A preliminary analysis for efficient laser wakefield acceleration in plasmas

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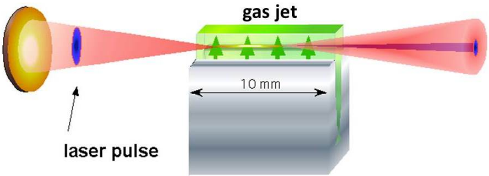
Introduction

Nowadays accelerators have paramount applications in research (particle physics, FEL, material science, biology, inertial fusion), medicine, industry, environmental remediation, cultural heritage study,...

Huge investments (e.g. the EU project *Eupraxia* [Assman et al '20]) are devoted to develop table-top ones based on new acceleration mechanisms of charged particles, e.g. those using laser-plasma interactions.

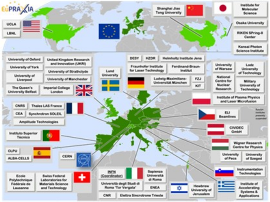
In the *Wake-Field Acceleration* (WFA) [Tajima, Dawson '79] ultrarelativistic electrons (e^-) accelerate (up to 1 GeV per cm in the *blowout regime* [Wang et al 2013]) "surfing" a plasma wave (PW) driven by a very short laser pulse (or charged particle beam), e.g. in a supersonic diluted gas jet.



The EuPRAXIA Consortia Today

Funded by the European Union

- **54 institutes** *(in addition > 6 asked to join us presently)*
- from **18 countries** plus CERN
- signed on one or several presently **active EuPRAXIA consortia**:
  - **ESFRI** consortium (funding in-kind)
  - **Preparatory Phase** consortium (funding EU, UK, Switzerland, in-kind)
  - **Doctoral Network** (funding EU, UK, in-kind)



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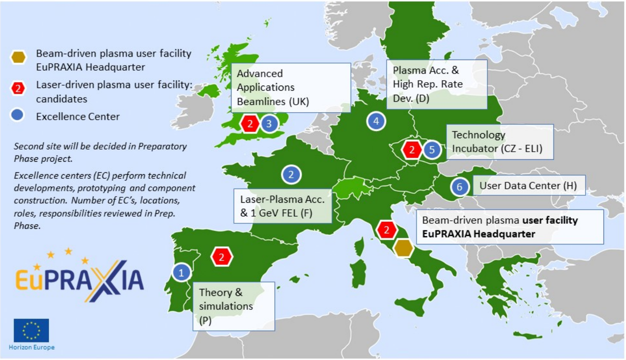


