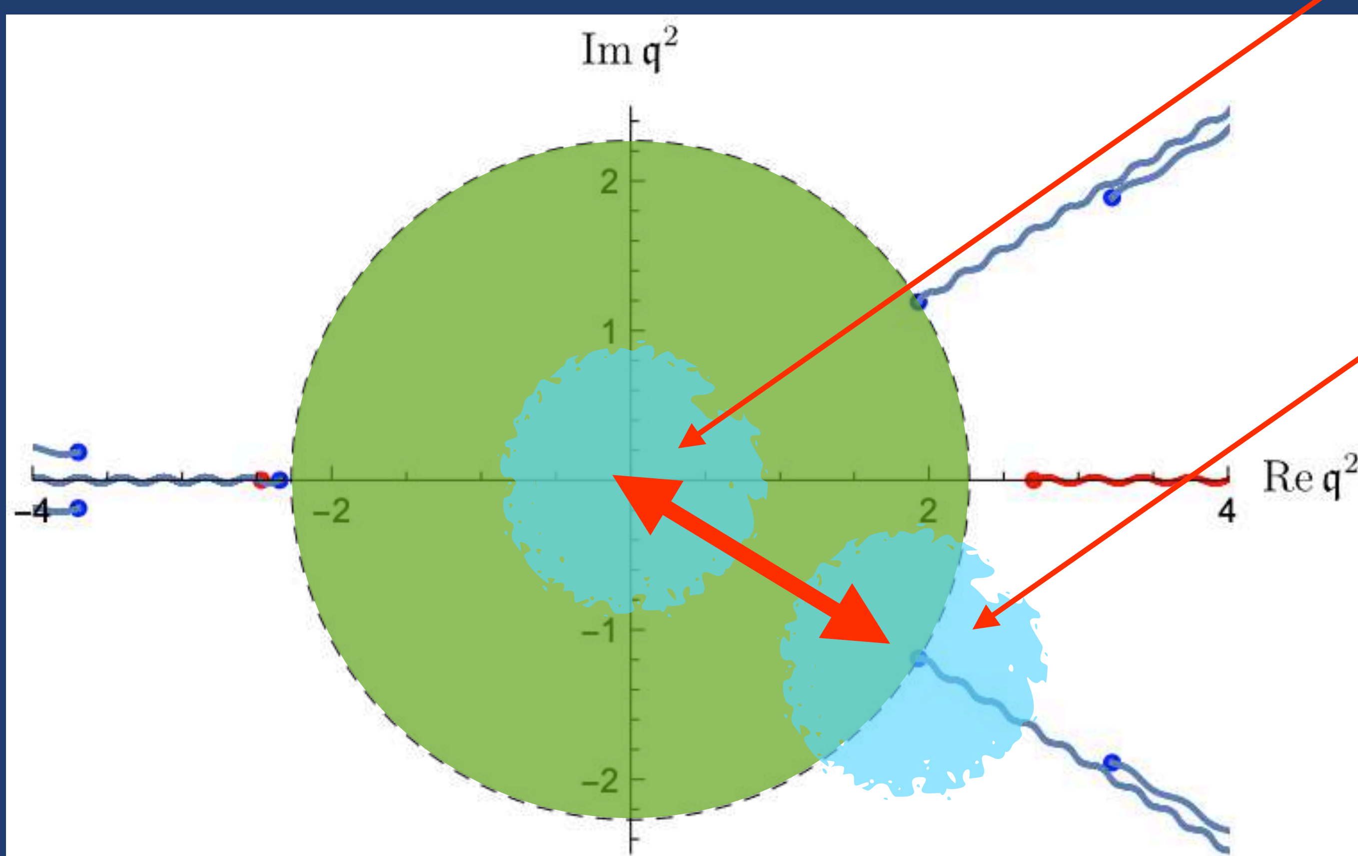
claim: systematic reconstruction of *all* modes connected via *level-crossing* is possible by exploration (analytic continuations) of the Riemann surface connecting physical modes

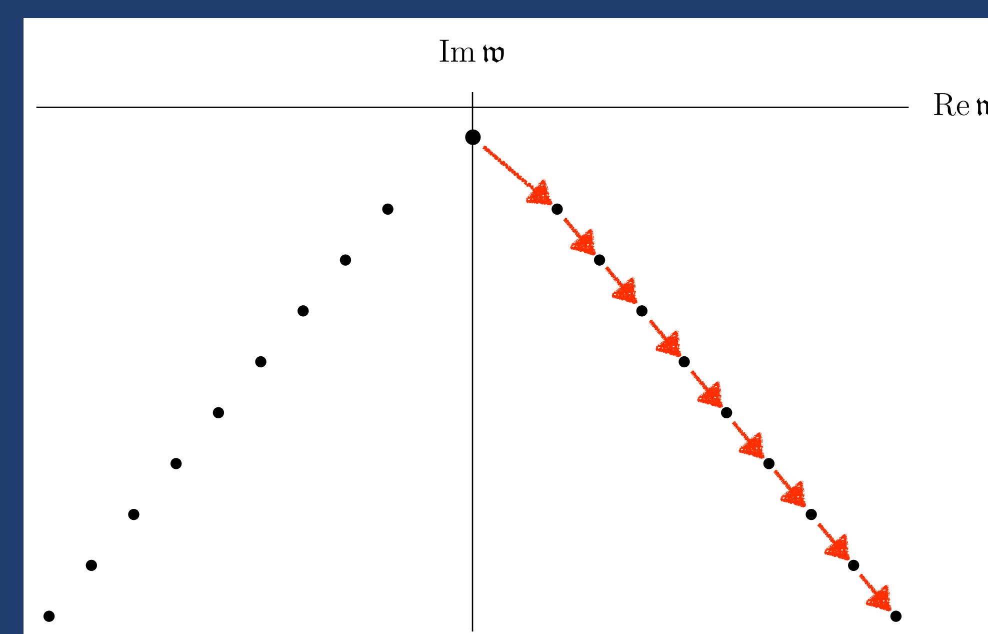
- statement should hold for spectra that are 'sufficiently complicated' – Heun function
- momentum space analogue of resurgence in position space – everything is convergent

