UNIVERSITY OF BIRMINGHAM

How It's Made

Extreme-mass-ratio inspiral waveforms

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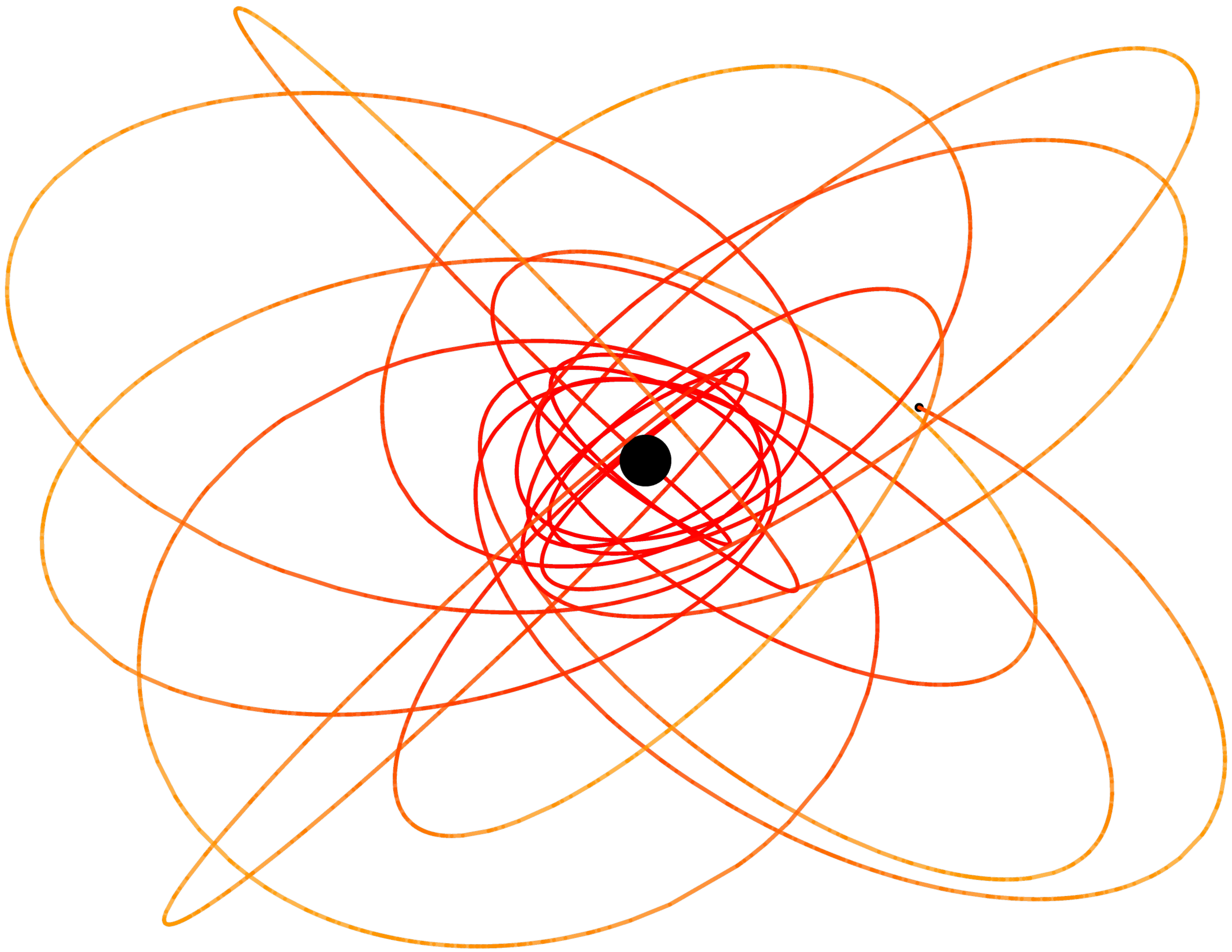
GraSP24, University of Pisa

Extreme-mass-ratio inspirals (EMRIs)

- Long-lived gravitational wave sources driven by strong-field relativistic dynamics.
- Stellar-mass compact object (CO) inspirals into massive black holes (MBHs) are a primary target of the LISA mission.
- Their intricate dynamics enable high measurement precision, making them astrophysical probes of enormous potential.
  - **Christopher Berry's talk**
- However, such complexity is challenging to model accurately at feasible cost.
  - **This talk (to some extent!)**



To build a waveform…

EMRIs are ill-suited to equal-mass binary modelling techniques.

- The computational cost of numerical relativity simulations of compact binaries scales with the (large) mass ratio of the system.
  - EMRIs are very expensive per orbital cycle, and the system must be modelled for **tens of thousands** of orbital cycles (with eccentricity).
- The inspiralling body is relativistic - standard Post-Newtonian expansions converge extremely poorly.
- **Gravitational self-force:** treats the metric deformation of the smaller body as a small perturbation in the spacetime of the larger body.

Multivoice decomposition framework

Hughes *et al.* (2021)

$$h(t) = \frac{\mu}{d_{\mathrm{L}}} \sum_{lmkn} H_{lmkn}(t, \theta, \phi) e^{-i\Phi_{mkn}(t)}$$

Multivoice decomposition framework

Hughes *et al.* (2021)

(Angle-dependent) mode amplitude

CO mass

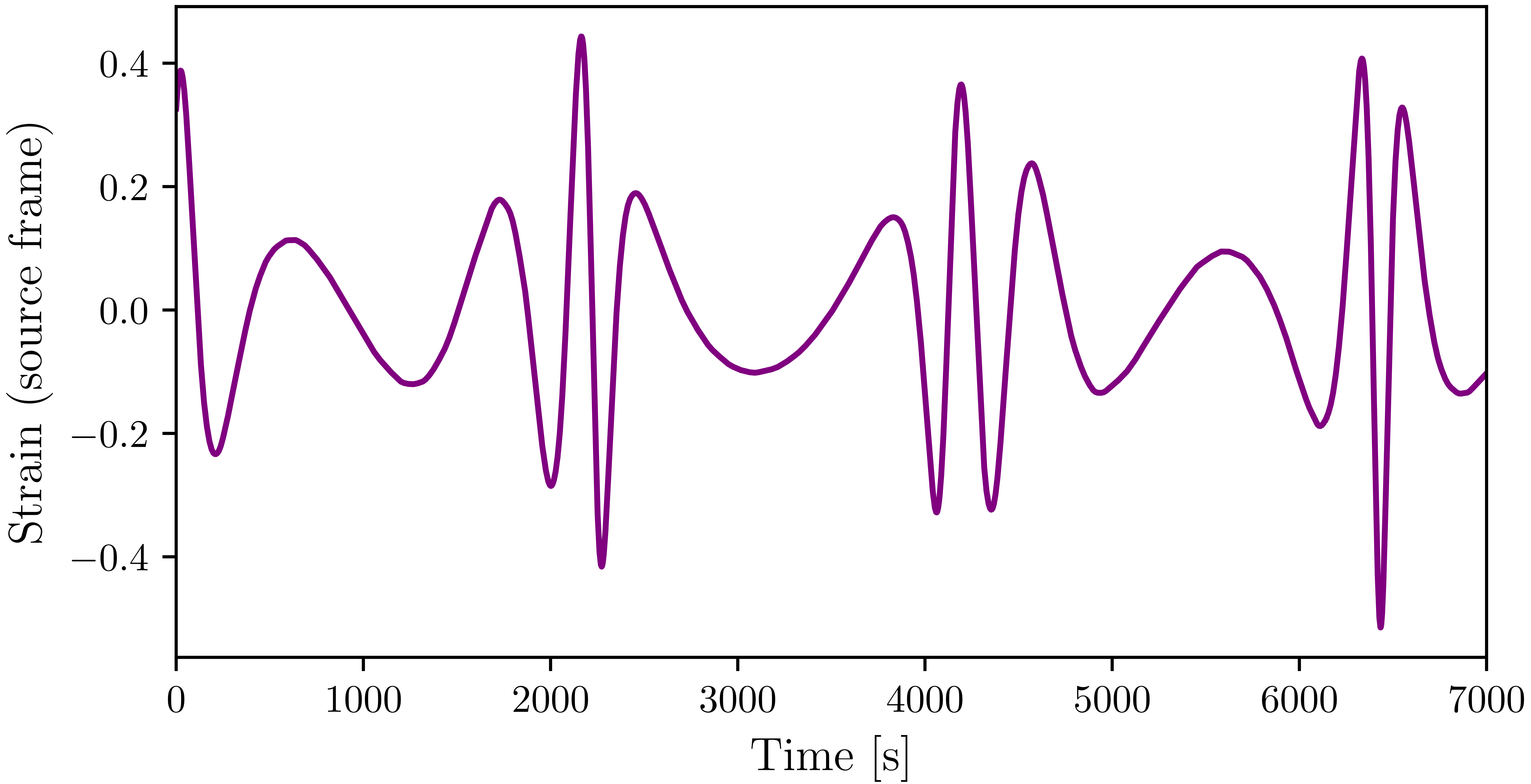
$$h(t) = \frac{\mu}{d_{\mathrm{L}}} \sum_{lmkn} H_{lmkn}(t, \theta, \phi) e^{-i\Phi_{mkn}(t)}$$

Luminosity distance

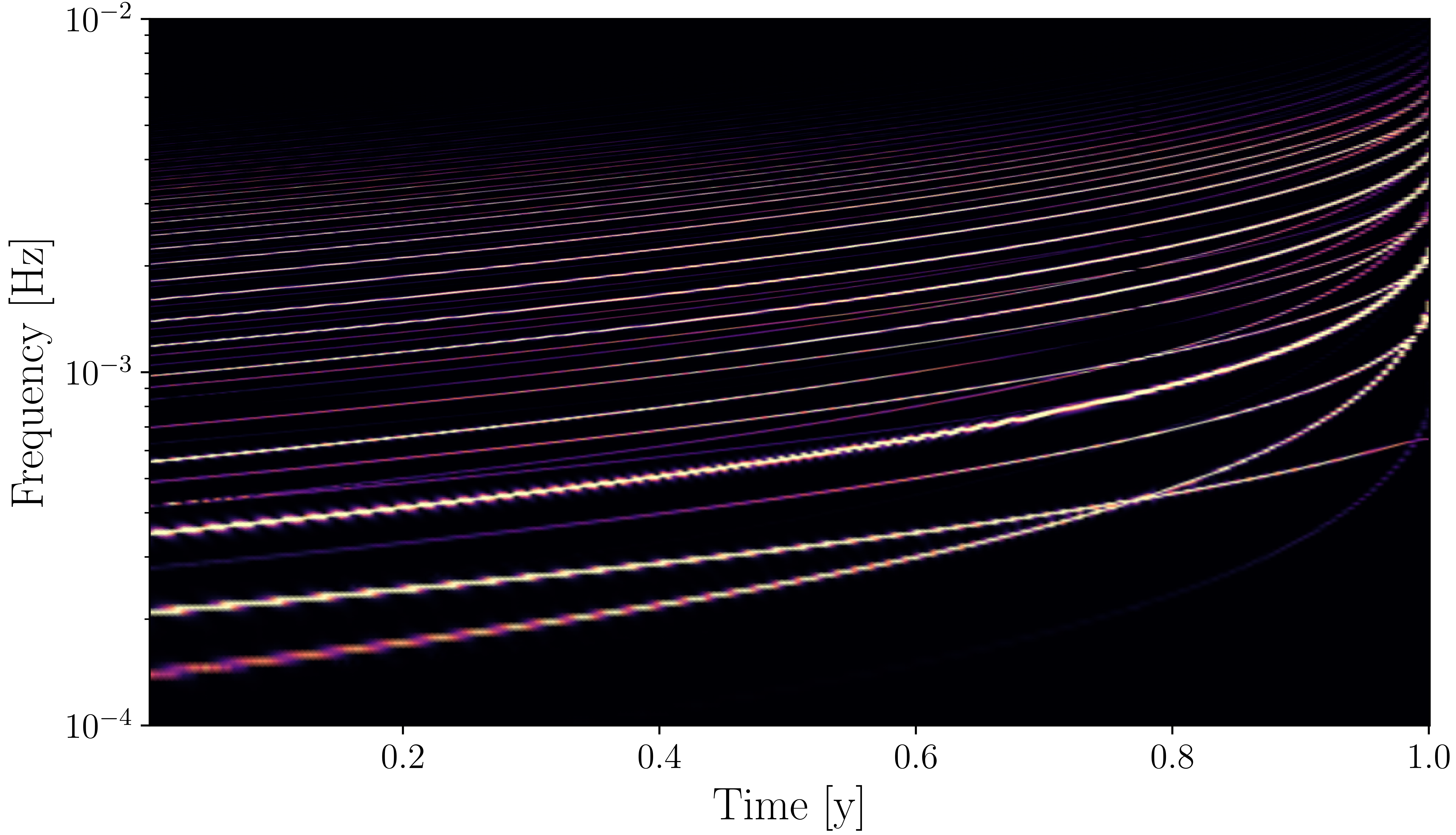
Harmonic of orbital phases

$$\Phi_{mkn} = m\Phi_\phi + k\Phi_\theta + n\Phi_r$$

(l, m, k, n) are harmonic mode indices



0.4
0.2
0.0
−0.2
−0.4
Strain (source frame)
0
1000
2000
3000
4000
5000
6000
7000
Time [s]



