Cosmic Variance of Hellings-Downs Curve and Source Anisotropies

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Physical Review D 110, 043044

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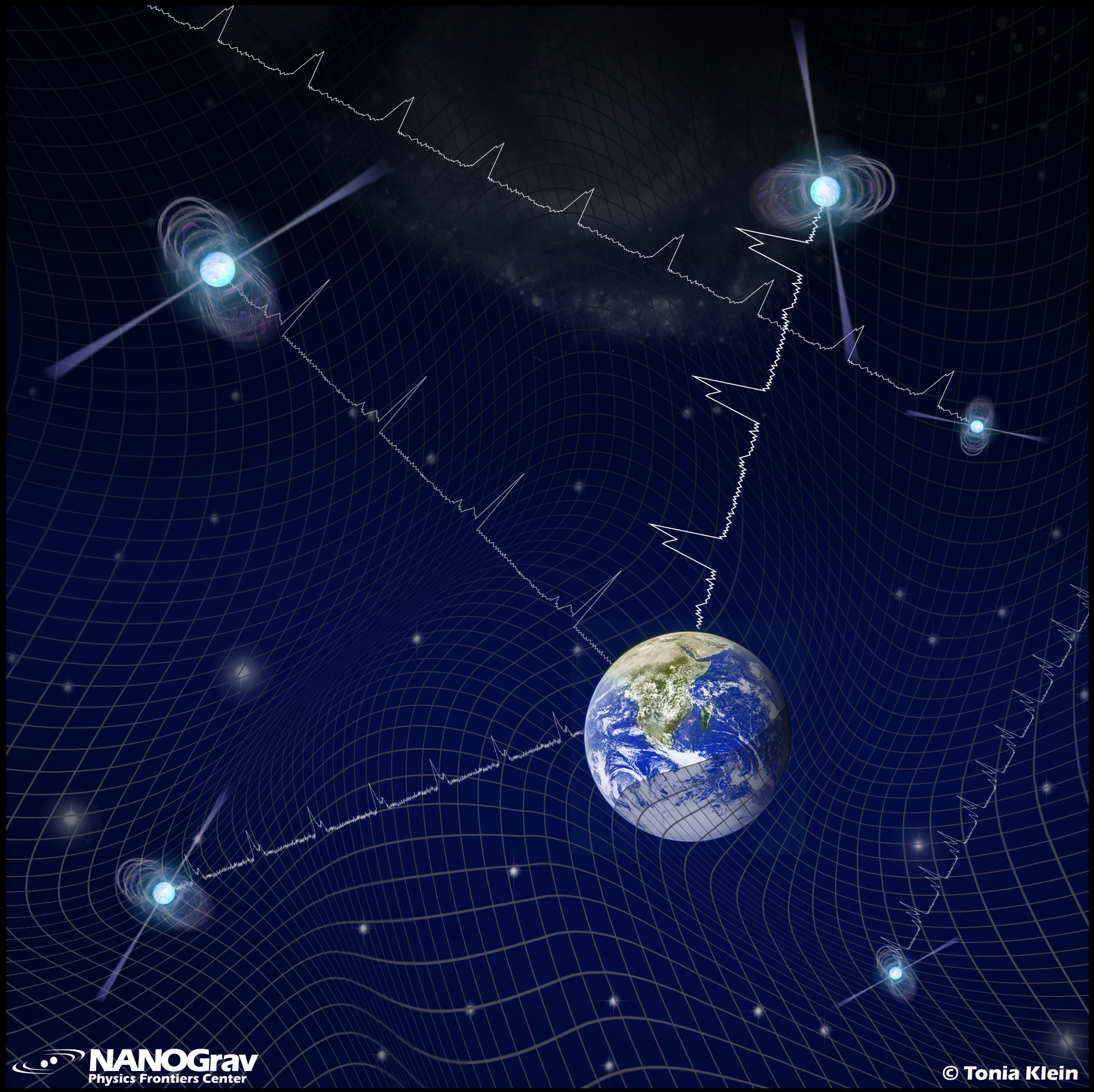
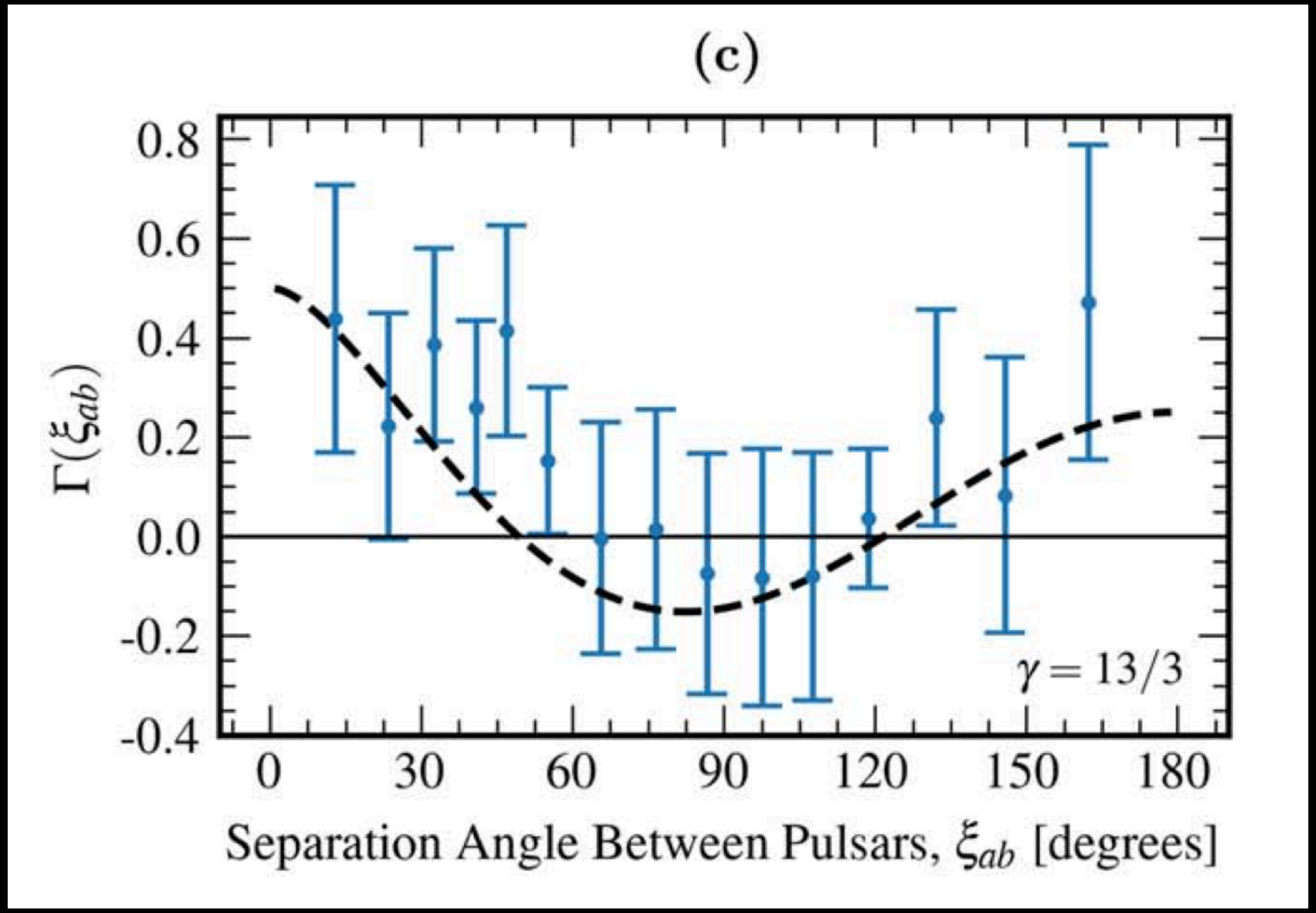
October 25, 2024

Pulsar Timing Array and HD Curve



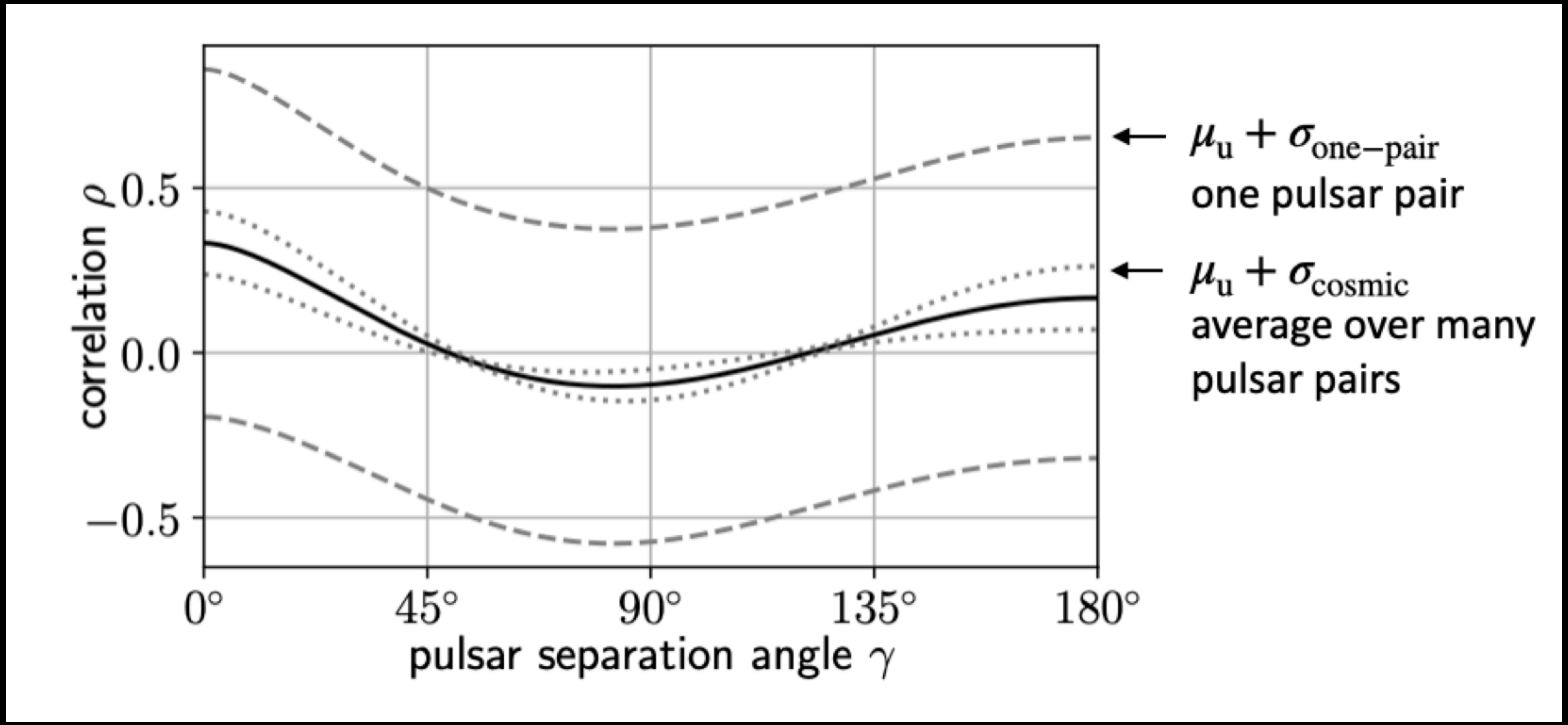


Gabriella Agazie *et al* 2023 *ApJL* **951** L8

Question: Will we recover HD curve in noise-free + infinite pulsar case?