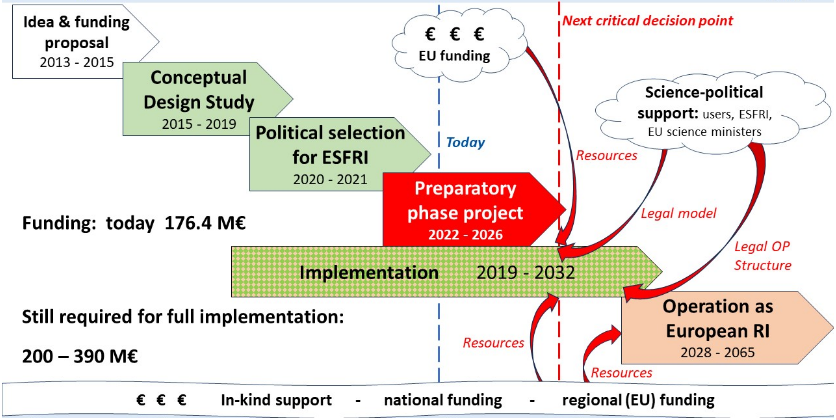
Today`s status

Excellence centers: **several** (6 – 10) assumed to be realized

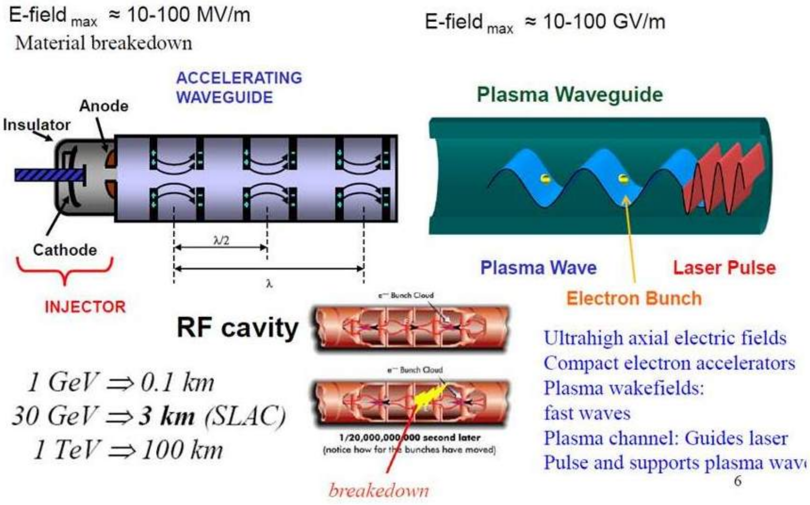
Second site: **one** to be selected

Connect with WP´s to Horizon Europe and national funding lines

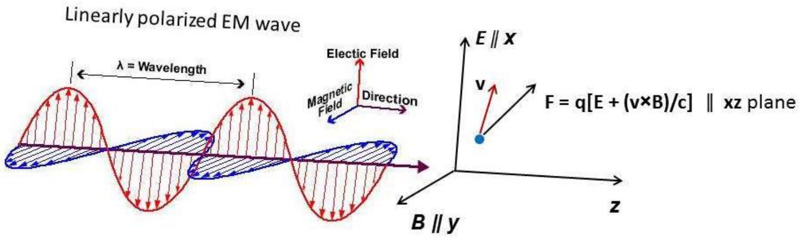
EuPRAXIA European Project: Next Step



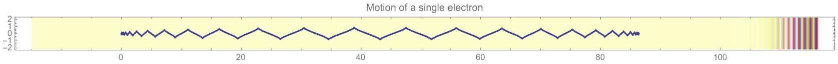
Classical vs. plasma wave accelerators



Laser pulse on a charged particle initially at rest in *vacuum*:



Oscillating $\mathbf{F}_e = q\mathbf{E}$ induces transverse oscillations with no average drift. Oscillating $\mathbf{F}_m = q\frac{\mathbf{v}}{c} \times \mathbf{B}$ is $\parallel \vec{z}$; its average on a cycle (*ponderomotive force*) $\mathbf{F}_p = \langle \mathbf{F}_m \rangle \neq \mathbf{0}$ causes a longitudinal drift forward. However



No net energy gain (**Lawson-Woodward theorem**), alas!

Pulse in *diluted plasma* displaces e^- w.r.t. ions; very intense $\Rightarrow$ huge E^z!

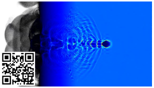
Δn_e arrange in a **plasma wave** (PW) traveling with phase velocity $\simeq c$; again the e^- remain in the plasma, in spite of huge accelerations (alas!). As water molecules in water waves.

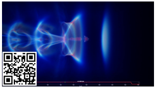
However, if some e^- are **injected** faster than their neighbours, they can increase their speed "surfing" a PW. As foam at the crest of water waves.

These e^- are finally expelled out of the plasma just *behind* the beam.

Plasma waves can be induced also by particle - rather than laser - beams.

Phenomena ruled by Maxwell eqs coupled to a kinetic theory for plasma e^-, ions; solvable via more & more powerful particle-in-cell (PIC) codes.





But simulations involve huge costs for each choice of the input data (ID).

Better: run PIC after a **preliminary selection of ID via simpler models** **= the subject of our research in this talk**.

Plan

Setup & Plane Hydrodynamic model

$\mathbf{v}_e(0,\mathbf{x})=\mathbf{0}$. **Input = nontrivial initial data (ID), i.e.:**

a) the function $\widetilde{n}_0(z) \geq 0$, with $\widetilde{n}_0(z)=0$ if $z<0$, $\widetilde{n}_0(z)\leq n_b\in\mathbb{R}^+$ if $z>0$, yielding the initial electron (e^-) and proton densities n_e, n_p:

$$n_e(0,\mathbf{x})=n_p(0,\mathbf{x})=\widetilde{n}_0(z); \tag{1}$$

b) the vector-valued function $\boldsymbol{\epsilon}^\perp(\xi)$ yielding the initial laser-pulse EM fields:

$$\mathbf{E}(t,\mathbf{x}) = \mathbf{E}^\perp(t,\mathbf{x}) = \boldsymbol{\epsilon}^\perp(ct-z), \qquad \mathbf{B} = \mathbf{B}^\perp = \mathbf{k}\times\mathbf{E}^\perp \qquad \text{if } t\leq 0, \tag{2}$$

support$(\boldsymbol{\epsilon}^\perp) \subseteq [0,l]$ with $l \lesssim \sqrt{\pi mc^2/n_b e^2}$: the pulse reaches the plasma at $t=0$ & overshoots all e^- before their z reach the 1^{st} minimum<0 (ES pulse).

