Odd viscoelasticity

Piotr Surówka
Wrocław University of Science and Technology

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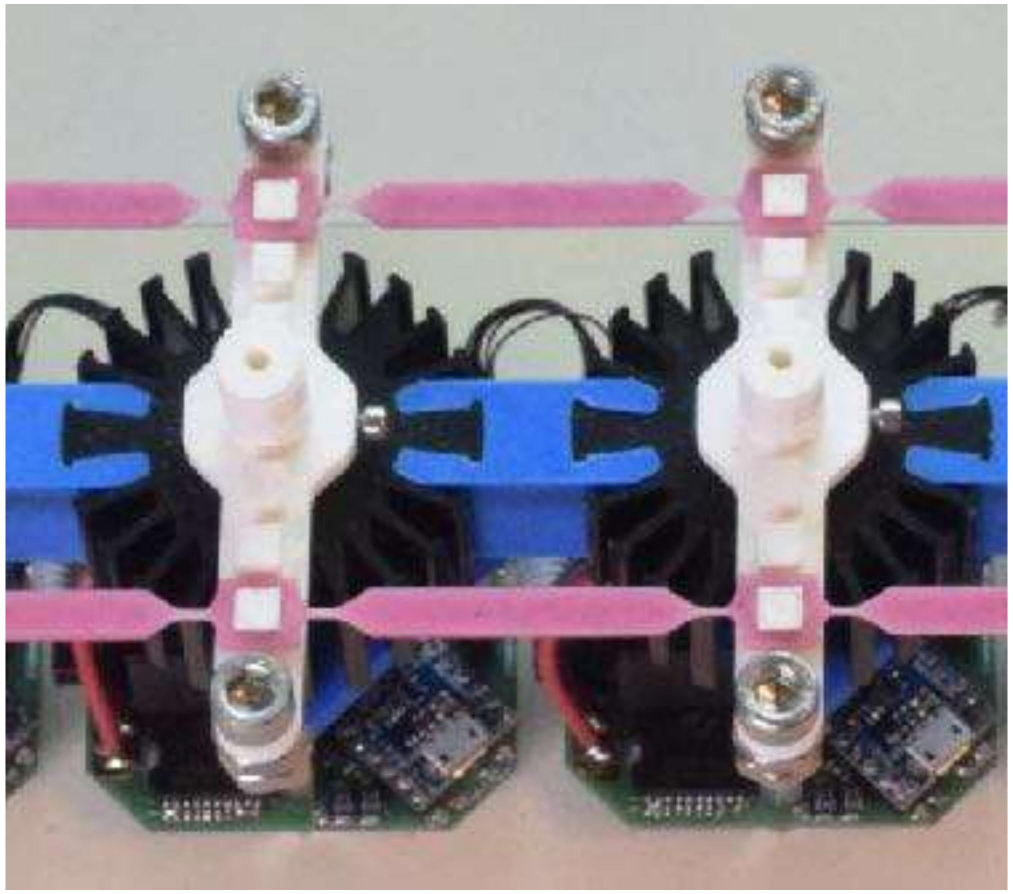
Motivation- composite materials

Concrete is cheap and relatively light, but it breaks apart easily under tension. By contrast, steel is strong but expensive and heavy. By pouring the concrete around prestressed metal bars one obtains a composite, namely, reinforced concrete, that is cheap, relatively light, and strong.



Coulais Lab

Modern day composites can also involve materials made of programmable robots. Macroscopic description of such materials requires a modified viscoelastic description.

Granular matter

A granular material is a conglomeration of discrete solid, macroscopic particles characterized by a loss of energy whenever the particles interact. Examples of granular materials are snow, nuts, coal, sand, rice, coffee, corn flakes, fertilizer.

Wikipedia

Active matter

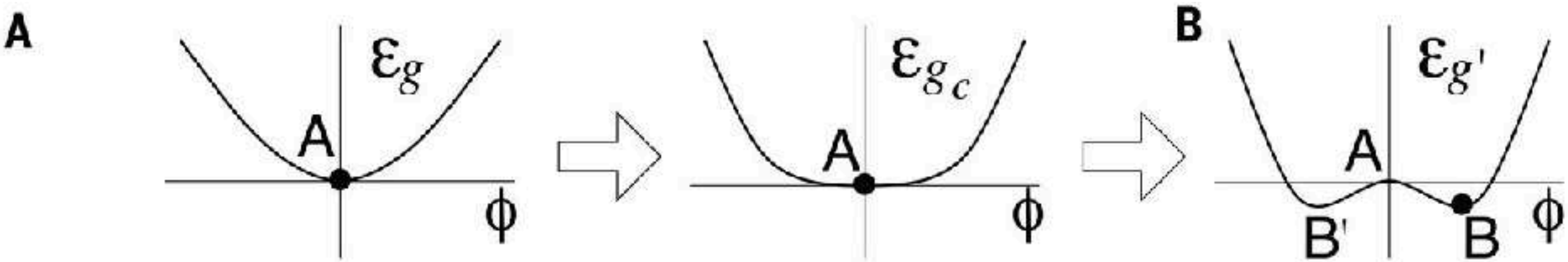
Hans Overduin/NIS/Minden/Getty

SardiTemp

We want to understand the principles behind systems, whose microscopic constituents are not in equilibrium (flocks, fish schools)

Active matter systems are made up of units that consume energy. Physicists group flocks of birds, molecular motors and layers of vibrating grains together in this category because they all extract energy from their surroundings at a single particle level and transform it into mechanical work. By studying the behaviors that emerge, our understanding of these systems can be enhanced and new frameworks for investigating the statistical physics of out-of-equilibrium systems can be built.

