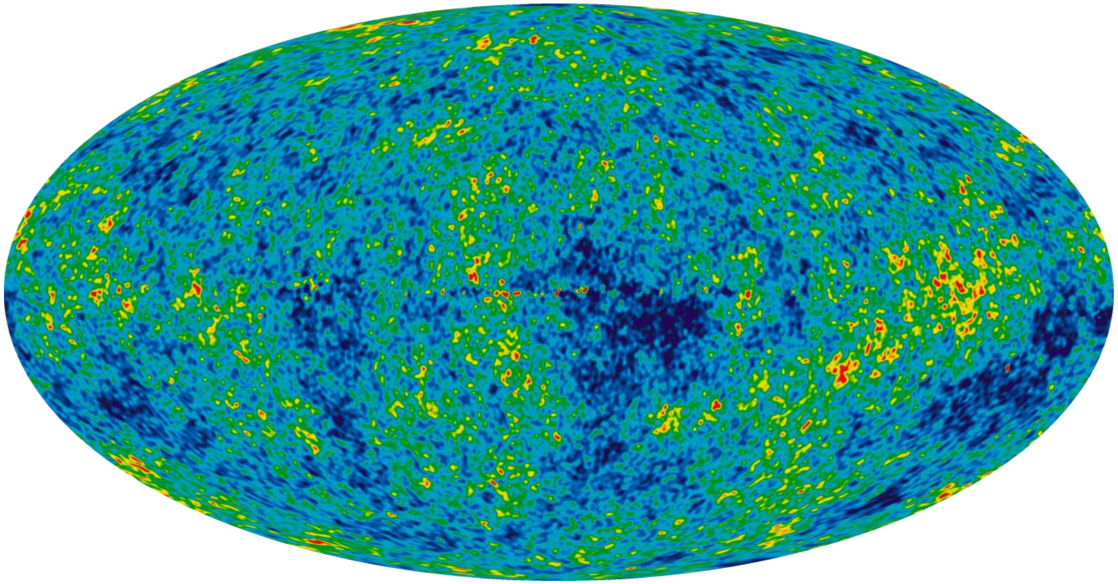
# Cosmic Ray Anisotropy



Part 1
Introduction

**Part 2**
**Methods and Applications**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MAPSES, Lecce 2011

# PART 2
# METHODS AND APPLICATIONS

# Anisotropy search methods

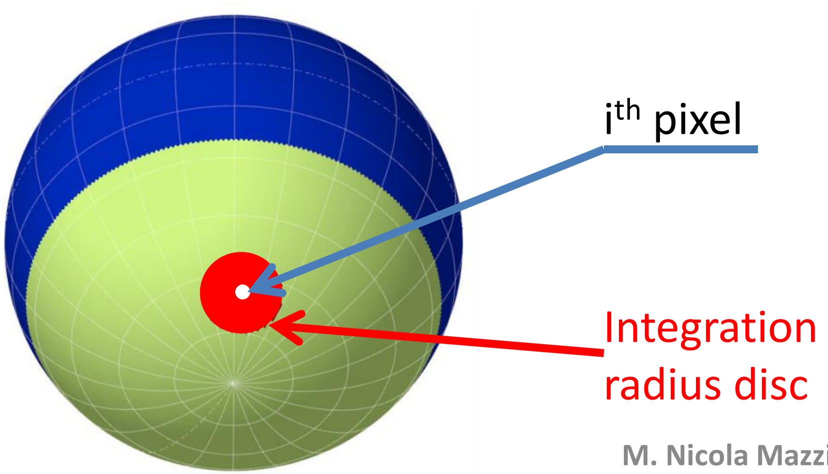
- The starting point of an anisotropy study is the construction of a sky map corresponding to the case of no-anisotropy (null hypothesis), i.e. case of a perfectly isotropic distribution, taking into account the instrument observation mode
- Comparison of the no-anisotropy sky map with the actual sky map can reveal the presence of any anisotropies in the data
  - Sometime the no-anisotropy map is called background map, since it represents the map to be used to search a possible significant excess signal when is compared with the real one
- Usually the signal is very weak, i.e. $I = I_{iso} + I_{aniso}, with\ I_{aniso} \ll I_{iso}$, so it is important to estimate the no-anisotropy map in an unbiased manner, trying to avoid to include any systematic uncertainties
  - A flawed no-anisotropy map estimate can mistakenly create an apparently significant signal, or bury a real signal

# Search at large angular scale

- Once the no-anisotropy map has been built, the next step is to search for any possible excess or deficit at different angular scales, since a large scale anisotropy is expected for charged cosmic rays
- One way to search for anisotropies at some angular scale is to use sky maps composed of independent bins with bin size similar to the angular scale of the anisotropy under search.
- However, when using independent bins, a potential anisotropic signal might become too weak to be detected since it will probably be distributed among multiple adjacent bins
  - spillover effects reduce sensitivity
- A more sensitive way to perform the search, is to use sky maps consisting of a large number of correlated bins.
  - The effective number of trials involved in evaluating all possible directions in integrated sky maps need to be taken into account

# Integrated sky maps

- The starting point are real and the no-anisotropy maps made with small independent-bins
- The content of a correlated bin is equal to the integrated number of events in a circular region around that bin.
- Using such ‘‘integrated sky maps,’’ it is very likely that there will be at least one bin with its center roughly aligned with the direction of the center of a potential anisotropy, reducing spillover effects and increasing sensitivity.
- In general, the sensitivity for detecting an anisotropy of given angular scale is greater when an integration radius close to that scale is chosen.
  - If the integration radius is too small or too large compared to the angular scale of the prospective anisotropy, the sensitivity becomes suboptimal since either the signal can be split among several adjacent bins or there can be too much ‘‘background’’ (isotropic signal) contamination.




$$N_i = \sum_{p \in disc} N_p$$

Both data and bkg maps are smoothed in this way

# Spherical harmonic analysis

- A function $f$ (e.g. fluctuation map) on the sphere can be expanded in spherical harmonics, $Y_{lm}$, as $f(d) = \sum_{l=0}^{l_{max}} \sum_{m=-l_{max}}^{l_{max}} a_{lm} Y_{lm}(d)$
- Where $d$ is a unit vector pointing at polar angle $\theta$ and azimuth $\varphi$, and under the assumption that there is insignificant signal power in modes with $l > l_{max}$, and $a_{lm} = \int f(\Omega) Y_{lm}^*(\Omega) d\Omega$ ($d \Leftrightarrow \Omega$)
- $< Y_{lm}^* Y_{kq} > = \int Y_{lm}^*(\Omega) Y_{kq}(\Omega) d\Omega = \delta_{lk}\delta_{mq}$
- Pixelising $f(d)$ corresponds to sampling it at $N_{pix}$ locations $d_p$, $p \in [0, Npix - 1]$
- The sample function values $f_p$ can then be used to estimate $a_{lm}$

$$\hat{a}_{lm} = \frac{4\pi}{N_{pix}} \sum_{p=0}^{N_{pix}-1} Y_{lm}^*(d_p) f(d_p)$$

- A statistically isotropic sky means that all *m*s are equivalent, *i.e.*, there is no preferred axis
- The average variance of these coefficients is used to construct an angular power spectrum at angular scale $\theta \sim 180°/l$
  - Dipole $\Leftrightarrow l = 1$
- The variance of the $f(d)$ (or equivalently the power spectrum in $l$) then fully characterizes the anisotropies

# Angular power spectrum

- The angular power spectrum can be defined as

  $$C_l = \frac{1}{2l+1} \sum_{m=-l_{max}}^{l_{max}} < |a_{lm}|^2 >$$

- The $\hat{a}_{lm}$ can be used to compute estimates of the angular power spectrum $\hat{C}_l$ as

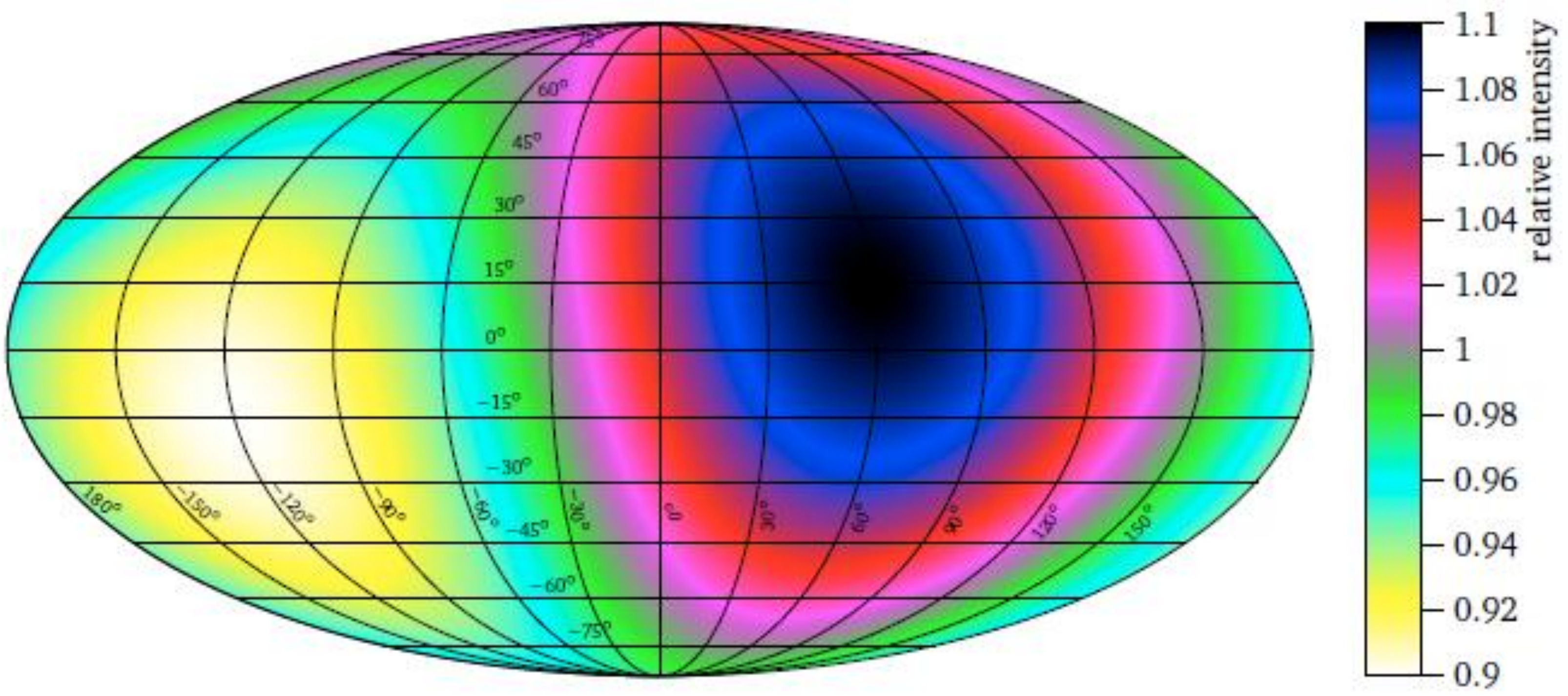
  $$\hat{C}_l = \frac{1}{2l+1} \sum_{m=-l_{max}}^{l_{max}} |\,\hat{a}_{lm}|^2$$