$$\omega_0(z) = \sum_{n=0}^{\infty} a_n z^n$$

$$\omega_0(z) = -i \sum_{n=0}^{\infty} e^{\frac{i\pi n}{2}} b_n (z - z_1)^{n/2}$$

$$\omega_1(z) = -i \sum_{n=0}^{\infty} e^{-\frac{i\pi n}{2}} b_n (z - z_1)^{n/2}$$





all UV modes from one IR mode

EXAMPLE: MOMENTUM DIFFUSION OF M2 BRANES

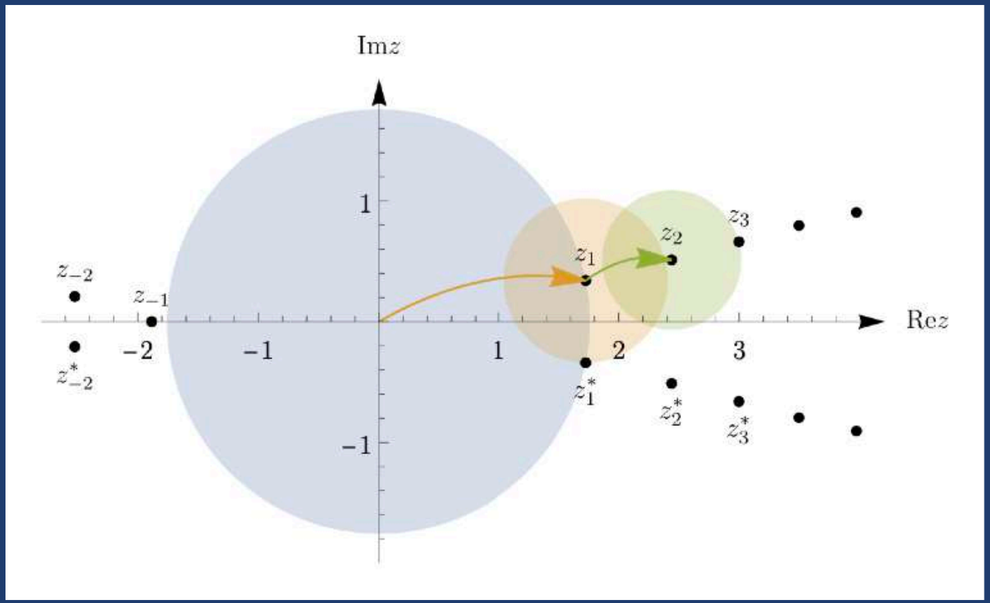
- start from 300 hydrodynamic coefficients $\omega_0(z) = \sum_{n=0}^{\infty} a_n z^n$
- use algorithm with 2 c.c. critical points, 'recover' 12 coefficients and compute the gap with analytic continuation on the same sheet (Padé approximant, …)

$$\boxed{\mathfrak{w}_1(z) = \sum_{n=0}^{(N_1=12)-1} b_n (z-z_1)^{n/2}} \longrightarrow \boxed{\begin{aligned} \mathfrak{w}_1^{\mathrm{calc}}(0) &= 1.23506 - 1.76338i \\ \mathfrak{w}(0) &= 1.23455 - 1.77586i \end{aligned}}$$

- (re)compute the first 300 coefficients, use algorithm with 2 general critical points, 'recover' 12 coefficients and compute the gap

$$\boxed{\mathfrak{w}_2(z) = \sum_{n=0}^{(N_2=12)-1} c_n (z-z_2)^{n/2}} \longrightarrow \boxed{\begin{aligned} \mathfrak{w}_2^{\mathrm{calc}}(0) &= 2.16275 - 3.25341i \\ \mathfrak{w}_2(0) &= 2.12981 - 3.28100i \end{aligned}}$$

- … exploration continues …
- conceptually useful and instructive, practically not quite (yet)…



