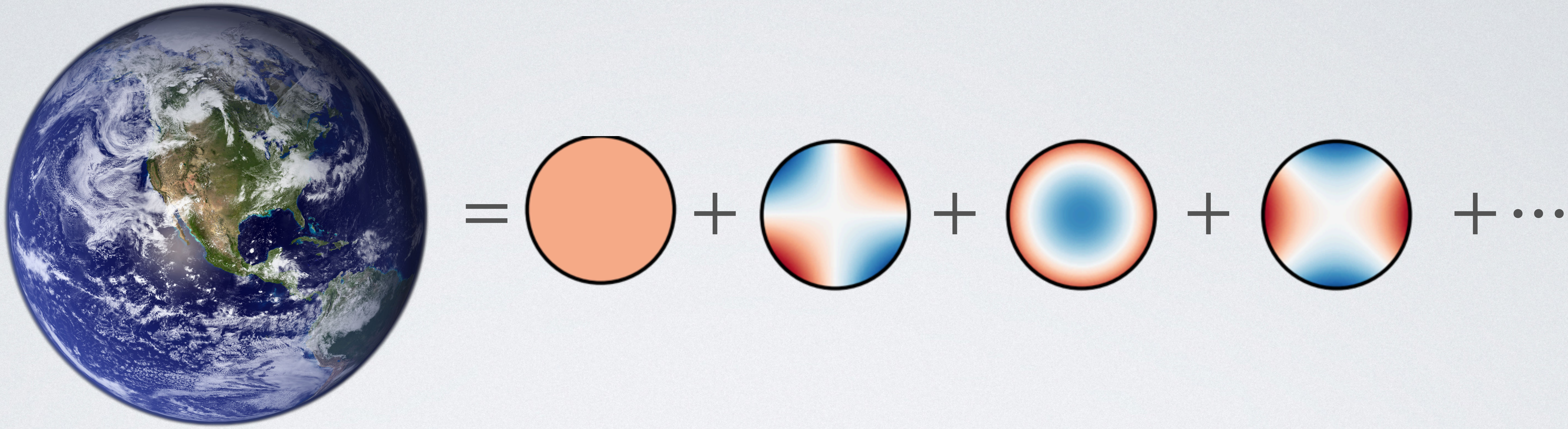
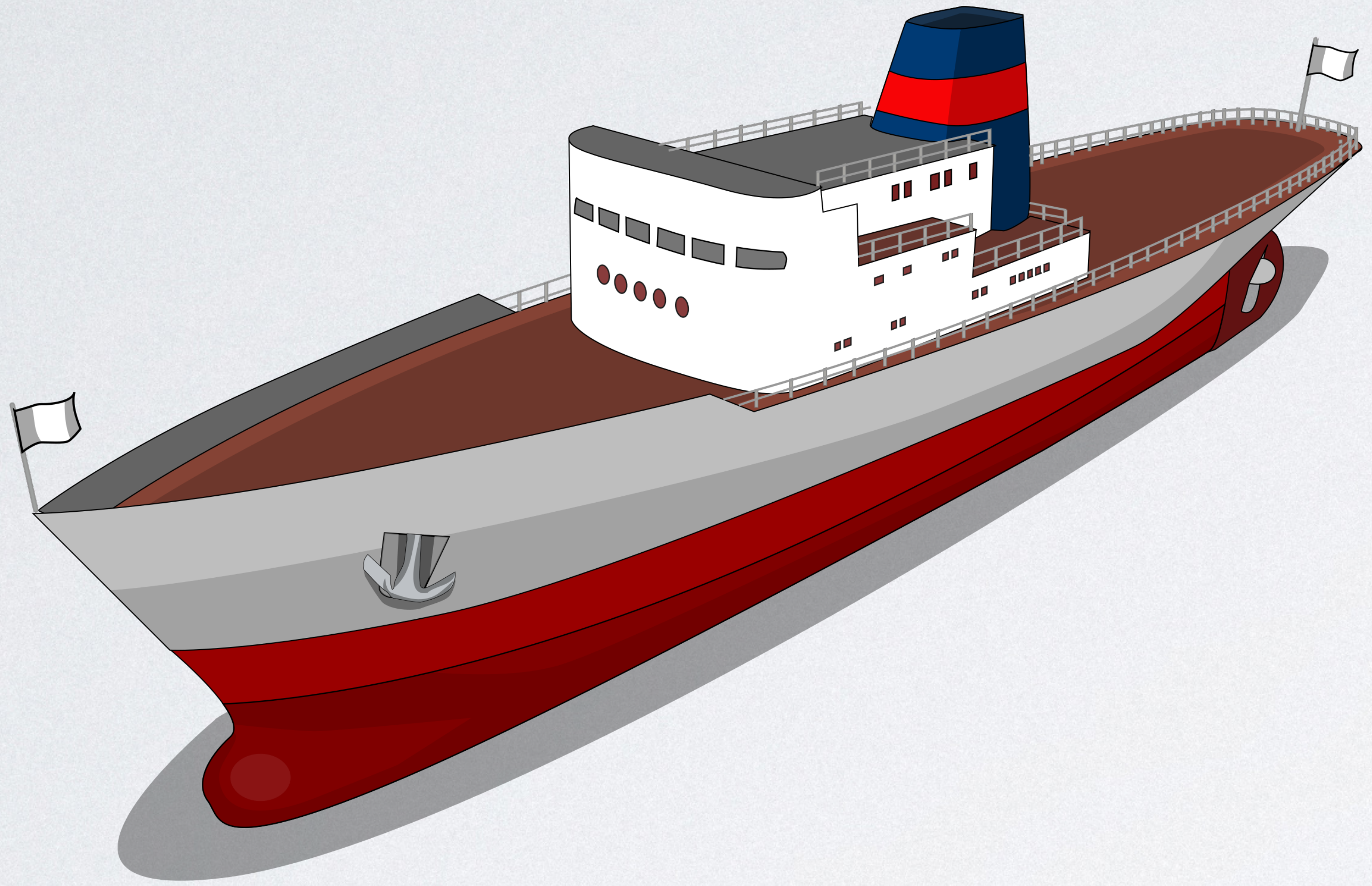