Pulsar and Cosmic Variance

Allen, Frascati Physics Series Vol. 74 (2022)
Allen, Phys. Rev. D 107, 043018, 2023
Allen and Romano, Phys. Rev. D 108, 043026, 2023



Answer : Probably not - due to stochasticity in GWB signal

Redshift Correlation

Redshift in Radio Pulses

$$Z(t) = \frac{1}{2}\frac{\hat{p}^i\hat{p}^j}{1+\hat{\Omega}\cdot\hat{p}}\left[h_{ij}(t,\vec{0}) - h_{ij}(t-L, L\hat{p})\right].$$

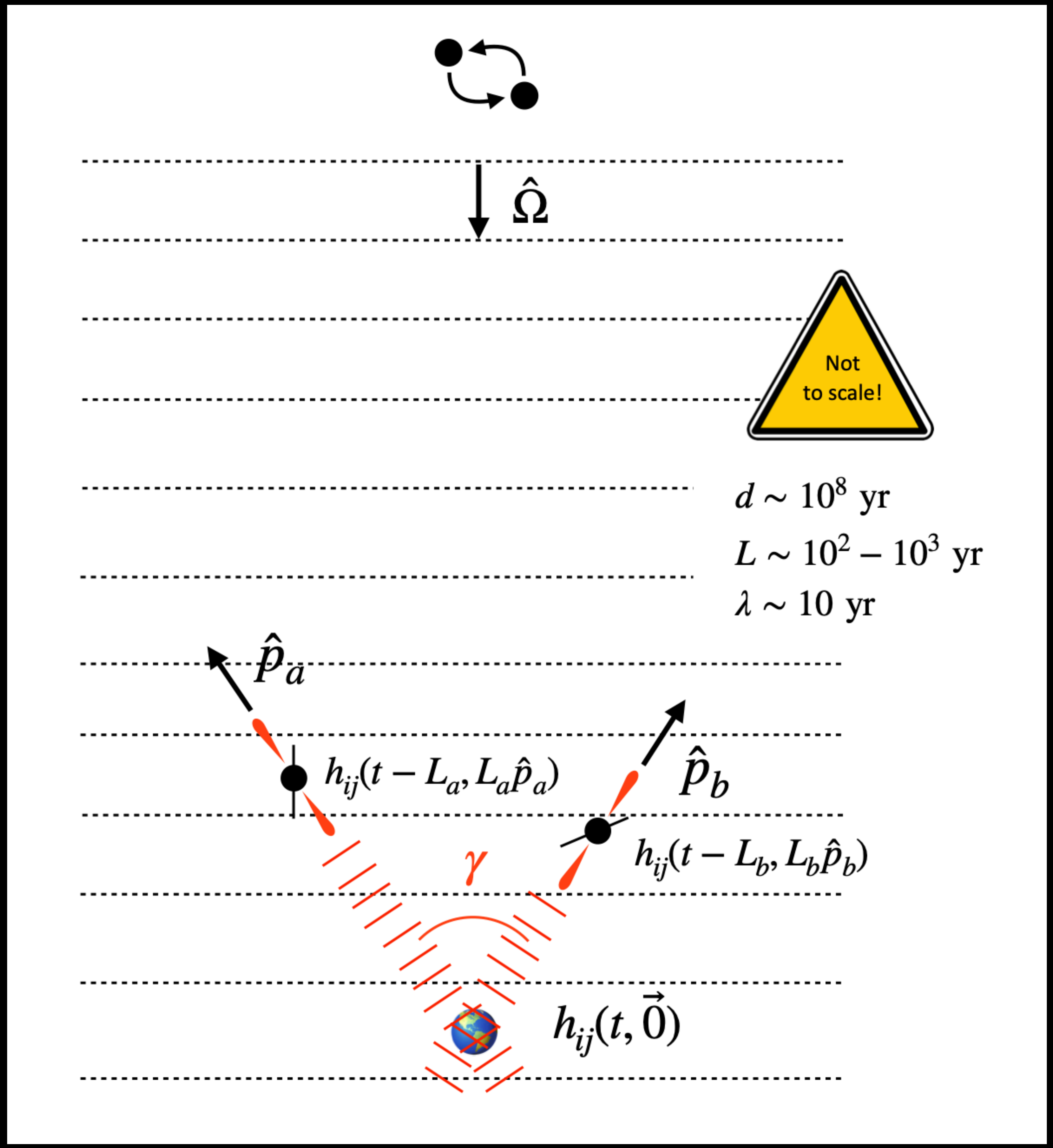
Earth Term

Pulsar Term

Correlation

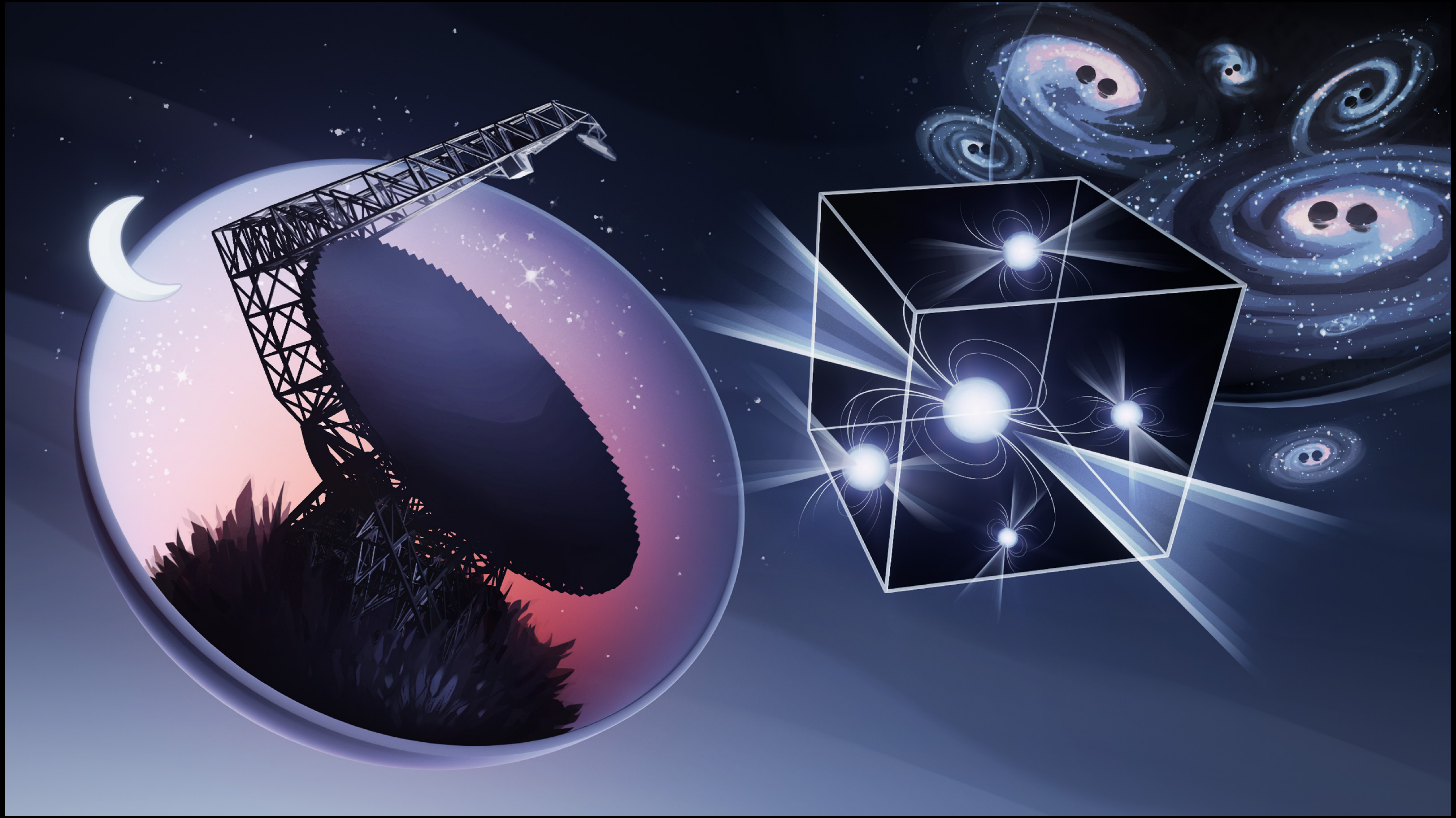
$$\rho_{12} = \overline{z_1(t)\, z_2(t)}$$

- Correlation depends upon source-pulsar-earth geometry.
- We will ignore contribution from pulsar term.

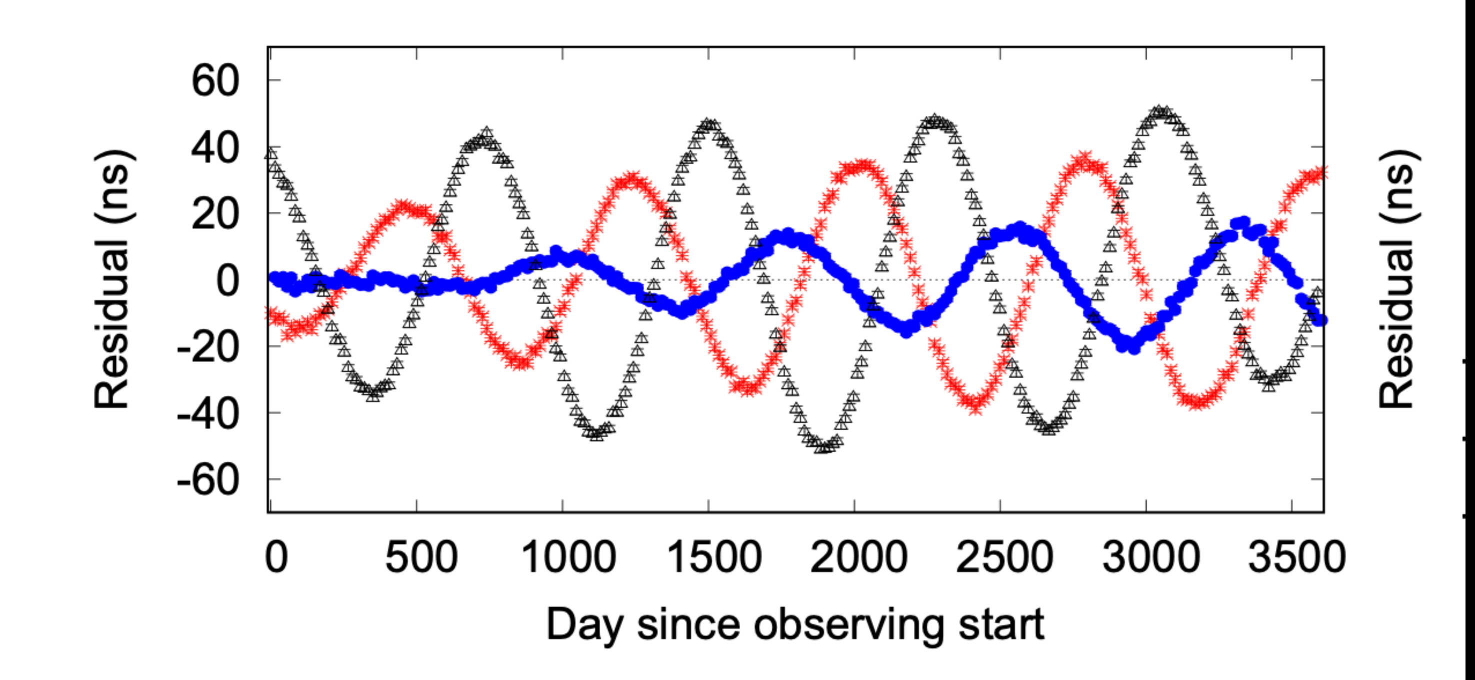


Romano and Allen, Class. Quantum Grav. 41 175008 (2024)