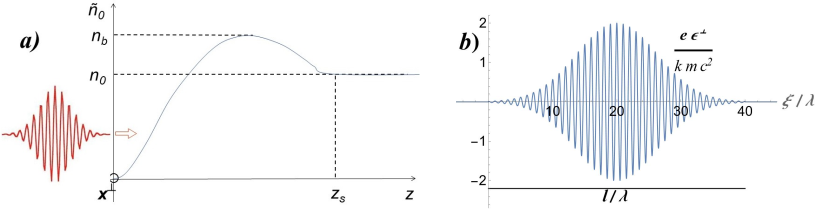
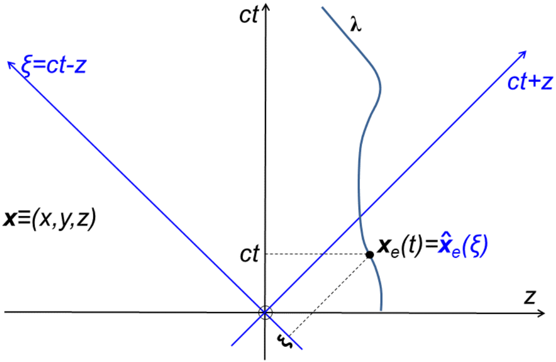
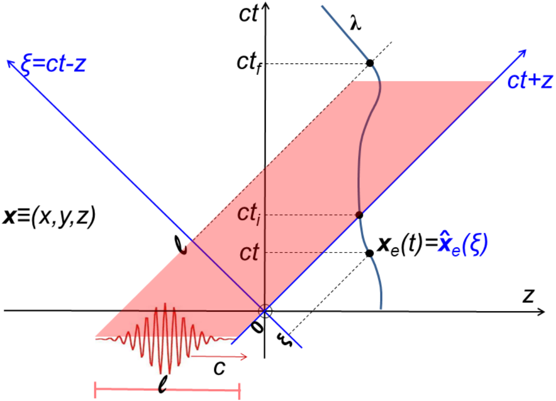


Figure 1: Here $\widetilde{n}_0(z)$ with a down-ramp + plateau as a), ES, SMM pulse as b)

How to to simplify $\dot{\mathbf{p}}(t) = q\epsilon^{\perp}[ct - z(t)] + q\mathbf{v}(t)/c \times \{\mathbf{k} \times \epsilon^{\perp}[ct - z(t)]\}$?



As every particle travels slower than light, $\tilde{\xi}(t) = ct - z(t)$ grows strictly, and $\xi = ct - z$ can replace t as the independent parameter along its worldline (WL) λ (in Minkowski space) and in its equation of motion [GF 2016].



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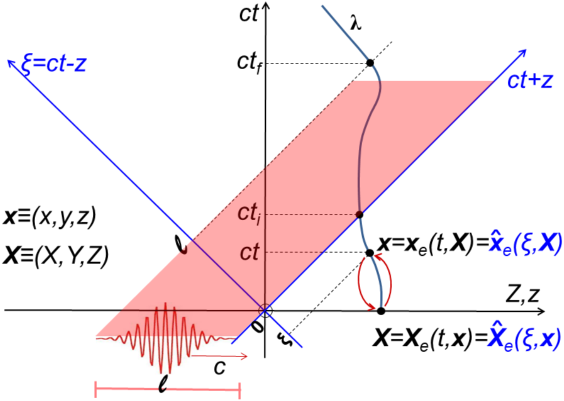

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Fluid: $\mathbf{X} \mapsto \mathbf{x}$ are 1-to-1 at all t, ξ, i.e. worldlines (WLs) do not intersect.

Eulerian observables $f(t,\mathbf{x}) = \tilde{f}(t,\mathbf{X}) = \hat{f}(\xi,\mathbf{X})$ Lagrangian observables.

Reducing the fluid-regime dynamics to decoupled ODEs...