MULTIPOLE EXPANSION IN GRAVITY: FROM NEWTON TO EINSTEIN

Claudio Gambino, University of Rome "La Sapienza"
March 6th, PhD Seminars, Season 9, Episode 4

IS A SHIP A MATERIAL POINT?



$$\stackrel{?}{=}$$

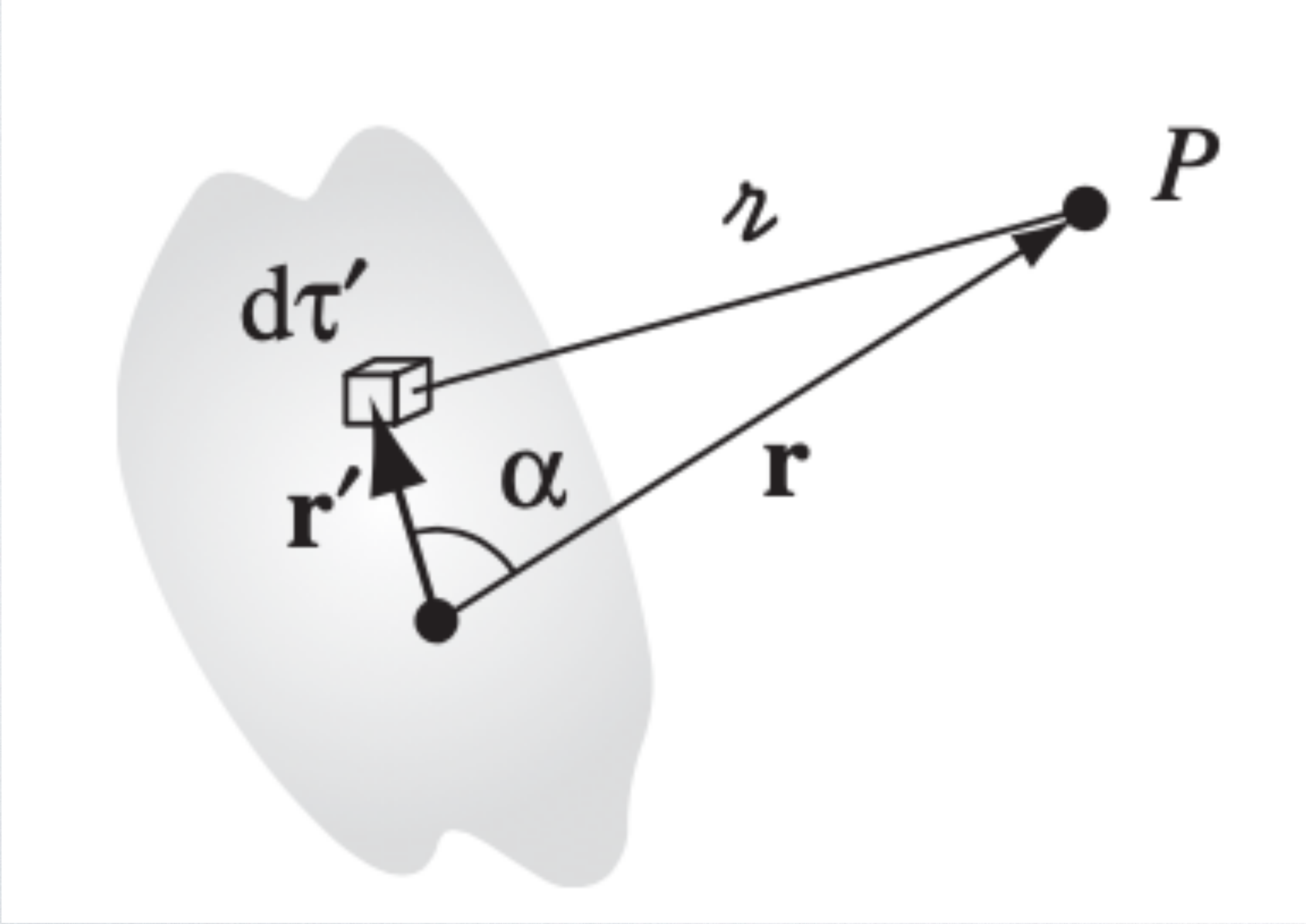
IT DEPENDS…

NEWTONIAN GRAVITY

Gravitational field

$$\nabla^2\Phi(t,\vec{x}) = 4\pi G\,\rho(t,\vec{x}) \qquad \Rightarrow \qquad \Phi(t,\vec{x}) = -\,G\int \frac{\rho(t,\vec{x})}{|\vec{x}-\vec{x}'|}d^3x'$$