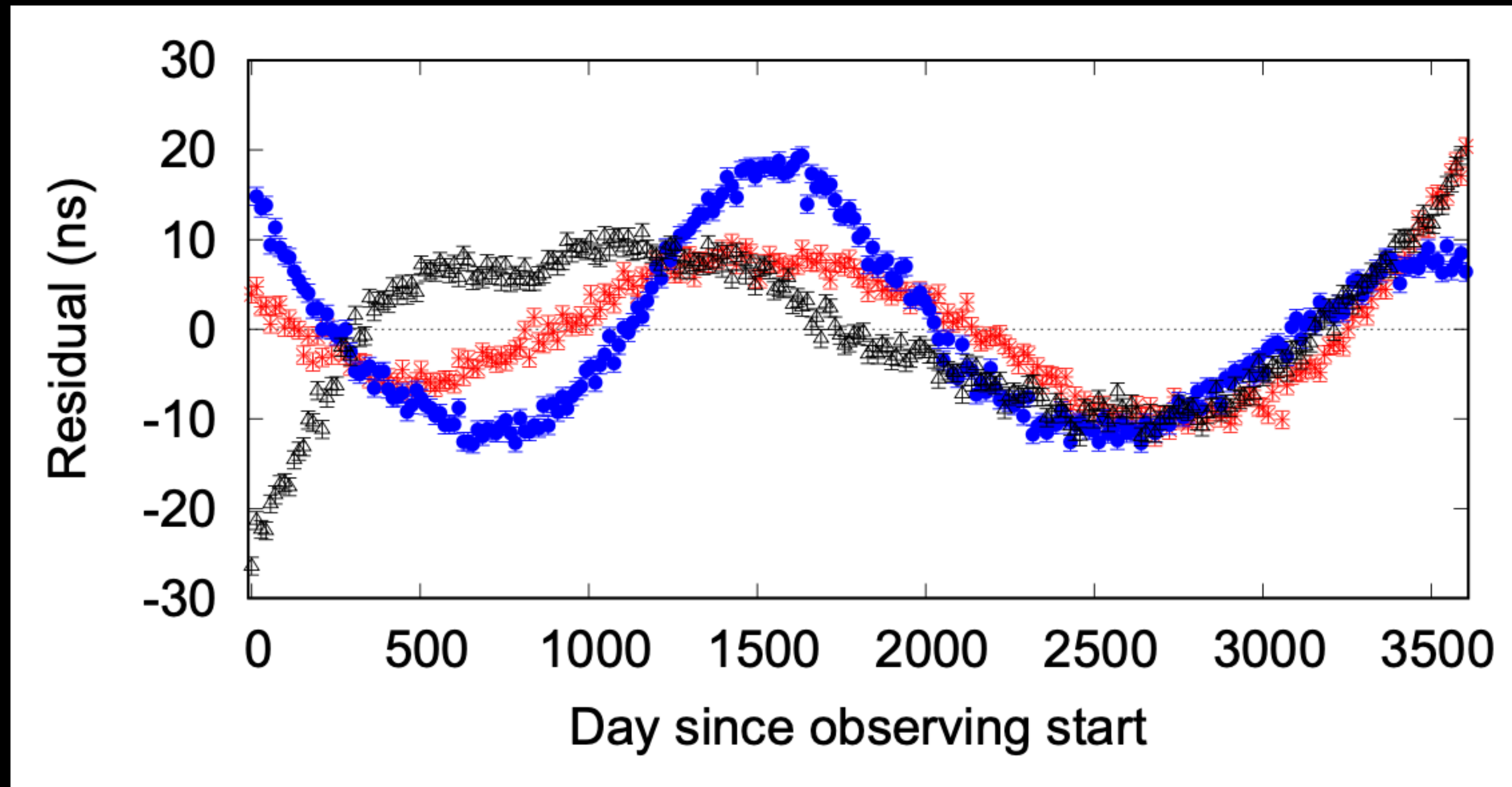
Stochastic Background



Credit: Olena Shmahalo for NANOGrav



(a) Continuous wave



(b) Background

Burke-Spolaor, Astron Astrophys Rev 27, 5 (2019)

Characterization of Stochastic Background

- Strain Fourier Coefficients $\tilde{h}_A(f, \hat{\Omega})$ can be assumed to Gaussian random variables.

$$\langle \tilde{h}_A(f, \hat{\Omega}) \rangle_h = 0$$

$$\langle \tilde{h}_A(f, \hat{\Omega})\, \tilde{h}^*_{A'}(f', \hat{\Omega}') \rangle_h = \psi(\hat{\Omega})\, H(f)\, \delta_{AA'} \delta(f - f') \delta^2(\hat{\Omega}, \hat{\Omega}')$$

Anisotropy & Source Power Spectral Density

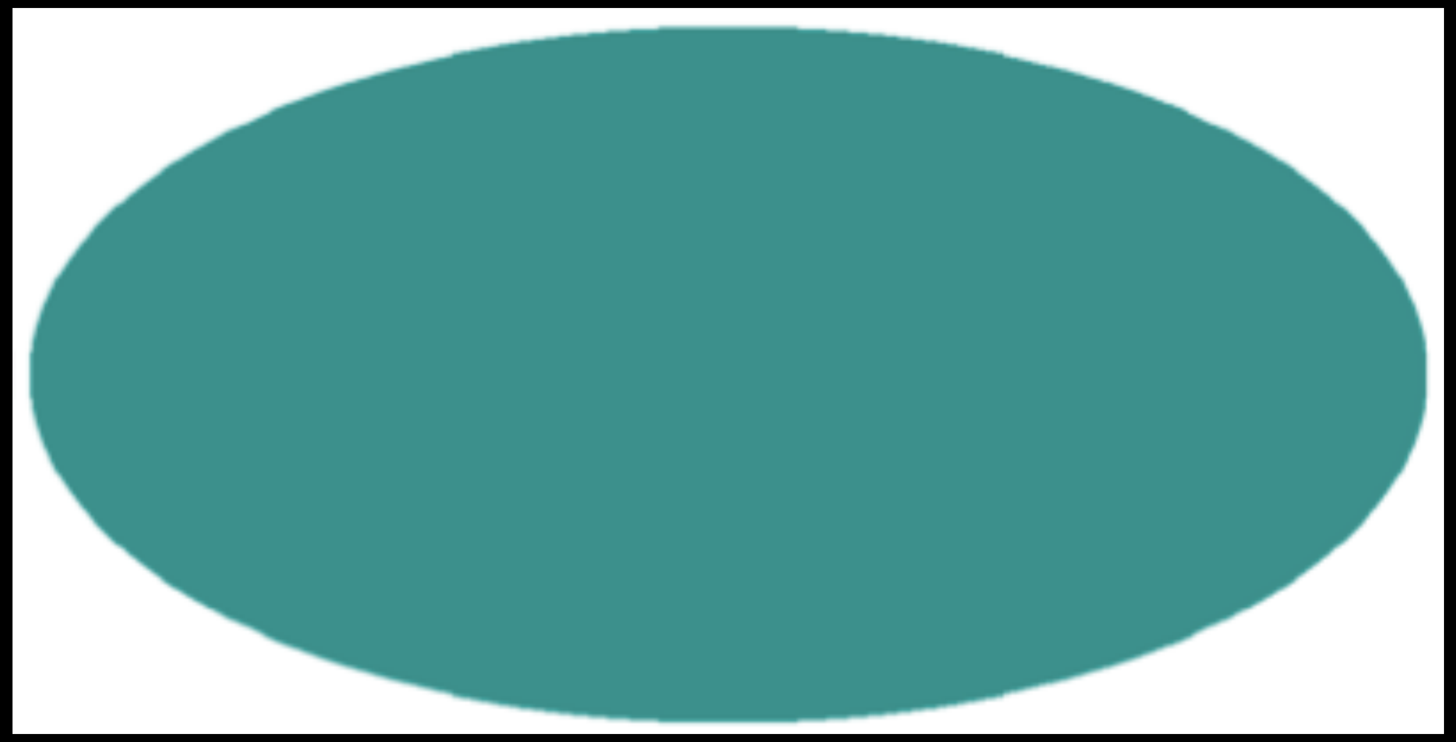
(Gaussian, Unpolarized, Stationary in space and time)

- Effect of discreteness (see Allen 2023, Allen and Valtolina 2024, Lamb and Taylor 2024, Allen et al 2024).

HD Curve

- Isotropic Universe

$$\psi(\hat{\Omega}) = 1$$

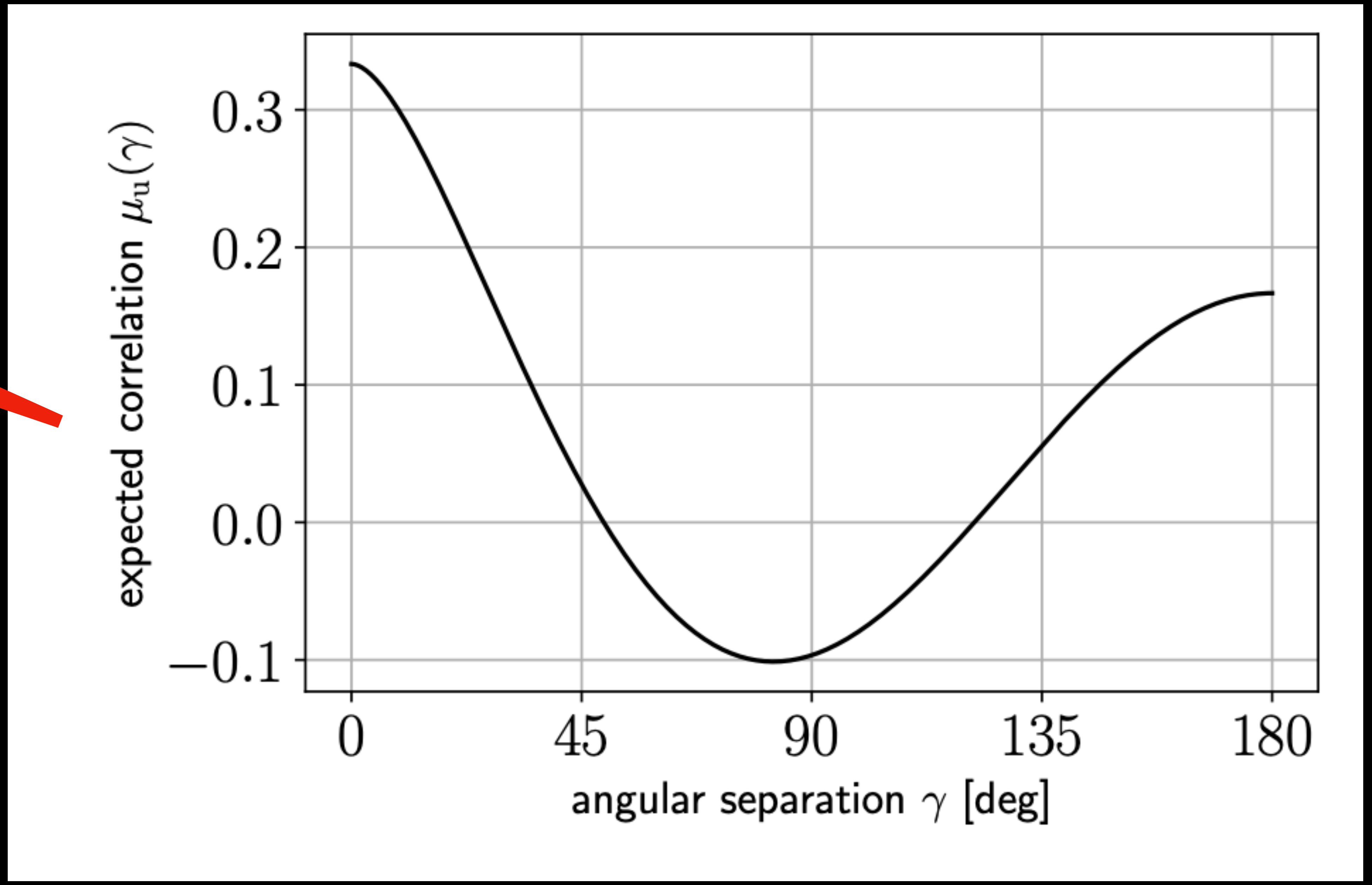


Correlation

$$\rho_{12} = \overline{z_1(t)\, z_2(t)}$$

$$\langle \rho_{12} \rangle_h = h^2 \mu_u(\gamma)$$

$$h^2 = 4\pi \int df\, H(f)$$



- Ensemble Averaged Correlation = HD curve.
- Independent of Pulsar-pair position.