Use CGS units. Let $\boldsymbol{\beta}\equiv\frac{\dot{\mathbf{x}}}{c}$, $\gamma\equiv\frac{1}{\sqrt{1-\boldsymbol{\beta}^2}}$, $u=(u^0,\mathbf{u})\equiv(\gamma,\gamma\boldsymbol{\beta})=\left(\frac{p^0}{mc^2},\frac{\mathbf{p}}{mc}\right)=$ 4-velocity, $s\equiv\gamma-u^z>0$ (all dimensionless). $s\to 0$ implies $u^z\to\infty$.

PDEs: **Lorentz-Maxwell & continuity eq. for the electron fluid**; + in. cond. Are reduced to the family (parametrized by Z) of ordinary Cauchy problems

$$\hat{\Delta}'=\frac{1+v}{2\hat{s}^2}-\frac{1}{2},\qquad \hat{s}'=K\left\{\widetilde{N}\left[Z+\hat{\Delta}\right]-\widetilde{N}(Z)\right\},\tag{3}$$

$$\hat{\Delta}(0,Z)=0,\qquad \hat{s}(0,Z)=1\tag{4}$$

[GF2018] ($\hat{f}'\equiv\partial\hat{f}/\partial\xi$) in the unknowns $\hat{\Delta}(\xi,Z)\equiv\hat{z}_e(\xi,Z)-Z$, $\hat{s}(\xi,Z)$, in the spacetime region where $\hat{\mathbf{x}}_e(\xi,\cdot):\mathbf{X}\mapsto\mathbf{x}$ is one-to-one, we can neglect 2-particle collisions+pulse depletion, and regard ions as immobile. Here $K:=\frac{4\pi e^2}{mc^2}$, and

$$\begin{aligned} v(\xi)&:=\left[\frac{e\boldsymbol{\alpha}^{\perp}(\xi)}{mc^2}\right]^2, &\qquad \boldsymbol{\alpha}^{\perp}(\xi)&:=-\int_{-\infty}^{\xi}d\zeta\,\boldsymbol{\epsilon}^{\perp}(\zeta),\\ \widetilde{N}(Z)&:=\int_0^Z d\zeta\,\widetilde{n}_0(\zeta), &\qquad \mathcal{U}(\Delta;Z)&:=K\int_0^{\Delta}d\zeta\,(\Delta-\zeta)\,\widetilde{n}_0(Z+\zeta)\,. \end{aligned}\tag{5}$$

Clearly, $v\geq 0$, and $\widetilde{N}(Z)$ grows with Z.

...which are Hamiltonian for 1-dim systems

For each $Z \geq 0$ (3) are **Hamilton equations** $q' = \partial\hat{H}/\partial p,\ p' = -\partial\hat{H}/\partial q$ of a **1-dim system**: $\xi, \hat{\Delta}, -\hat{s}$ play the role of t, q, p, and the Hamiltonian reads

$$\hat{H}(\hat{\Delta}, \hat{s}, \xi; Z) := \frac{\hat{s}^2 + 1 + v(\xi)}{2\hat{s}} + \mathcal{U}(\hat{\Delta}; Z) \tag{6}$$

up to mc^2. For $\xi > l$ v=const, $\hat{H} = h$=const, (3) are autonomous and can be solved **by quadrature**; if $Z > 0$ the solutions are periodic in ξ; $\xi_H(Z) \equiv$ period.

All other unknowns can be expressed via $\left(\hat{\Delta}, \hat{s}\right)$:

$$\hat{\mathbf{u}}^\perp = \frac{e\,\boldsymbol{\alpha}^\perp(\xi)}{mc^2}, \qquad \hat{u}^z = \frac{1 + \hat{\mathbf{u}}^{\perp 2} - \hat{s}^2}{2\hat{s}}, \qquad \hat{\gamma} = \frac{1 + \hat{\mathbf{u}}^{\perp 2} + \hat{s}^2}{2\hat{s}}, \tag{7}$$

$$\hat{\mathbf{x}}_e^\perp(\xi, \mathbf{X}) - \mathbf{X}^\perp = \int_0^\xi d\eta\, \frac{\hat{\mathbf{u}}^\perp(\eta)}{\hat{s}(\eta, Z)}, \qquad \hat{z}_e(\xi, \mathbf{X}) - Z = \hat{\Delta}(\xi, Z). \tag{8}$$