- It is an unbiased estimator of the true power spectrum *Cl*, i.e. $< \hat{C}_l > = C_l$
- Each coefficient $\hat{C}_l$ characterizes the intensity of the fluctuations on an angular scale of $180°/l$
- The variance of each measured $C_l$ (i.e., the variance of the variance) is $var(C_l) = \frac{2}{2l+1} C_l^2$
- Each $C_l$ is $\chi^2$ distributed with $(2l + 1)$ degrees of freedom, i.e. the random variable $(2l + 1)\frac{\hat{C}_l}{C_l}$ follows a ${\chi^2}_{2l+1}$ distribution.

# Dipole anisotropy

- The most basic approach to a large scale variation of the cosmic ray flux is the assumption of a dipole anisotropy
- $I(\varphi,\theta) = I_0 + I_1 \hat{d}(\varphi,\theta) \cdot \hat{d}(\varphi_{dip},\theta_{dip}) \Rightarrow$
- $I = a_{00}Y_{00} + \sum_{-1}^{1} a_{1m}Y_{1m}$, since $a_{lm}=0$ for $l>1$
- $Y_{00} = \frac{1}{2}\sqrt{\frac{1}{\pi}}$
- $Y_{10} = \frac{1}{2}\sqrt{\frac{3}{\pi}}\cos(\theta)$
- $Y_{1\pm1} = \mp\frac{1}{2}\sqrt{\frac{3}{\pi}}\sin(\theta)e^{\pm i\varphi}$
- $I = I_0 + I_1 2\sqrt{\frac{\pi}{3}}Y_{10} \Rightarrow a_{00} = I_0\sqrt{4\pi}\ and\ a_{10} = I_1\sqrt{\frac{4\pi}{3}}$
- Setting $f(\theta) = \frac{I(\theta)-I_0}{I_0} \Rightarrow a_{10} = \frac{I_1}{I_0}\sqrt{\frac{4\pi}{3}}$
- $C_1 = \frac{|a_{10}|^2}{3} = \frac{1}{3}\left(\frac{I_1}{I_0}\right)^2\frac{4\pi}{3} \Rightarrow \delta = 3\sqrt{\frac{C_1}{4\pi}}$

# Dipole map example




Example of a dipole distribution:

- The dipole vector points to 15◦ latitude and 60◦ longitude
- The magnitude is 0.1.
- The Mollweide projection has been used for visualization. As features of this projection technique, latitudes are straight horizontal parallel lines, and equal solid angles are represented by equal areas in the projection.

# Poisson white noise

- The map resulting from a random realization of N equally weighted points can be written as

$$f(\Omega) = \frac{4\pi}{N} \sum_k \delta(\Omega - \Omega_k)$$

normalized to its mean value, i.e.

$$< f > = \int f(\Omega) d\Omega = 1$$

to be adimensional

- The coefficients $a_{lm}$ will follow (apart the constant monopole contribution) a gaussian distribution with $<a_{lm}> = 0$ and

$$< |a_{lm}|^2 > = \frac{4\pi}{N} \; and$$

$$C_l = C_N = \frac{1}{2l+1} \sum_{m=-l_{max}}^{l_{max}} < |a_{lm}|^2 > = \frac{4\pi}{N}$$

- That is, the APS is uniform (white or Poisson noise)
- The Poisson noise due to finite statistic represents a truly isotropic signal, so any anisotropy can be detected if the fluctuations (i.e. the angular power spectrum) are above the Poisson noise level

# No-anisotropy map: the exposure method

- Given a carefully calculated exposure map it is possible to calculate the intensity distribution from a set of measured arrival directions
  - The exposure takes the experiment's geometry (effective area or acceptance) and the actual measurement times into account.
  - For a given direction (expressed i.e. in terms of declination and right ascension), the exposure corresponds to the product of the surface area that has been exposed to the cosmic rays multiplied by the measurement time (i.e. livetime) during which the given direction has been visible (i.e. field of view)
- The effective area, calculated from a Monte Carlo simulation of the instrument, could suffer from systematic errors
- The integrated livetime over a given sky region could suffer from systematic uncertainties too
- Of course, any systematic errors involved in the calculation of the exposure will propagate to the flux, possibly affecting its directional distribution. If the magnitude of these systematic errors is comparable to or larger than the statistical power of the available data set, their effects on the flux's directional distribution might masquerade as a real detectable anisotropy

# No-anisotropy map: data based method

- Assuming that the detector is responding to an isotropic particle intensity, an equivalent data set corresponding to the null hypothesis can be constructed artificially from the measured data set
- Two approaches are usually implemented to build the no-anisotropy sky map:
  - Direct integration technique
  - Shuffling technique

# Calculation of the Exposure

- The detector instrument response functions depend on the angles between a given sky direction (i.e. right ascension $\alpha$ and declination $\delta$) to and the instrument X-axis an Z-axis (i.e. instrument polar and azimuth angles)
- The exposure can be most generally calculated as follows (to keep the notations simple, we will omit the energy dependence): $\mathcal{E}(\alpha,\delta) = \int A(\varphi,\theta,t) f_{livetime}(\varphi,\theta,t)\, dt$
- $A(\varphi,\theta,t)$ denotes the area that is exposed to the particular viewing direction at the time $t$.
- $f_{livetime}(\varphi,\theta,t)$ marks the fraction of livetime, i.e. the accumulated time during which the detector is actively taking data, from the given viewing direction and at the given time.
- The livetimes are therefore a function of the four dimensional space comprising the sky position and instrument angles

# Livetime calculation

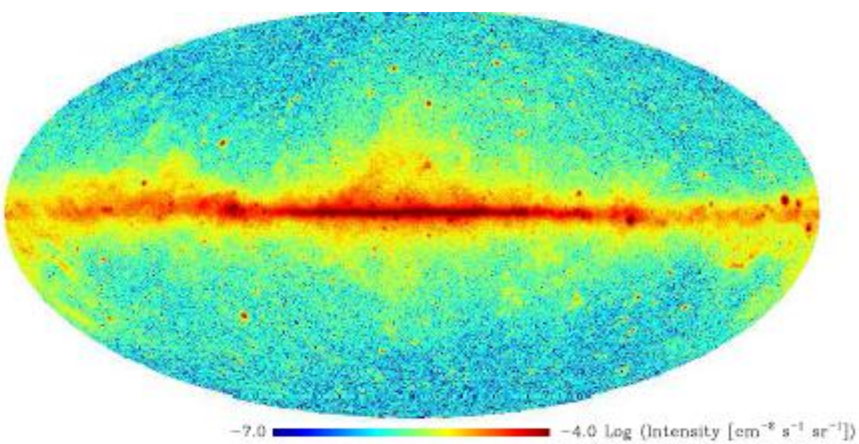
- As a practical matter, the livetime cannot be provided as a continuous function of inclination angles or position on the sky
- The livetime is evaluated on a grid on the sky and in inclination angles bins (livetime hypercube)
- To evaluate the livetime hypercube the detector pointing history is needed along with the time range and Good Time Interval (GTI) selections
  - any change in data selection affects the GTIs
- Since livetimes are additive, the livetime hypercube for a given epoch can be obtained by co-adding the livetime hypercubes for the subset of non-overlapping time ranges that it comprises
- Once the livetime is made, the exposure can be evaluated

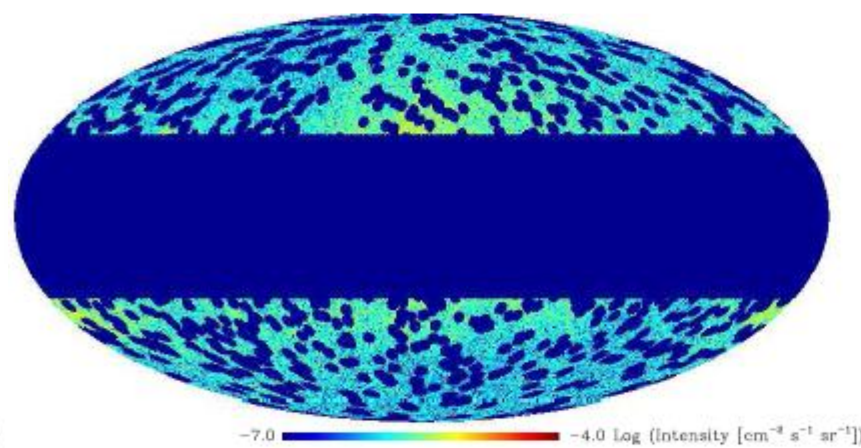
# Counts and intensity map

- The exposure and the counts map are made with finely-gridded sky pixel and energy bins,
- The intensity map can be then evaluated as $I(pix, E) = Counts(pix, E)/Exposure(pix, E)$
- The pixel size should be sufficiently small compared to the instrument's resolution to avoid any excessive, and pixel shape dependent, signal smoothing
- The finely-gridded energy bins of the intensity can be also summed to build maps covering larger energy bins

# Angular Power Spectrum

- Once the intensity map is built, the fluctuation map is evaluated as $f(\Omega_p) = \frac{I(\Omega_p) - \langle I \rangle}{\langle I \rangle}$
- The (raw) angular power spectrum can be then evaluated (outside of the mask, for a masked sky map)
  - anafast code belong the HEALPix packages can be used
  - $l_{max} = (2 \div 3)\, N_{side}$
  - In case the masked sky map pixel should be also provided