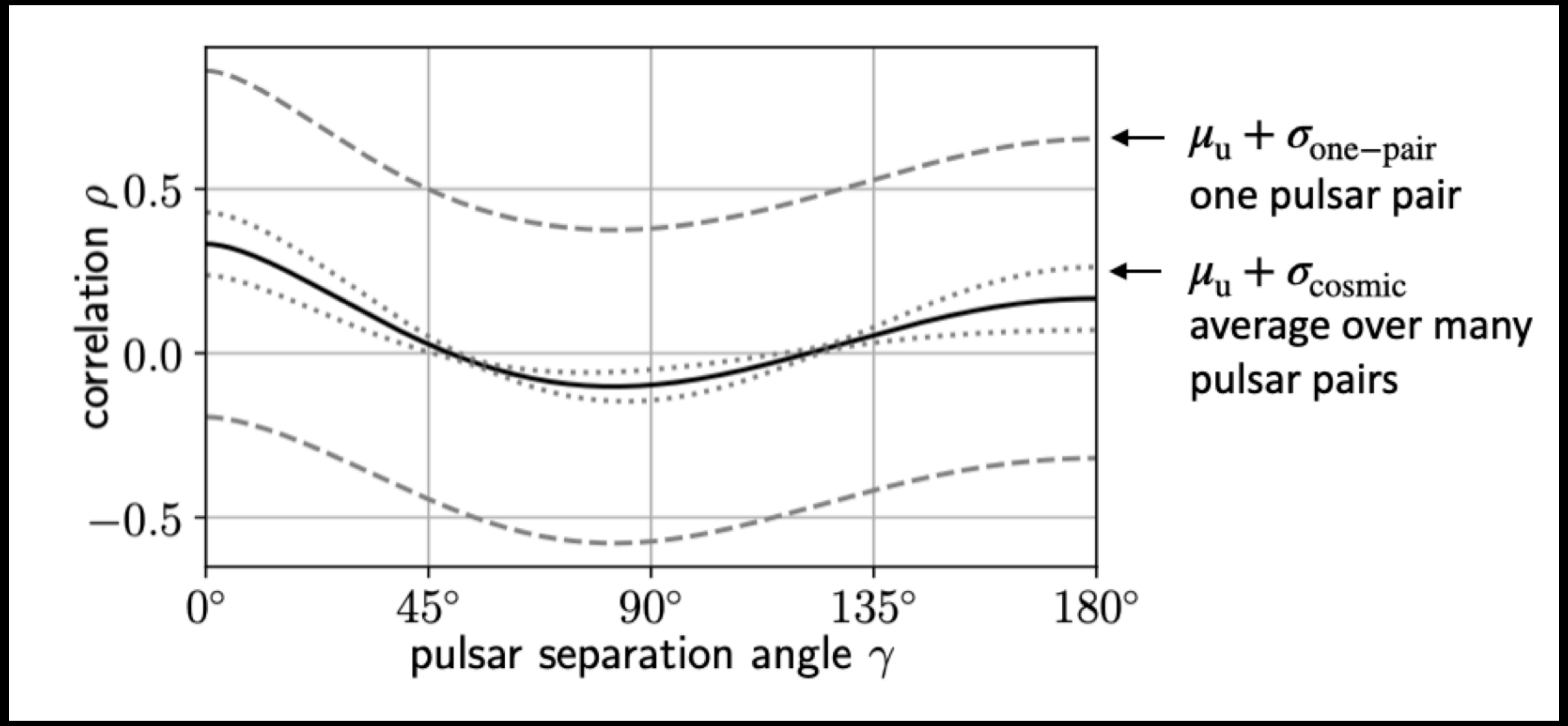
Romano and Allen, Class. Quantum Grav. 41 175008 (2024)

Pulsar Variance

Allen, Phys. Rev. D 107, 043018, 2023

Fluctuations in correlation curve away from HD curve:

Due to variation in pulsar pair's position in sky

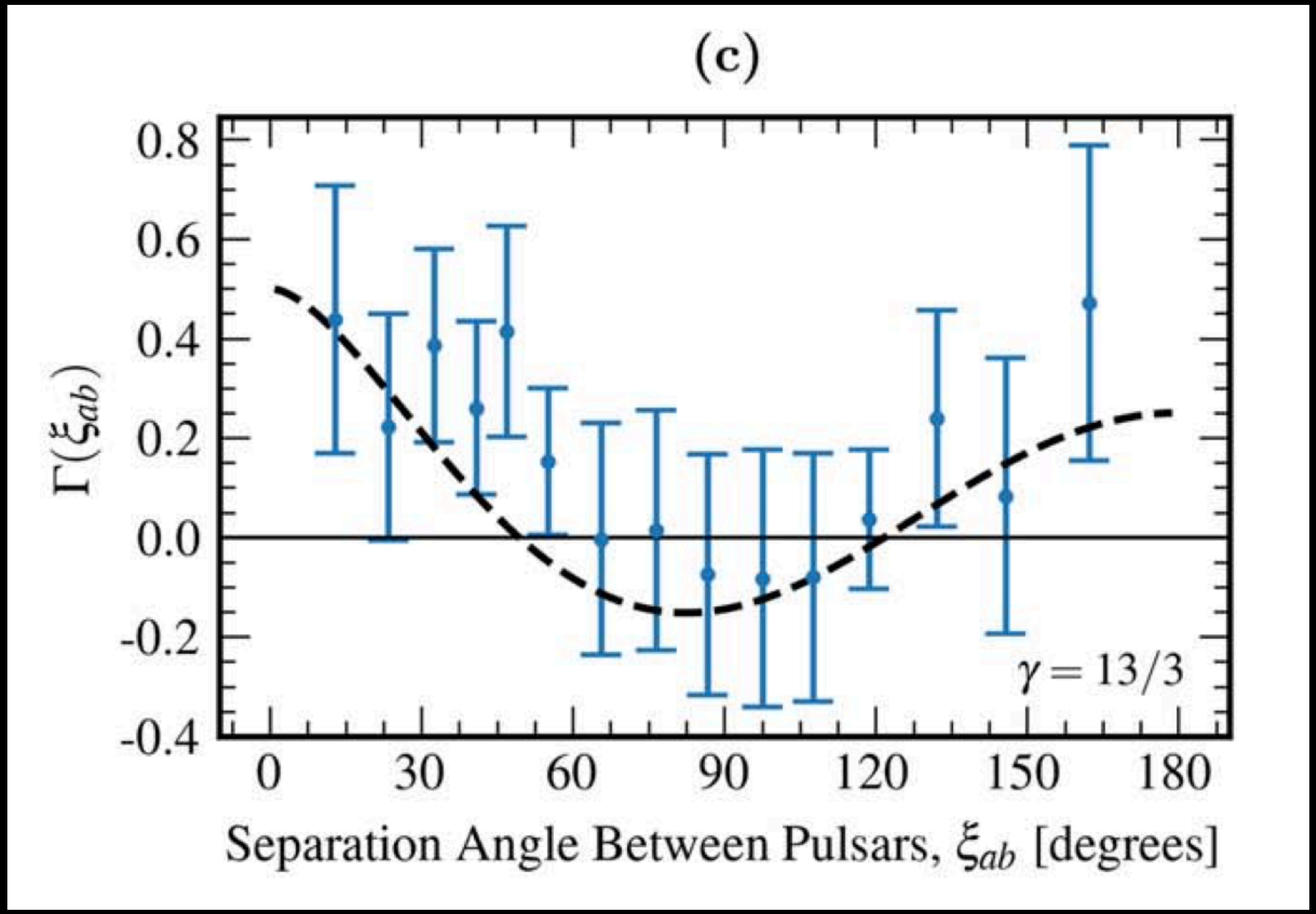


Allen, Frascati Physics Series Vol. 74 (2022)

Bin the pulsars Optimally combine the correlation

Pulsar Averaging

$$\Gamma(\gamma) \equiv \langle \rho_{12} \rangle_{12 \in \gamma}$$



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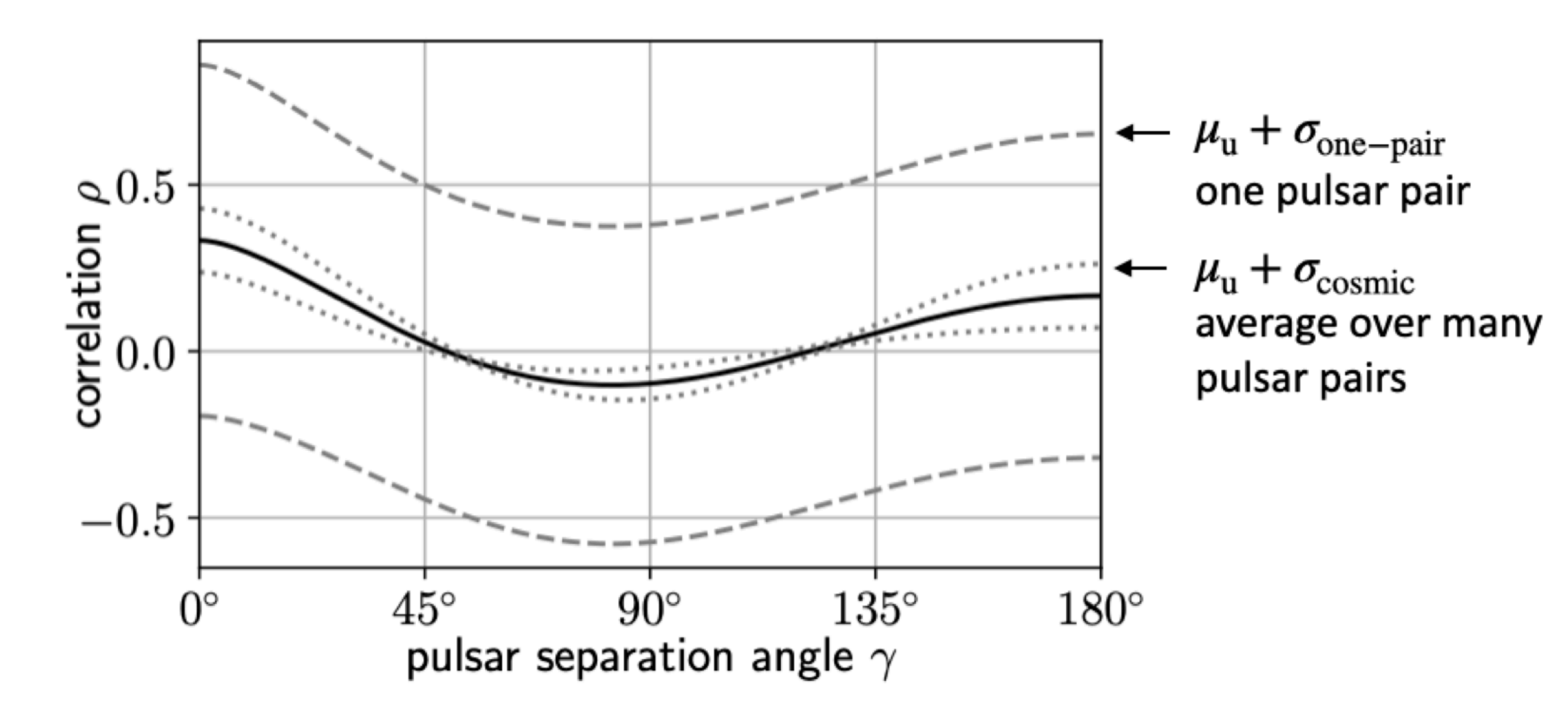
Cosmic Variance

Allen, Phys. Rev. D 107, 043018, 2023
Allen and Romano, Phys. Rev. D 108, 043026, 2023
Allen, Frascati Physics Series Vol. 74 (2022)

Fluctuations in correlation curve away from HD curve:

Due to variation in realization of GWB signal



Cosmic Variance

Geometric Factor

$$\sigma^2_{\rm cosmic}(\gamma) = \langle \Gamma(\gamma)^2 \rangle - \langle \Gamma(\gamma) \rangle^2 = 2\hbar^4 \tilde{\mu}^2(\gamma).$$