10^{-2}
10^{-3}
10^{-4}
Frequency [Hz]
0.2
0.4
0.6
0.8
1.0
Time [y]

Some other parameter conventions…

M : MBH mass

ϵ : Small mass ratio (μ/M)

a : MBH (dimensionless) spin parameter

p : Inspiral semi-latus rectum

e : Inspiral eccentricity

x_I : Cosine of inspiral inclination

Some other parameter conventions…

M : MBH mass

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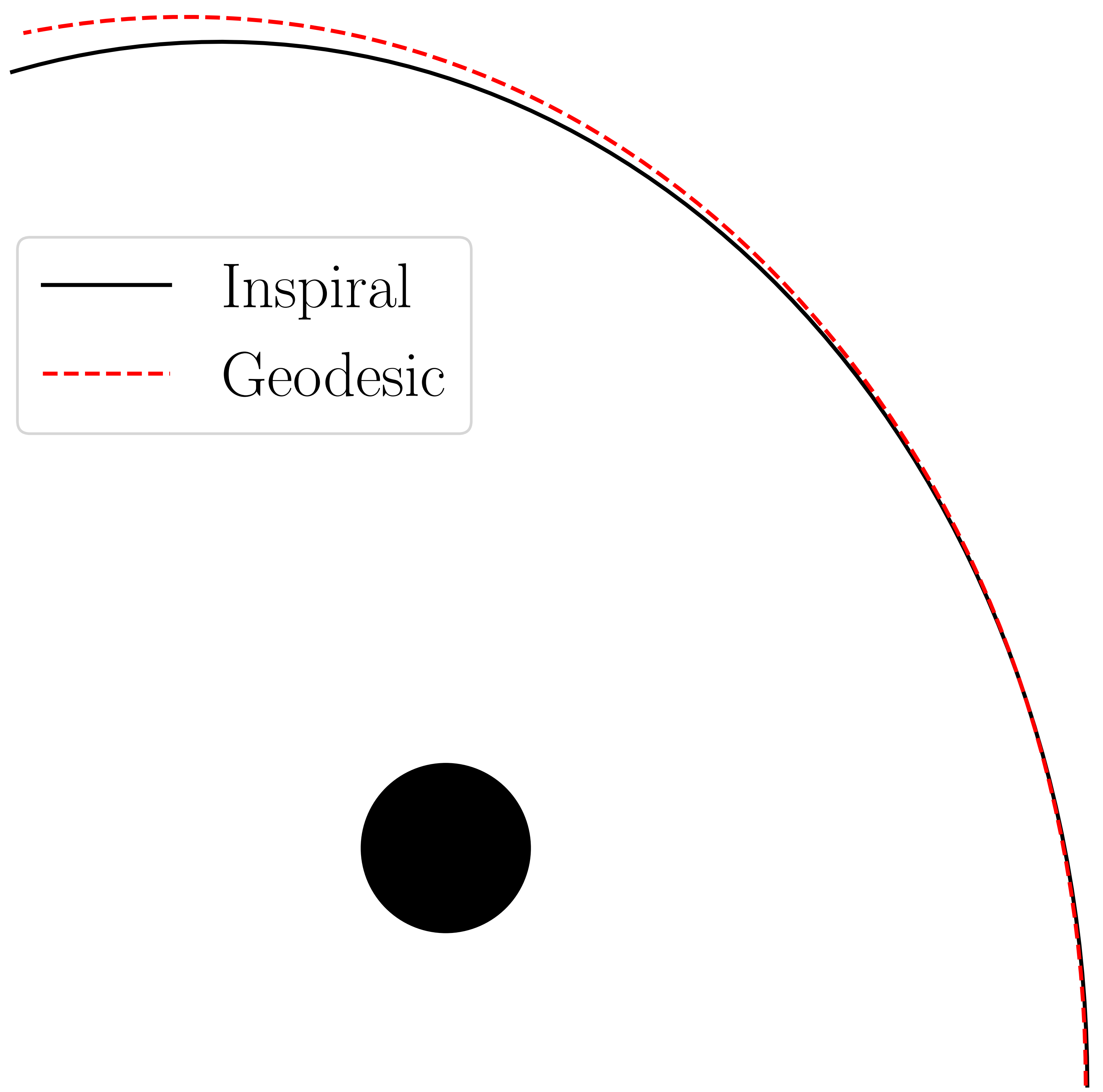
e : Inspiral eccentricity

x_I : Cosine of inspiral inclination

(p, e, x_I): **Orbital elements** (evolve with time)

Hughes *et al.* (2021)

Orbital trajectory modelling



Osculating geodesics: At any time, orbit is described by parameters of tangent geodesic

$$\frac{\mathrm{d}p}{\mathrm{d}t} = \epsilon f_p(a,p,e,x_I) + \mathcal{O}(\epsilon^2)$$

$$\frac{\mathrm{d}e}{\mathrm{d}t} = \epsilon f_e(a,p,e,x_I) + \mathcal{O}(\epsilon^2)$$

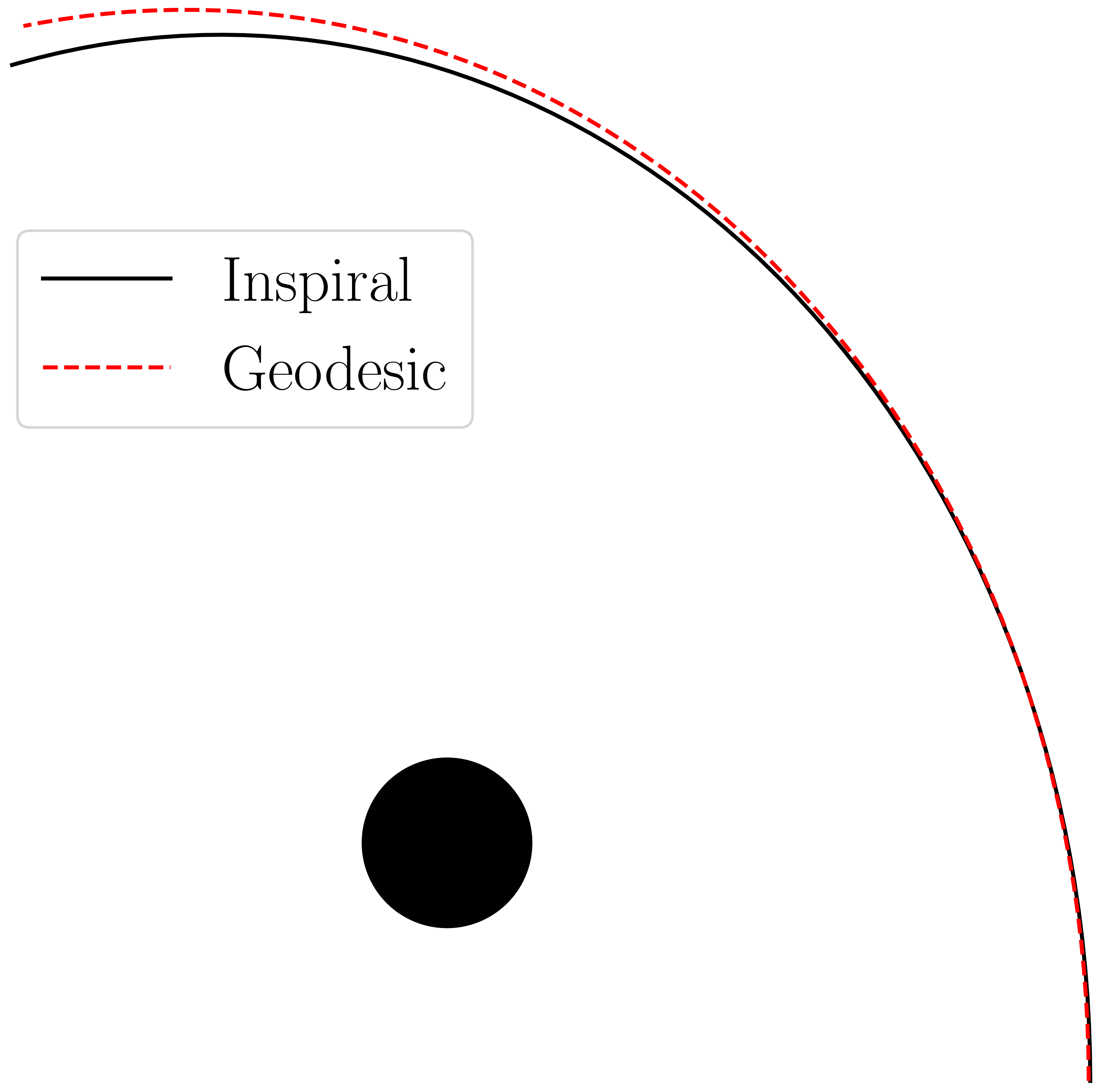
$$\frac{\mathrm{d}x_I}{\mathrm{d}t} = \epsilon f_{x_I}(a,p,e,x_I) + \mathcal{O}(\epsilon^2)$$

$$M\frac{\mathrm{d}\Phi_{\phi,\theta,r}}{\mathrm{d}t} = \Omega_{\phi,\theta,r}(a,p,e,x_I) + \mathcal{O}(\epsilon)$$

ODE system: integrate numerically

Hughes *et al.* (2021)

Orbital trajectory modelling



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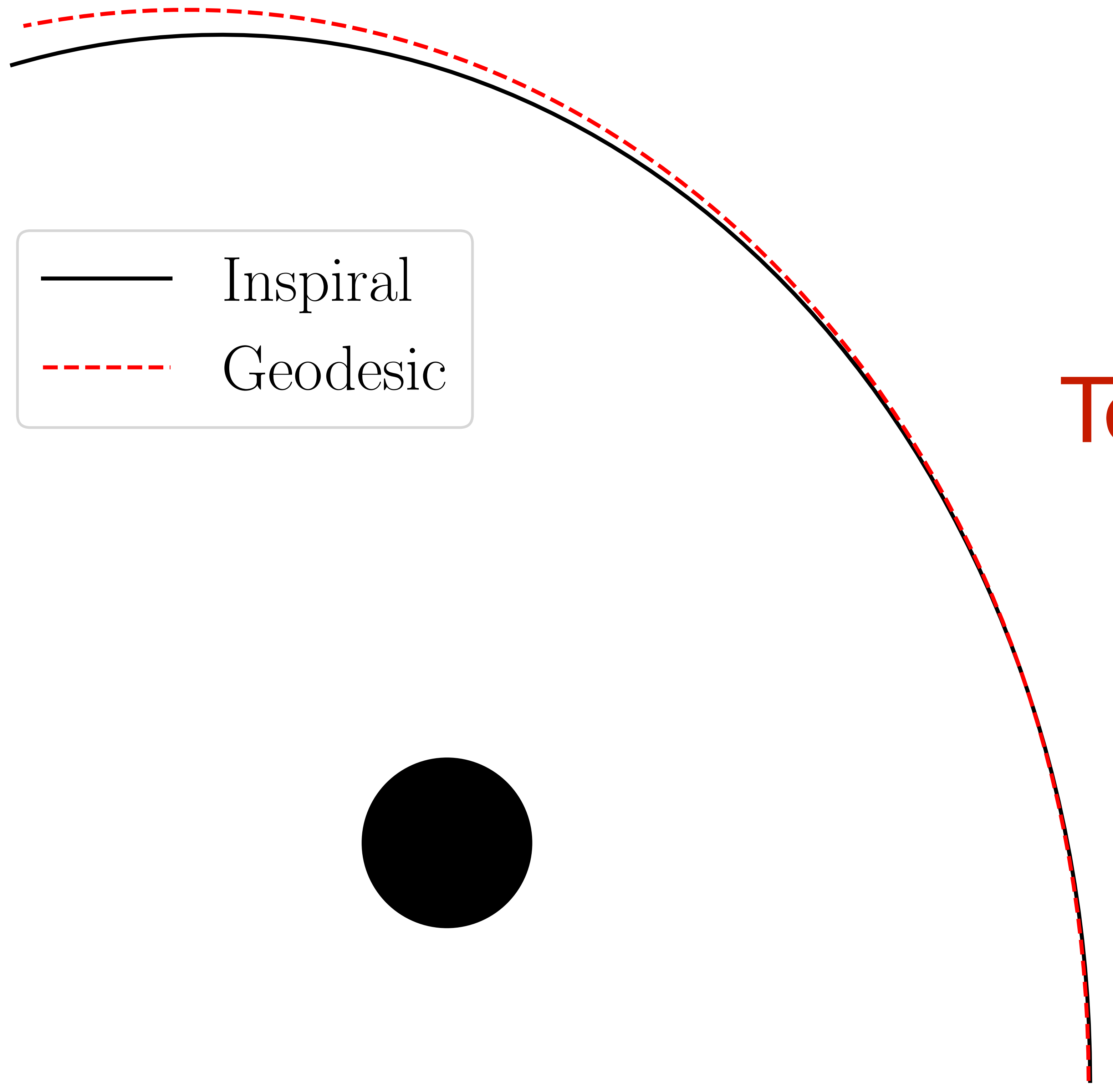