# Partial sky map coverage

- When a fraction of the sky is masked, the measured spherical harmonics coefficients are related to the true, underlying spherical harmonics coefficients, $(a_{lm})^{true}$, via the so called coupling matrix
- anafast returns a raw angular power spectrum $(C_l)^{raw}$ that is related to the true one as $(C_l)^{raw} = (C_l)^{true} f_{sky}$, whre $f_{sky}$ is the portion of unmasked sky used in the analysis
  - The Poisson noise for masked sky is $C_N = \frac{4\pi}{N} f_{sky}$

# Pixel and Beam Window Functions

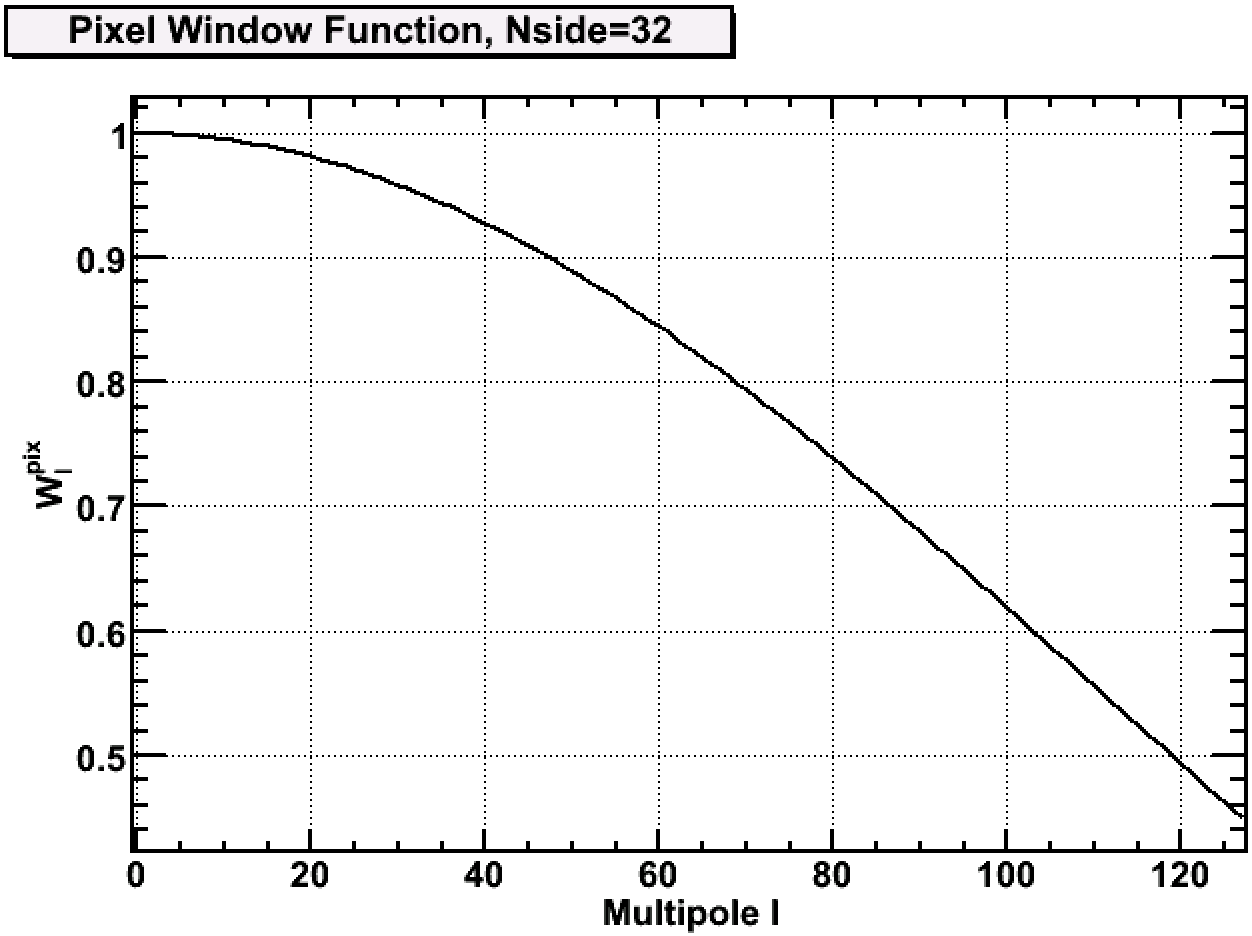
- The angular power spectrum calculated from a map is affected by the point spread function (PSF) (i.e. detector angular resolution) of the instrument and the pixelization of the map (i.e. size of the pixel), encoded in the beam window function $W_l^{beam}$ and the pixel window function $W_l^{pix}$ respectively, both of which can lead to a multipole-dependent suppression of angular power that becomes stronger at larger multipoles
  - $W_l^{pix}$ is provided with the HEALPix package for $l \leq 4N_{side}$ for each resolution parameter $N_{side}$
- In case of Gaussian angular resolution, σ, the beam window function is

$$W_l^{beam} = W_l^{pix} e^{-\frac{l(l+1)\sigma^2}{2}}$$

- Depending upon whether the power spectrum originates from signal or noise, corrections for the beam and pixel window functions must be applied to the measurement differently
- If we assume all the pixels to be identical, the power spectrum of the signal, $C_l^{signal}$, is related to the raw angular power spectrum $C_l^{raw}$, by

$$C_l^{signal} = \frac{\frac{C_l^{raw}}{f_{sky}} - C_N}{(W_l^{beam})^2} \text{ where } C_N = \frac{4\pi}{N} f_{sky}$$

# Example of pixel window function



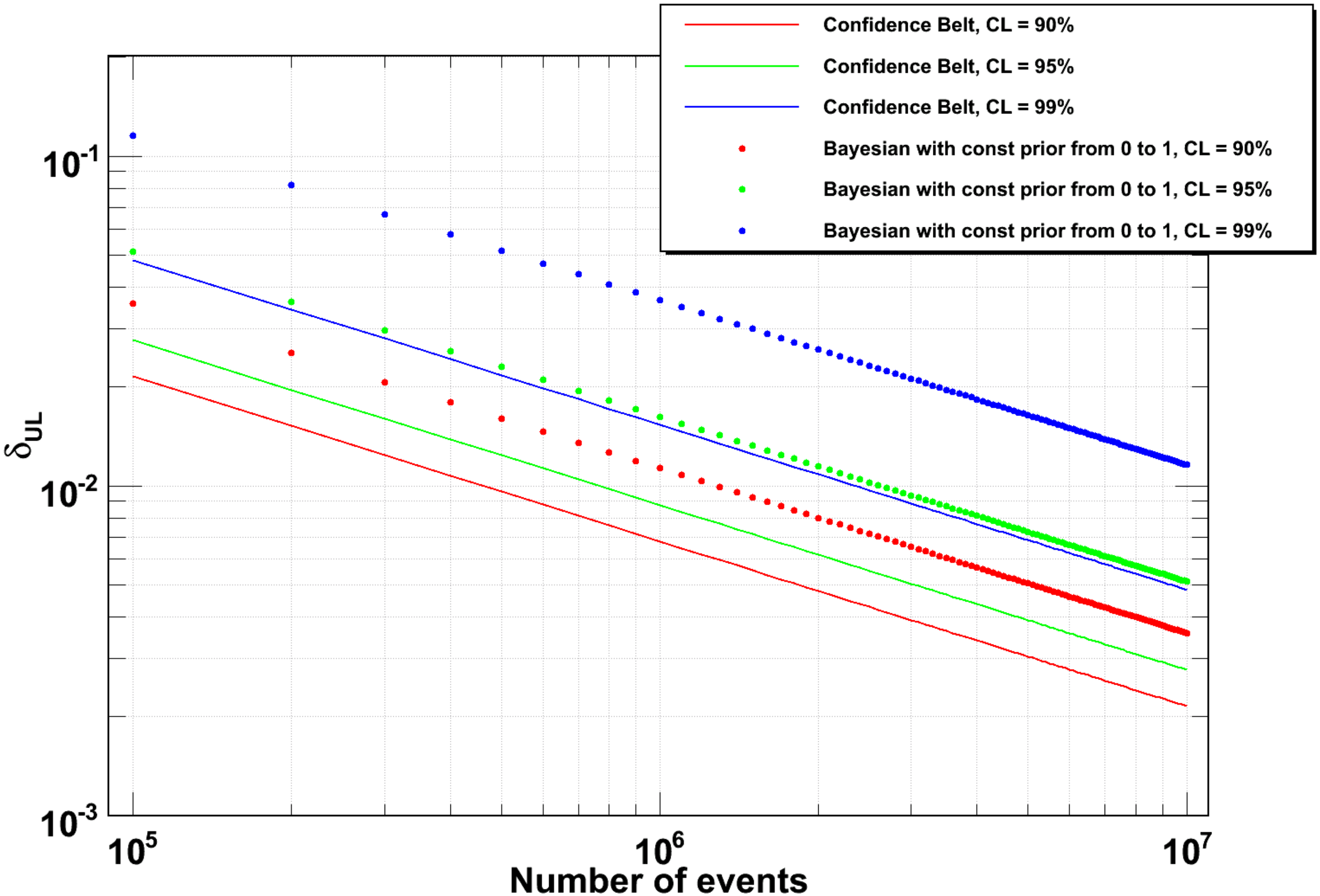
# UL on dipole from APS

- From fluctuation maps is possible to evaluate the amplitude of dipole anisotropy as $\delta = \frac{I_1}{I_o} = 3\sqrt{\frac{C_1}{4\pi}}$ where $C_1$ is the angular power spectrum for $l=1$
  - In case of Poisson noise $\delta = 3\sqrt{\frac{1}{N}}$ (i.e. sensitivity)
- To set an upper limit on δ we start by calculating the probability distribution function (pdf) of $\hat{\delta} = 3\sqrt{\frac{\hat{C}_1}{4\pi}}$ by a change of variable on the probability distribution function of $\hat{C}_1$ ($\chi_3^2$ centered on $C_1$)
- The probability density function to observe $\hat{C}_1$ given the true dipole power $C_1$ is $P\left(\hat{C}_1 \middle| C_1\right) = \frac{3\sqrt{3}}{\sqrt{2\pi}C_1}\sqrt{\frac{\hat{C}_1}{C_1}}\, e^{-\frac{3\hat{C}_1}{2C_1}}$
- With a change of variables we obtain $P\left(\hat{\delta} \middle| \delta\right) = \frac{3\sqrt{6}}{\sqrt{\pi}}\frac{\hat{\delta}^2}{\delta^3}\, e^{-\frac{3\hat{\delta}^2}{2\delta^2}}$