POLE-SKIPPING AND
THE RECONSTRUCTION

POLE-SKIPPING

- precise analytic relation between hydrodynamics and quantum chaos
  [SG, Schalm, Scopelliti, PRL (2017); Blake, Lee, Liu, JHEP (2018); Blake, Davison, SG, Liu, JHEP (2018); SG, JHEP (2019)]
- resumed all-order hydrodynamic series (e.g. sound) $\omega(q) = \sum_{n=1}^{\infty} \alpha_n \left(T, \mu_i, \langle \mathcal{O}_j \rangle, \lambda\right) q^n$

  passes through a 'chaos point' at where the associated *2*-pt function is "0/0":

$$\boxed{\omega(q = i\lambda_L / v_B) = i\lambda_L = 2\pi T i}$$

$$\boxed{G_R = \frac{0}{0} = N(\delta\omega/\delta q)}$$

- relation to quantum chaos as measured by the out-of-time-ordered correlation functions

$$\boxed{C(t, \mathbf{x}) = \left\langle [W(t,\mathbf{x}), V(0,\mathbf{0})]^\dagger [W(t,\mathbf{x}), V(0,\mathbf{0})] \right\rangle_T \sim \epsilon\, e^{\lambda_L (t - |\mathbf{x}|/v_B)}}$$

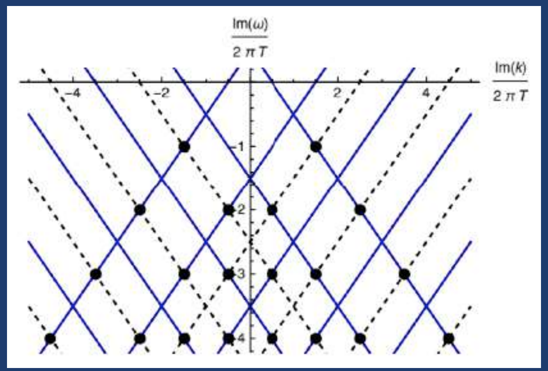
'quantum' Lyapunov exponent

butterfly velocity

- triviality of Einstein's equations at the horizon

- infinite number of such '0/0' points:
  [SG, Kovtun, Starinets, Tadić, JHEP (2019); Blake, Davison, Vegh, JHEP (2019)]

$$\boxed{\omega_n(q_n) = -2\pi T i n}$$



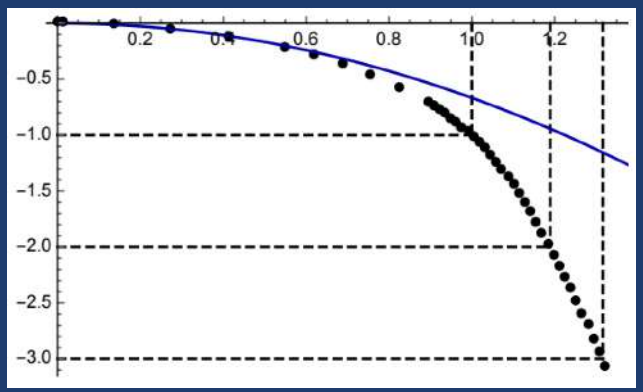
[from Blake, Davison, Vegh, JHEP (2019)]

DIFFUSION AND SPECIAL POLE-SKIPPING POINTS

- consider diffusion in a neutral 3*d* CFT dual to AdS$_4$-Schwarzschild

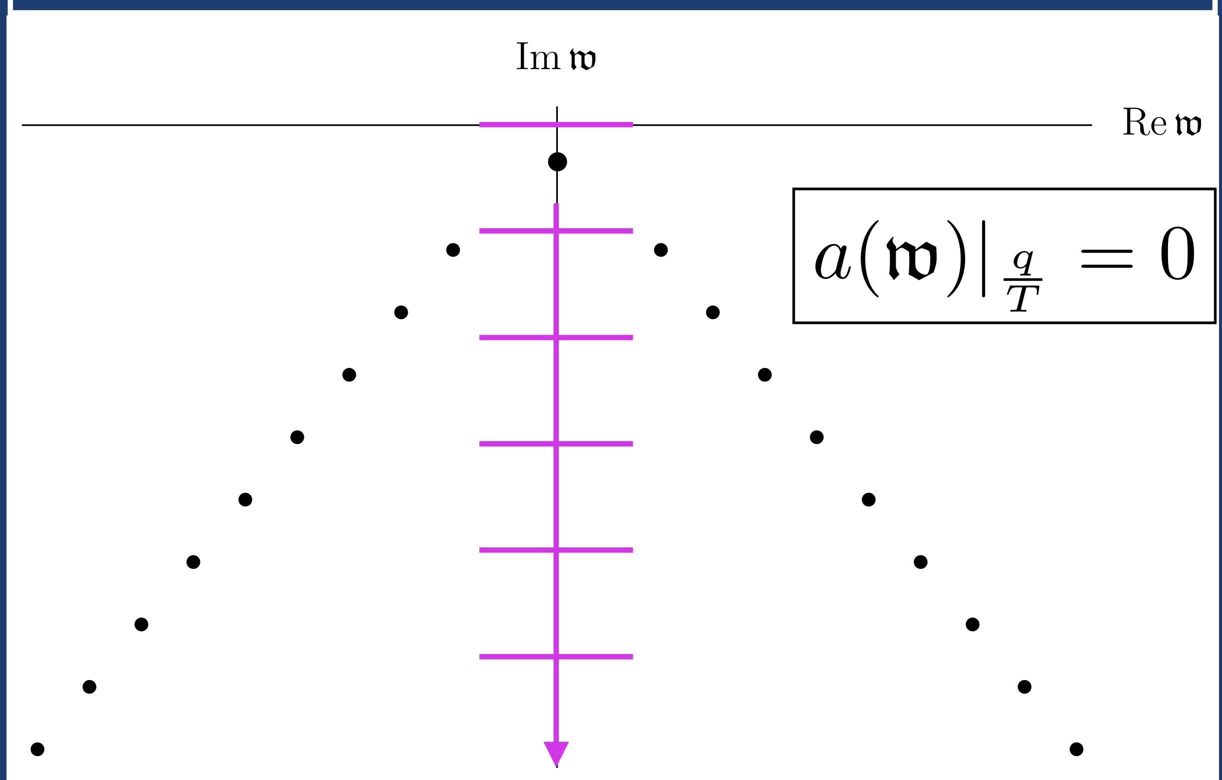
$$\boxed{\mathfrak{w}_{\text{shear}} = -i \sum_{n=1}^{\infty} c_n \left(\mathfrak{q}^2\right)^n = -i\mathfrak{D}\mathfrak{q}^2 + \ldots}$$