Fluxes from Teukolsky equation

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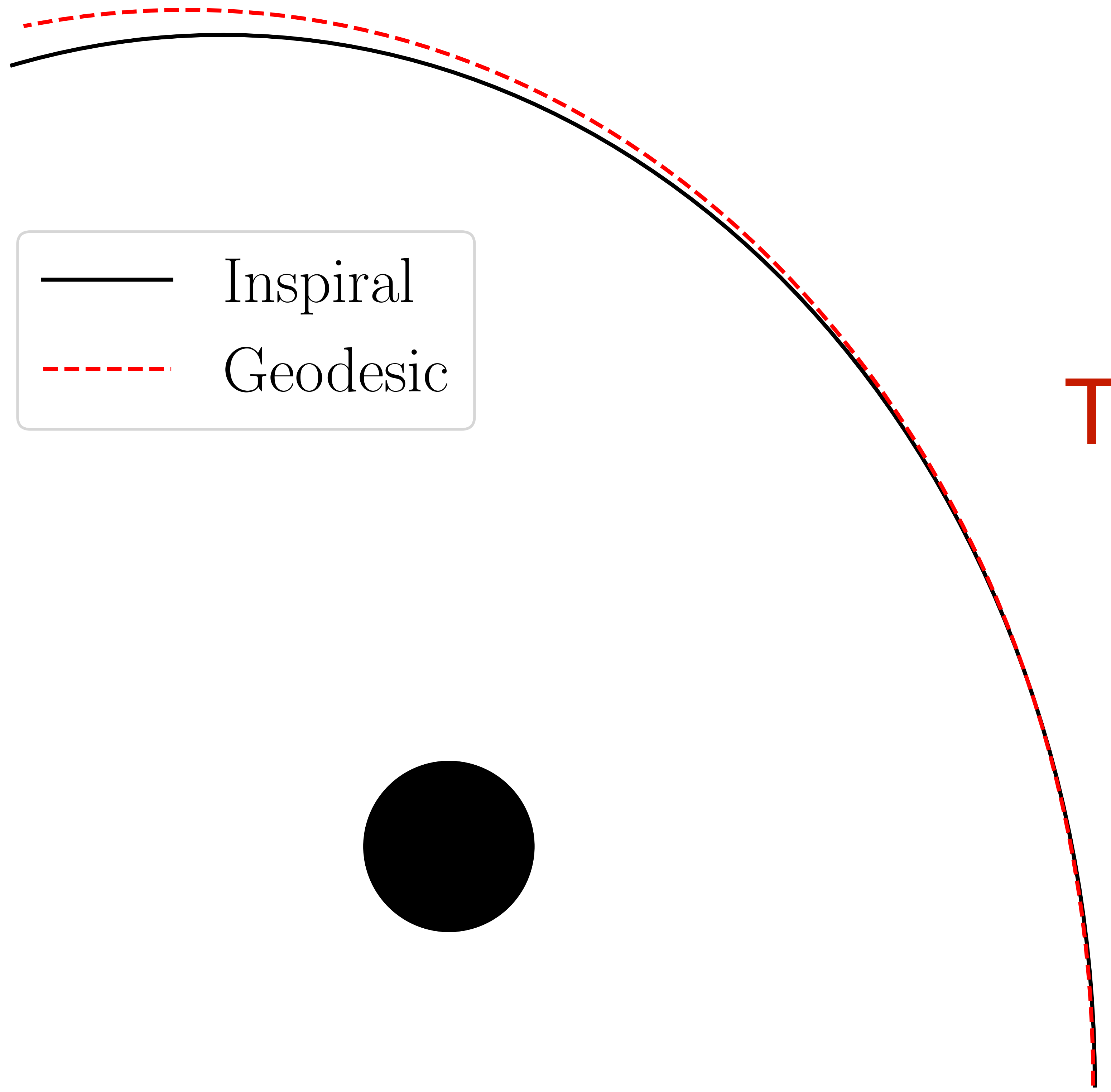

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Geodesic orbital frequencies

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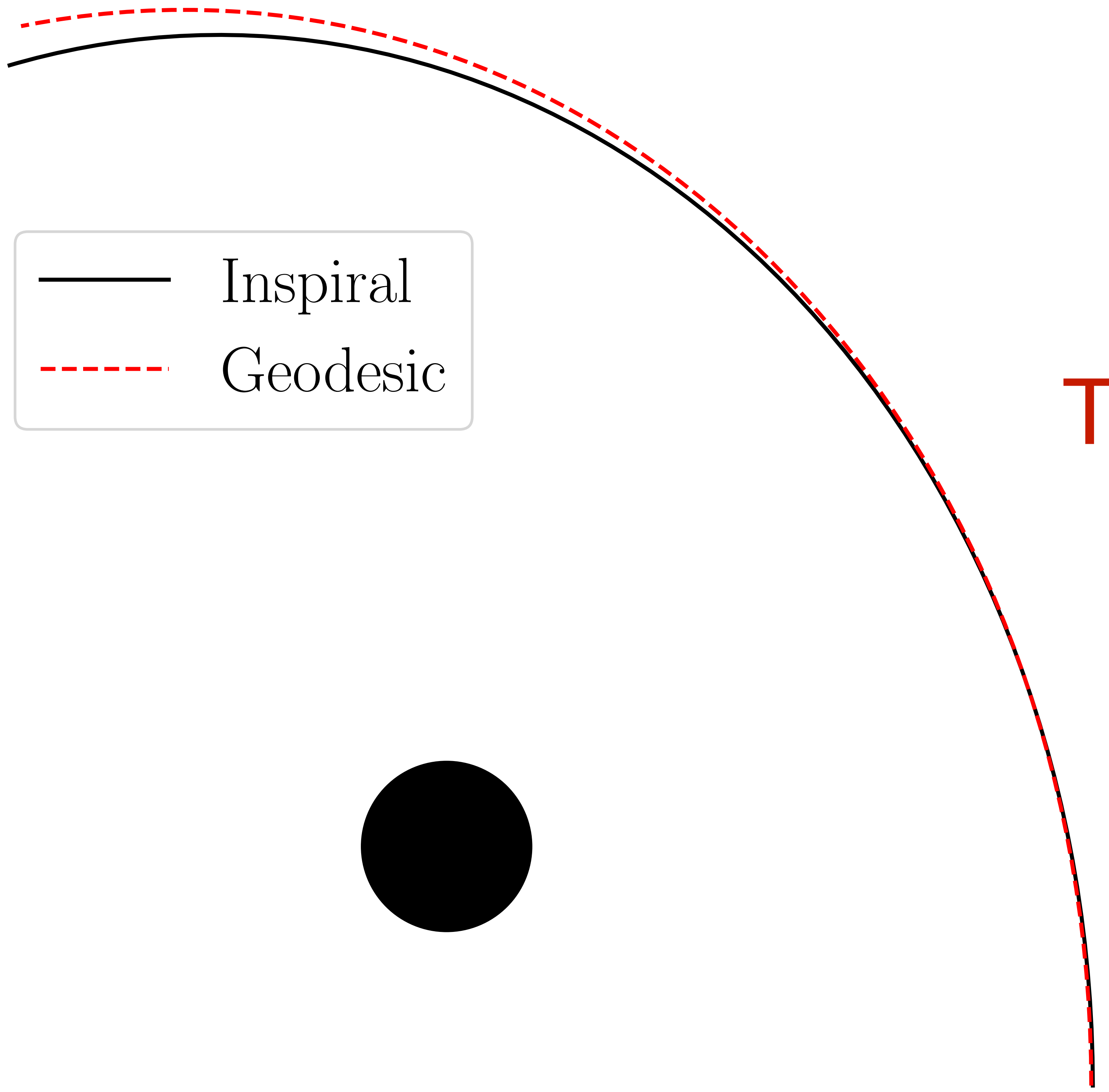
Hughes *et al.* (2021)

Orbital trajectory modelling



Fluxes from Teukolsky equation

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Adiabatic order

Geodesic orbital frequencies

$$M\frac{\mathrm{d}\Phi_{\phi,\theta,r}}{\mathrm{d}t} = \Omega_{\phi,\theta,r} + \cancel{\mathcal{O}(\epsilon)}$$

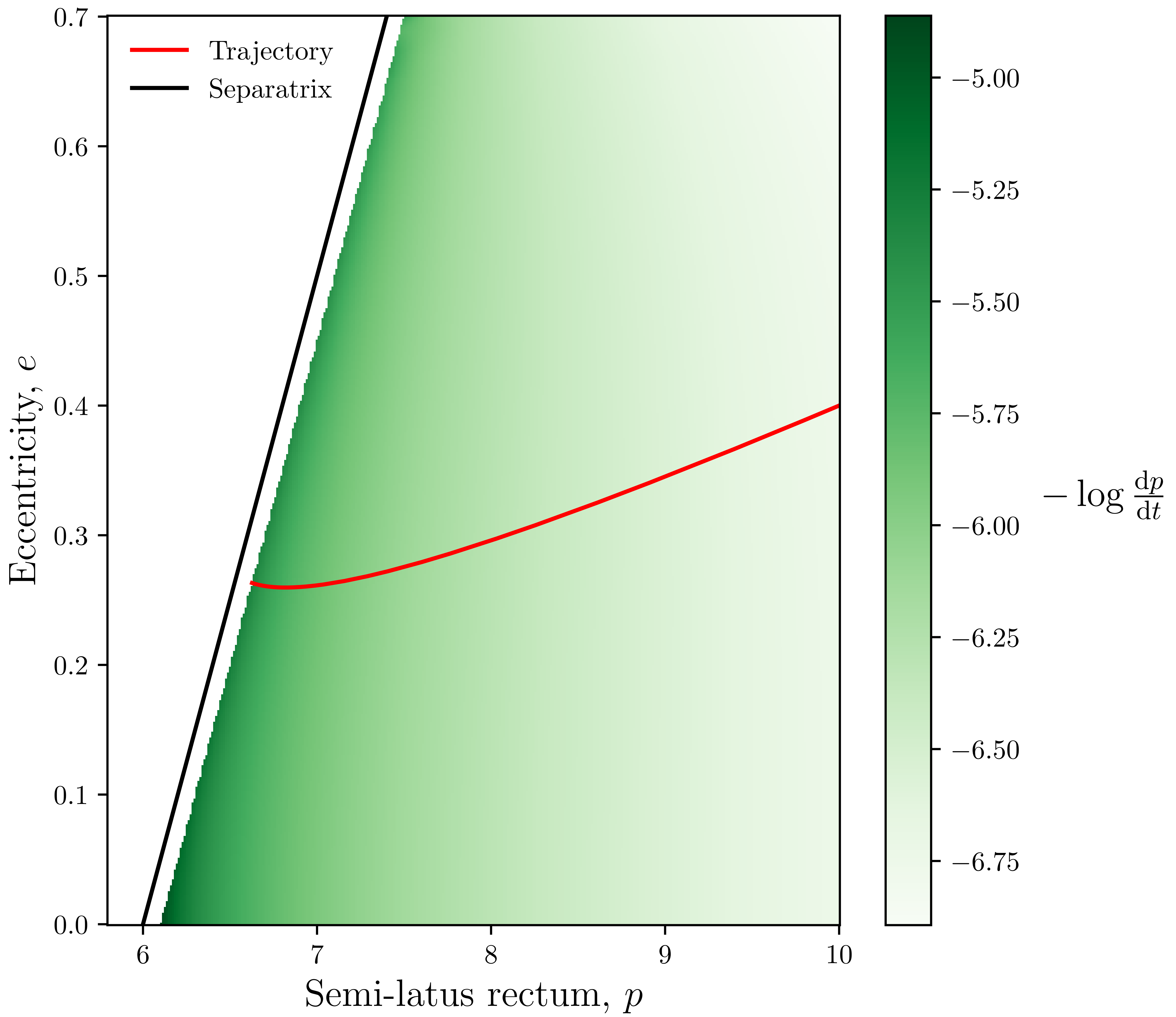
Osculating geodesics: At any time, orbit is described by parameters of tangent geodesic