# UL on dipole from APS (cont'd)

- The upper limit of the true anisotropy $\delta_{UL}$ is the value of for which the integrated probability of measuring a value of $\hat{\delta}$ at least as large as the one we measured is equal to the confidence level, i.e. $\int_0^{\delta_{UL}} P(\delta|\hat{\delta})d\delta = CL$, where $P(\delta|\hat{\delta}) = \frac{P(\hat{\delta}|\delta)P(\delta)}{\int P(\hat{\delta}|\delta)P(\delta)d\delta}$
- Assuming $P(\delta)=1$ *for* $0\leq\delta\leq 1$ we get $\delta_{UL} = \frac{1}{\sqrt{1-\frac{2}{3}\frac{\ln(CL)}{\hat{\delta}^2}}}$
- On the other hand, it is possible to evaluate the upper limit by using frequentist confidence interval (confidence belt) by solving $\int_0^{\hat{\delta}} P(\hat{\delta}|\delta)d\hat{\delta} = 1 - CL$

# UL on dipole from APS due to Poisson noise



# No-anisotropy map: direct integration method

- The number of (background) events expected to fall within a sky map pixel with right ascension $\alpha$ and declination $\delta$, (to keep the notations simple, we will omit the energy dependence), is given by: $N(pix) = \int dt \int d\Omega\, R(\varphi,\theta,t)\, \varepsilon(t)$ where
  - $R(\varphi,\theta,t)$ is the (background) event rate per solid angle as a function of the instrument zenith angle, $\theta$, azimuthal angle, $\varphi$, and time, t
  - $\varepsilon(t)=1$ when $\varphi,\theta$ and t are such that they fall within the sky pixel, and $\varepsilon(t)=0$ otherwise
- $R(\varphi,\theta,t)$ is evaluated from the distribution of detected events, with the assumption that events from a possible anisotropy signal represent, at most, a minor perturbation to this distribution
- The local-angle dependence and the time dependence of R are assumed to be independent: $R(\varphi,\theta,t) = E(\varphi,\theta)\mathcal{R}(t)$
- This recognizes the fact that the overall event rate, $\mathcal{R}(t)$, changes often but that $E(\varphi,\theta)$ is nearly constant except when changes in the detector configuration are made.

# Direct integration method (cont'd)

- The main idea of this method is to first extract from the data set the values of $\mathcal{R}(t)$ and $E(\varphi,\theta)$, and then construct the associated no-anisotropy sky map.
- The presence of any anisotropies in the data would create transient fluctuations in the instantaneous values of these functions, as these anisotropies passed through the instrument field of view.
- However, these anisotropies would have no effect on the longer-term average values of these functions, since any transient fluctuations would be averaged out
  - In this application, a constant-over-time $E(\varphi,\theta)$ is used
- Similarly, any temporal variations of the effective area of the detector would create fluctuations of the instantaneous value and the longer-term averages of $\mathcal{R}(t)$, depending on the time scales of the effective-area variations
  - The all-sky rate exhibited fluctuations on multiple time scales caused by varying background rates occurring for instance as the spacecraft was moving through regions of different geomagnetic coordinates and by changes in the instrument's hardware settings.

# Direct integration method (cont'd)

- Any such instrumental effects affecting the all-sky rate could be evstimated, and given an averaged over multiple sky observation value of the all-sky rate, the all-sky rate at some shorter duration segment could be accurately predicted.
- These parameterizations are necessary because the direct-integration method constructs the no-anisotropy-sky map incrementally, adding the results from observations of short enough duration that the instrument pointing can be assumed quasi-constant
- This choice has the benefit of automatically taking care of any temporal variations of the effective area, avoiding the need to apply any corrections and, most importantly, avoiding any systematic errors introduced by such corrections.
- It should be noted however that not performing an averaging in the all-sky rate weakens the power of this method to smear out the presence of any anisotropies in the data, with the result of any stronger anisotropies possibly leaking in the no-anisotropy sky map.

# No-anisotropy map: shuffling method

- One way of generating the no-anisotropy map is to randomize the reconstructed directions of the detected events
- A set of isotropic simulated events can be built by randomly coupling the times and the directions of real events in local instrument coordinates.
- The randomization is performed starting with the position of a given event in the instrument frame and exchanging it with the direction of another event, which was selected randomly from the data set with a uniform probability.
- Starting with this information, the sky direction is re-evaluated for the simulated (random) event.
- In case the direction distribution of the CR flux is perfectly isotropic, a time-independent intensity should be detected when looking at any given detector direction.
- Possible time variation of the intensity would be due only to changes in the operating conditions of the instrument.

# Shuffling method (cont'd)

- The random coupling preserves the exposure and the total number of events.
- This process is repeated multiple times, with each time producing a sky map that is compatible with an isotropic CR direction distribution.
- The final no-anisotropy sky map is produced by taking the average of these sky maps.
- By this construction, the simulated data set preserves exactly the energy and angular (with respect to the instrument reference frame) distributions, and also accounts for the detector dead times.
- To minimize the possible effects of a varying CR event rate due to any changes in the detector configuration, the data set can be first split into sub-segments, and the technique is applied to each of these sets separately.

# Remarks on no-anisotropy map from real events

- Since the real events are used to estimate the no-anisotropy map for the Direct integration and Shuffling method, this map could be an overestimate of the “true” isotropic map if a signal is present
- This leads to an underestimation of the signal strength, in particular for large angular scale anisotropy
- The coordinate system has to fixed a priori for the background map evaluated with Direct Integration method, while the Shuffling Method allows to evaluate the background map in all systems at the same time
- As general comment: a cross-check among different analysis is appreciated!

# Significance: Direct bin-to-bin comparison

- Once the real and the no-anisotropy maps are made, a simple direct bin-to-bin (i.e. pixel to pixel) comparison of the two maps can be performed to search for statistically significant deviations between the number of actually detected and the number of expected events under the assumption of isotropy is performed.
  – The comparison is also performed between the contents of the integrated actual and no-anisotropy sky maps

# Calculating the significance of an excess/deficit

- Once the best estimate has been obtained for the number of background events in a given pixel, $n_b = n_b(pix)$, there remains the question of the best way to estimate the statistical significance of any excess or deficit.
- For the case of the direct-integration technique, since its produced no-anisotropy sky map is the result of a direct calculation, there were no associated statistical errors and the significance can be calculated using simple Poisson probabilities with known value of the background, i.e. $\mathrm{P}(\geq n_r) = \sum_{k=n_r}^{\infty} \frac{e^{-b} b^k}{k!}$, where $b = n_b$ and $n_r$ is the number of events in the real map for that pixel, i.e. $n_r = n_r(pix)$
  - For large values of $n_b$ and $n_r$ the Gaussian pdf can be used instead of Poisson one

# Li and Ma significance

- For the case of the shuffling method, the statistical errors involved in construction of the no-anisotropy sky maps should be taken into account
- The method suggested by Li and Ma, Astrophys. J. 272, 317 (1983), is widely adopted.
- This prescription involves the determination of the likelihood functions for the null hypothesis $H_0$ i.e. no signal ($n_s=0$) and all observed events are due to background, and for the signal hypothesis $H_1$, i.e. $n_s \neq 0$
- The significance of the excess is obtained from the maximum likelihood ratio λ of these likelihood functions, $\lambda = \frac{L(X|H_0)}{L(X|H_1,)}$
- In this case the observed data $X=\{n_r, n_b\}$, estimated unknown parameter $\Theta=\{n_s, b\}$
- In case of large number of events $-2ln\lambda$ follows a $\chi^2$ distribution with one degree of freedom

# Li and Ma significance (cont'd)

- For $H_1$, the maximum likelihood estimates $b=\alpha n_b$ and $n_s=n_r- \alpha n_b$
- For $H_0$, $n_s=0$ and $b = \frac{\alpha}{1+\alpha}(n_r + n_b)$
- $\alpha$ is the ratio of the source signal region and the background region, i.e. $\alpha$=1/number of random maps evaluated with the shuffling method
- The significance S (in sigma units) is evaluated as $S = \sqrt{-2ln\lambda}$ while the probability $p$ that the significance is not less than S is produced by the backgrund can be evaluated by the Normal pdf.
- Then the probability that a real excess exists, that is the confidence level CL, is $CL=1-p$

# Significance with small numbers

- In case of small number of events, the significance can be evaluated from a Poisson pdf including the fluctuations of both observed and background events
- This involves a calculation of the probability of observing $n_r$ events or more, given each possible fluctuation in the total number of observed background events, $n_b$
- Assuming that $n_b \sim Poisson(b), P(n_b|b) = \frac{e^{-b} b^{n_b}}{n_b!}$ and $n_s \sim Poisson(\alpha b), P(n_s|\alpha b) = \frac{e^{-\alpha b} (\alpha b)^{n_s}}{n_s!}$
- $\mathrm{P}(\geq n_r | \, n_b, \alpha) = \sum_{n_s=n_r}^{\infty} \int_0^{\infty} db \, P(n_b|b) \, P(n_s|\alpha b) = \sum_{n_s=n_r}^{\infty} \frac{\alpha^{n_s}}{(1+\alpha)^{n_b+n_s+1}} \frac{(n_b+n_s)!}{n_b! \, n_s!} = 1 - \sum_{n_s=0}^{n_r-1} \frac{\alpha^{n_s}}{(1+\alpha)^{n_b+n_s+1}} \frac{(n_b+n_s)!}{n_b! \, n_s!}$
- Once the probability P is evaluated, the significance in sigma units can be evaluated from a Normal pdf, i.e. $S=\Phi^{-1}(1-P)$ where $\Phi$ is the cumulative distribution of the Normal and $\Phi^{-1}$is it inverse (quantile) function.