As $\boldsymbol{\alpha}^\perp(\xi)$ is independent of $\mathbf{X}$ so are $\hat{\mathbf{p}}^\perp, \hat{\mathbf{u}}^\perp$; as $\hat{s}, \hat{\Delta}$ are independent of X, Y so are $\hat{p}^z, \hat{u}^z, \Delta\hat{\mathbf{x}}_e$. Replacing $(\xi, \mathbf{X}) \mapsto \left(ct - z, \hat{\mathbf{X}}_e(ct - z, \mathbf{x})\right)$ in the arguments we get their Eulerian counterparts, e.g. $n_e(t, z) = \left[\frac{\hat{\gamma}\,\widetilde{n_0}}{\hat{s}\,\hat{J}}\right]_{(\xi, Z) = \left(ct - z, \hat{Z}_e(ct - z, z)\right)}$.

Special case: $\widetilde{n}_0(Z) \equiv n_0 = \text{const}$

If $\widetilde{n}_0(Z) \equiv n_0 = \textbf{const}$, then (3) and its solution are in fact *Z-independent*:

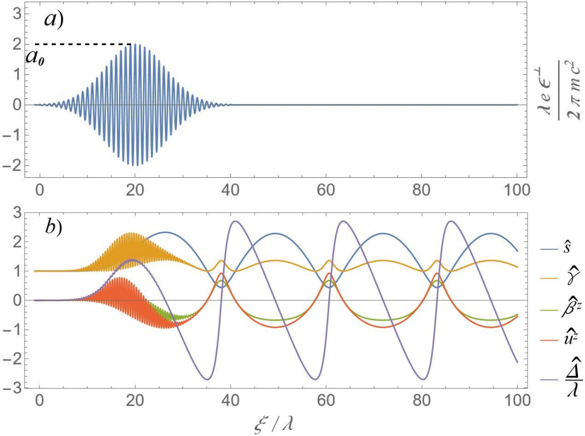
$$\Delta' = \frac{1+v}{2s^2} - \frac{1}{2}, \qquad s' = M\Delta, \qquad \Delta(0)=0, \quad s(0)=1, \quad (9)$$

where $M \equiv Kn_0 = \omega_p^2/c^2$, $\mathcal{U}(\Delta, Z) \equiv M\Delta^2/2$: relativistic harmonic oscillator.

a) Linearly polarized gaussian pulse with peak amplitude $a_0 \equiv \lambda e E_M^\perp / 2\pi mc^2 = 2$, $l_{fwhm} = 10\lambda$. We consider $l = 40\lambda$ and cut tails outside $|\xi - l/2| < l/2$.

b) Corresponding solution of (9) if $\widetilde{n}_0(z) = n_0^j \equiv n_{cr}/267$ ($n_{cr} = \pi mc^2/e^2\lambda^2$ is the critical density); as a result, $E/mc^2 \equiv h = 1.28$.

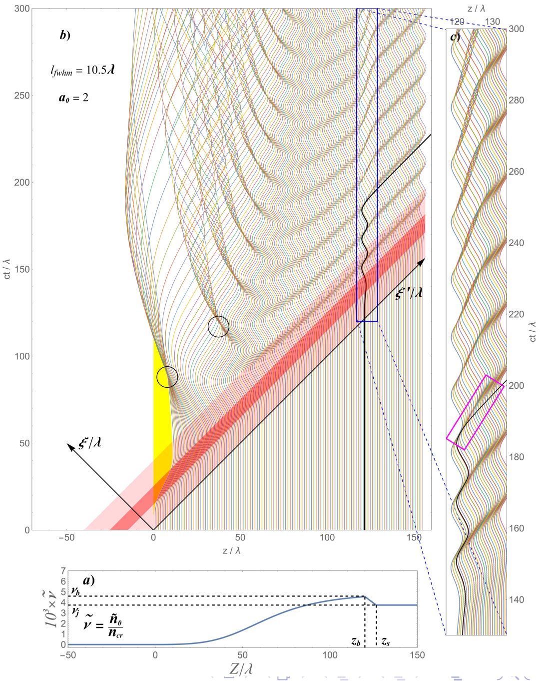
$\hat{s}$ is insensitive to fast oscillations of $\epsilon^\perp$!



a) "Optimal" $\widetilde{n}_0(z)$ for the above pulse: $n_0 = n_0^j = n_{cr}/267$, $n_b = 1.28 \times n_0^j$, $z_b = 120\lambda$, $z_s - z_b = 6.6\lambda$ [GF 2023].

b) WLs of e^- with $Z = 0, \lambda, ..., 156\lambda$ are obtained solving (3-4) and look as plot until they first intersect (circles), $\Rightarrow$ WBs. The black WL of the e^- self-injected by the earliest WB holds **for all** t; after WB it is ruled by (15). The yellow region is filled only by ions; in the pink region ($0 < \xi < 40\lambda$) the pulse modulating intensity ϵ_s^2 is nonzero; in the red region ($|\xi - 20\lambda| < 5.25\lambda$) ϵ_s^2 is above half maximum.

c) Zoom of the blue box in a).