ODE system: integrate numerically

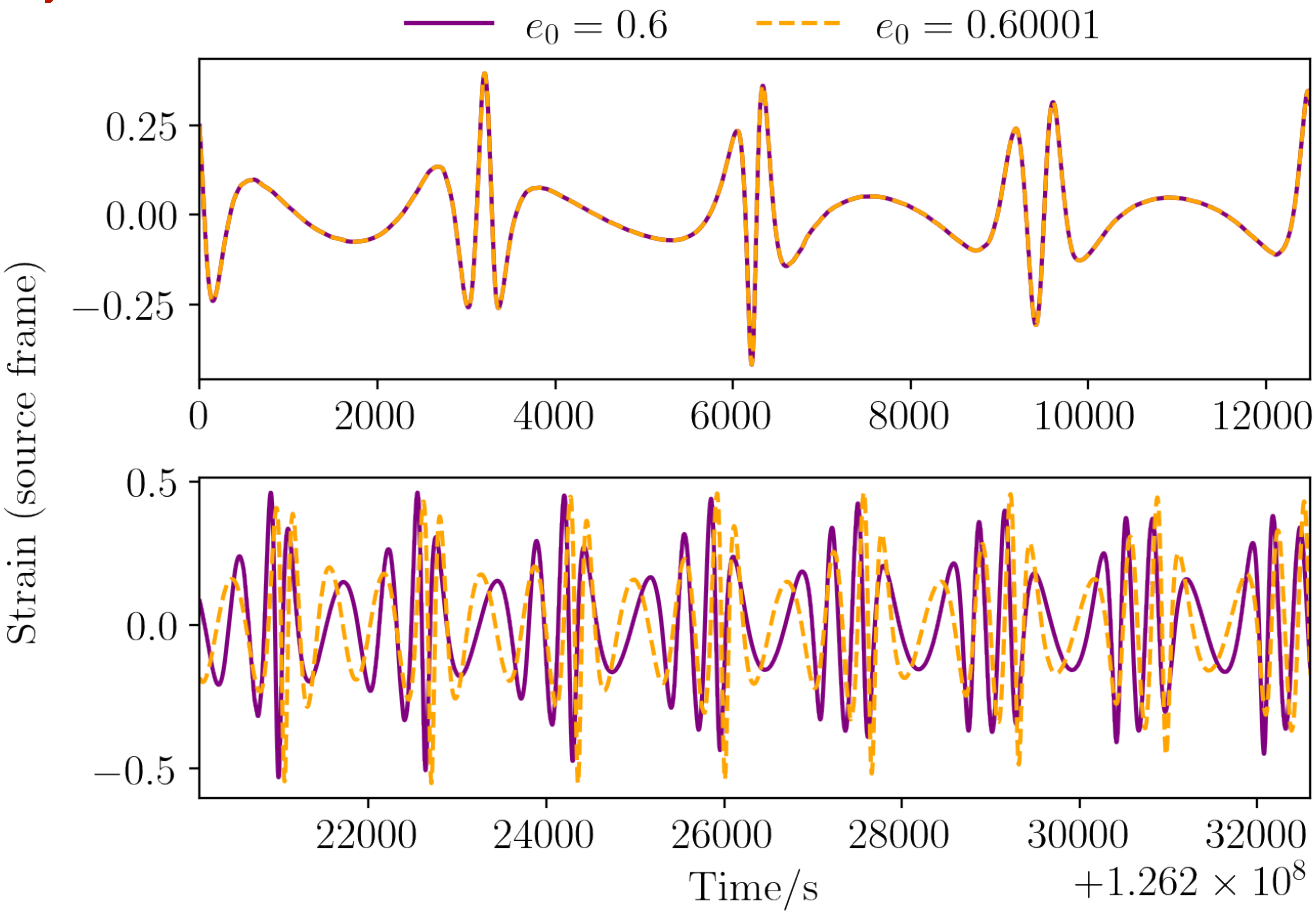
Katz *et al.* (2021)

Orbital trajectory modelling… quickly

- The computationally expensive fluxes f_{p,e,x_I} are typically evaluated $\mathcal{O}(10^4)$ times per trajectory integration.
- We require millisecond trajectory evaluation → **microsecond flux evaluation!**
- Only practical solution is to approximate these quantities via interpolation.
- Adaptive time-stepping yields trajectories of $\mathcal{O}(10^2)$ points.



Phase accuracy is **essential**!



Mode amplitudes

$$H_{lmkn}(t, \theta, \phi) = A_{lmkn}(a, p, e, x_I) \times S_{lmkn}(\theta, \Omega_{mkn})e^{im\phi}$$

- A_{lmkn} : Obtained by solving Teukolsky equation (expensive!)
- S_{lmkn} : Spin-weighted spheroidal harmonics (also expensive!)

One can show that A_{lmkn} can be projected onto a spherical harmonic basis:

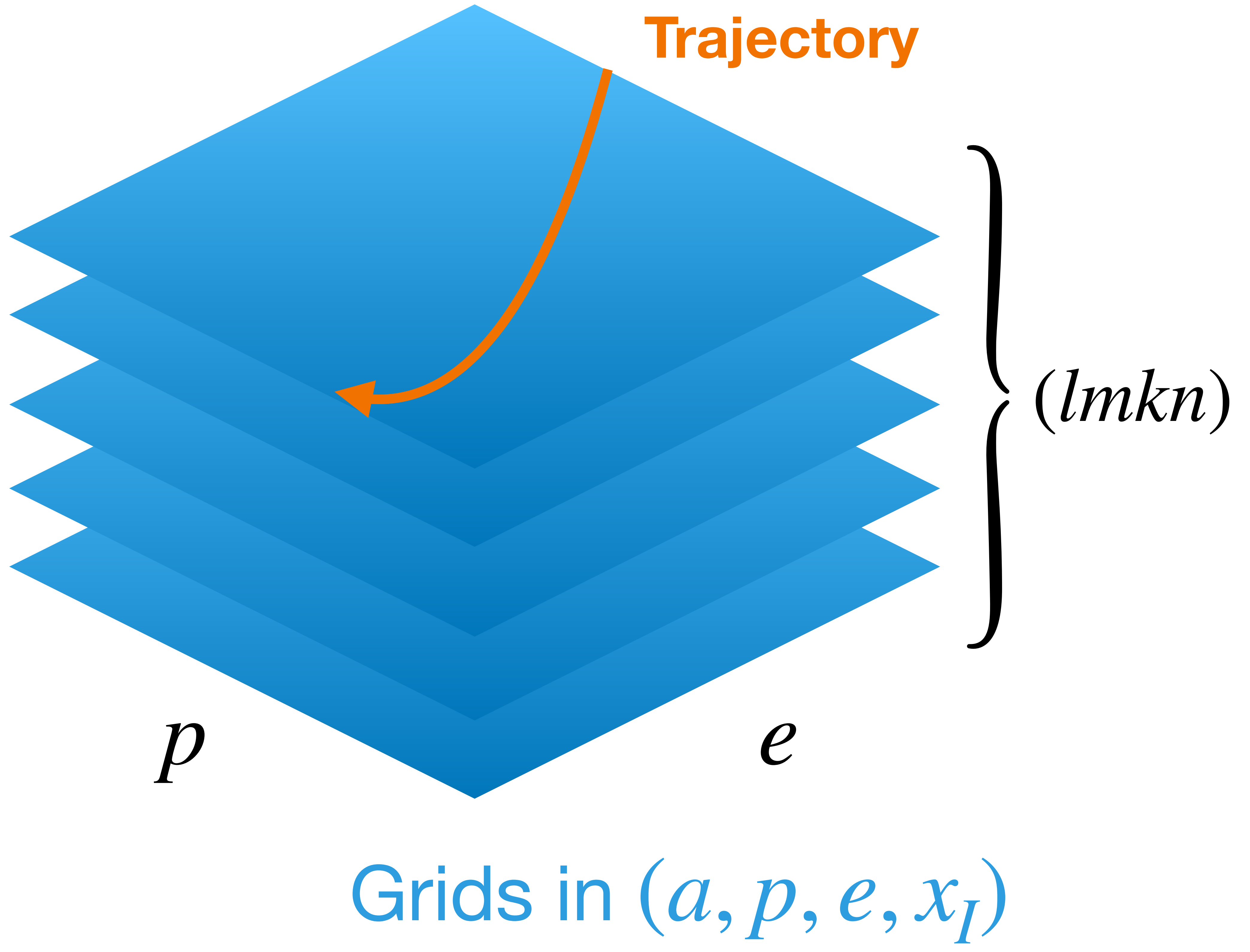
Hughes (2000)

$$H_{lmkn}(t, \theta, \phi) \approx \mathscr{A}_{lmkn}(a, p, e, x_I)_{-2}Y_{lm}(\theta, \phi)$$

Mode amplitude interpolation

- $\mathscr{A}_{lmkn}$ are smoothly varying and interpolated well with tensor cubic splines.
- As $N(\mathscr{A}_{lmkn})$ is large, they must be interpolated very efficiently.
- Embarrassingly parallelisable problem, so we can use graphics processing units (GPUs) to great effect.
- Compute $\mathcal{O}(10^4)$ mode amplitudes at each sparse trajectory point in $\sim 10\,\mathrm{ms}$.

Vectorised amplitude interpolation



Katz *et al.* (2021)