$$\boxed{\omega_n(q_n) = -2\pi T i n}$$



[plot from Blake,
Davison, Vegh, JHEP (2019)]

$$\langle T_{\mu\nu}(-\omega,-q), T_{\rho\sigma}(\omega,q)\rangle_R \sim \frac{b(\omega,q)}{a(\omega,q)}$$



for increasing real q

analytic result known for 4*d* bulk
[SG, PRL (2021)]

$$q_n = \frac{4\pi T}{\sqrt{3}} n^{1/4}, \quad n = 0, 1, 2, \ldots$$

algebraically special points

4*d* odd-even duality: Darboux transform
[Chandrasekhar (1983); SG, Vrbica (2023)]

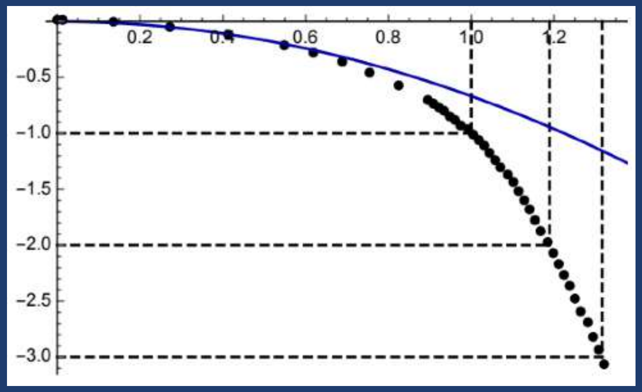
RECONSTRUCTION FROM POLE-SKIPPING

- how much information is required to reconstruct a QFT spectrum?
  so far in the talk: the knowledge of one dispersion relation

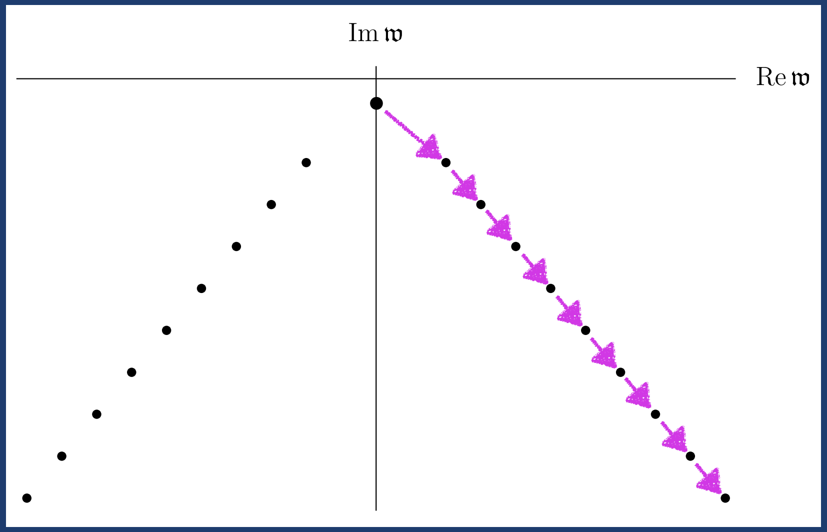
claim: in holographic theories of the type discussed here (N=4 SYM, M2, M5, …),
the entire spectrum can be computed from only a discrete set of pole-skipping points

[SG, Lemut, Pedraza, PRD (2023)]

$$\boxed{\omega_n(q_n) = -2\pi T i n} \longrightarrow \boxed{\mathfrak{w}_{\text{shear}} = -i\sum_{n=1}^{\infty} c_n \left(\mathfrak{q}^2\right)^n = -i\mathfrak{D}\mathfrak{q}^2 + \dots}$$



[plot from Blake, Davison, Vegh, JHEP (2019)]