Hydrodynamic regime up to wave-breaking

The map $\hat{\mathbf{x}}_e(\xi,\cdot):\mathbf{X}\mapsto\mathbf{x}$, is invertible, and the HR is justified, as long as

$$\hat{J}\equiv\left|\frac{\partial\hat{\mathbf{x}}_e}{\partial\mathbf{X}}\right|=\frac{\partial\hat{z}_e}{\partial Z}>0. \qquad (10)$$

$\hat{J}(\xi,Z)\leq 0$: $\exists Z'\neq Z$, s.t. $\hat{z}_e(\xi,Z')=\hat{z}_e(\xi,Z)$, i.e. Z,Z' e^- layer cross, $\exists$ WB.

$$n_e(t,z)=\left[\frac{\hat{\gamma}\,\widetilde{n_0}}{\hat{s}\,\hat{J}}\right]_{(\xi,Z)=\left(ct-z,\hat{Z}_e(ct-z,z)\right)} \quad \text{diverges where } \hat{J}=0. \qquad (11)$$

For $\xi>l$ then $\hat{J}$ satisfies [GF, DeNicola, Akhter, Fedele, Jovanović '23]

$$\begin{aligned}\hat{J}(\xi+n\xi_H,Z) &= \hat{J}(\xi,Z)-n\frac{\partial\xi_H}{\partial Z}\Delta'(\xi,Z), \qquad \forall n\in\mathbb{N},\ Z\geq 0, &(12)\\ \Leftrightarrow \quad \hat{J}(\xi,Z) &= a(\xi,Z)+\xi\,b(\xi,Z), &(13)\end{aligned}$$

where $b\equiv-\frac{\partial\log\xi_H}{\partial Z}\hat{\Delta}'$, $a\equiv\hat{J}-\xi\,b$ are ξ_H-periodic in ξ, and b has zero mean over a period (apply ∂_Z to $\Delta[\xi+n\xi_H(Z),Z]=\Delta(\xi,Z)$, use ξ_H-periodicity of Δ').

By (12) we can extend our knowledge of $\hat{J}$ from $[l,l+\xi_H[$ to all $\xi\geq l$.

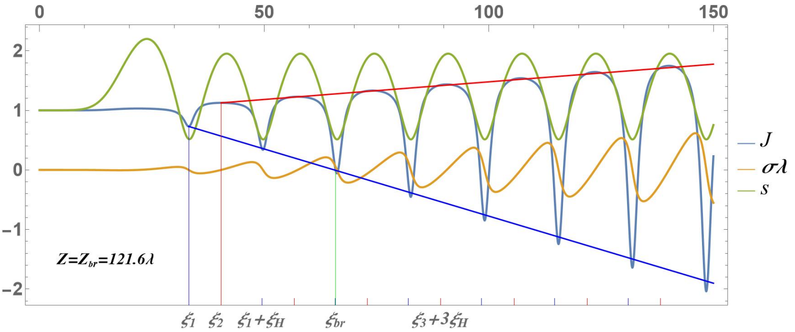


Figure 4: $\hat{J}, \hat{\sigma}$ vs. ξ for $Z = Z_{br} \simeq 121.6\lambda$ and input data as in Fig. 3.

Differentiating (3-4) w.r.t. Z one finds that $\hat{J}$, $\sigma \equiv \frac{\partial \hat{s}}{\partial Z}$ fulfill

$$\begin{aligned} &\hat{J}' = -\frac{1+v}{\hat{s}^3}\hat{\sigma}, \qquad && \hat{\sigma}' = K\left(\check{n}\,\hat{J} - \widetilde{n_0}\right), \\ &\hat{J}(0,Z) = 1, && \hat{\sigma}(0,Z) = 0, \end{aligned} \tag{14}$$

where $\check{n}(\xi, Z) \equiv \widetilde{n_0}\,[\hat{z}_e(\xi, Z)]$. Studying (14) one finds sufficient conditions on $\widetilde{n_0}, \epsilon^{\perp}$ [GF et al 2022-23] for the first WB to occur *after* the laser-plasma interaction ($\xi > l$) and be controlled via (12).