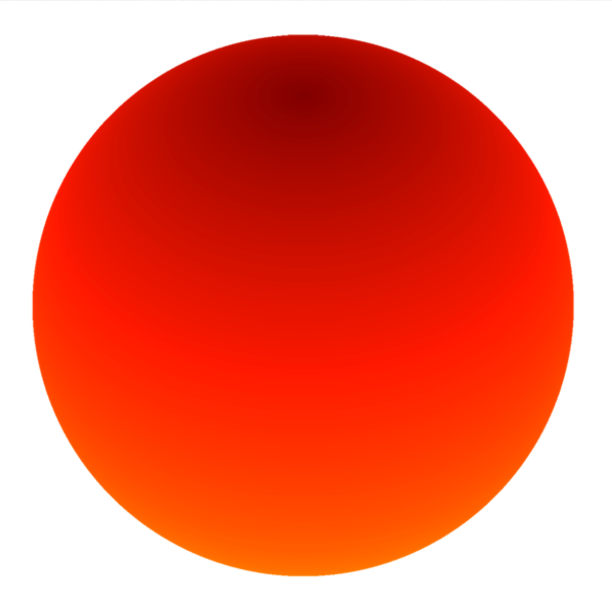
Mass distribution



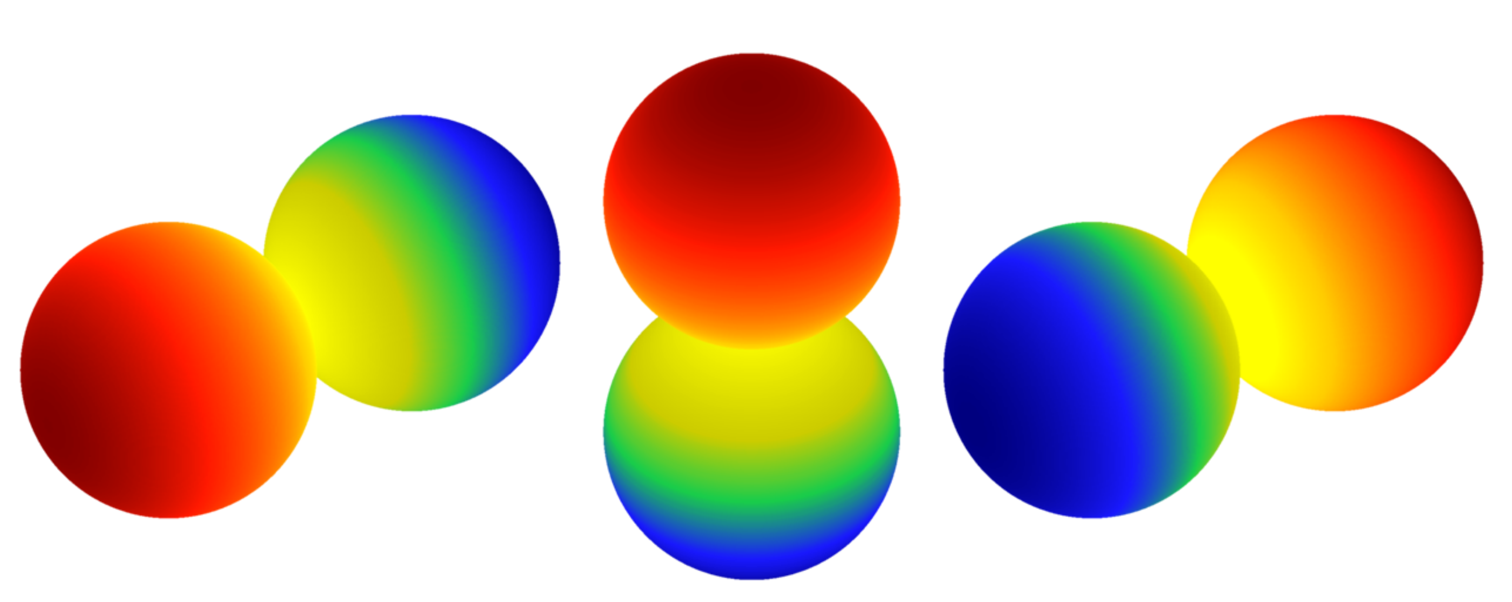
$$\Phi(t,\vec{x}) = -\,G\sum_{\ell=0}^{+\infty}\sum_{m=-\ell}^{\ell}\frac{1}{r^{\ell+1}}I_{\ell m}(t)Y_{\ell m}(\theta,\varphi)$$