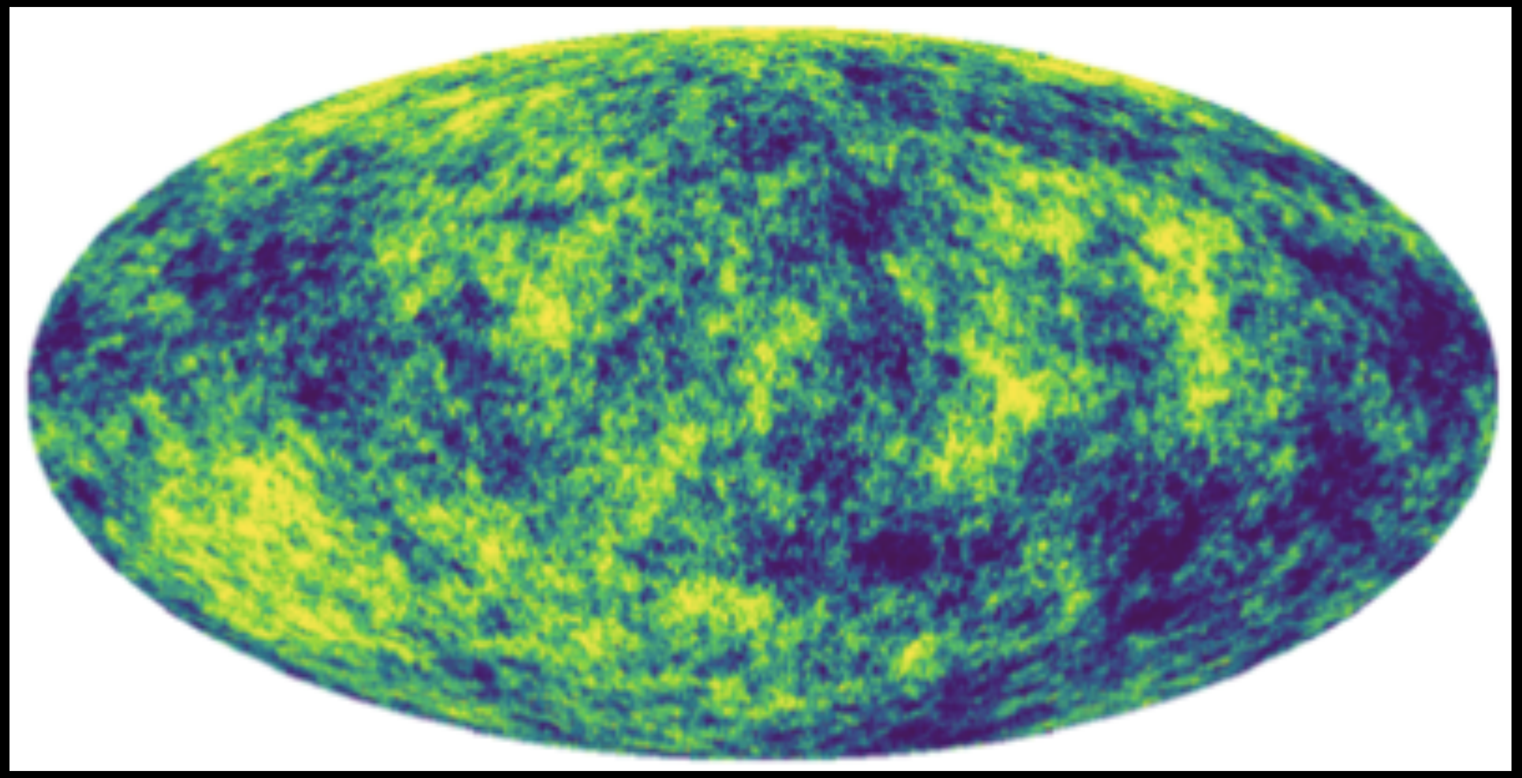
$$\Gamma(\gamma) \equiv \langle \rho_{12} \rangle_{12 \in \gamma}$$

Interferring Sources

$$\hbar^4 = (4\pi)^2 \int df_1 \int df_2 \, \mathrm{sinc}^2 \left[\pi T(f_1 - f_2)\right] H(f_1) H(f_2)$$

But Universe has Anisotropies !!

- Astrophysical sources are located in cosmic structure and may produce anisotropic background.
  - Different mechanism of GW production
  - Astrophysical distribution of sources in galaxies
  - Galaxy Formation and Large scale structure distribution
  - Spacetime geometry along line of sight



Credit: Alexander C. Jenkins

- Point-like (hot spot) and extended anisotropy

- Anisotropy can be useful in distinguishing between astrophysical and Cosmological sources!!

Outline of Talk

- Rotationally Invariant Ensemble with Angular Correlations
    - Description of ensemble
    - Spherical Harmonic Basis and Angular power spectrum
- Harmonic Decomposition of Antenna Pattern
- Cosmic Moments of Correlation
    - Mean
    - Cosmic Variance

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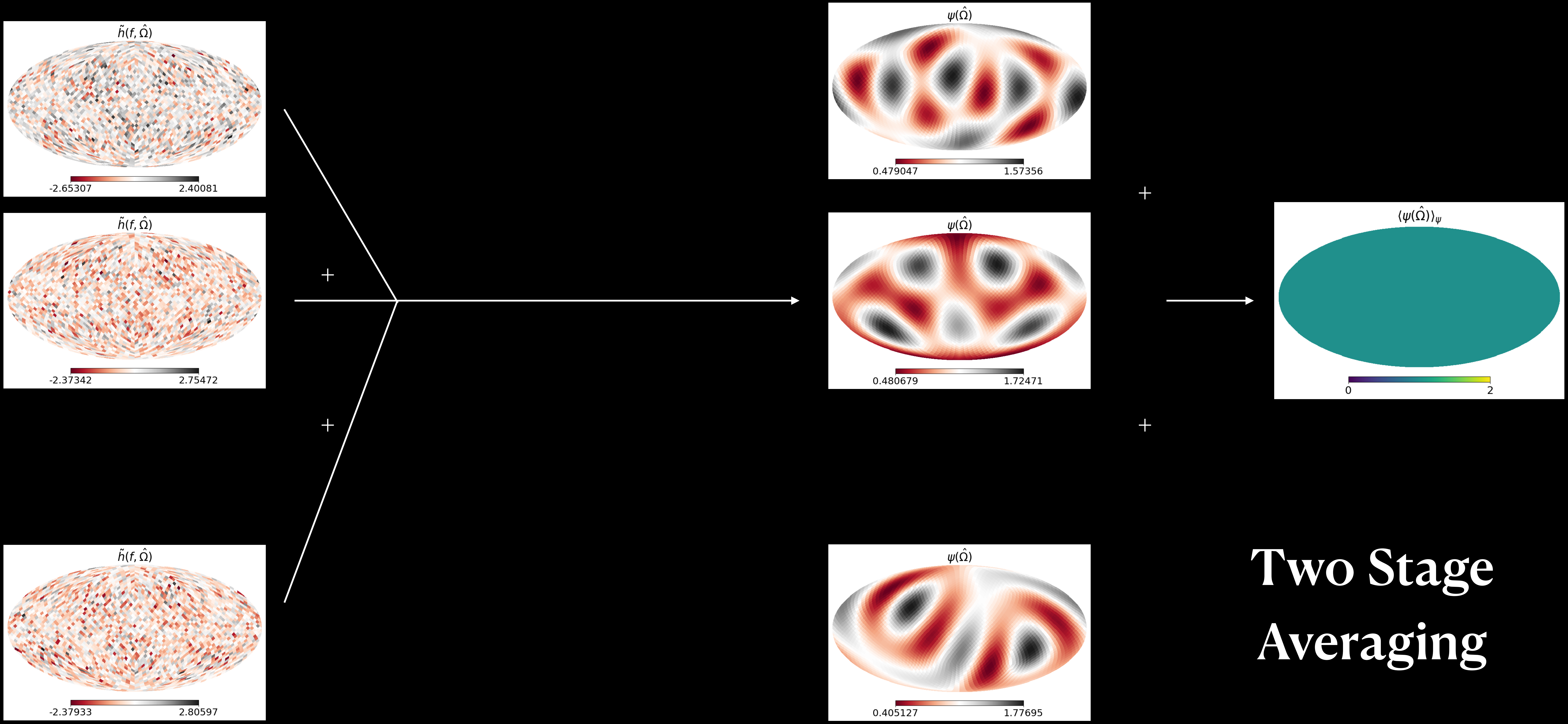
Rotationally Invariant Universe ($C_4 \neq 0, H(f) = 1$)



Two Stage Averaging

Rotationally Invariant Universe

- Gaussian ensemble for strain with fixed $\psi(\hat{\Omega})$

$$\langle \tilde{h}_A(f, \hat{\Omega}) \rangle_h = 0$$

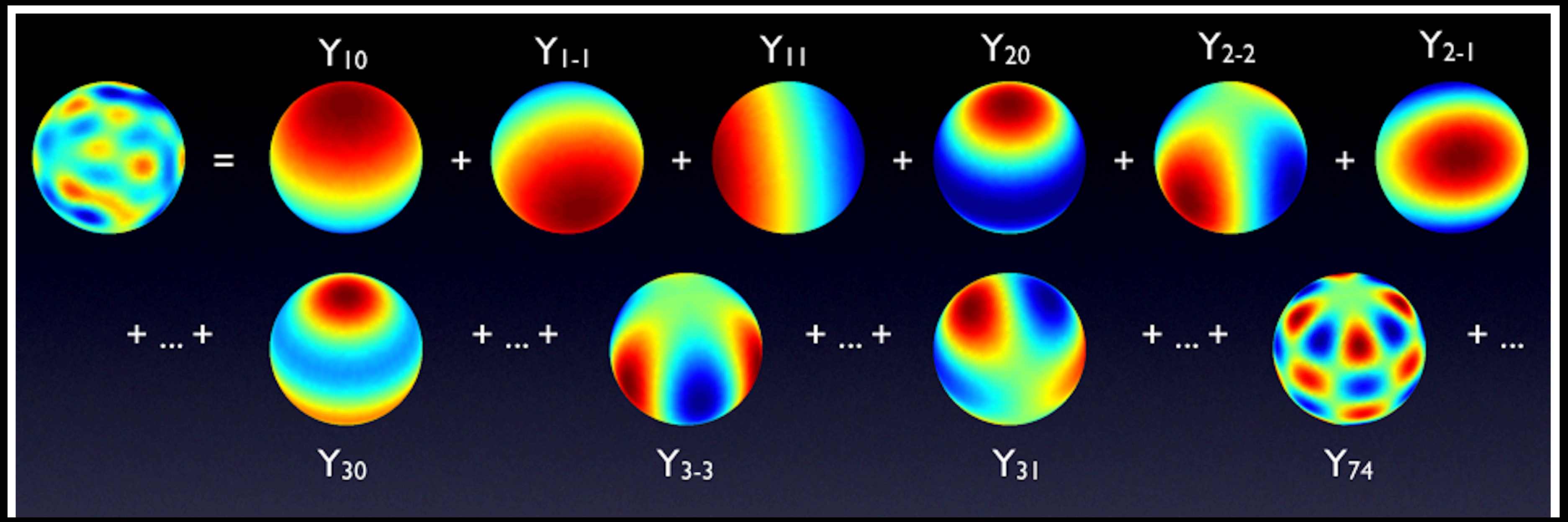
$$\langle \tilde{h}_A(f, \hat{\Omega})\, \tilde{h}^*_{A'}(f', \hat{\Omega}') \rangle_h = \psi(\hat{\Omega}) H(f)\, \delta_{AA'} \delta(f - f') \delta^2(\hat{\Omega}, \hat{\Omega}')$$

- Rotationally invariant ensemble for $\psi(\hat{\Omega})$

$$\langle \psi(\hat{\Omega}) \rangle_\psi = 1$$

$$\langle \psi(\hat{\Omega})\, \psi(\hat{\Omega}') \rangle_\psi - \langle \psi(\hat{\Omega}) \rangle_\psi \langle \psi(\hat{\Omega}') \rangle_\psi = C(\hat{\Omega} \cdot \hat{\Omega}')$$

- Spherical Harmonic Basis



doi: https://doi.org/10.1371/journal.pone.0044439.g012

$$\langle \psi_{lm} \rangle_\psi = \sqrt{4\pi}\, \delta_{l0} \delta_{m0}\,,$$

$$\langle \psi_{lm} \psi^*_{l'm'} \rangle_\psi - \langle \psi_{lm} \rangle_\psi \langle \psi^*_{l'm'} \rangle_\psi = C_l\, \delta_{ll'} \delta_{mm'}\,,$$

Angular Power Spectrum

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Reminder of Pulsar and Sky Averaging

$$\rho_{12} = \overline{z_1(t)\, z_2(t)}$$

$$= \sum_{A,A'} \int df \int df' \int_{S^2} d^2\hat{\Omega} \int_{S^2} d^2\hat{\Omega}'\, \tilde{h}_A(f,\hat{\Omega})\tilde{h}^*_{A'}(f',\hat{\Omega}')\, F_1^A(\hat{\Omega})F_2^{A'}(\hat{\Omega}')\, \mathrm{sinc}[\pi T(f-f')]$$