Maximizing the WFA of (self-)injected e^-

Motion of a test electron in the plasma wave

If a *test* e^- is injected in the PW behind the pulse its $\hat{z}_i, \hat{s}_i$ evolve after

$$\hat{z}_i' = \frac{1-\hat{s}_i^2}{2\hat{s}_i^2}, \qquad \hat{s}_i'(\xi) = K\left\{\widetilde{N}[\hat{z}_i(\xi)] - \widetilde{N}\left[\hat{Z}_e(\xi,\hat{z}_i(\xi))\right]\right\}. \tag{15}$$

(15b) reduces to $\hat{s}_i' = M\Delta$, cf. (9b), and $\hat{s}_i(\xi) - s(\xi) = \delta s \equiv s_{i0} - s(\xi_0) = \text{const}$ along the density plateau. If $\delta s < -s_m$ (**trapping condition**), then $\exists \xi_f > \xi_0$ s.t. $\hat{s}_i(\xi_f) = 0$, e^- **is trapped & accelerated in a trough of the PW**. As $t \to \infty$

$$z_i \sim ct, \qquad \gamma_i \simeq F\, z_i/\lambda \xrightarrow{z_i \to \infty} \infty, \tag{16}$$

$F \equiv K n_0 \lambda\, |\Delta(\xi_f)|$; reliable as long as pulse depletion is negligible, $0 \leq z_i \leq z_{pd}$.

Fixed n_0, if $\delta s = -1$, then $|\Delta(\xi_f)| = |\Delta_m|$, F is maximal:

$$\gamma_i(z_i, n_0) \simeq \sqrt{j(\nu)}\; z_i/\lambda; \quad (17)$$

$j(\nu) \equiv 8\pi^2\nu\left[\bar{h}(\nu) - 1\right]$, $\bar{h}(\nu) =$ final energy transfered by the pulse to the e^- if $\widetilde{n}_0(z) = n_0$, vs. $\nu \equiv n_0/n_{cr}$.

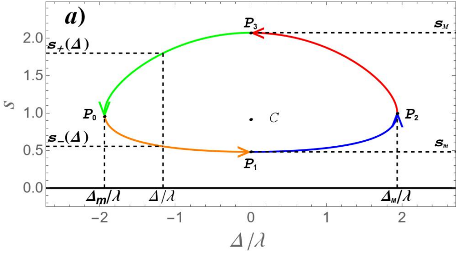


Figure 5: Phase portrait of plateau e^- of fig. 3

Self-injection & maximal WFA by fixing $\widetilde{n}_0$ in 4 steps

Step 1: Computing $\bar{h}(\nu), j(\nu)$. (We interpolate 200 points; few seconds via *Mathematica*).

Step 2: Optimal plateau density n_0. If the depth available for WFA is $z_i \leq z_{pd}(\nu_j)$, set $n_0/n_{cr} = \nu_j \equiv \max\{j(\nu)\}$:

$$\gamma_i^M(z_i) \simeq \sqrt{j(\nu_j)}\, z_i/\lambda. \qquad (18)$$

Step 3: $\widetilde{n}_0$ with optimal down-ramp for self-injection, LWFA.

$$\widetilde{n}_0(Z) = n_0 + \Upsilon(Z - z_s), \qquad z_b \leq Z \leq z_s,$$

$\Upsilon = \frac{n_0 - n_b}{z_s - z_b}$. Let (ξ_{br}, Z_{br}) be the pair (ξ, Z) with smallest ξ s.t. $\hat{J}(\xi, Z) = 0$. The Z_{br} e^- are the fastest injected & trapped in a PW trough by the 1st WB. We fix Υ, z_b requiring: $\delta s = -1$, so that (17) applies; no WBDLPI.