Summation

With Φ_{mkn} and H_{lmkn} at sparse points along the trajectory…

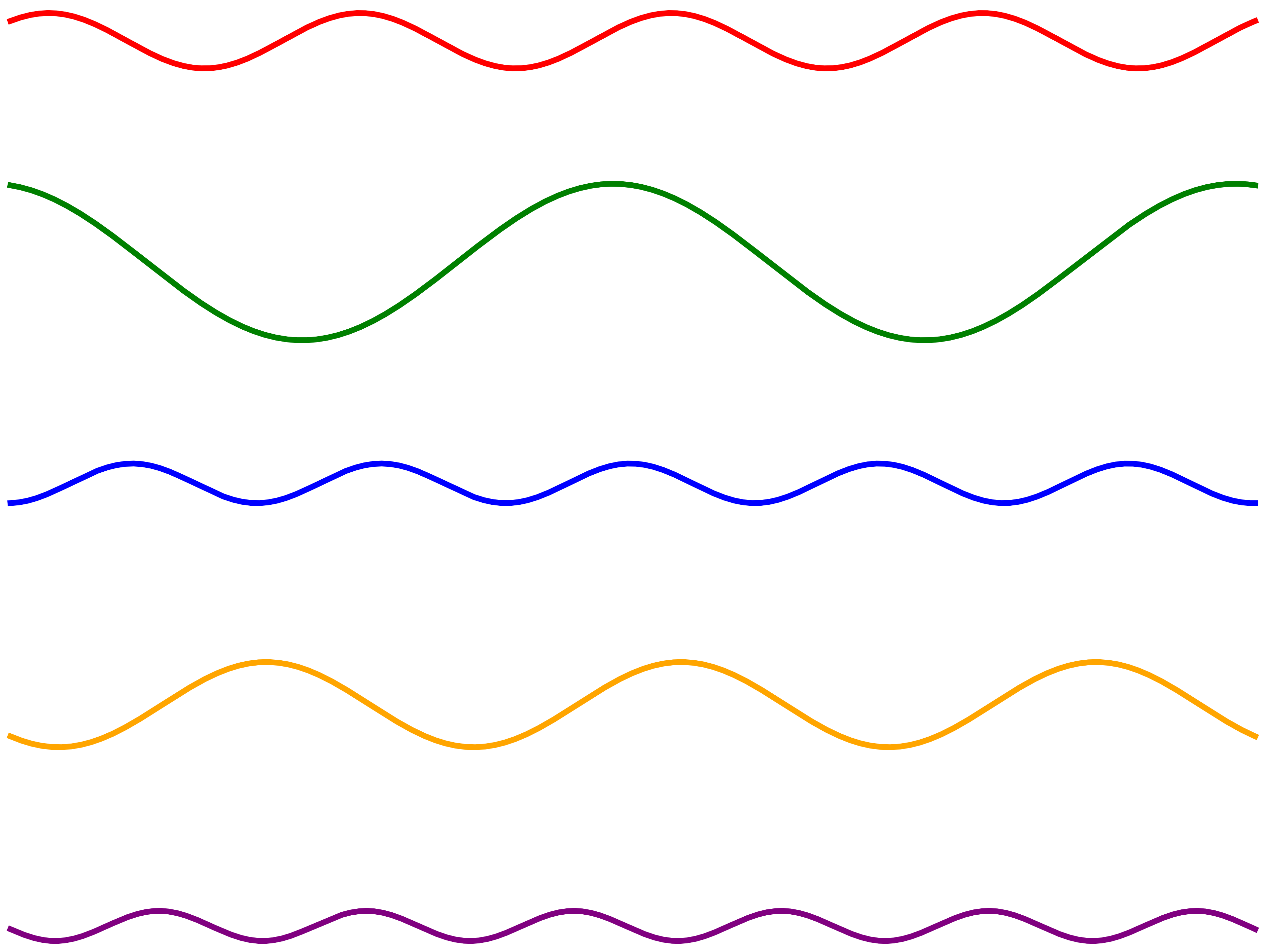
$$h(t) = \frac{\mu}{d_\mathrm{L}} \sum_{lmkn} H_{lmkn}(t, \theta, \phi) e^{-i\Phi_{mkn}(t)}$$

1. Interpolate these quantities with splines to obtain continuous representation
2. Sum over spline outputs at each requested time

Very expensive for LISA waveforms → use GPUs for $\mathcal{O}(10^4)$ times speedup.

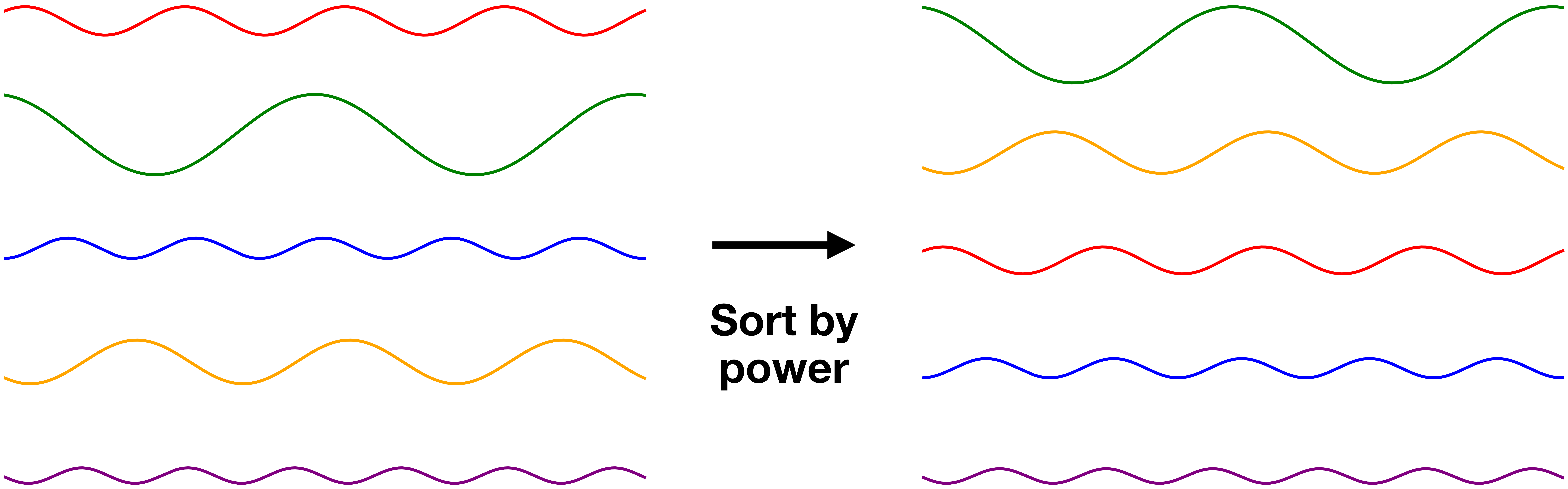
Katz *et al.* (2021)

Summation - mode selection



Katz *et al.* (2021)

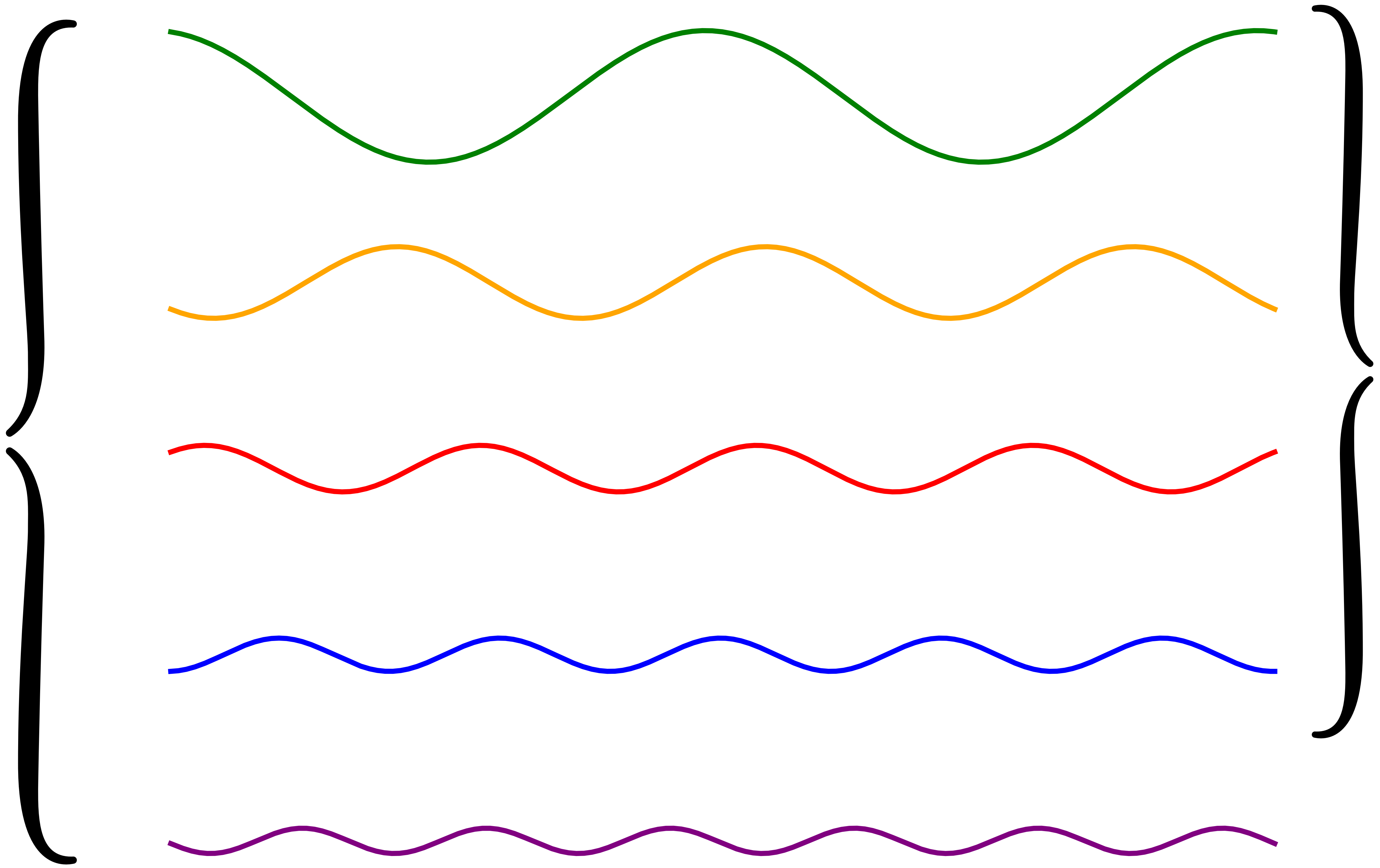
Summation - mode selection



Katz *et al.* (2021)

Summation - mode selection

$$P_{\mathrm{tot}} = \sum_{lmkn} P_{lmkn}$$



$$P_{\mathrm{cut}} = \varepsilon P_{\mathrm{tot}}$$

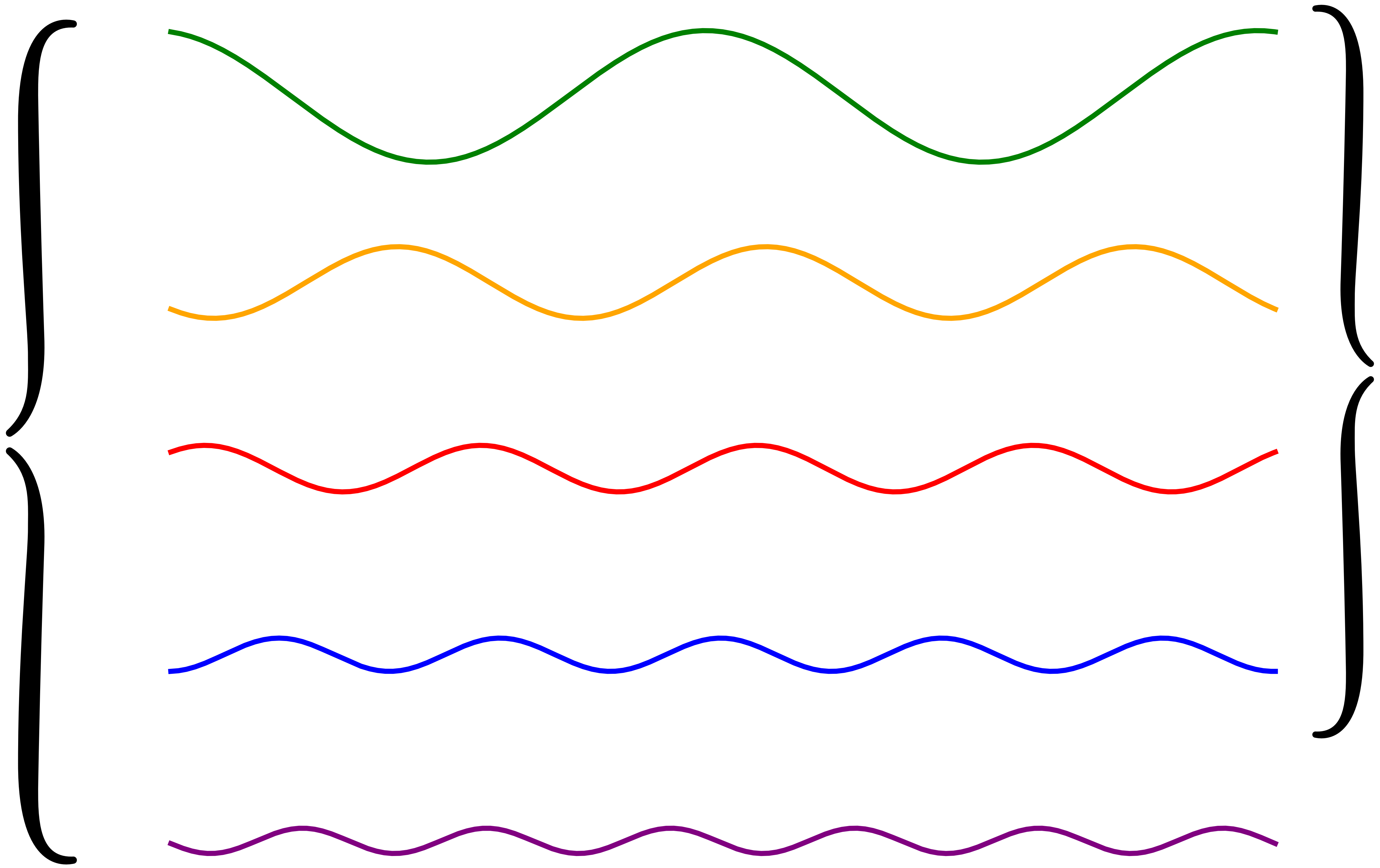
Truncate **cumulative summation** at some threshold fraction ε of the total mode power

Summation - mode selection

Sub-optimal!