- Pulsar Averaging: $\Gamma = \langle \rho_{12} \rangle_{12\in\gamma}$
- Cosmic Mean: $\langle\langle \Gamma \rangle_h \rangle_\psi$
- Cosmic Variance: $\sigma^2_{\mathrm{cosmic}} = \langle\langle \Gamma^2 \rangle_h \rangle_\psi - \langle\langle \Gamma \rangle_h \rangle^2_\psi$

Harmonic Decomposition of Antenna Pattern Functions and Pulsar Averaging

Bernardo et al 2023, Phys. Rev. D 107, 044007

Bernardo and Ng 2022, JCAP11(2022)046

- $$F^A(\hat{\Omega}, \hat{p}) = \sum_{l,m} F^A_{lm}(\hat{\Omega}) Y_{lm}(\hat{p})$$

- $$\langle Y_{lm}(\hat{p}_1) Y^*_{l'm'}(\hat{p}_2) \rangle_{12 \in \gamma} = \delta_{ll'} \delta_{mm'} \frac{P_l(\cos\gamma)}{4\pi}$$

Pulsar Averaging

- $$\mu_{AA'}(\gamma, \hat{\Omega}, \hat{\Omega}') = \frac{1}{4\pi} \sum_{l,m} F^A_{lm}(\hat{\Omega}) F^{A'*}_{lm}(\hat{\Omega}')\, P_l(\cos\gamma)$$

- $$F^{A=+,\times}_{lm}(\hat{\Omega}) = -2\pi i (-i)^{2l-1}_{2l} \sqrt{\frac{(l-2)!}{(l+2)!}} \left[{}_{-2}Y^*_{lm}(\hat{\Omega}) e^{-i2\alpha(\hat{\Omega})} \pm {}_{2}Y^*_{lm}(\hat{\Omega}) e^{i2\alpha(\hat{\Omega})} \right] \quad \text{for } l \geq 2$$

Spin Weighted Spherical Harmonics

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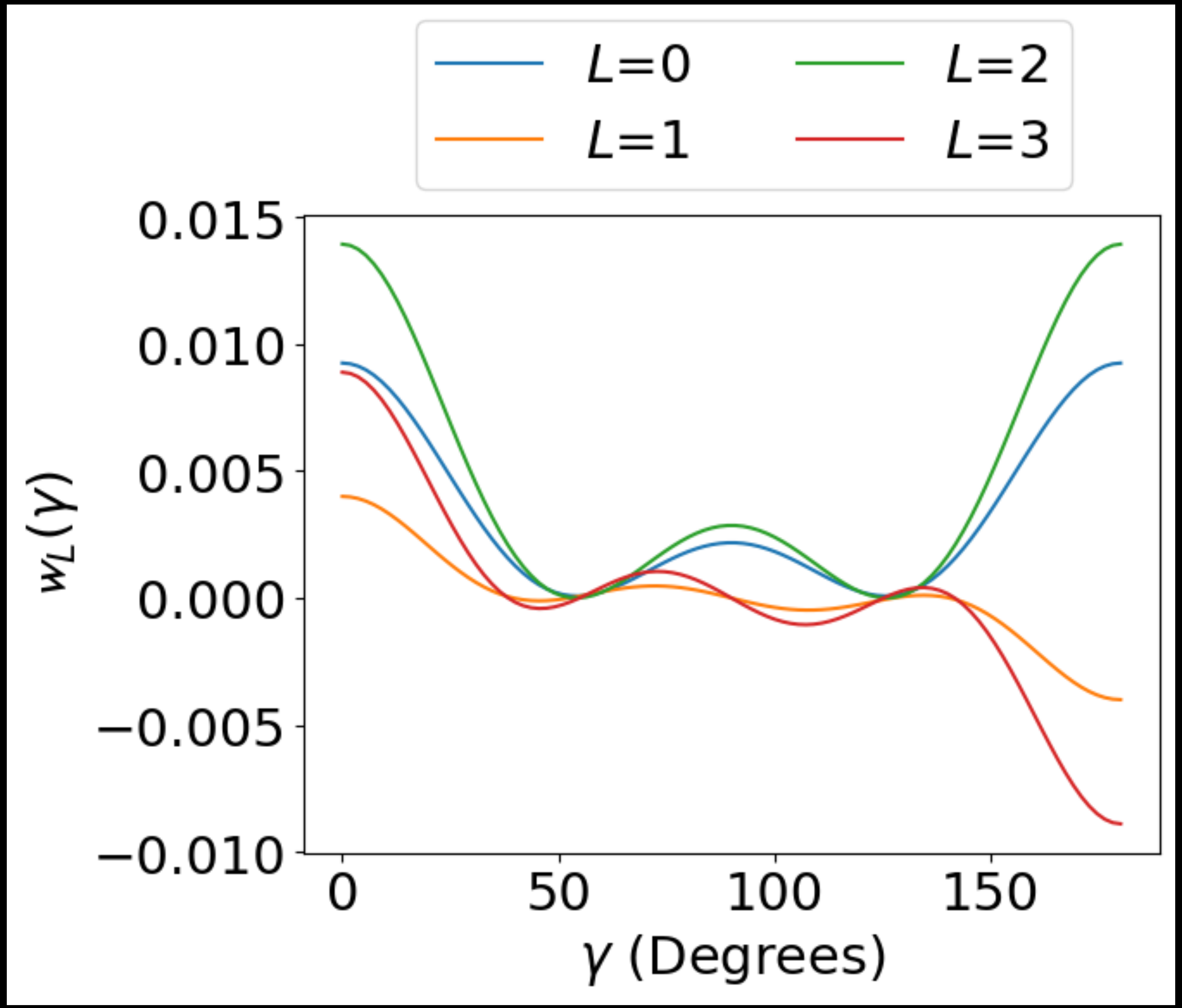
Cosmic Mean and Variance

- $\mu_{\text{cosmic}} = \langle\langle\Gamma\rangle_h\rangle_\psi = h^2\mu_u(\gamma)$ [No effect of Angular Correlation]
- $$\sigma^2_{\text{cosmic}}(\gamma) = 2\hbar^4\widetilde{\mu^2}(\gamma) + \frac{h^4}{4\pi}\mu_u^2(\gamma)\,C_0 + \frac{2\hbar^4}{4\pi}\sum_{L=0}^{\infty} \mathrm{w}_L(\gamma)\,C_L$$