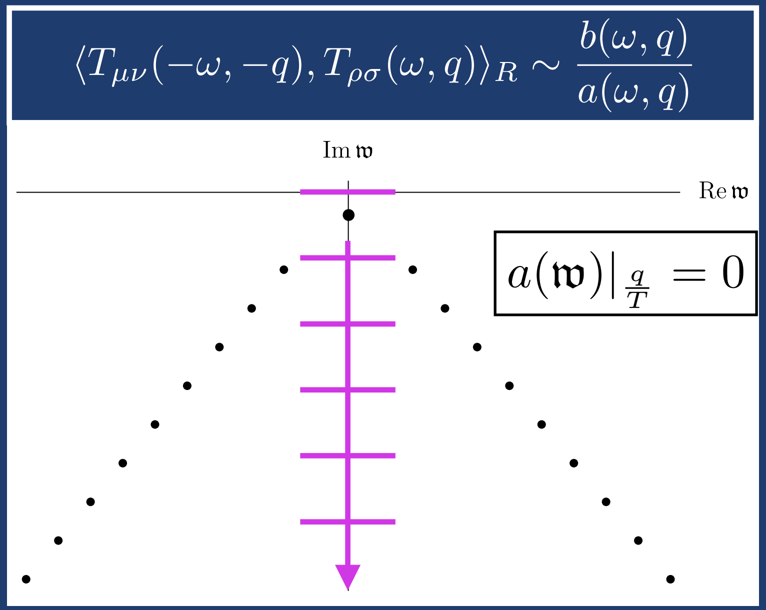


RECONSTRUCTION FROM POLE-SKIPPING

- interpolation problem:

$$\boxed{\omega_n(q_n) = -2\pi T i n}$$

$$\boxed{\mathfrak{w}_{\text{shear}} = -i \sum_{n=1}^{\infty} c_n \left(\mathfrak{q}^2\right)^n = -i\mathfrak{D}\mathfrak{q}^2 + \dots}$$



for increasing real q

- unique solutions to interpolation problems are extremely hard in the absence of special known properties of the function (e.g. Weierstrass, Hadamard, Nevanlinna-Pick, …)
- trick: ‘analytic continuation’ to *d* spacetime dimensions and expansion around infinite *d*
- general relativity in large *d* drastically simplifies [review by Emparan, Herzog, 2003.11394]

$$V \sim 1/r^d$$

- recall: large-*d* limit of quantum mechanics allows a solution of the Helium problem
- convergence of such series depends on the details

RECONSTRUCTION FROM POLE-SKIPPING

- interpolation: $\omega_n(q_n) = -2\pi T i n \longrightarrow \mathfrak{w}_{\text{shear}} = -i \sum_{n=1}^{\infty} c_n \left(\mathfrak{q}^2\right)^n = -i\mathfrak{D}\mathfrak{q}^2 + \dots$

- 'analytic continuation' to *d* spacetime dimensions and expansion around infinite *d*

$$\omega_0(q) = -i\left(\frac{q}{\sqrt{d}}\right)^2 - i\sum_{m=2}^{\infty}\frac{1}{d^m}\sum_{j=2}^{m} c_{m,n}\left(\frac{q}{\sqrt{d}}\right)^{2j} \qquad \frac{q_n}{\sqrt{d}} = \sqrt{\frac{nd}{2}}\left(1 + \sum_{m=1}^{\infty}\frac{b_{n,m}}{d^m}\right)$$

$$b_{n,1} = -\sum_{m=2}^{\infty}\frac{n^{m-1}c_{m,m}}{2^m}$$

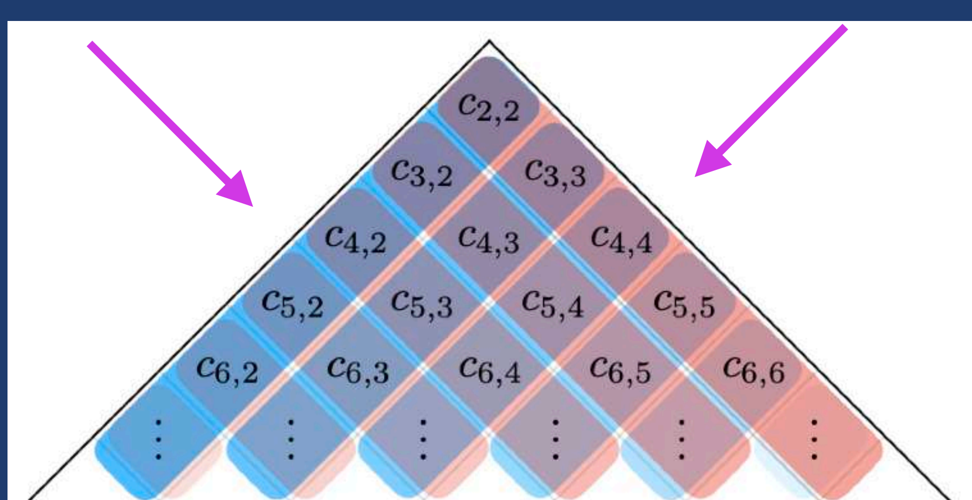
$$b_{n,2} = -\frac{b_{n,1}^2}{2} - \sum_{m=2}^{\infty}\frac{n^{m-1}\left(c_{m+1,m} + 2m b_{n,1} c_{m,m}\right)}{2^m}$$