$$I_{\ell m}(t) = \int \rho(t,\vec{x})r^{\ell}Y_{\ell m}(\theta,\varphi)d^3x$$

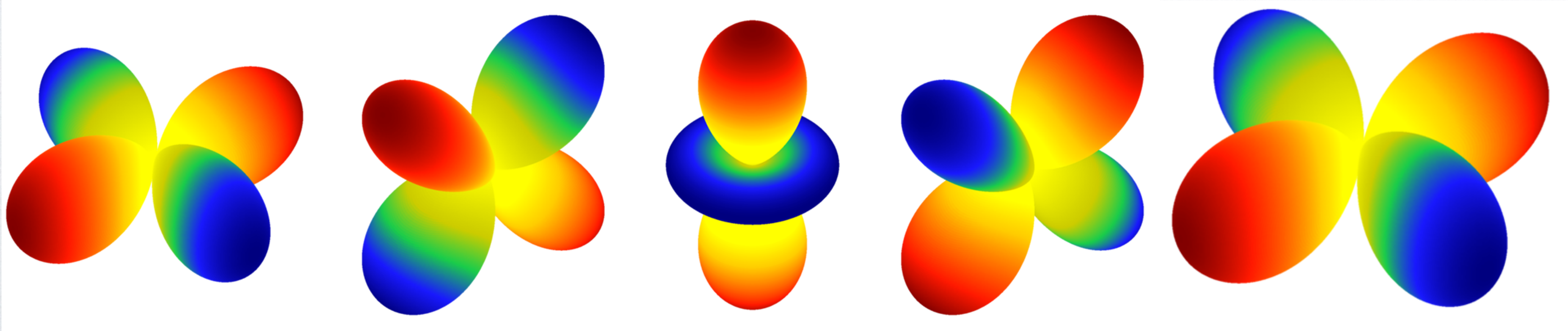
$I_{\ell m}(t)$ are the ***multipole moments*** of the potential



Monopole, $\ell = 0$



Dipole, $\ell = 1$



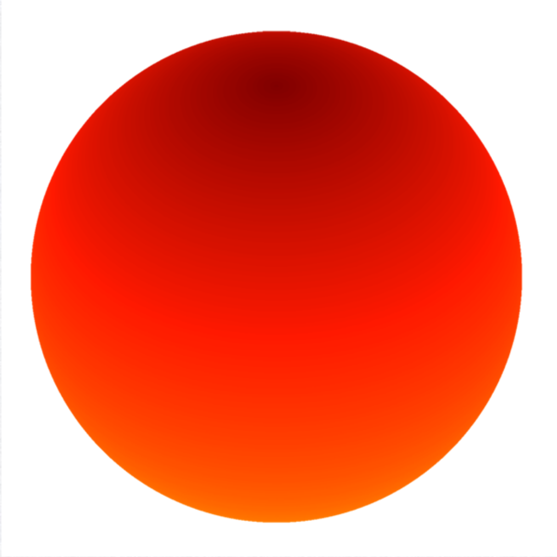
Quadrupole, $\ell = 2$

If the source is **stationary** and ***axisymmetric*** the analysis simplifies

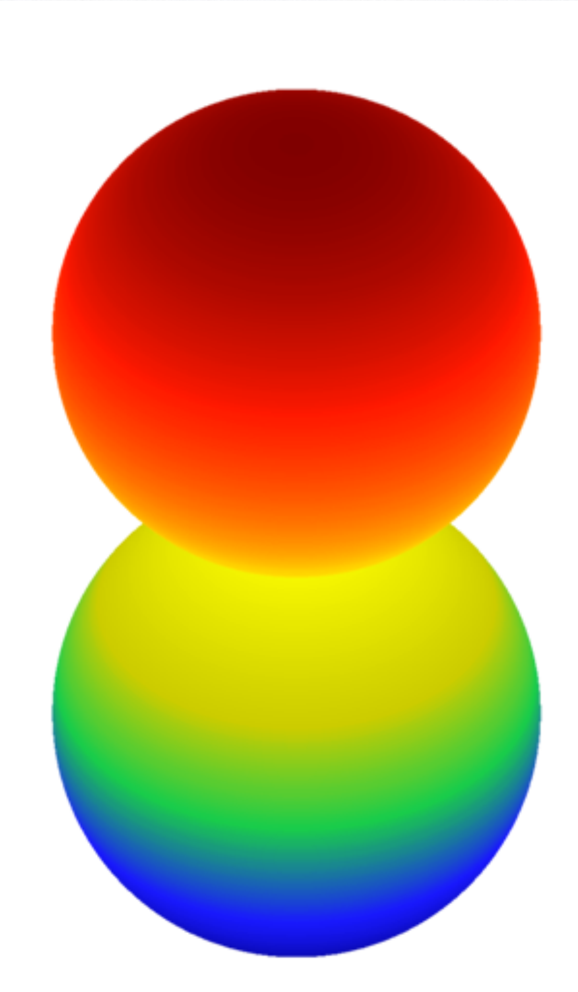
$$M_\ell = \int \rho(\vec{x}) r^\ell Y_{\ell 0}(\theta) d^3x \qquad \Rightarrow \qquad \Phi(\vec{x}) = -G \sum_{\ell=0}^{+\infty} \frac{M_\ell}{r^{\ell+1}} Y_{\ell 0}(\theta)$$