# Independent Trials

- Sometime many observation could be made to search any excess from a given region in the sky
- In this case the significance $S$ after $T$ attempts (trails) being made, the confidence level is not only dependent on its significance, but also on the total number of trials
- In this context the a priori significance $S_{pre}$ and the posterior significance $S_{post}$ are introduced to take into account the number of trials
- Consider an outcome X that occurs with probability $P_{pre}(X)$ in a single trial.
- Then the probability of producing $t$ such events is $C_T^t P_{pre}^t (1 - P_{pre})^{T-t}$, where $C_T^t$ is the binomial coefficient.
- The probability that none of such events (t=0) is produced in all $T$ independent trials is $(1 - P_{pre})^T$
- Consequently, the confidence level that a real excess is observed in $T$ uncorrelated trials is $P_{post} = 1 - (1 - P_{pre})^T$

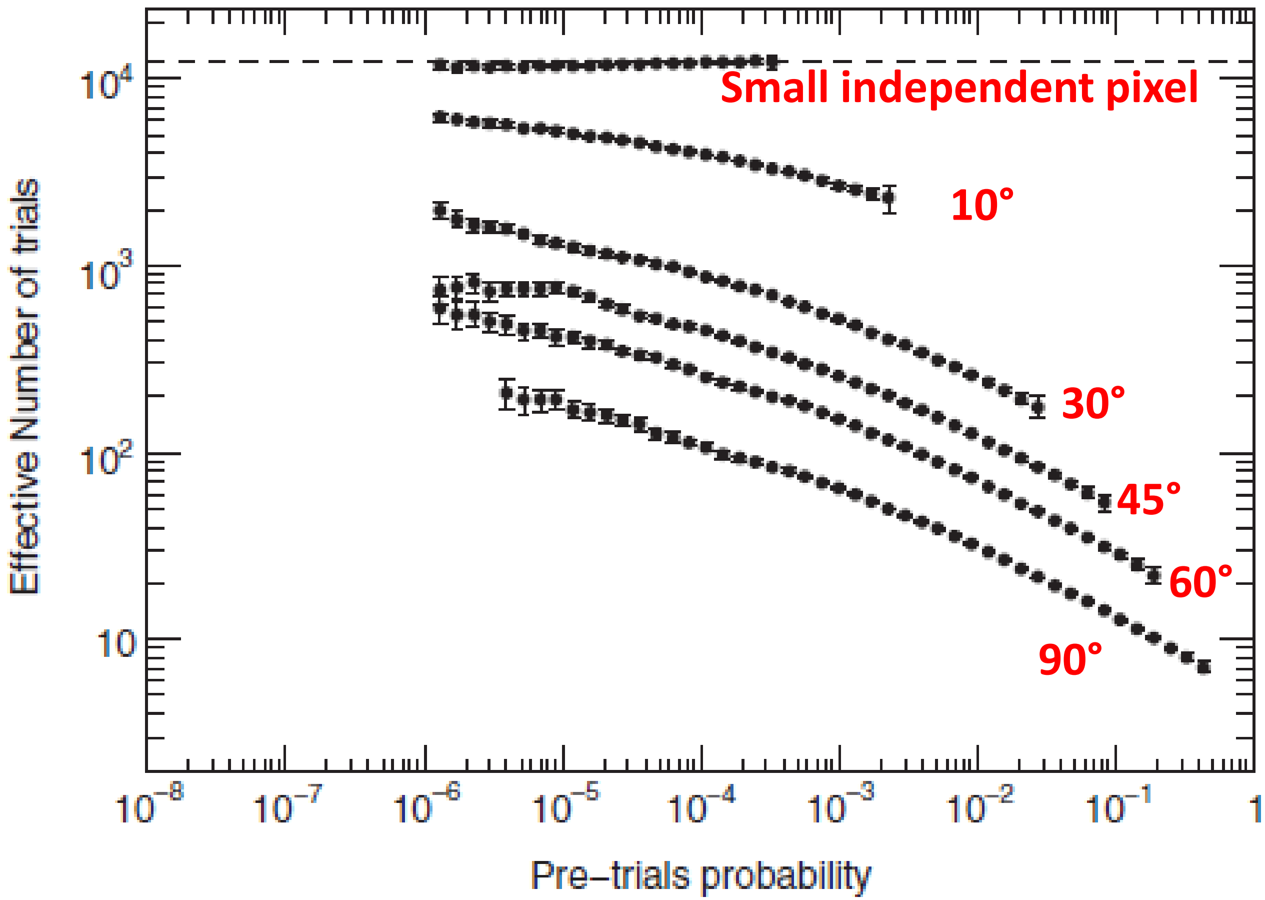
# Effective Trials

- Sometime the whole sky was searched for anisotropies with no a priori assumptions, that is from any direction in the whole sky
- As such, it involves a large number of trials (independent tests), that have to be accounted for when judging the statistical significance of its results.
- For a whole-sky search performed using independent bin sky maps, the number of trials is equal to the number of pixels in the sky map.
- On the other hand, searches that use correlated-bin sky maps, such as this one, involve a number of effective trials that is in general smaller than the number of bins in one such sky map.
- For correlated trials $T$ is replaced by its effective number $T_{eff}$, i.e. $P_{post} = 1 - (1 - P_{pre})^{T_{eff}}$

# Effective Trials (cont'd)

- The number of effective trials involved in evaluating the contents of an integrated map can be evaluated using a Monte Carlo simulation
- Fake significance maps were built corresponding to various integration radii and to a perfectly isotropic signal.
- By counting the fraction of such sky maps $P_{post}$ that contained at least one bin with a probability smaller than some value $P_{pre}$, the effective number of trials involved in the search of sky maps of some integration radius is calculated as
$$T_{eff} = \frac{\log(1-P_{post})}{\log(1-P_{pre})}$$
- Using this effective number of trials, the post-trials probability $\hat{P}_{post}$ corresponding to an measured pre-trials $\hat{P}_{pre}$ probability can be calculated as:
$$\hat{P}_{post} = 1 - (1 - \hat{P}_{pre})^{T_{eff}(\hat{P}_{pre})}$$

# Example of Number of effective trials




**HEALPix pixelization scheme with 12288 pixels ( ≈3deg$^2$) used for the sky maps**

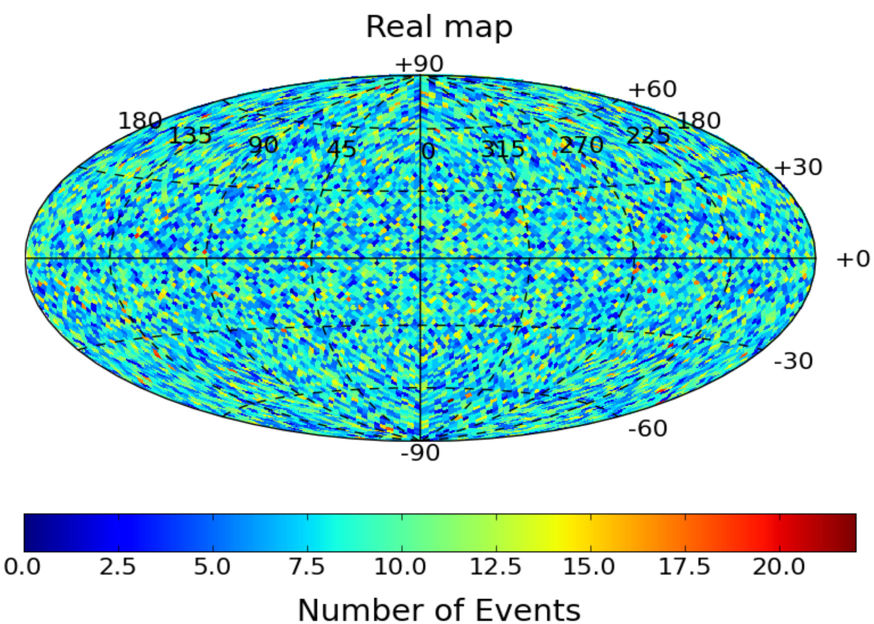
# Fractional excess

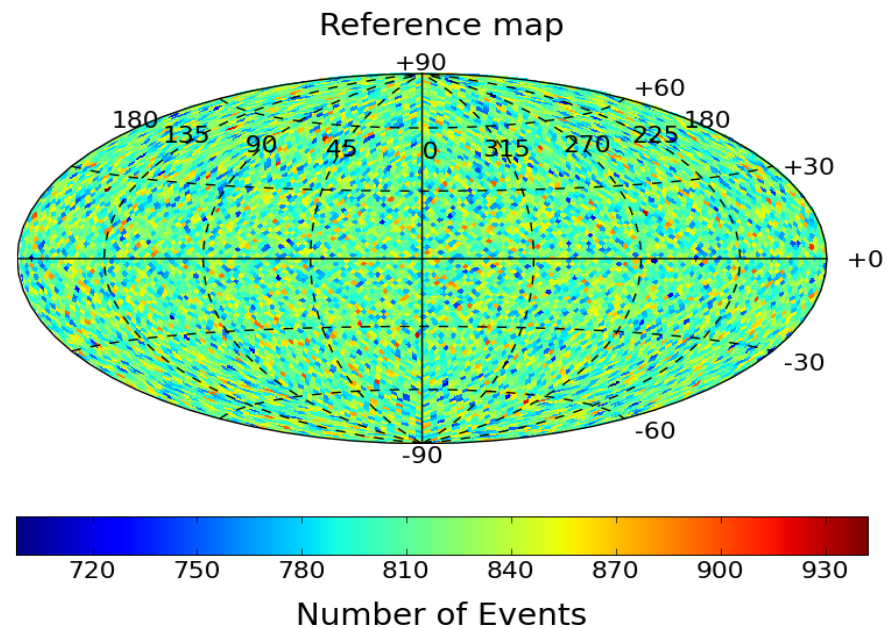
- The fractional excess (aka fluctuations) is defined as the number of excess anisotropy events from a given pixel (even from a circular region) in the sky over the number of events expected to be detected from the same pixel (region) if the sky was perfectly isotropic, minus 1
- To evaluate the sensitivity, that is the fractional excess needed to detect an anisotropy with a given post-trials significance, the UL on the number of events at that significance can be used starting from the number events expected  for the isotropic sky