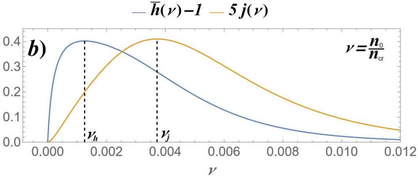


Figure 6: $\bar{h} - 1$ (energygain per plasma e^-) and j by the pulse of fig. 2a, vs. ν

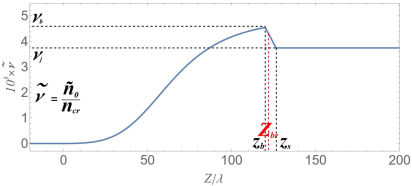


Figure 7: The optimal initial density associated to the pulse of fig. 2a.

Step 4: Choosing an up-ramp of $\widetilde{n}_0$ out of the ∞-ly many ones growing from 0 to n_b and preventing WB for $\xi < \xi_{br}$; $\widetilde{n}_0(z) \simeq O(z^2)$ [GF et al 2022-23].

3D effects, discussion and conclusions

Summarizing, the steps of our preliminary optimization process are:

1. finding the final energy $\bar{h}$ transfered by the pulse to the plateau plasma electrons and $j = 8\pi^2\left[\bar{h}-1\right] n_0/n_{cr}$ as functions of the density n_0;
2. finding the 'optimal' value n_0^j of n_0 maximizing $j(n_0)$;
3. finding the 'optimal' length $z_b - z_s$ and slope Υ of the density down-ramp;
4. adjusting the up-ramp $(z < z_b)$ of $\widetilde{n_0}(z)$ to avoid WB for $\xi < \xi_{br}$.

Range of applicability of the model?

The depletion of the pulse is negligible in the tilted (rather long) rectangle

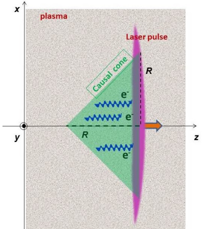
$$0 \le ct - z \le l, \qquad 0 \le ct + z \lesssim mc^2/e^2 n_0\lambda \tag{19}$$

Pulse cylindrically symmetric around $\vec{z}$ with waist R: by causality our results hold strictly in the green causal cone trailing the pulse, approximately nearby.

In particular, if the pulse has maximum at $\xi = \frac{l}{2}$, and

$$R > \xi_{br} - \frac{l}{2}, \quad R \gg \frac{a_0\lambda}{2\pi}\left[\bar{h} + \sqrt{\bar{h}^2 - 1}\right] \tag{20}$$

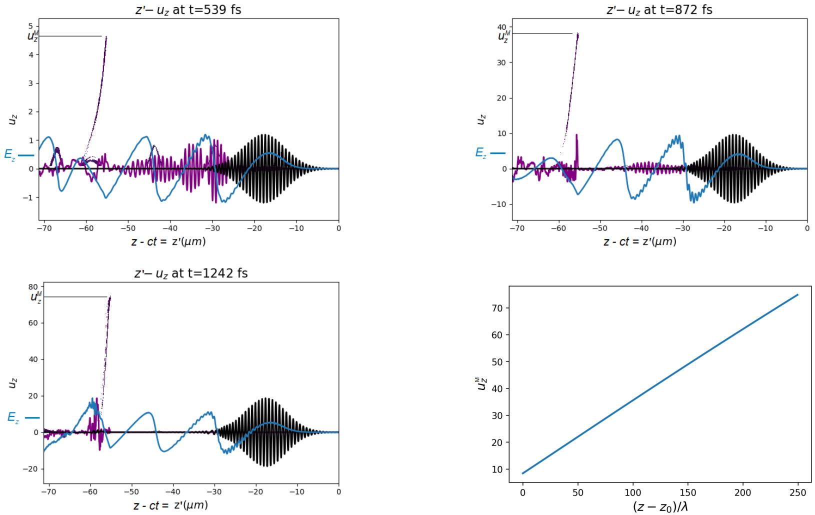
then the $\mathbf{X} \simeq (0,0,Z_{br})$ e^- keep in that cone and move as above: same maximal WFA, as far as pulse not depleted.



Apply our optimization procedure to the pulse of Fig. 2a ($a_0 = 2$, $l_{fwhm} = 10\lambda$): we find the initial density $\widetilde{n}_0(z)$ and the WLs of Fig. 3; $F = 0.28$.

Ti-sapphire laser: $\lambda \simeq 0.8\mu$m; 'moderate' peak intensity $\mathcal{I} = 1.7 \times 10^{19}$W/cm^2 yields the remarkable energy gain 1.8 GeV/cm of the Z_{br} electron (black WL).

Good agreement with 2D FB-PIC simulations (courtesy of P. Tomassini):



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