and only the $m = 0$ harmonics survive

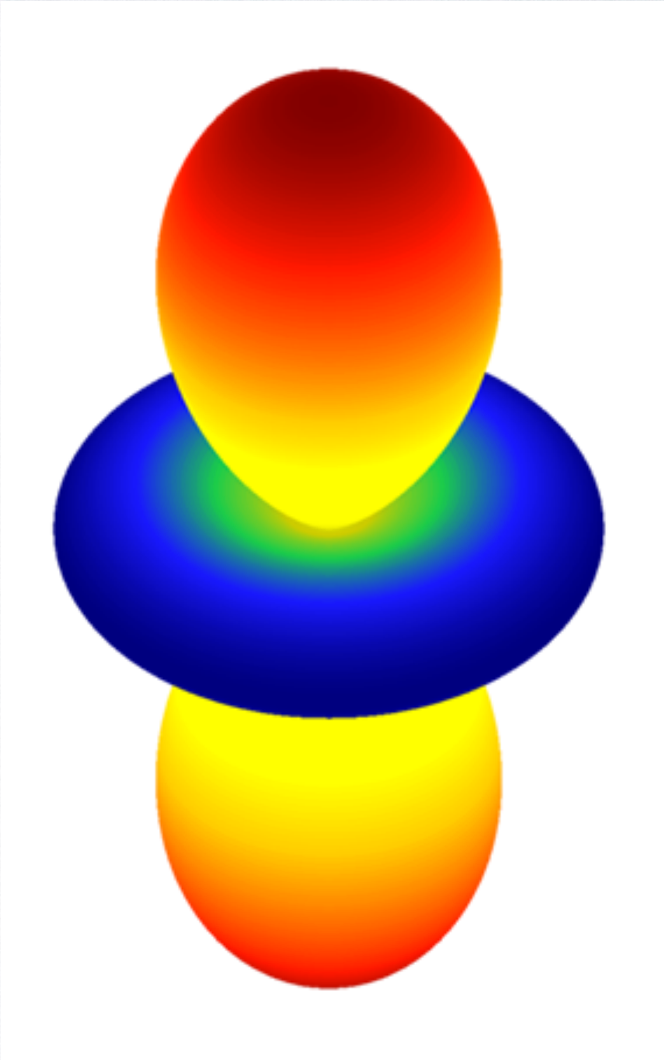
$$\Phi(\vec{x}) = -G\Bigg(\frac{M_0}{r}$$



$$+ \frac{M_1}{r^2}$$



$$+ \frac{M_2}{r^3}$$



$$+ \cdots \Bigg)$$

Since nothing in life is a perfect sphere, everything is a multipole expansion.