Chapman-Bird *et al.* (in prep)

$$P_{\mathrm{tot}} = \sum_{lmkn} P_{lmkn}$$



$$P_{\mathrm{cut}} = \varepsilon P_{\mathrm{tot}}$$

Truncate **cumulative summation** at some threshold fraction ε of the total mode power

Bringing it all together

FastEMRIWaveforms (FEW) Katz *et al.* (2021)

Part of the **Black Hole Perturbation Toolkit**

- Millisecond waveform generation with GPUs without compromising accuracy
- Modular framework → *build your own waveform models!*

- FEW is currently limited to eccentric inspirals into Schwarzschild MBHs.
- Lays the necessary groundwork for building LISA-ready EMRI waveforms.

What comes next?

Chapman-Bird *et al.* (in prep)

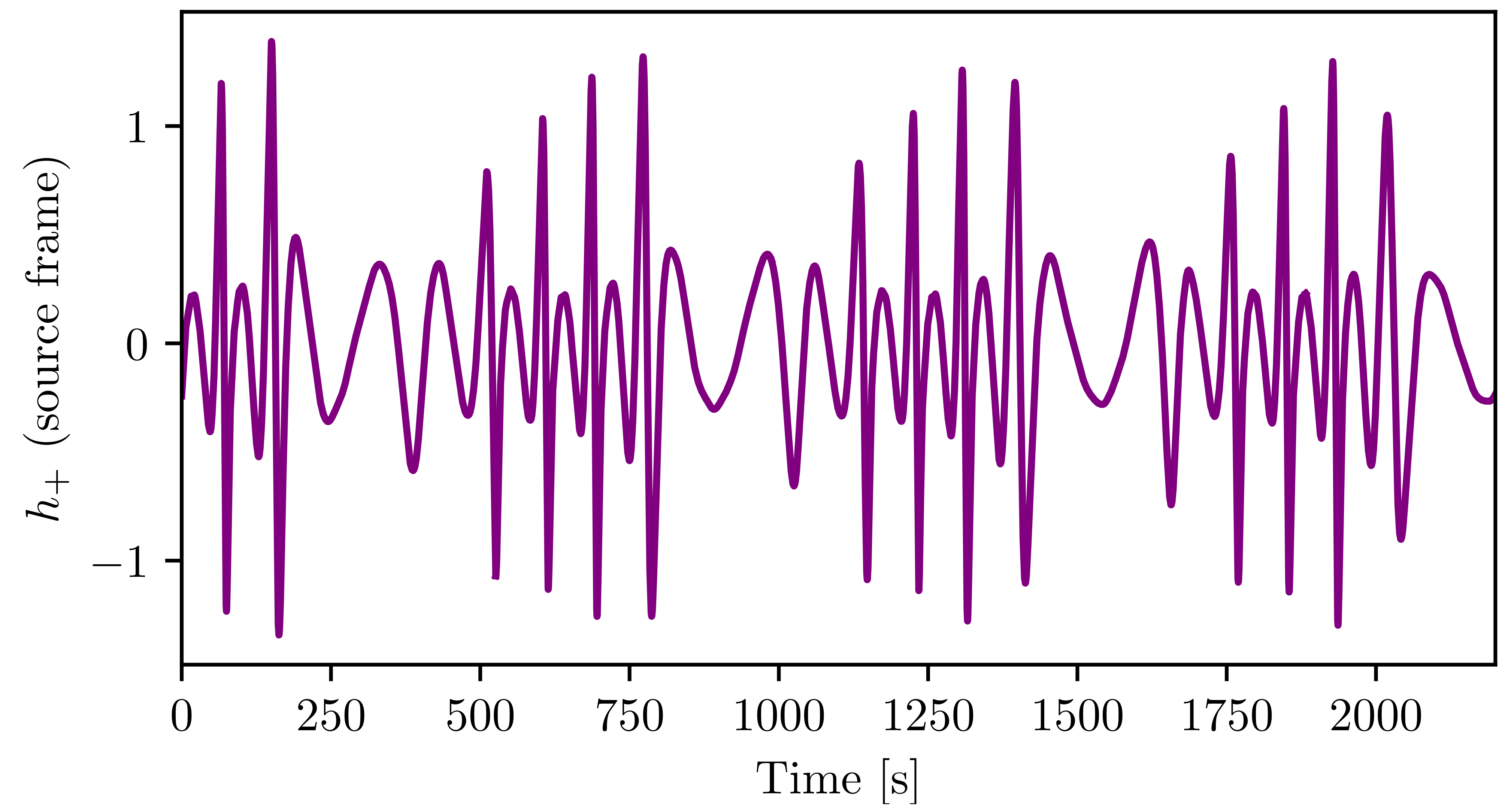
Kerr Eccentric Equatorial waveforms

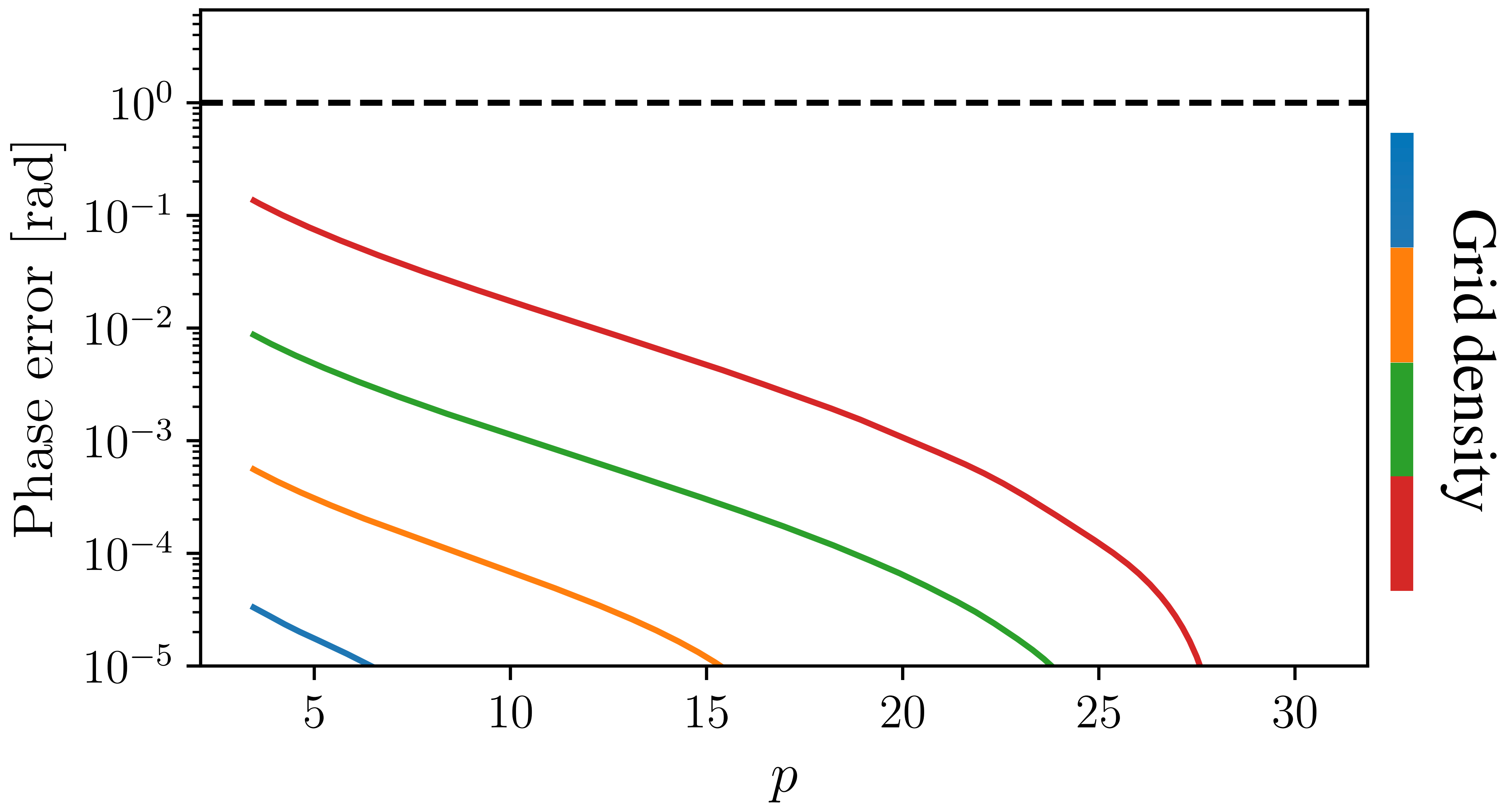
- We expect many MBHs to be spinning rapidly; this strongly impacts EMRI measurement precision. Babak *et al.* (2017)

- Including MBH spin means 2D → 3D data grids, requiring new and efficient interpolation codes for fluxes and mode amplitudes.

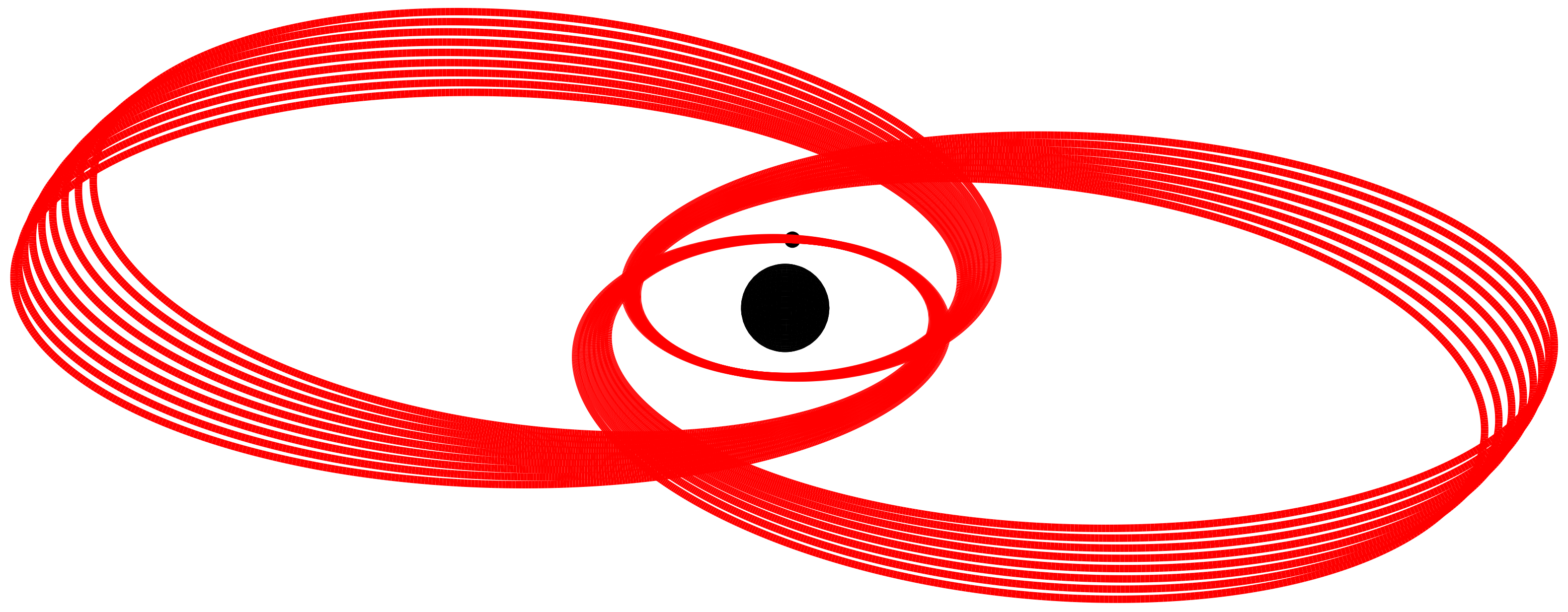
- Also plan to properly explore interpolation error and ensure that FEW waveforms are **accurate.**