second analytic continuation

$$n \in \mathbb{Z} \to x \in \mathbb{R}$$

hydrodynamics

pole-skipping



$$c_{m,m} = -\frac{2^m}{(m-1)!}\partial_x^{m-1} b_1(0)$$

$$c_{m+1,m} = -\frac{2^m \partial_x^{m-1} b_2(0)}{(m-1)!} + \sum_{j=2}^{m-1}\left(j - \frac{1}{4}\right) c_{j,j} c_{m-j+1,m-j+1}$$

generating functions

RECONSTRUCTION FROM POLE-SKIPPING

$$\boxed{\omega_0(q) = -i\left(\frac{q}{\sqrt{d}}\right)^2 - i\sum_{m=2}^{\infty}\frac{1}{d^m}\sum_{j=2}^{m} c_{m,n}\left(\frac{q}{\sqrt{d}}\right)^{2j}} \qquad \boxed{\frac{q_n}{\sqrt{d}} = \sqrt{\frac{nd}{2}}\left(1+\sum_{m=1}^{\infty}\frac{b_{n,m}}{d^m}\right)}$$

- first level: $b_{n,1} = -\frac{1}{2}H_n = -\frac{1}{2}\sum_{k=1}^{n}\frac{1}{k}$ $\longrightarrow$ $H_n \to H(x) = \sum_{k=1}^{\infty}\frac{x}{k(x+k)}$

$\longrightarrow$ $\boxed{c_{m,2} = c_{m,m} = (-1)^m 2^{m-1}\zeta(m)}$ $\quad$ $\boxed{\omega_0(q) = -i\bar{q}^2 - i\frac{\bar{q}^2}{d}H_{2\bar{q}^2/d} + \ldots}$

partial (convergent) resummation

- second level benefits from numerical approach (Padé interpolation):

$$\begin{aligned}
b_{n,2} &= a_n + b_n H_n + c_n H_n^{(2)} + d_n H_n H_n^{(2)} + e_n H_n^2 + \sum_{k=1}^{n-3}\left(j_{n,k} + k_{n,k}H_n + l_{n,k}H_k + m_{n,k}H_k^{(2)}\right) \\
&+ \frac{n}{8(n+1)}\left[\sum_{k=1}^{n-3}\frac{1}{k+1}\left(\frac{n+2}{n}\right)^k\right]\left[\sum_{k=1}^{n-3}\frac{1}{k+1}\left(\frac{n}{n+2}\right)^k\right] - \frac{3n}{8(n+1)}\sum_{k=2}^{n-3}\frac{1}{k+1}\left(\frac{n}{n+2}\right)^k\sum_{j=0}^{k-2}\frac{1}{j+1}\left(\frac{n+2}{n}\right)^j \\
&+ \frac{1}{2(n+1)}\sum_{k=2}^{n-3}H_k\left(\frac{n}{n+2}\right)^k\sum_{j=0}^{k-2}\frac{1}{j+1}\left(\frac{n+2}{n}\right)^j + \frac{n+2}{4n(n+1)}\sum_{k=2}^{n-3}\frac{1}{k-1}\left(\frac{n}{n+2}\right)^{k-1}\sum_{j=k}^{n-3}\frac{1}{j+1}\left(\frac{n+2}{n}\right)^j \\
&+ \frac{n+2}{4n(n+1)}\sum_{k=2}^{n-3}\frac{1}{k-1}\left(\frac{n}{n+2}\right)^{k-1}\sum_{j=k}^{n-3}H_j\left(\frac{n+2}{n}\right)^j + \frac{3n^2}{8(n+1)(n+2)}\sum_{k=1}^{n-3}\frac{1}{k}\sum_{j=k+1}^{n-3}\frac{1}{j}\left(\frac{n+2}{n}\right)^{n-j}, \qquad \text{(B17)}
\end{aligned}$$

$$\begin{aligned}
a_n &= \frac{1-2n}{4(n^2-1)}\left(\frac{n}{n+2}\right)^{n-3} + \frac{12-n(2(n-3)(n-1)n+11)}{8n^2(n^2-4)^2(n^2-1)}\left(\frac{n+2}{n}\right)^n \\
&\quad - \frac{n^9-17n^8+77n^7-103n^6-44n^5+4n^4+378n^3-544n^2+392n-96}{8(n-2)^2(n-1)^3n^2(n+1)(n+2)}, \\
b_n &= \frac{n^7-10n^6+21n^5-32n^4-28n^3+72n^2-96n+96}{16(n-2)^2(n-1)^2n(n+1)(n+2)} \\
&\quad + \frac{(5n+6)}{8(n+1)}\left(\frac{n}{n+2}\right)^{n-2} - \frac{3(n-2)}{16(n+1)}\left(\frac{n+2}{n}\right)^{n-2}, \\
c_n &= -\frac{n(6n^2-16n+9)}{8(n+1)(n^2-3n+2)}, \\
d_n &= -\frac{n}{4n+4}, \\
e_n &= \frac{5n^3-8n^2-10n+10}{8(n^3-2n^2-n+2)},
\end{aligned}$$