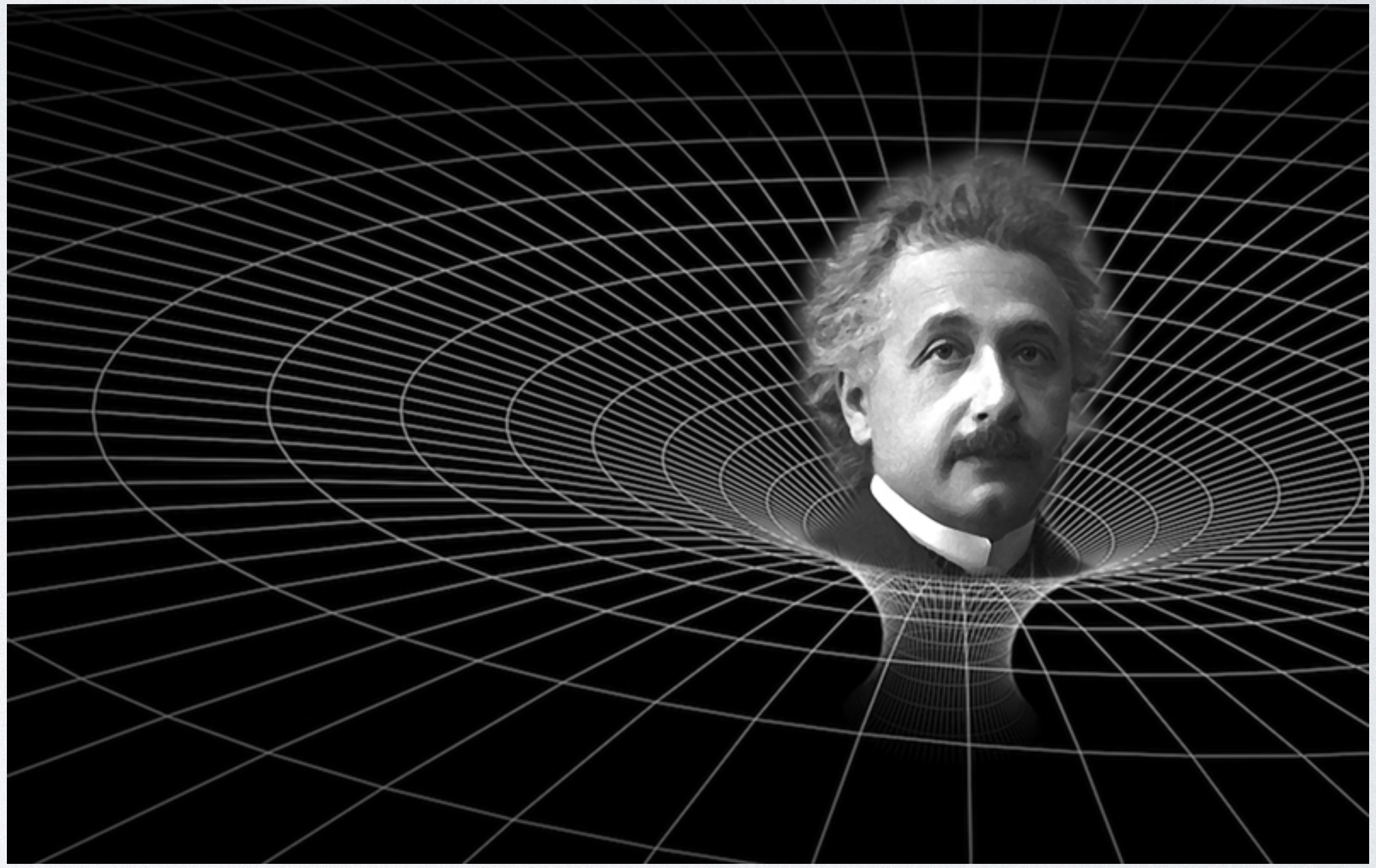
Cit. Claudio

With these hypotheses on the source, at every multipole order there is only **one degree of freedom**

GENERAL RELATIVITY… IN A NUTSHELL

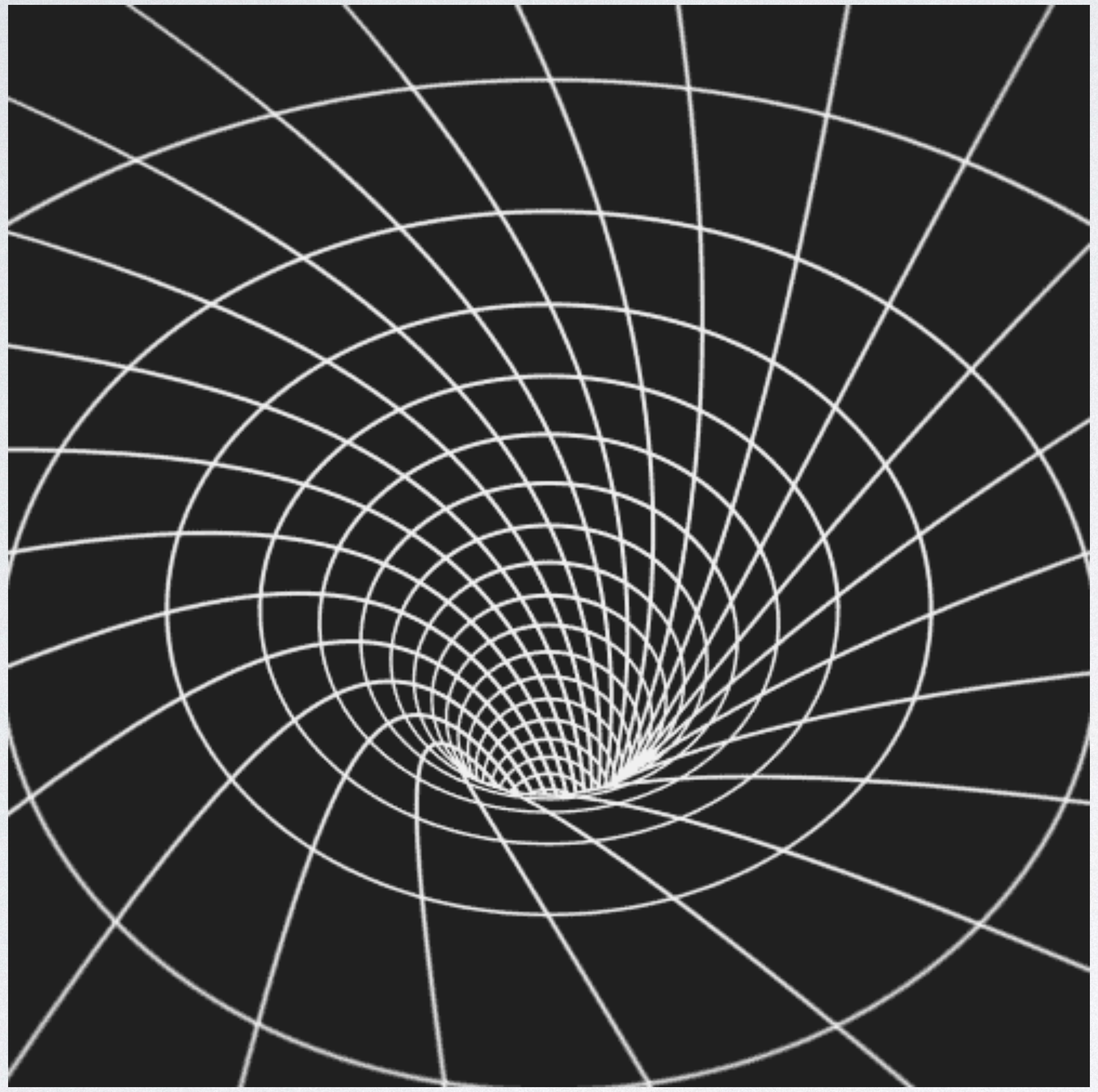
Matter tells space-time how to curve and space-time tells matter how to move.