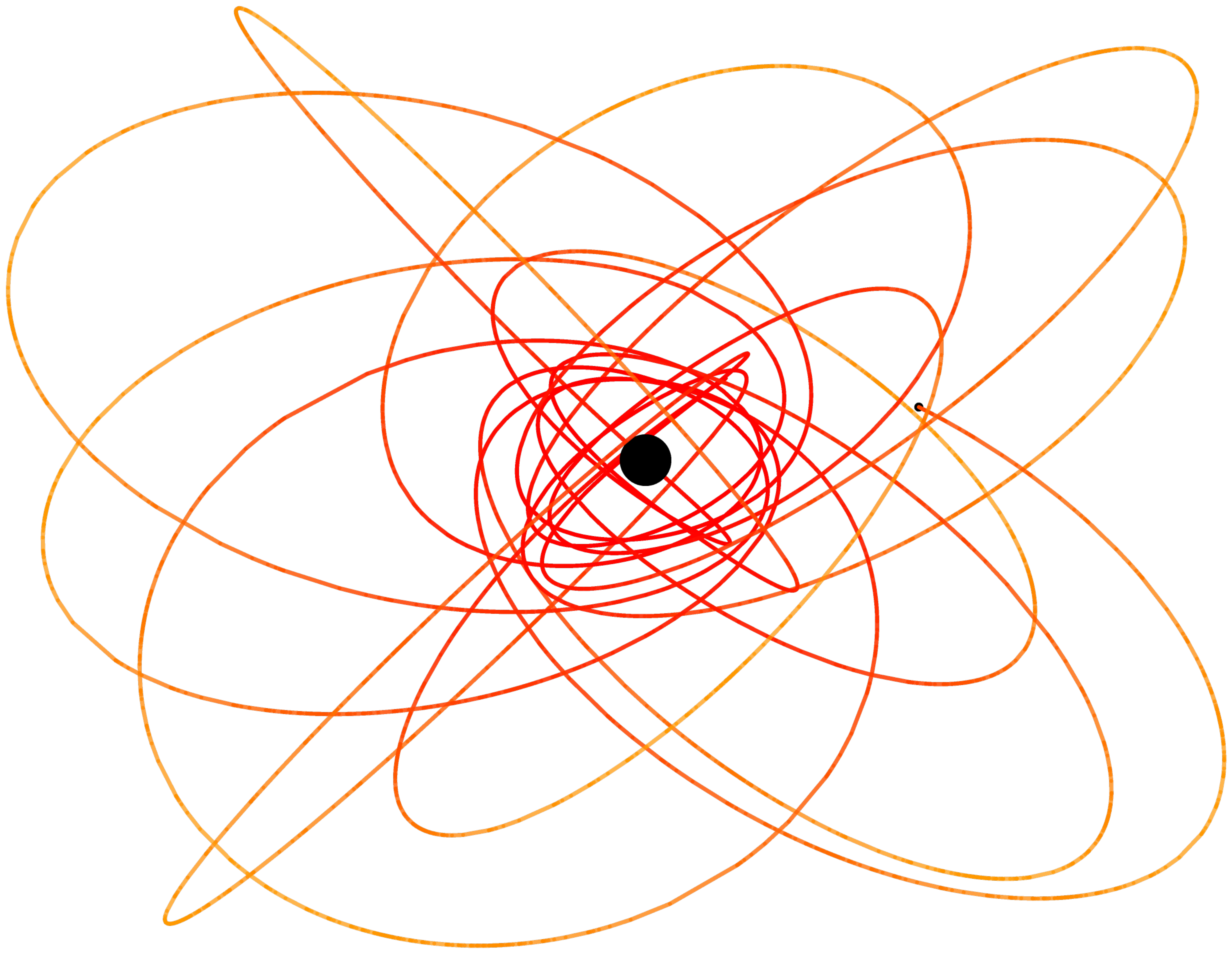
- Waveform model complete in $\mathcal{O}(\text{weeks})$.





And after that…





Add inclination: the final step for generic inspirals at adiabatic order.

Hughes *et al.* (2021)
Van de Meent (2017)

And after that…

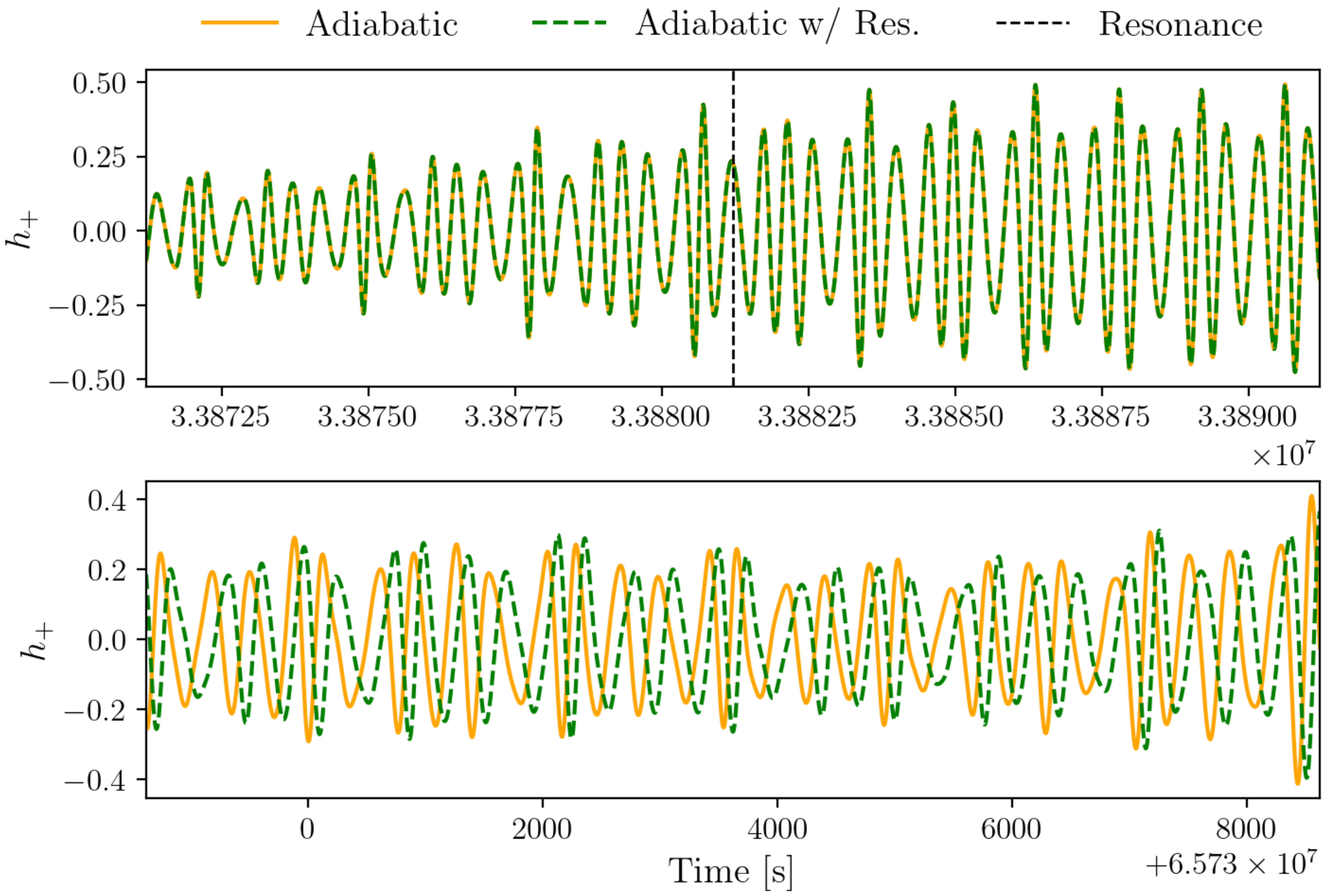
Orbital resonances

- Step-change in (p, e, x_I)
- Occurs when Ω_θ and Ω_r become commensurate:

$$a\Omega_\theta = b\Omega_r$$

- Only impactful for inclined and eccentric inspirals
- Phase error scales as $\epsilon^{-1/2}$

See Lynch *et al.* (2024) for further details.



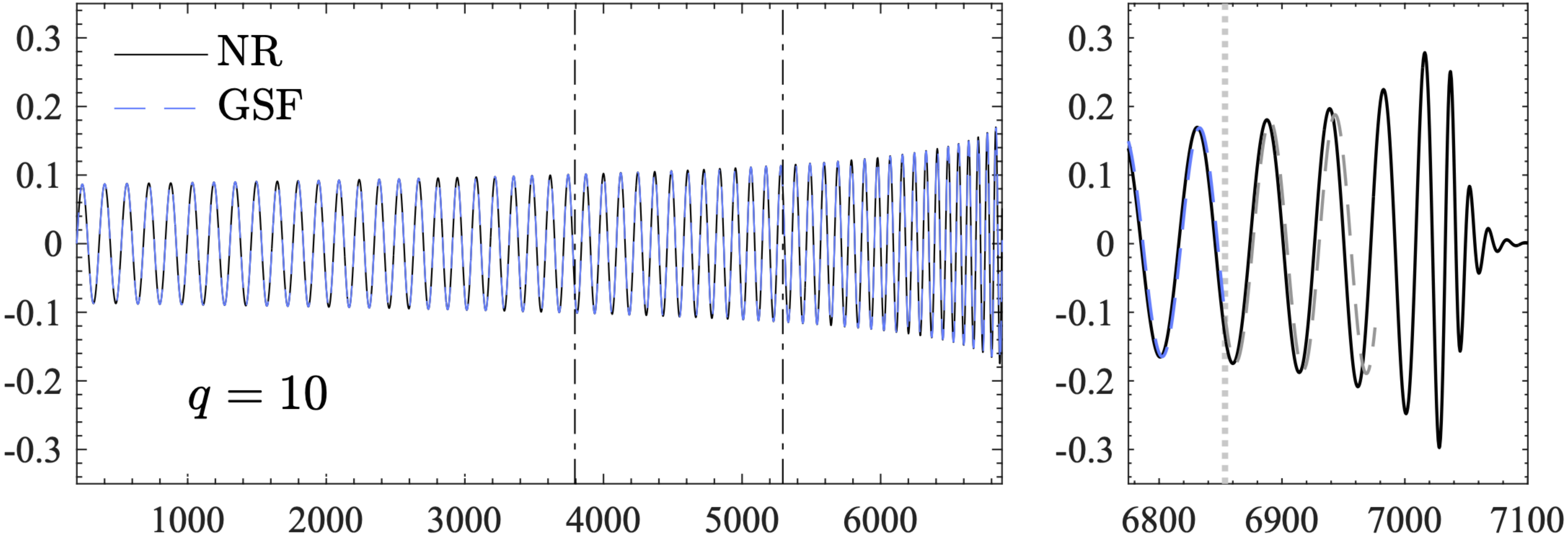
And after that…

$$M\frac{\mathrm{d}\Phi_{\phi,\theta,r}}{\mathrm{d}t} = \Omega_{\phi,\theta,r} + \mathcal{O}(\epsilon) \qquad \frac{\mathrm{d}e}{\mathrm{d}t} = \epsilon f_e + \mathcal{O}(\epsilon^2)$$

$$\frac{\mathrm{d}p}{\mathrm{d}t} = \epsilon f_p + \mathcal{O}(\epsilon^2) \qquad \frac{\mathrm{d}x_I}{\mathrm{d}t} = \epsilon f_{x_I} + \mathcal{O}(\epsilon^2)$$

Post-adiabatic (1PA) corrections

- Phase error scales as ϵ^0



Schwarzschild quasi-circular: Albertini *et al.* (2022)