$$\begin{aligned}
j_{n,k} &= \left(\frac{(k(k(5k+24)+23)+8)n^2-2(3k+1)(k+1)^2n+4k(k+1)^2}{8k(k+1)^2(n-2)(n-1)n(n+1)(k-n+1)}\right. \\
&\quad \left.+\frac{-k(k+2)n^5+(k(k(3k+10)+11)+3)n^4+(-k(k+2)(7k+12)-9)n^3}{8k(k+1)^2(n-2)(n-1)n(n+1)(k-n+1)}\right)\left(\frac{n+2}{n}\right)^k \\
&\quad + \frac{k(n(n(n+3)+6)-4)+n(6-(n-3)n(n+1))-4}{8(k+1)(n^2-1)(k-n+1)}\left(\frac{n}{n+2}\right)^k \\
&\quad + \frac{2k^2n^2-5k^2n-2kn^3+8kn^2-9kn+n^3-3n^2+2n}{8k^3(k+1)(n-2)(n-1)(n+1)}, \\
k_{n,k} &= \frac{2k^2(k+1)^2+(k(k+4)+1)n^2-2k(k+1)^2n-n^3}{8k(k+1)n(n+1)(k-n+1)}\left(\frac{n+2}{n}\right)^k \\
&\quad + \frac{-2k(n+1)(n+6)+n(n(2n+7)-18)-24}{8(n+1)(n+2)(-k+n-2)(-k+n-1)}\left(\frac{n}{n+2}\right)^k, \\
l_{n,k} &= \frac{n(4k(n+2)-3n^2+4n+8)}{8(k+1)(n+1)(n+2)^2}\left(\frac{n+2}{n}\right)^{n-k} - \frac{(n-2k)}{8k(k+1)(n+1)}\left(\frac{n+2}{n}\right)^k \\
&\quad + \frac{4(k(k+2)-2)(k(k+11)+6)n^2-8k(k+2)(4k-1)(k+1)n+16k(k+2)(k+1)^2}{8k^2(k+1)^2(k+2)n(n^4-5n^2+4)} \\
&\quad + \frac{(k(k(5(k-1)k+4)+102)+44)n^4+2(k(k(k(17k+33)-32)-32)+8)n^3}{8k^2(k+1)^2(k+2)n(n^4-5n^2+4)} \\
&\quad + \frac{4(k^3-5k-2)n^6+(k(10-3k(k(3k+7)-2))-4)n^5}{8k^2(k+1)^2(k+2)n(n^4-5n^2+4)}, \\
m_{n,k} &= \frac{(k-1)n}{4k(k+2)(n+1)}.
\end{aligned}$$

RECONSTRUCTION FROM POLE-SKIPPING

- results:

$$c_{2,2} = 2\zeta(2),$$
$$c_{3,2} = -4\zeta(3), \quad c_{3,3} = -4\zeta(3),$$
$$c_{4,2} = 8\zeta(4), \quad c_{4,3} = 7 \times 8\zeta(4), \quad c_{4,4} = 8\zeta(4)$$

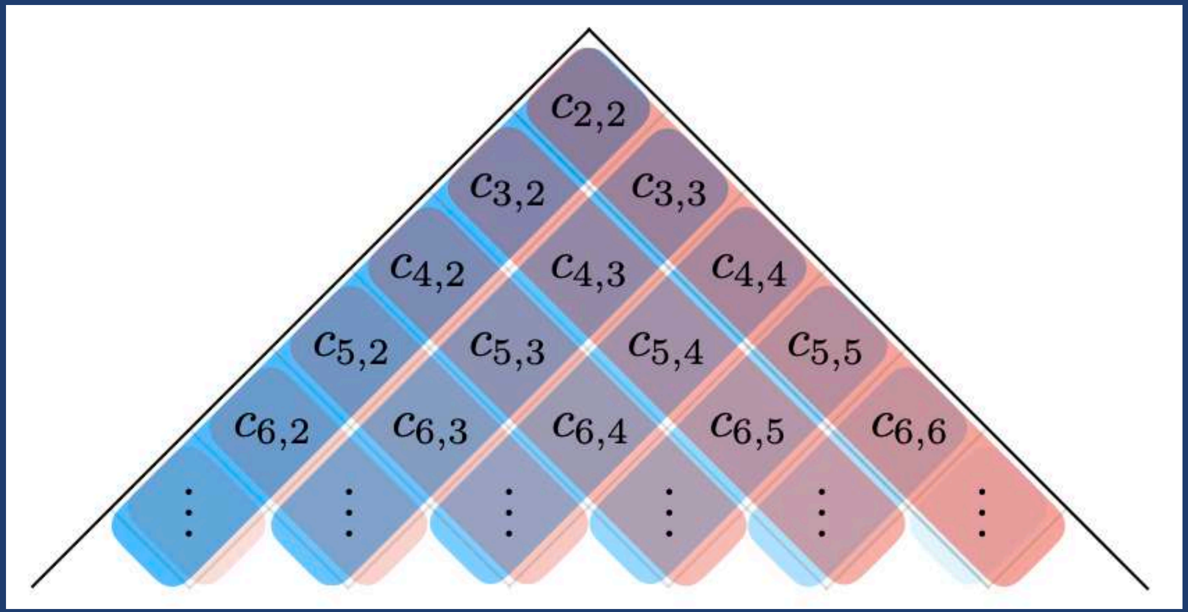
- symmetries

$$b_{n,1} \longrightarrow \boxed{c_{m,2} = c_{m,m} = (-1)^m 2^{m-1}\zeta(m)}$$

- 'polylogs'
  [see e.g. Aminov et.al. 2307.10141]