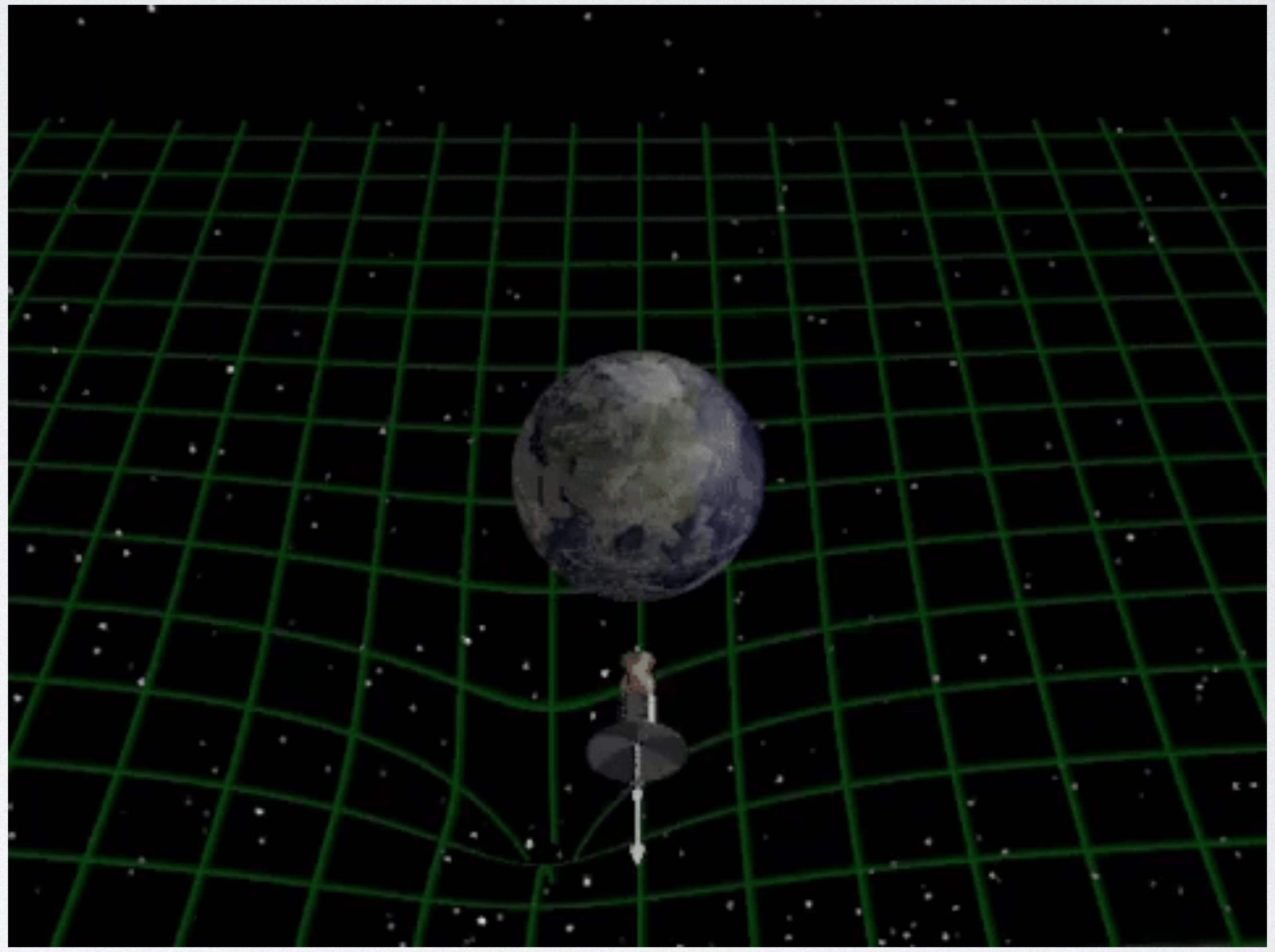
Cit. Wheeler



There is no more a potential. The gravitational field is encoded in the metric tensor $g_{\mu\nu}$.

The space-time rotates within the rotating object, and so the "gravitational field" does…





The rotating space-time drags the satellite to rotate with it. This introduces **new degrees of freedom** in the multipole expansion.

We can study the multipole expansion of the metric tensor separating it into three different contributions

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{0i} & g_{ij} \end{pmatrix}$$

$$g_{00} \sim G \sum_{\ell=0}^{+\infty} \frac{M_\ell}{r^{\ell+1}} Y_{\ell 0}(\theta) \longrightarrow$$ ***Mass Multipoles***

$$g_{0i} \sim G \sum_{\ell=0}^{+\infty} \frac{J_\ell}{r^{\ell+1}} Y_{\ell 0}(\theta) \longrightarrow$$ ***Current Multipoles***

$$g_{ij} \sim G \sum_{\ell=0}^{+\infty} \frac{M_\ell}{r^{\ell+1}} Y_{\ell 0}(\theta) \sim g_{00} \longrightarrow$$ Related to Mass Multipoles

In General Relativity there are **two** degrees of freedom for any multipole order.