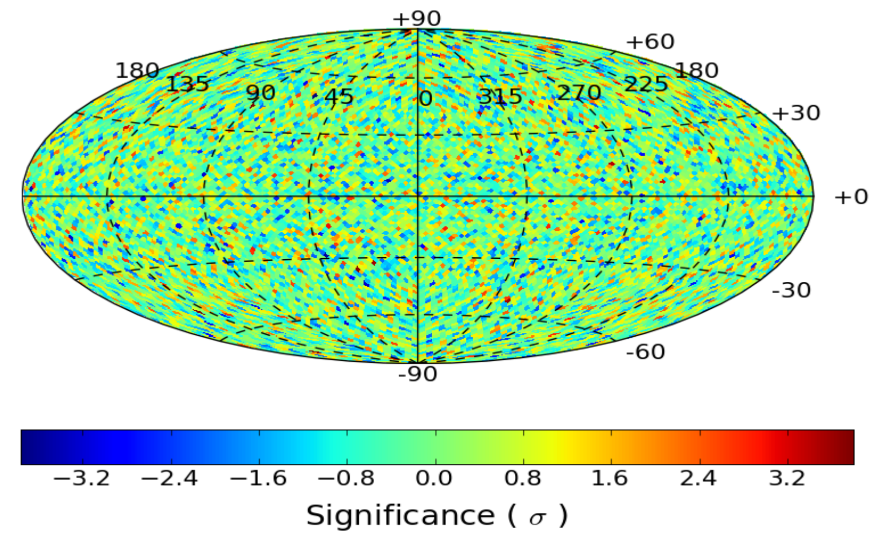
# Isotropic simulation

- A simple Monte Carlo can be performed to simulate an isotropic distribution on the sphere without any instrument
  - It can be performed to produce random $(\varphi,\theta)$ angles as $\varphi=2\pi r_1$ and $\theta=\mathrm{acos}(2r_2-1)$, where $r_1$ and $r_2$ are random numbers from uniform distribution in (0, 1)
  - Alternatively, once a pixelization is set with HEALPix it is enough to generate uniform integer number in $[0, N_{pix}-1]$
- A map (real) is made with integrated number of events from 600k to 3.6M with 6 steps with 600k each (600k ➔ 6 months)
- A reference (smoothed) map is made with x100 statistic

# Iostropic simulation maps

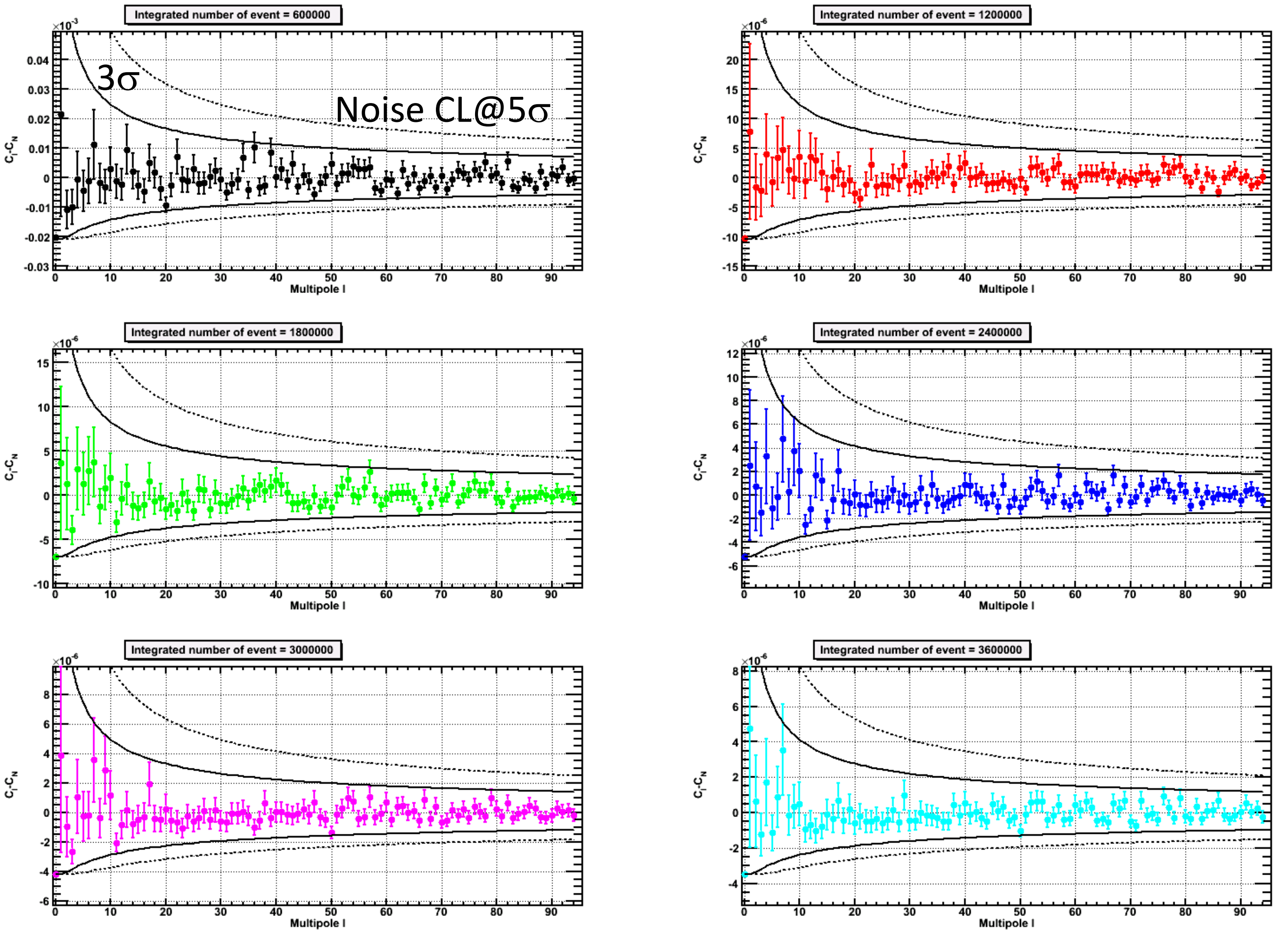


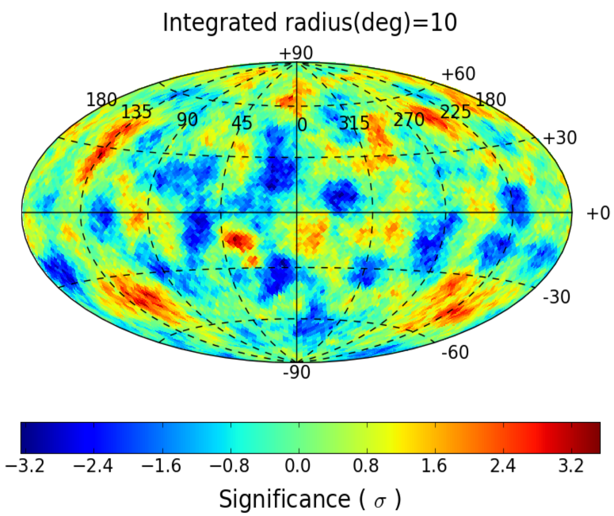



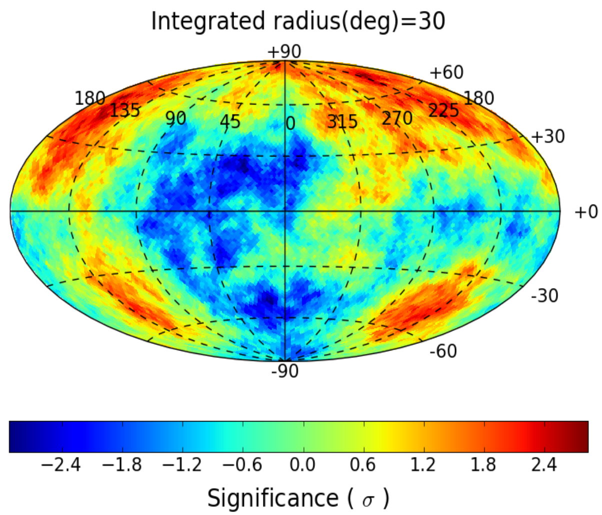

**HEALPix pixelization scheme with 12288 pixels (≈3deg$^2$) used for the skymaps**

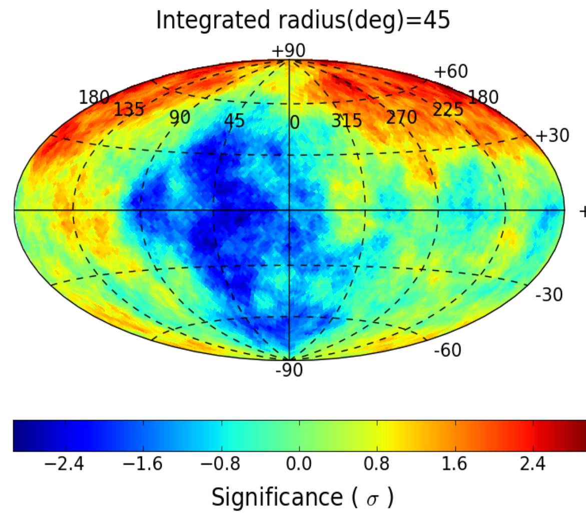
# Isotropic simulation APS (raw $C_l$)