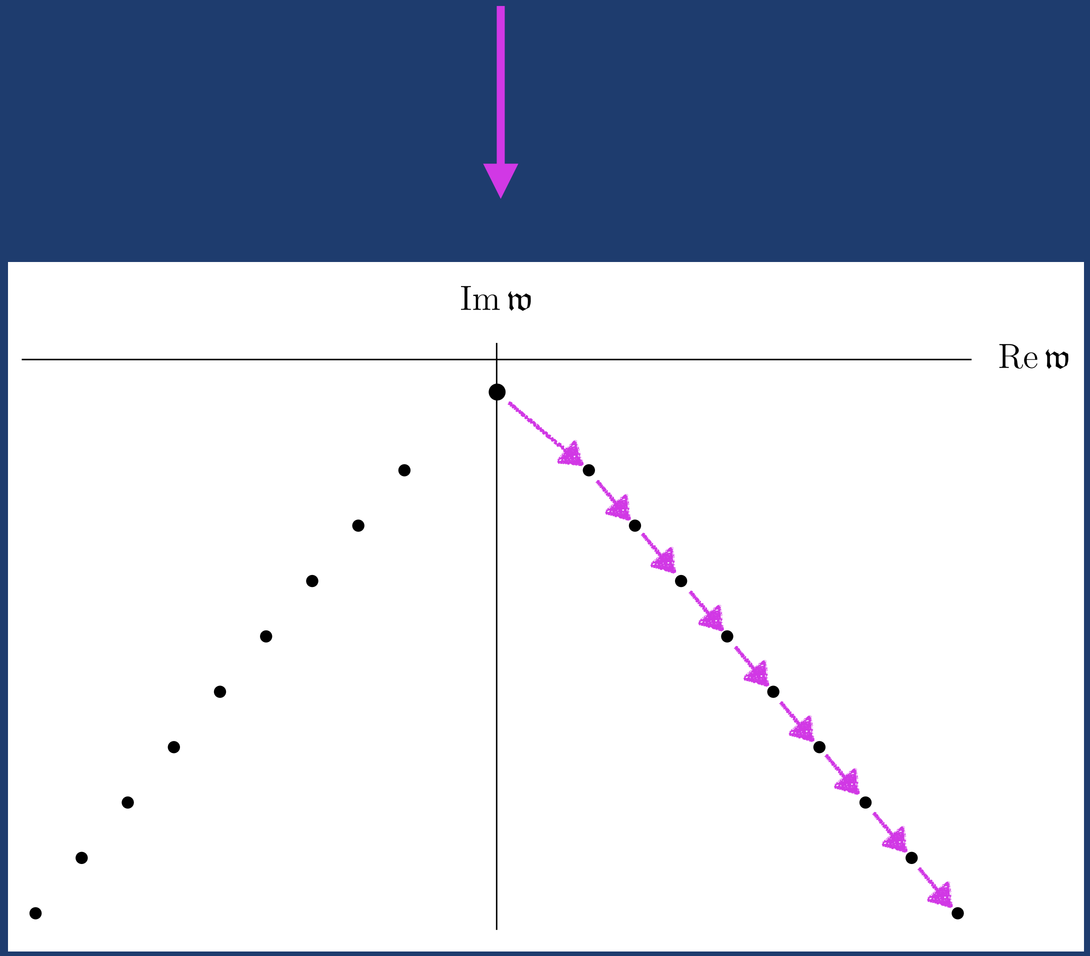
$$b_{n,2} \longrightarrow \begin{aligned} c_{3,2} &\approx -1.000 \times 4\zeta(3), \\ c_{4,3} &\approx 7.001 \times 8\zeta(4), \\ c_{5,4} &\approx -15.548 \times 16\zeta(5), \\ c_{6,5} &\approx 27.546 \times 32\zeta(6) \end{aligned}$$

$$b_{n,3} \longrightarrow \begin{aligned} c_{4,2} &\approx 1.000 \times 8\zeta(4), \\ c_{5,3} &\approx -15.502 \times 16\zeta(5) \end{aligned}$$



RECONSTRUCTION FROM POLE-SKIPPING

- the rest of the spectrum follows from a reconstruction discussed before



- complete reconstruction of the spectrum using only algebraic near-horizon manipulations (local instead of global (ODE/PDE) analysis)
- 'full' correlator follows from poles – product formula by Dodelson, et.al., 2304.12339

SUMMARY AND FUTURE DIRECTIONS

SUMMARY AND FUTURE DIRECTIONS

- complex analytic structures of transport are a powerful tool for exploring physics
- new results about the nature of thermal QFTs
- in momentum space we can deal with convergent series, but, ‘morally’, this is equivalent to resurgence in position space
- useful not only in QFTs, also for QNMs and general linearised operator spectra
- improve practical aspects of reconstructions given a limited number of known coefficient
- can these techniques be used in realistic QFTs (Euler-Heisenberg, chiral Lagrangian)?
- claim: reconstruction from IR to UV is possible from a discrete set of pole-skipping points
- large-N QFT calculation $\longrightarrow$ differential equations $\longrightarrow$ algebraic manipulations

THANK YOU!