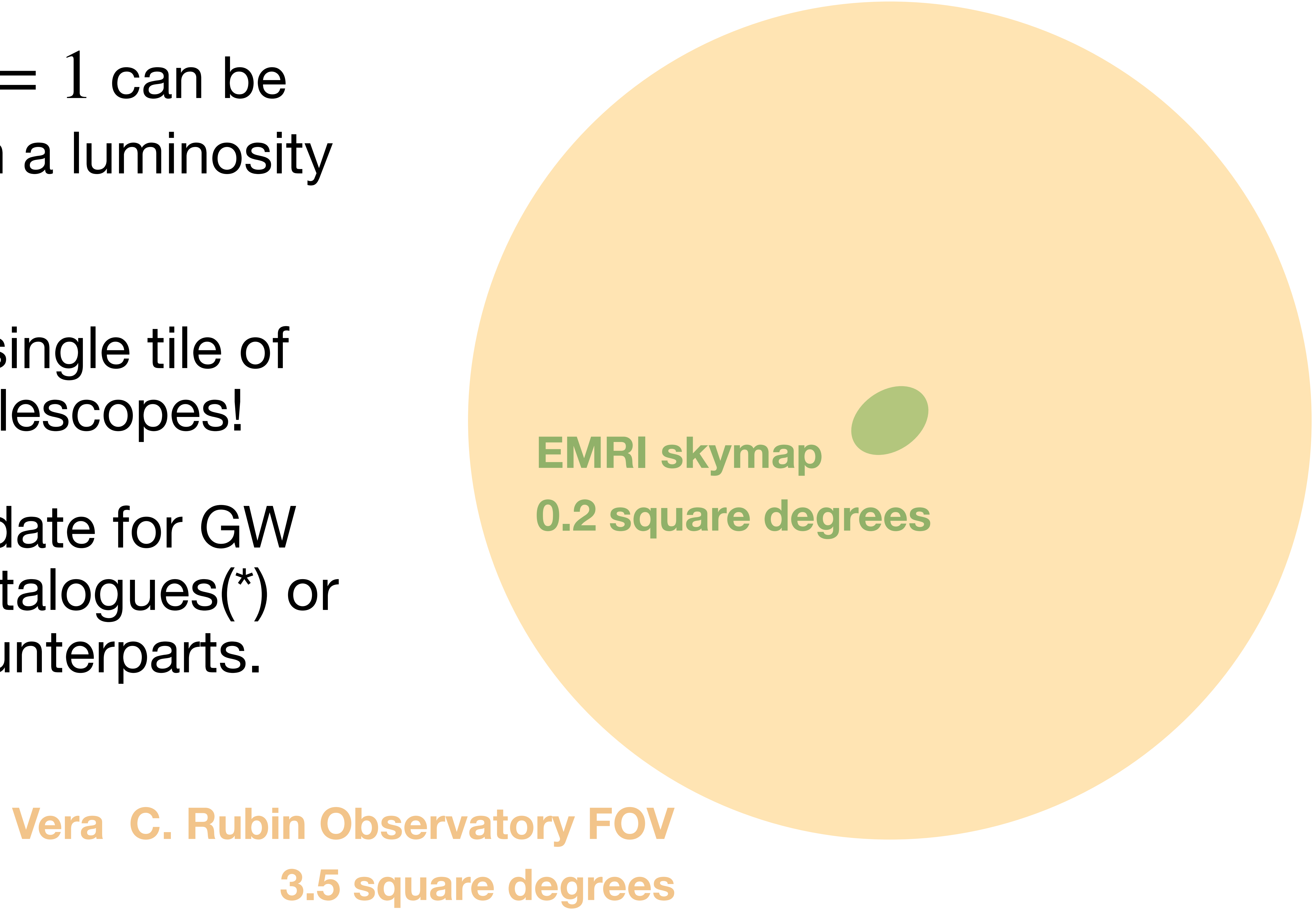
Some astrophysics…

EMRIs as cosmological probes

EMRIs are broad-band signals of long duration → excellent measurement of extrinsic parameters.

- A (fairly ideal) source at $z = 1$ can be localised to $< 1\,\mathrm{deg}^2$ with a luminosity distance error of $\mathcal{O}(1\%)$.
- Skymap fits neatly into a single tile of future wide-field survey telescopes!
- EMRIs an excellent candidate for GW cosmology with galaxy catalogues(*) or direct detection of EM counterparts.

Laghi *et al.* (2021)



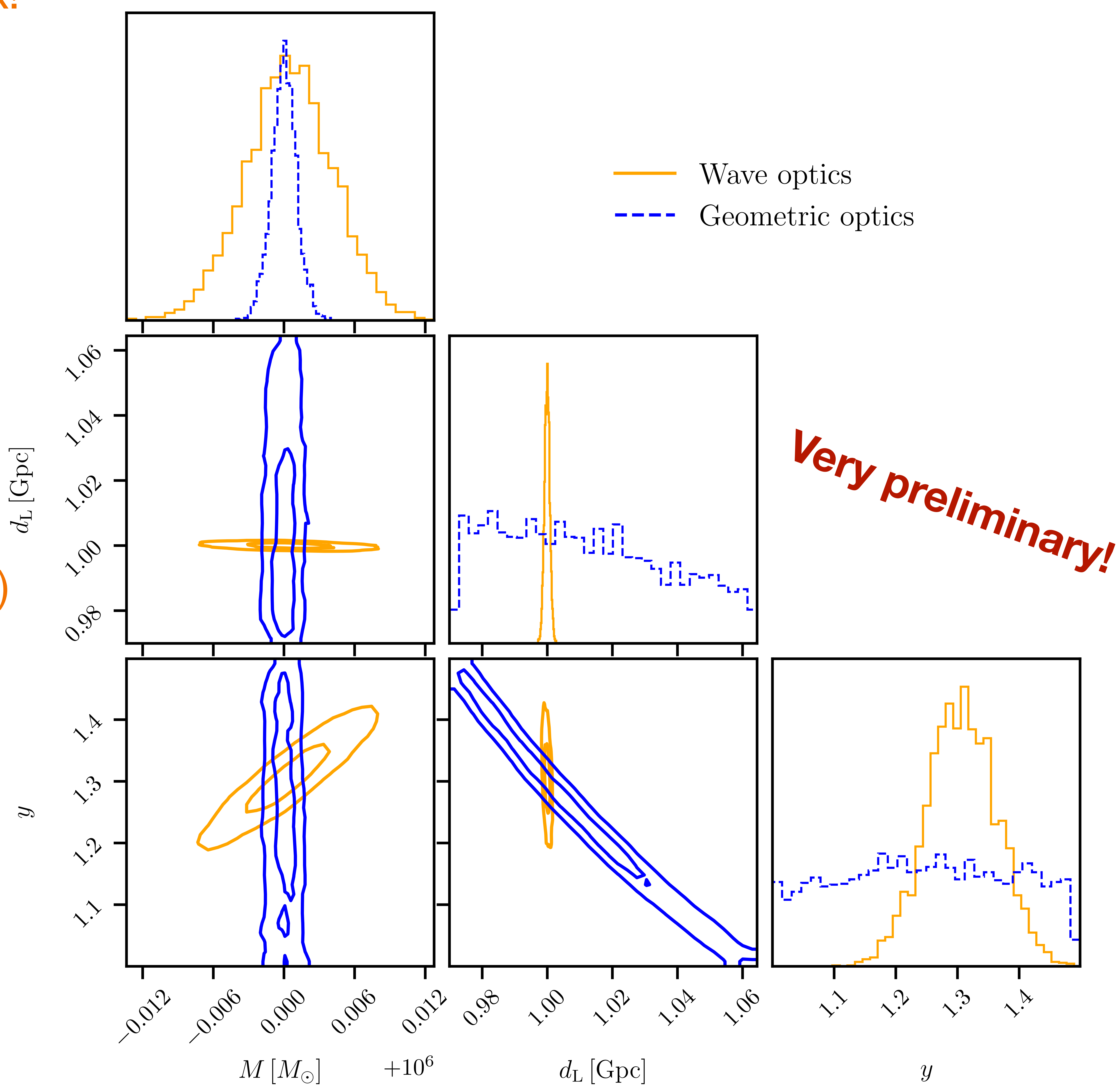
See Martina Toscani's talk!

Measuring wave-optics lensing effects

- EMRIs are also a good candidate for measuring lensing effects.
- Rates are fairly low (but EMRI rates are not well-known anyway!)

Toscani *et al.* (2024)

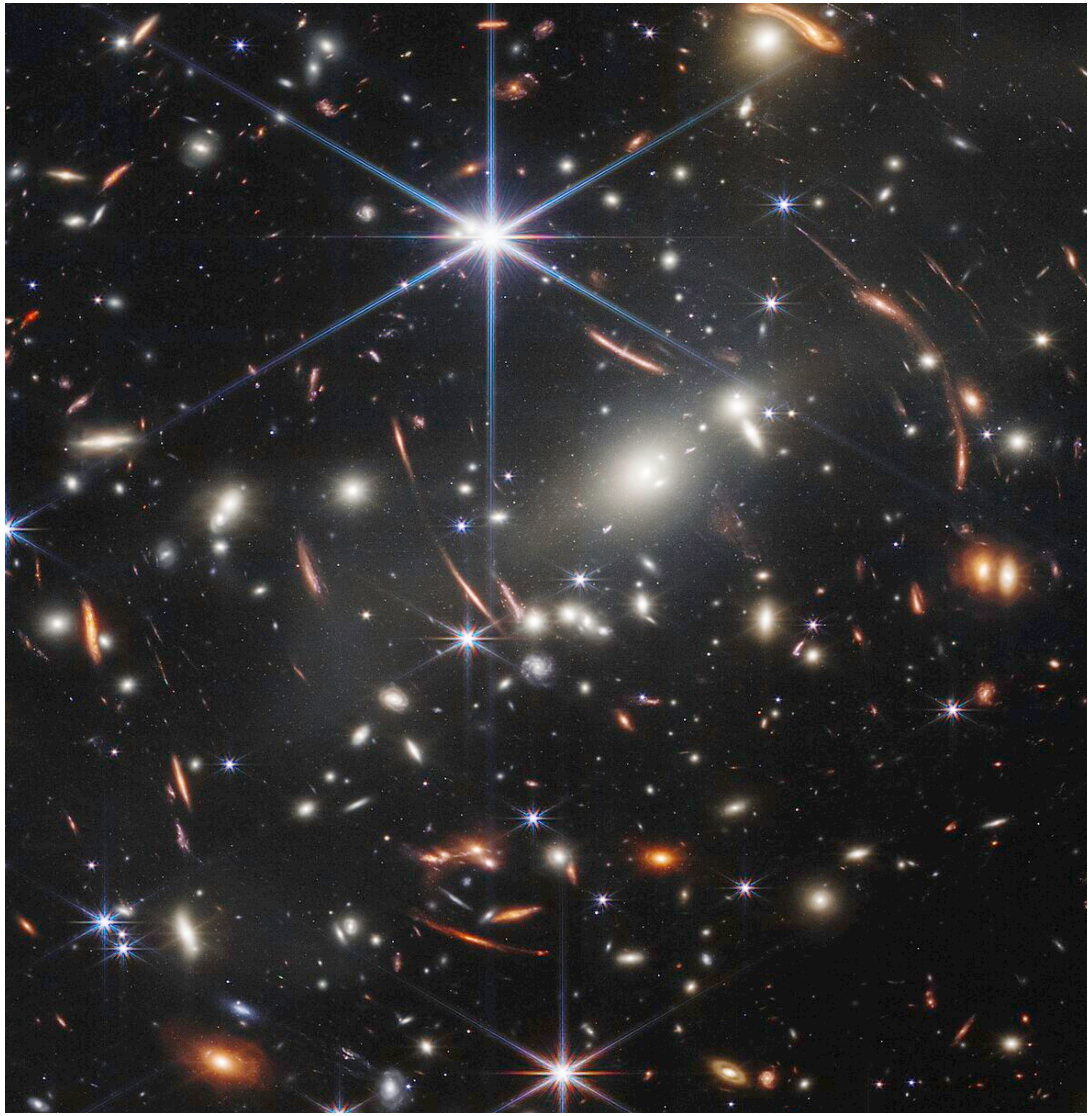
- An example (left): wave-optics lensing effects break position/distance degeneracy (lens mass of $10^5\, M_\odot$).
- Type-II lensing effects trivial to identify in EMRI signals - smoking gun for lensed source



Together: localising a gravitational lens

- A realistic prospect due to small localisation volume.
- Information from signal analysis (e.g. MBH mass, lens parameters) can be incorporated statistically.
- Even a single case can enable $\mathcal{O}(1\%)$ constraint on H_0.

Toscani *et al.* (2024)



Conclusions

FastEMRIWaveforms

Black Hole Perturbation Toolkit

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- EMRIs have complicated waveforms that are challenging to model.
- With GPU parallelism, we can produce EMRI waveforms in milliseconds (**FastEMRIWaveforms**) without loss of accuracy or generality.
  - A number of improvements to come in the near future, but the core framework is in place to implement these improvements.

    Chapman-Bird *et al.* (in prep)
  - **Kerr Eccentric Equatorial** waveforms will be available in $\mathcal{O}$(weeks)!
- The long duration and strong harmonic mode content of EMRIs makes them excellent candidates for both cosmological and lensing analyses.