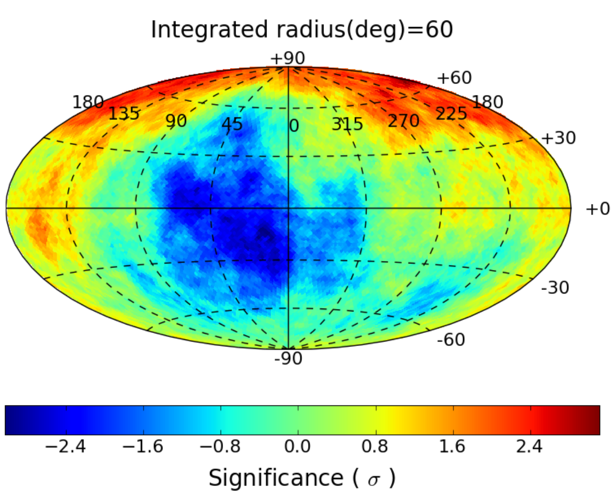


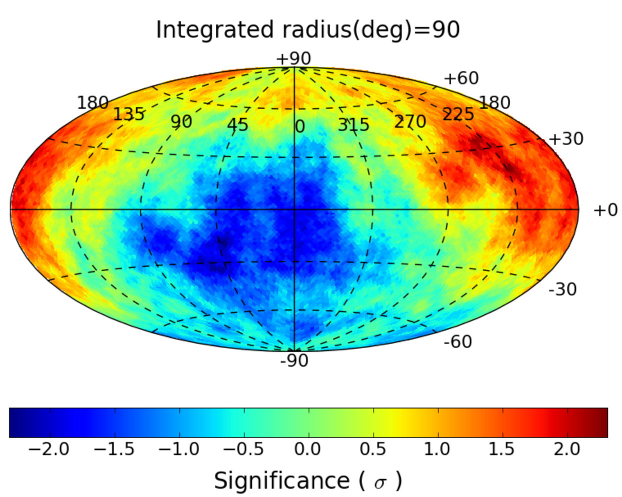
# Isotropic simulation: integrated maps



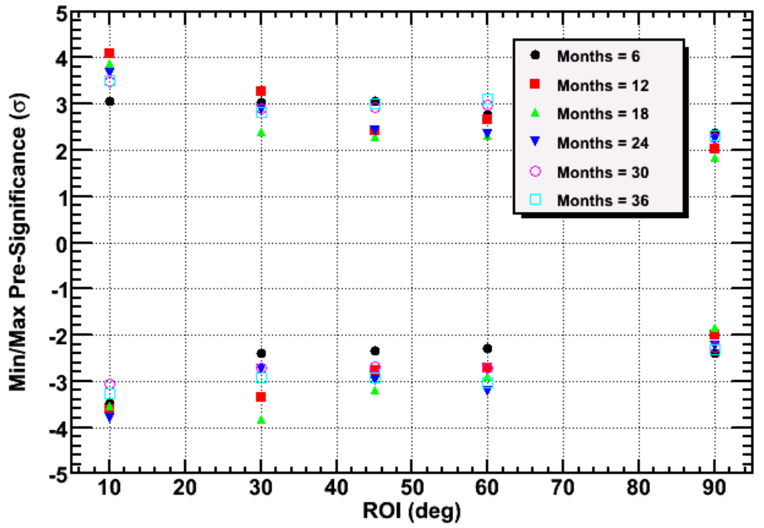



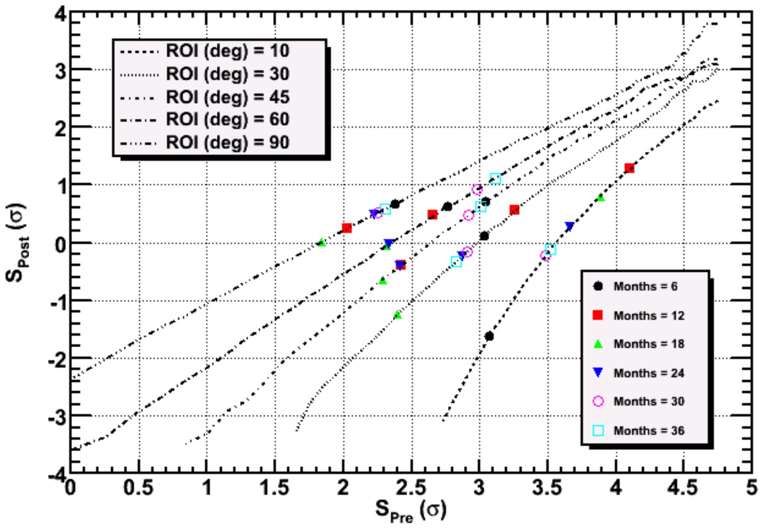



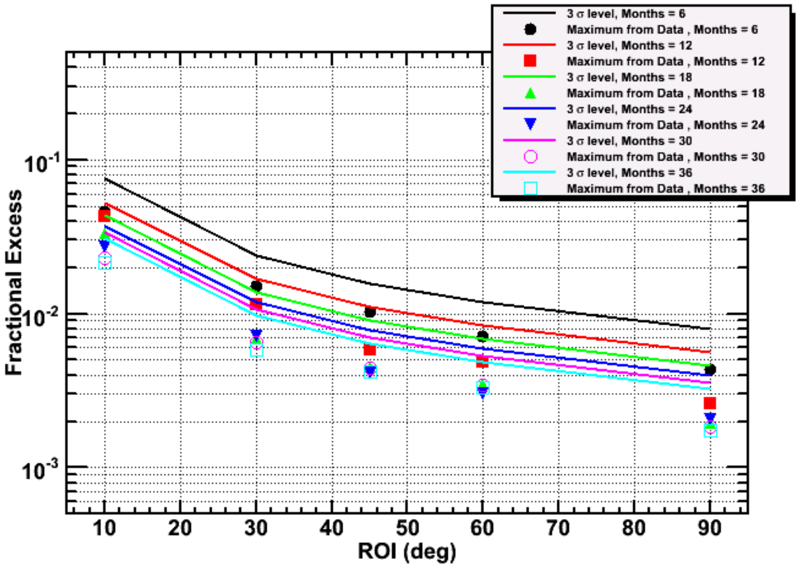



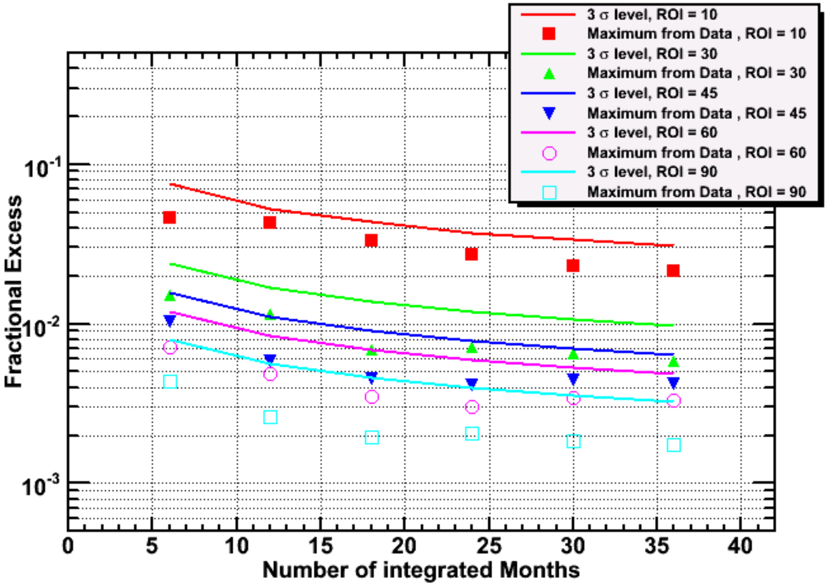

**HEALPix pixelization scheme with 12288 pixels,**

# Isotropic simulation: some results



# Dipole simulation

- While the probability per solid angle is constant for the isotropic distribution, the dipole distribution depends on the direction for any anisotropic distribution.
- By choosing the dipole vector to be parallel to the Z-axis (i.e. perpendicular to the equatorial plane of the coordinate system), the probability distribution is then given by:

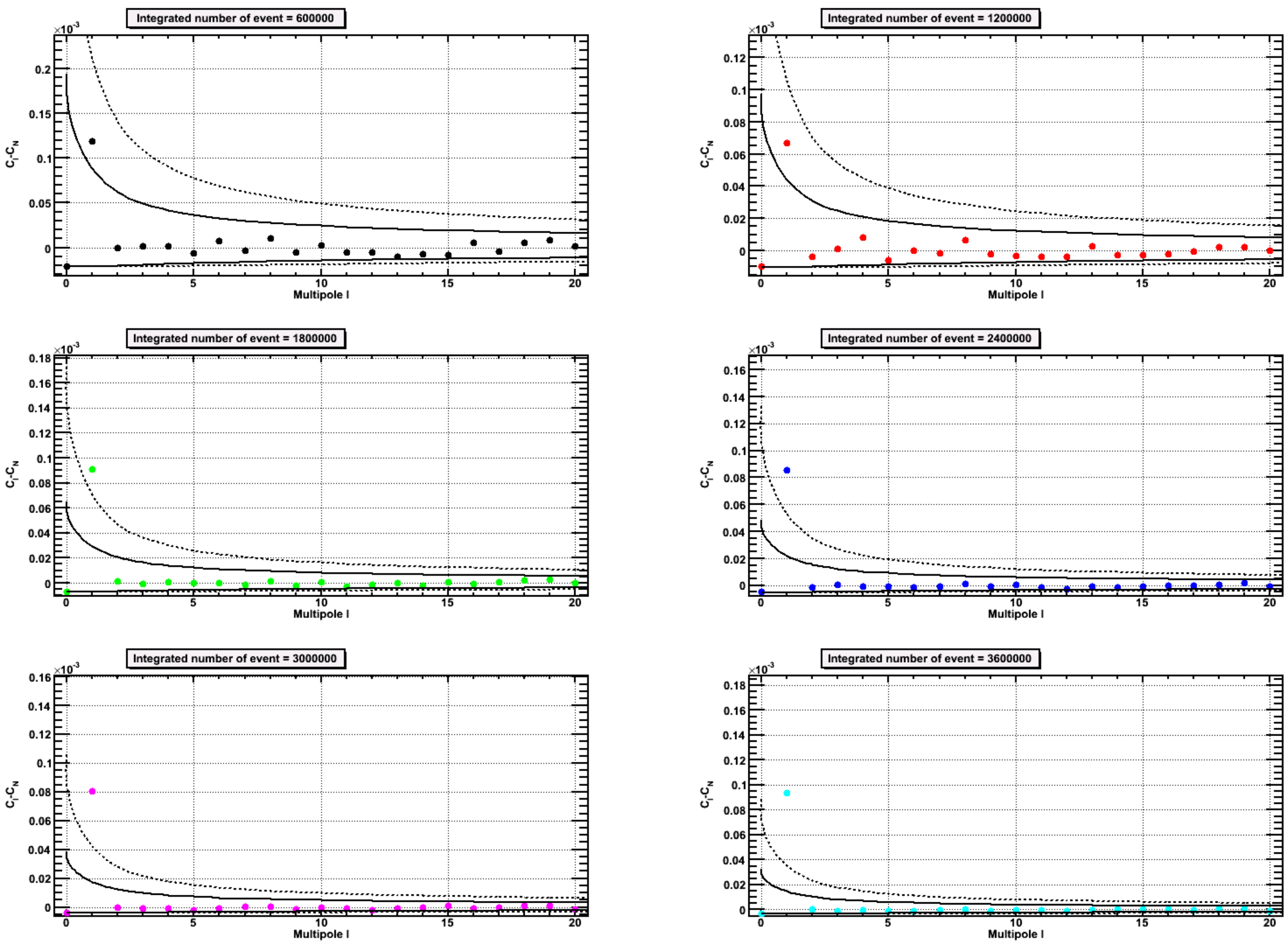
$$p(\Omega)d\Omega = \frac{1}{4\pi}(1 + \delta\cos(\theta))\, d\varphi\, dcos(\theta) =$$
$$= (p(\varphi)d\varphi)\,(p(\cos(\theta))\, dcos(\theta))$$