But what happens if we consider gravity in $D = d + 1 > 4$***?***

$$g_{00} \sim G \sum_{\ell=0}^{+\infty} \frac{M_\ell}{r^{\ell+d-2}} Y_{\ell 0}^{(d)} \qquad g_{0i} \sim G \sum_{\ell=0}^{+\infty} \frac{J_\ell}{r^{\ell+d-2}} Y_{\ell 0}^{(d)}$$

Spherical harmonic in d dimensions

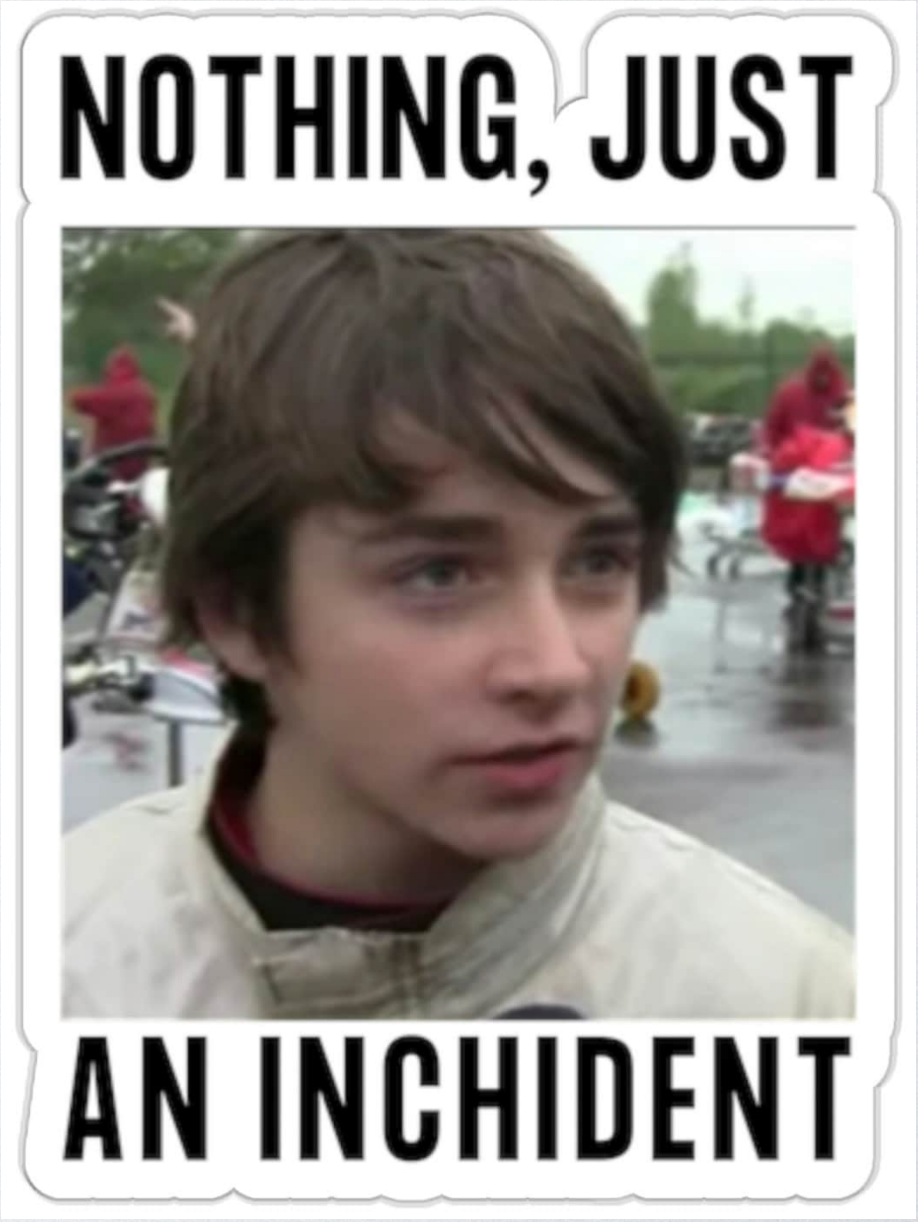
$$g_{ij} \sim G \sum_{\ell=0}^{+\infty} \frac{K_\ell}{r^{\ell+d-2}} Y_{\ell 0}^{(d)} \longrightarrow$$

A new independent multipole moment arises!

In arbitrary dimensions there are **three** degrees of freedom for any multipole order.

Gravity is richer in higher dimensions.

The fact that in $D = 4$ there are two degrees of freedom is not a universal property, but just an accident.



So why our four dimensional space-time is so special?

In physics the angular momentum is defined as an anti-symmetric rank-2 tensor S_{ij}.

- In $d = 3$ we can define a spin tensor $S_{ij} = \epsilon_{ijk} S^k$

$$\Rightarrow \quad M_\ell \sim K_\ell$$

- In $d > 3$ we cannot define a spin tensor (S_{ij} is an irreducible representation)

$\Rightarrow$ M_ℓ and K_ℓ ***are independent***

Take home message

- The multipole expansion is an important tool in physics.
- Some properties of gravity in four dimensions are not universal.

Well, that's relativity, folks.

Cit. Interstellar