Excess variance due to angular correlations

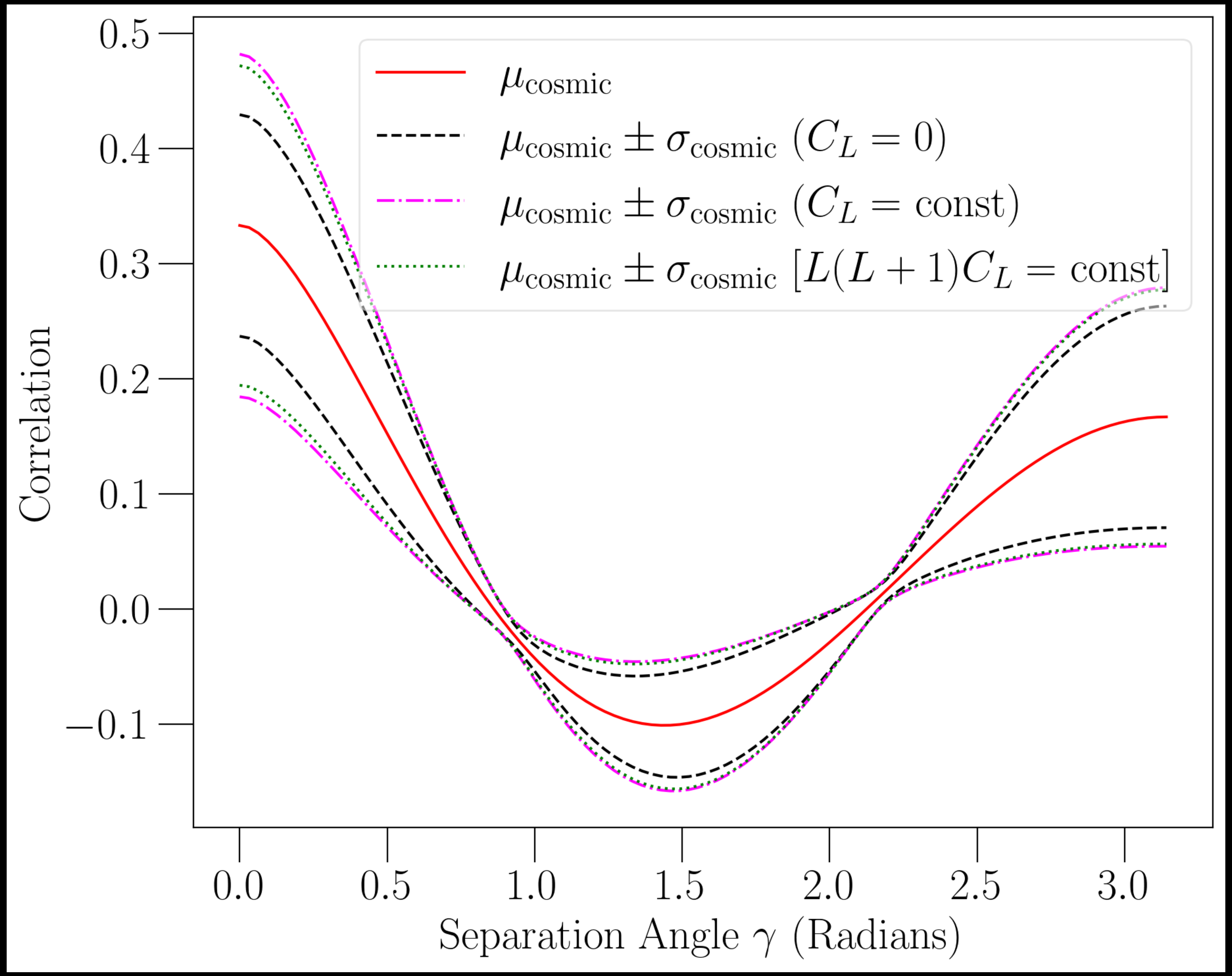


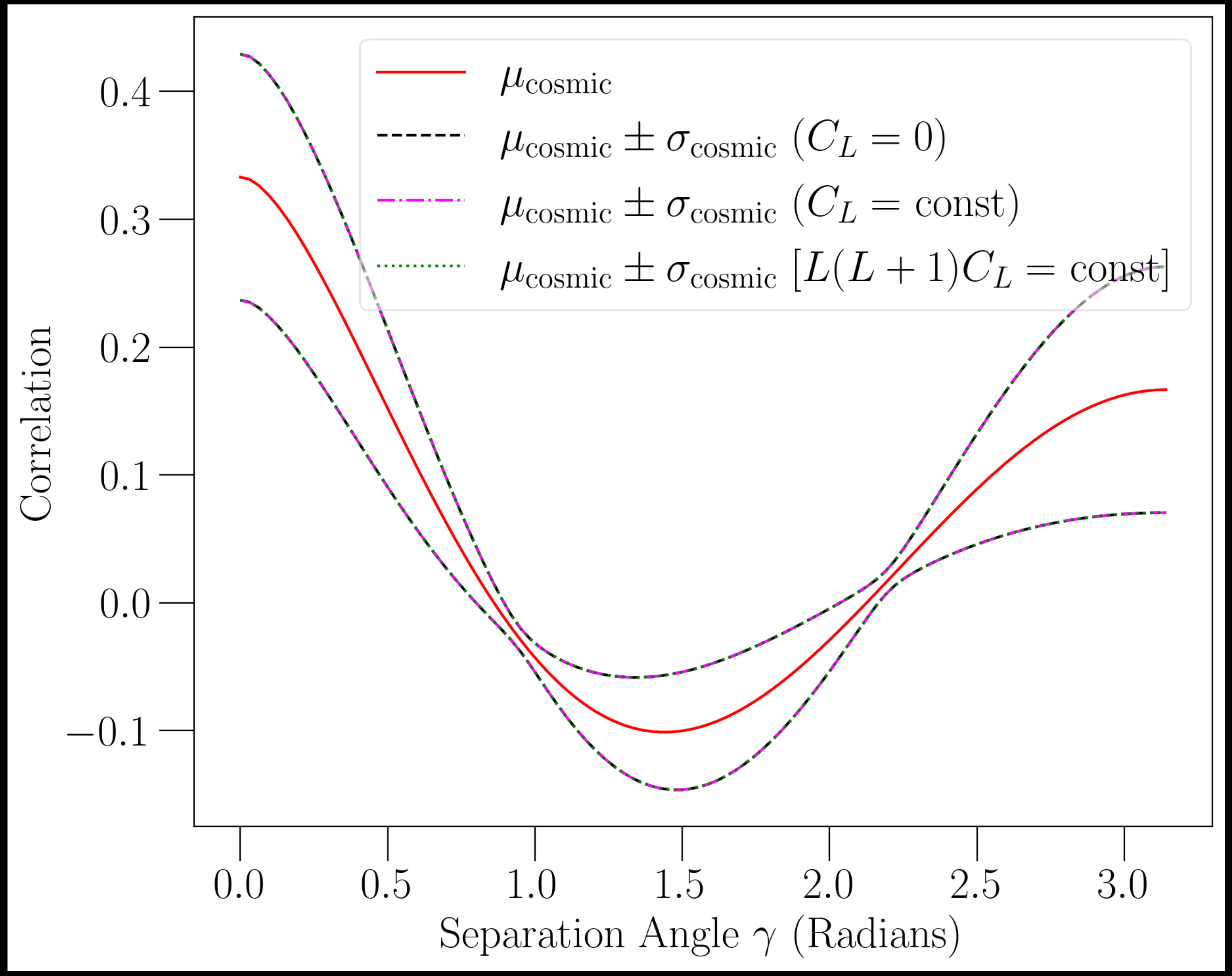
Effect of Angular Power Spectrum

Physical Review D 110, 043044

$\hbar^4/h^4 = 1/2$ $\quad h^2 = 1$ $\quad L_{\max} = 8$

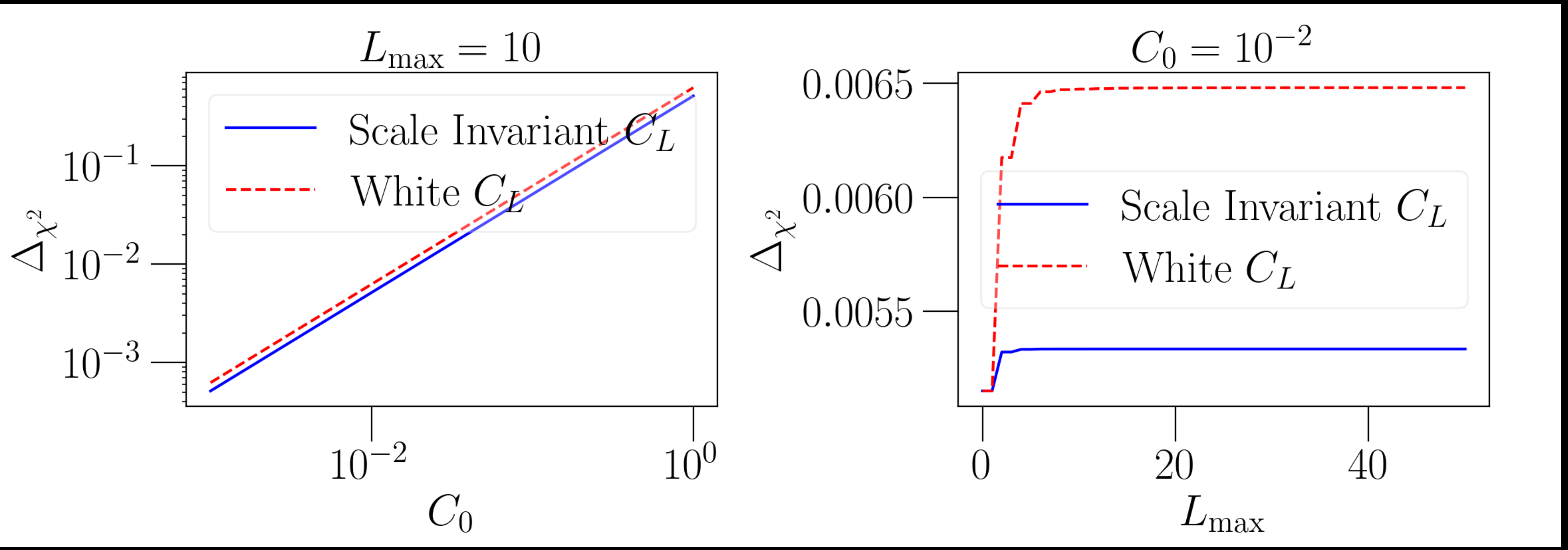
In the case of Non-zero C_L, **Left:** $C_0 = 1$, **Right:** $C_0 = 10^{-3}$





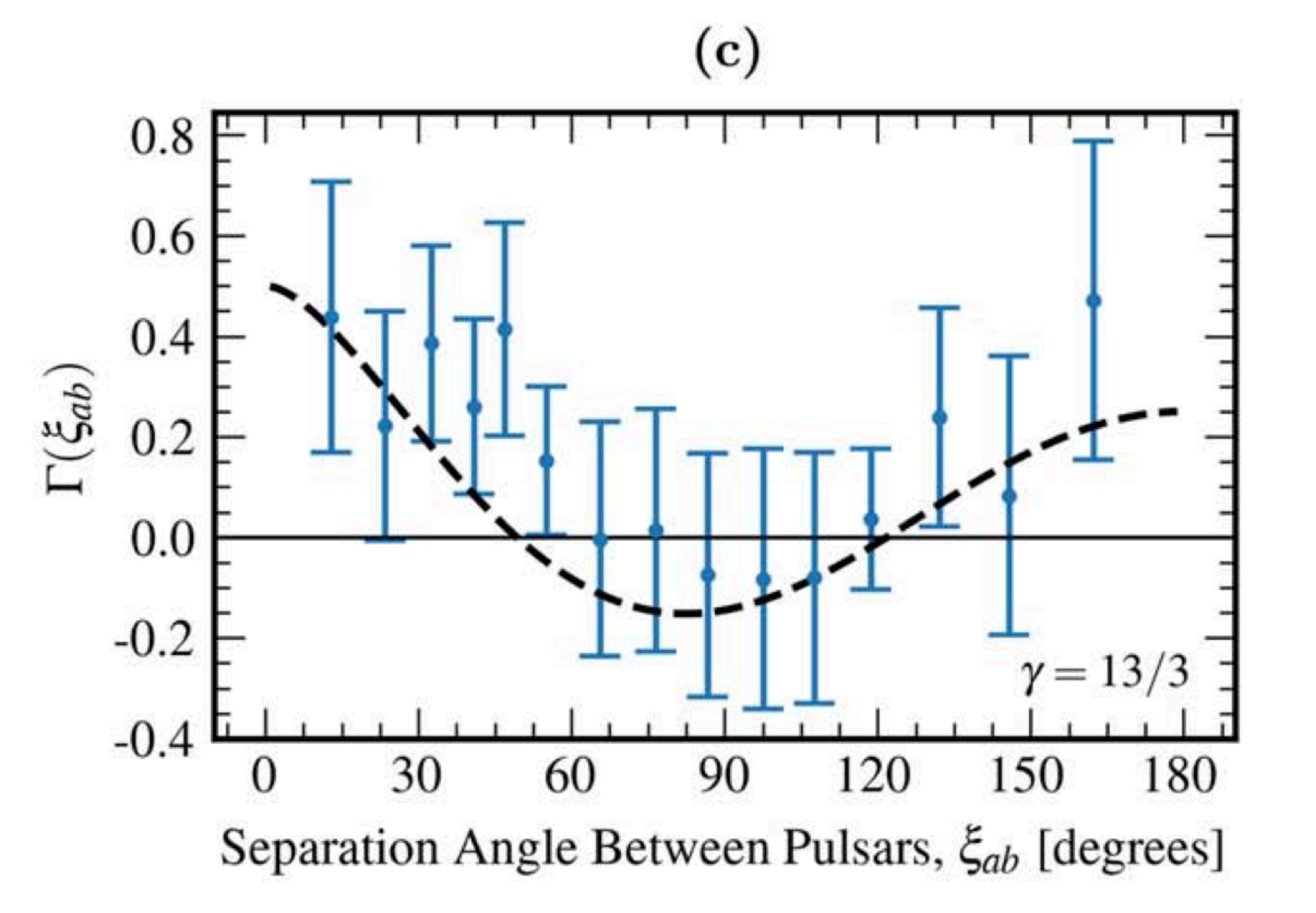
Difference in χ^2-statistic

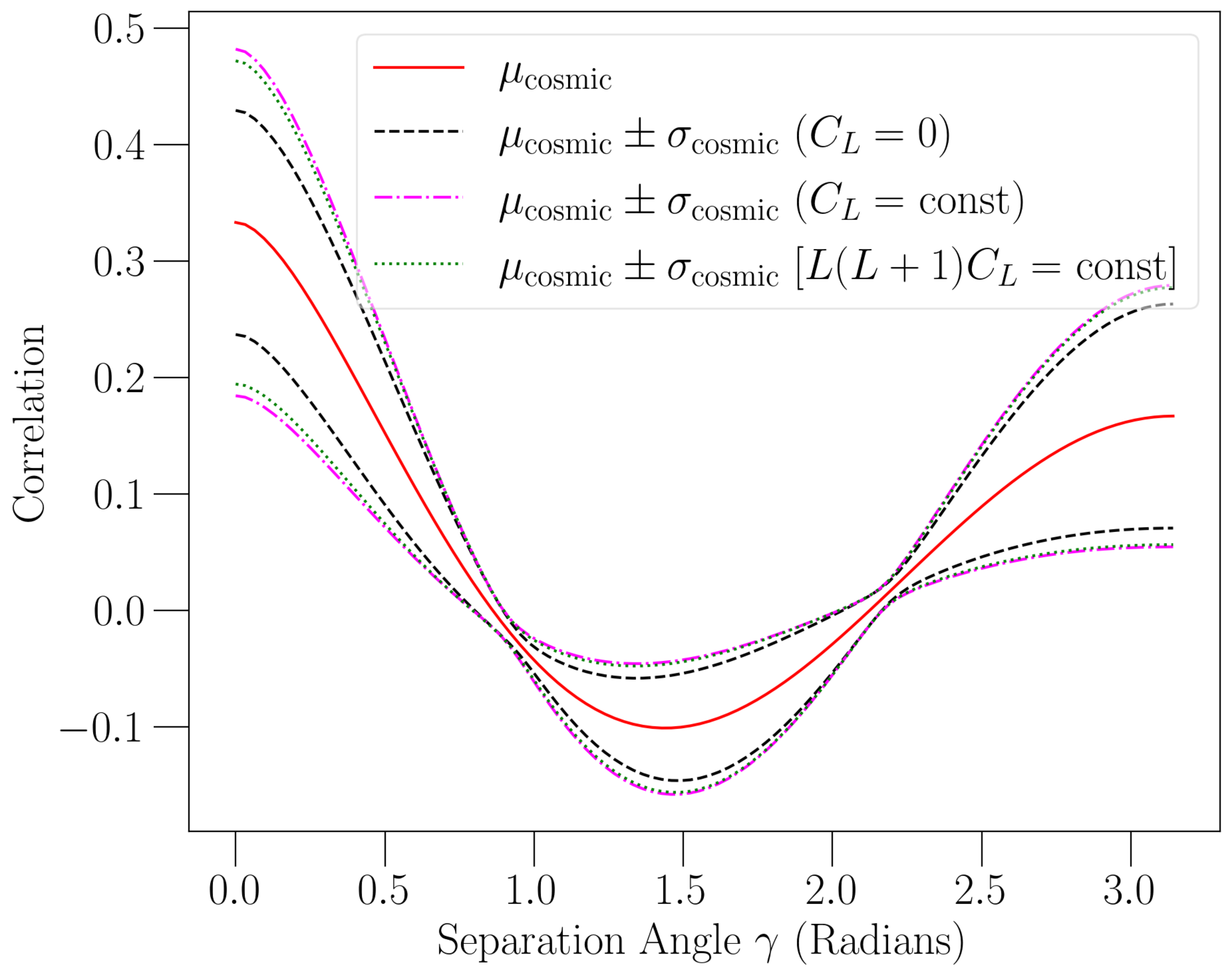
- We define a statistic
  - $$\chi^2 = \sum_{i=1}^{N} \frac{(x_i - \mu_{\text{cosmic},i})^2}{\sigma^2_{\text{cosmic},i}}$$
  - Source model with and without angular correlations



- The fractional difference in statistic
  - $$\Delta_{\chi^2} = \left| 1 - \frac{1}{N} \sum_{i=1}^{N} \frac{\sigma^2_{\text{cosmic},i}}{\sigma^2_{\text{cosmic,gauss},i}} \right|$$

Looking Forward to IPTA DR3 !!