- Where $p(\varphi) = \frac{1}{2\pi}$ and $p(\cos(\theta)) = \frac{1}{2}(1 + \delta\cos(\theta))$
- The dipole simulation can be performed to produce random ($\varphi$,cos($\theta$)) (azimuth and polar angles) as $\varphi=2\pi r_1$ and

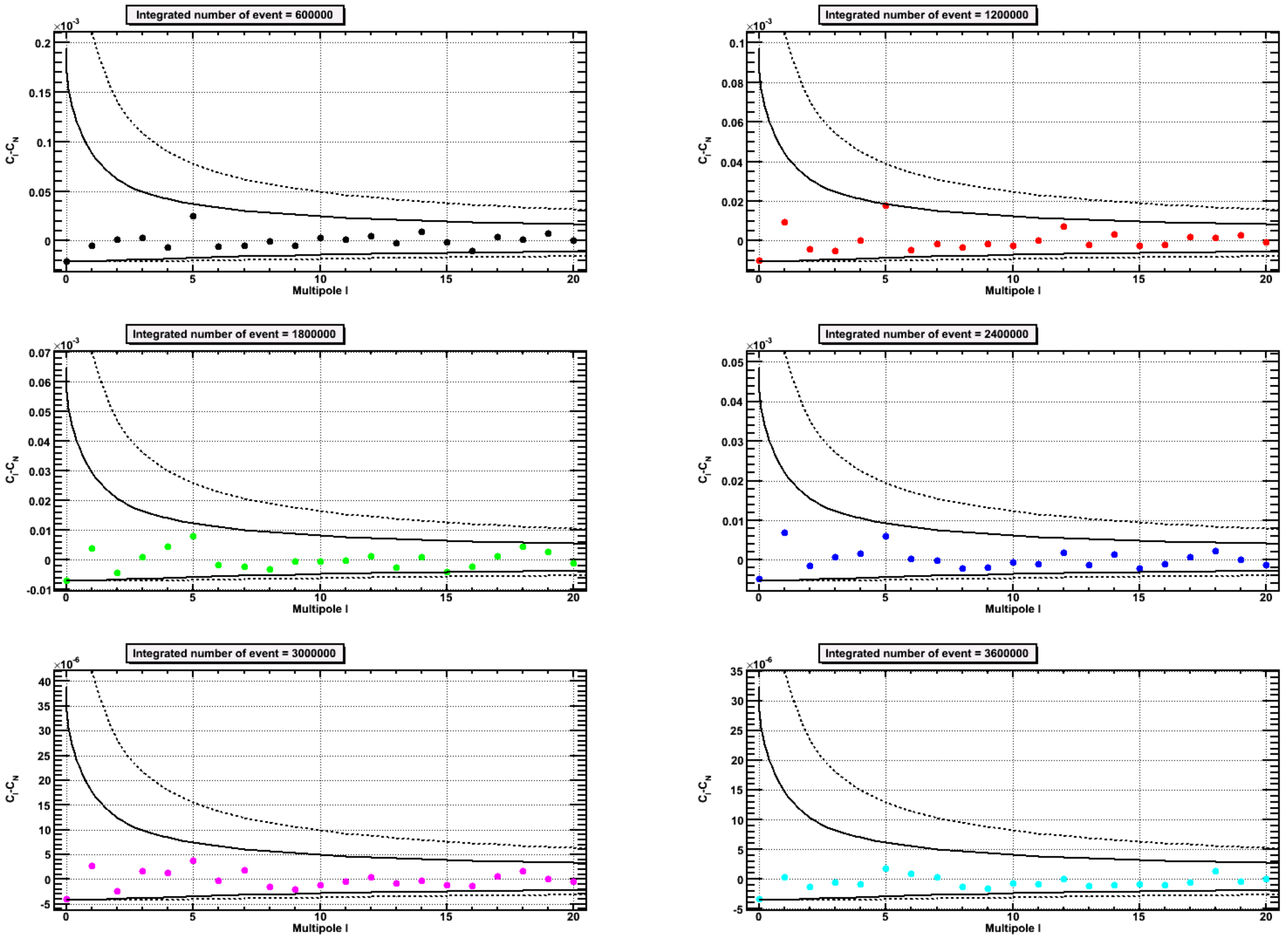
$\cos(\theta) = \frac{\sqrt{\delta^2+2\delta(2r_2-1)+1}-1}{\delta}$, where $r_1$ and $r_2$ are random numbers from uniform distribution in (0, 1)

- In order to generate dipoles pointing to an arbitrary direction (and different coordinate system), the coordinates obtained through the above mapping have to be rotated

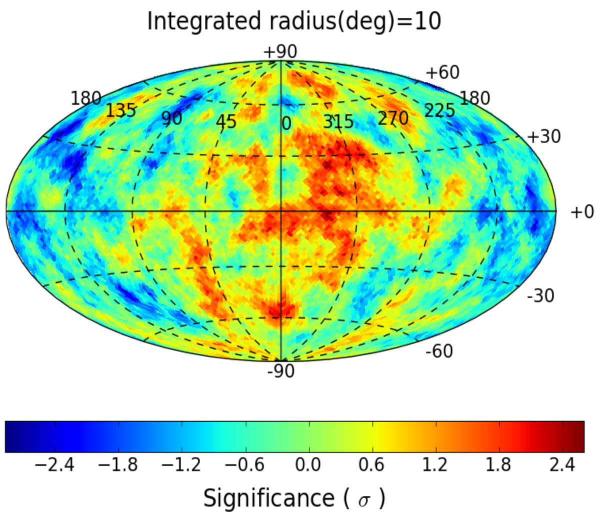
# Dipole anisotropy δ=0.01 APS

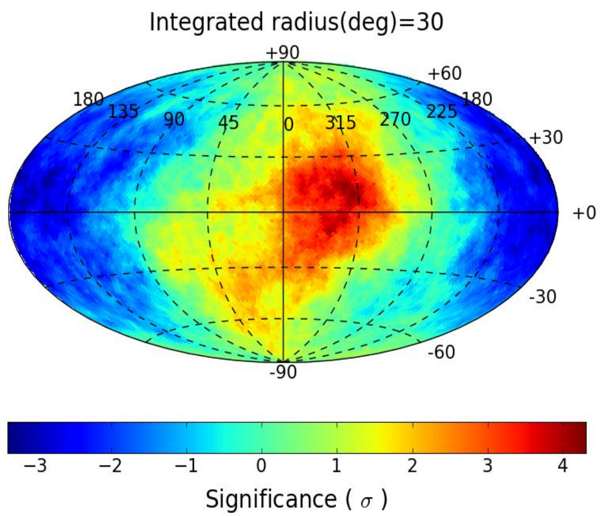


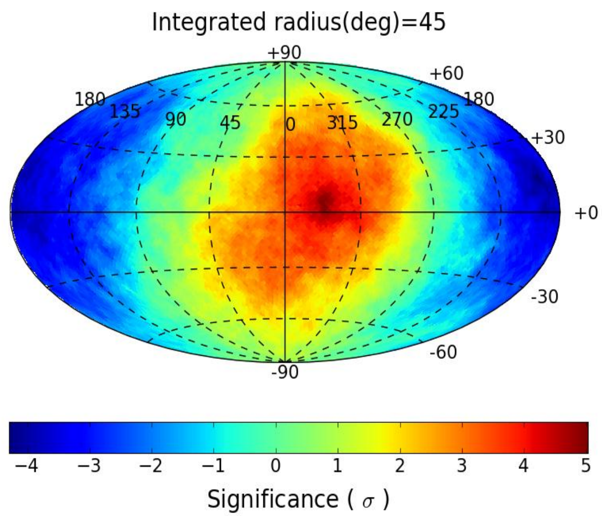
# Dipole anisotropy δ=0.001 APS

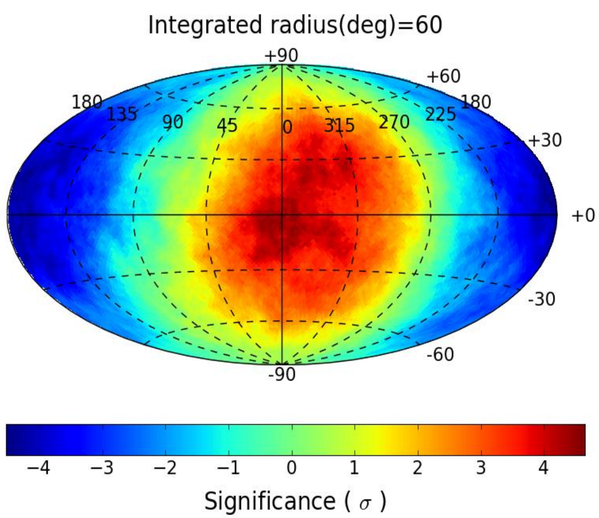


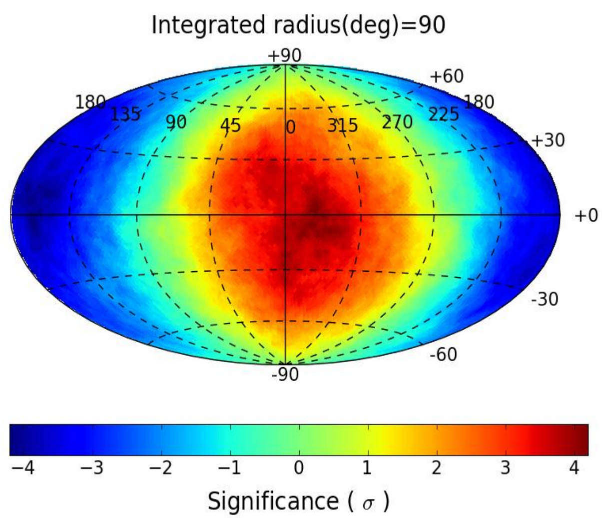
# Dipole δ=0.01 - Significance maps



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