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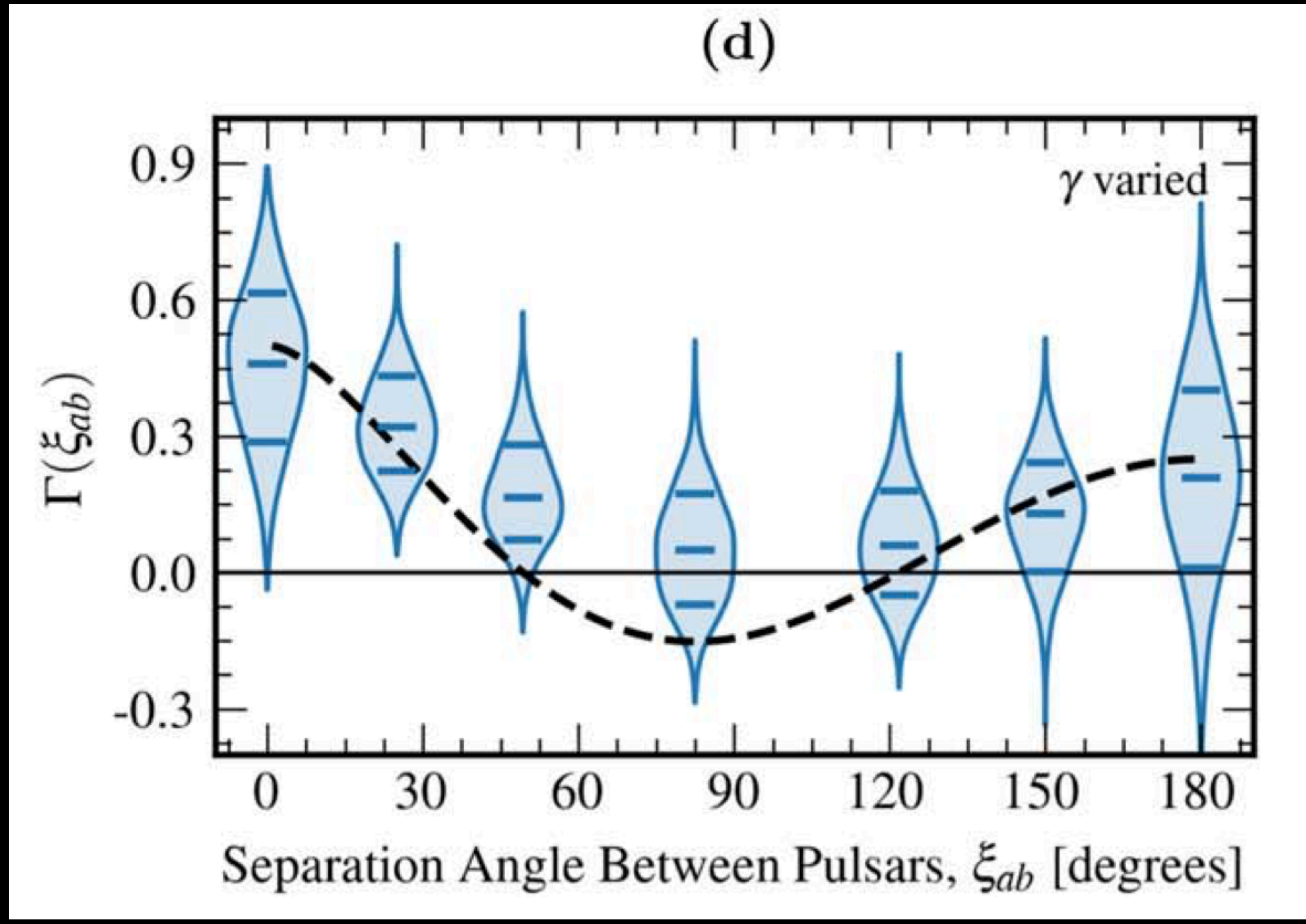
Summary

- In a universe described by Rotationally Invariant Ensemble, correlation curve is on-average HD curve.
- There will be variance due to angular power spectrum.
- The effect of angular correlations will be small for realistic values, i.e., $C_L/C_0 \lesssim 10^{-3}$.
- Harmonic space decomposition of antenna response to GWs simplifies the calculations.
- Our results are found to be consistent with work reported in Allen, Phys. Rev. D 110, 043043 (2024).

Extra Slides



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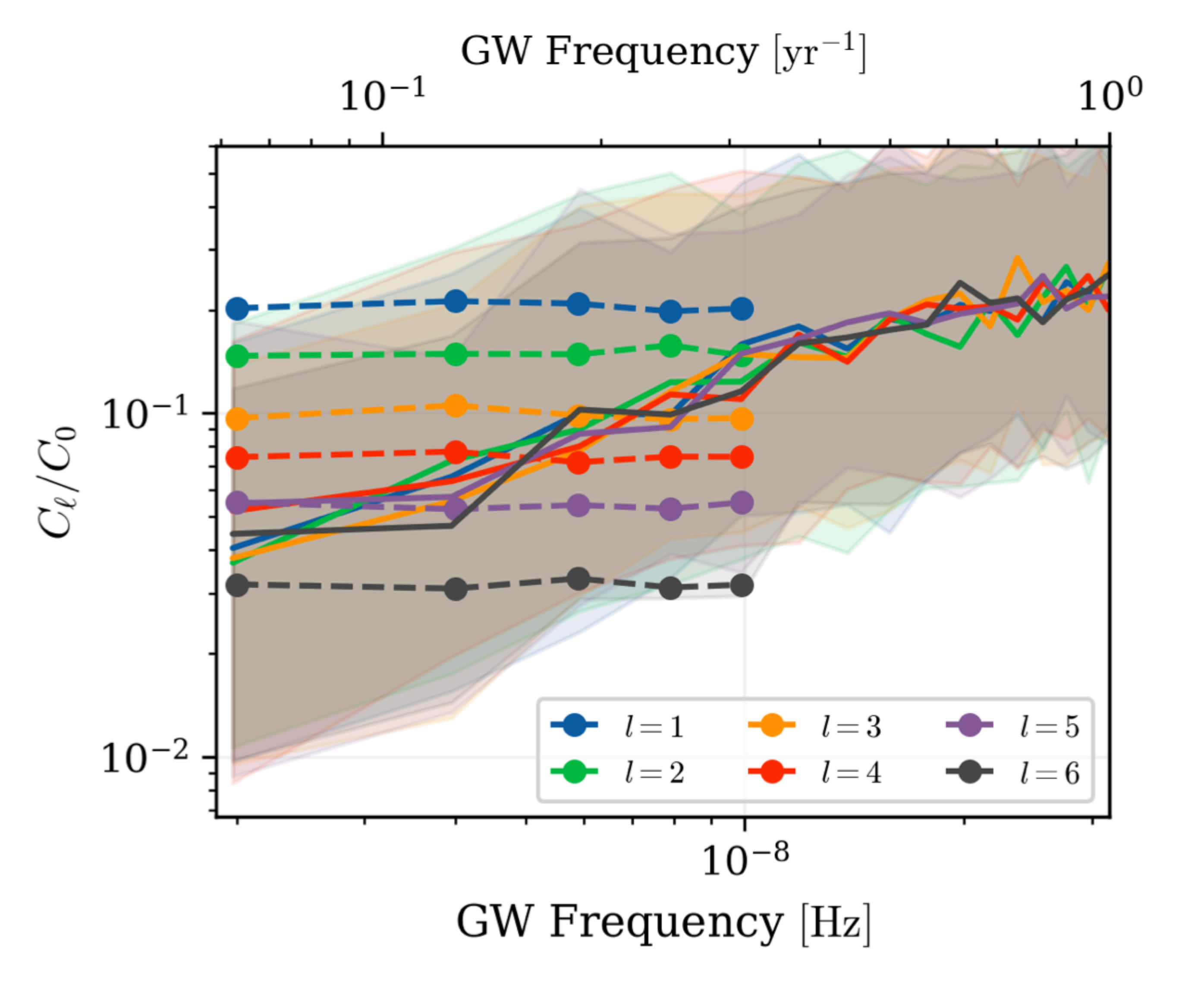

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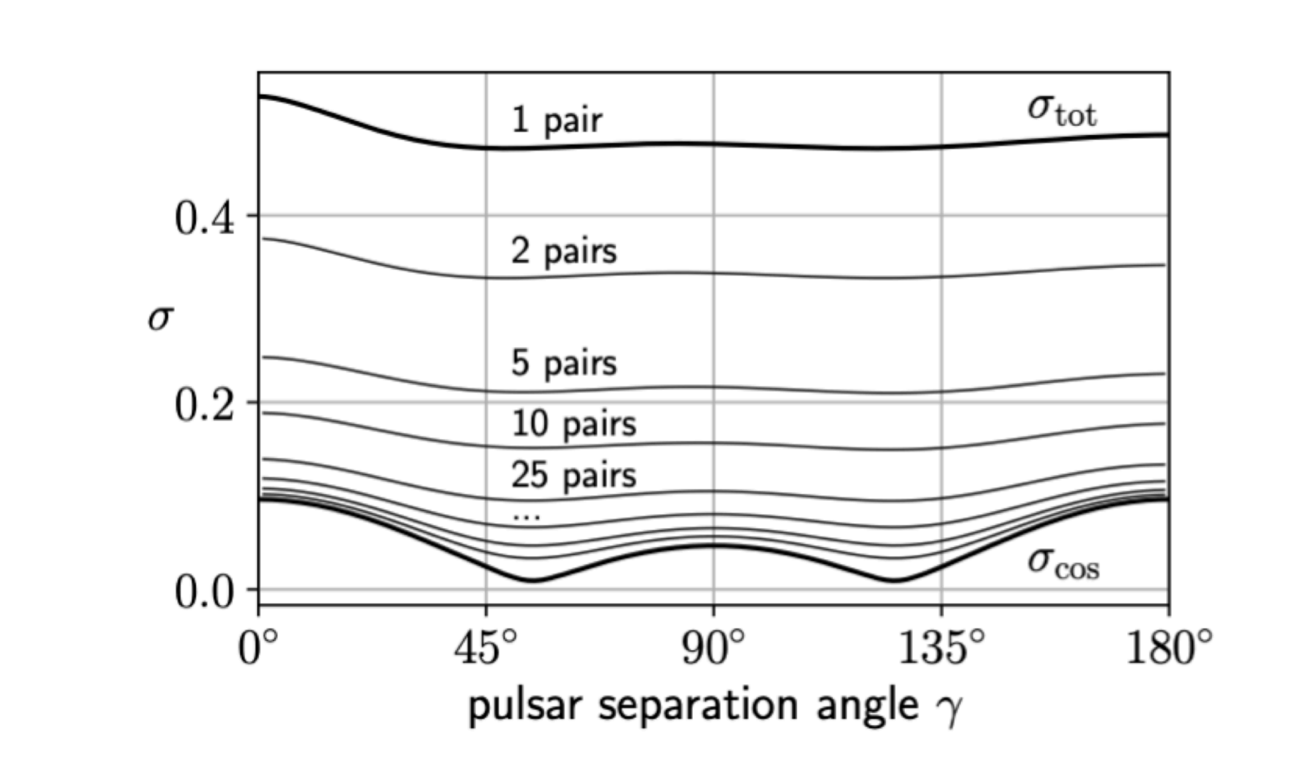

Figure 7: *The variance decreases as more pulsar pairs are added, but does not decrease to zero. Instead, it converges to the cosmic variance. (Plot is for a GW confusion-noise model, with $h^2 = 1$ and $\hbar^4/h^4 = 1/2$, see [2].)*

Allen, Frascati Physics Series Vol. 74 (2022)