Landau paradigm



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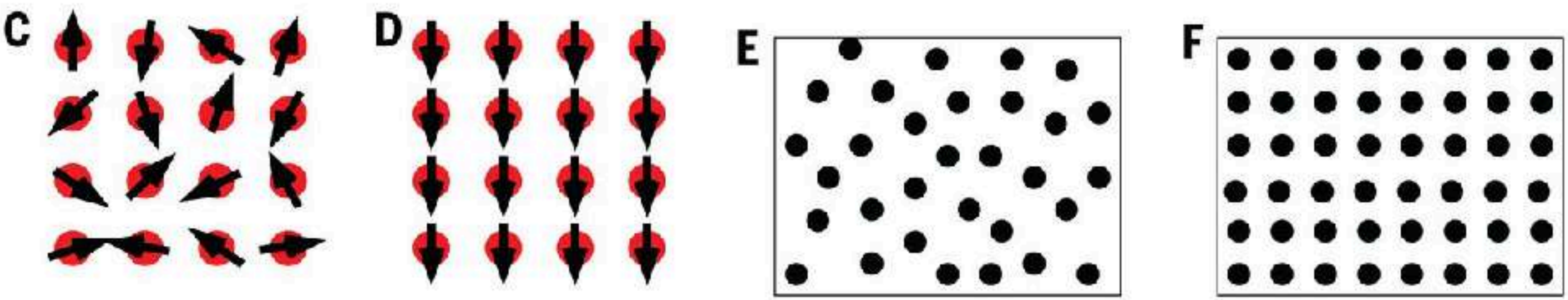
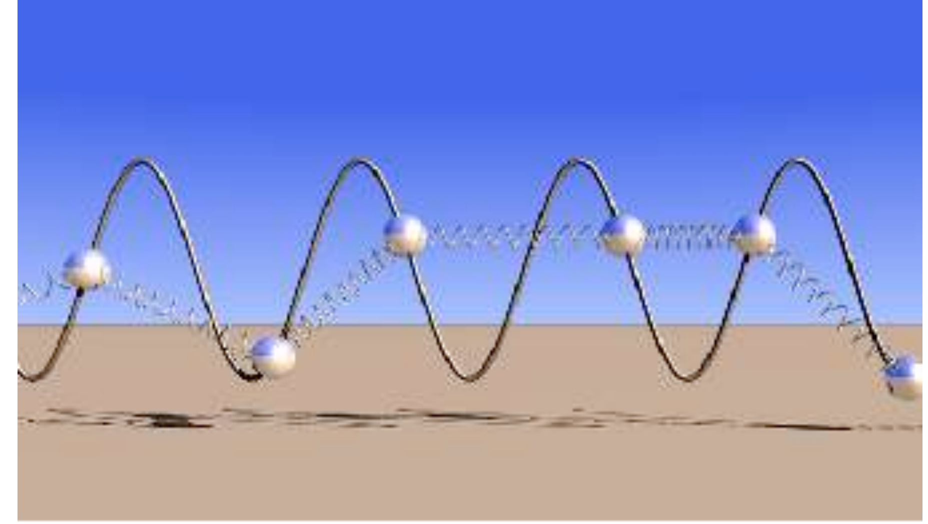


Disordered liquid states that do not break any symmetry [(A), (C), and (E)]. Ordered states that spontaneously break some symmetries [(B), (D), and (F)]. For example, the energy function has a symmetry $\phi \to -\phi$, $\varepsilon_g(\phi) = \varepsilon_g(-\phi)$. However, as the parameter g (e.g., magnetic field) changes, the minimal energy state (the ground state) sometimes respects the symmetry [(A), (C), and (E)] and other times must settle into a state that does not respect the symmetry. Landau theory generalizes the above picture to describe all phases and all phase transitions. Within this theory, the symmetry of the ordering of constituent particles distinguishes one phase from another.

Elasticity



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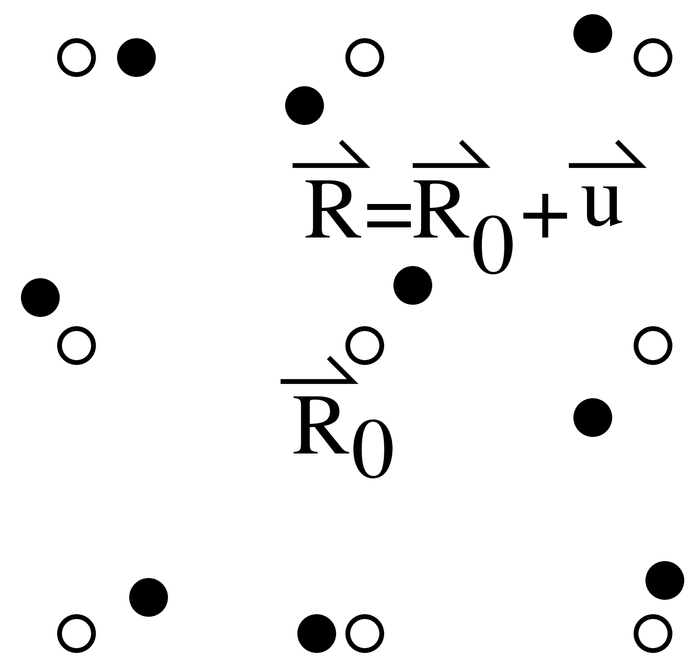
At distances large compared to the lattice constant, one can define a displacement field

$$\vec{u}(\vec{x},t)$$

such that

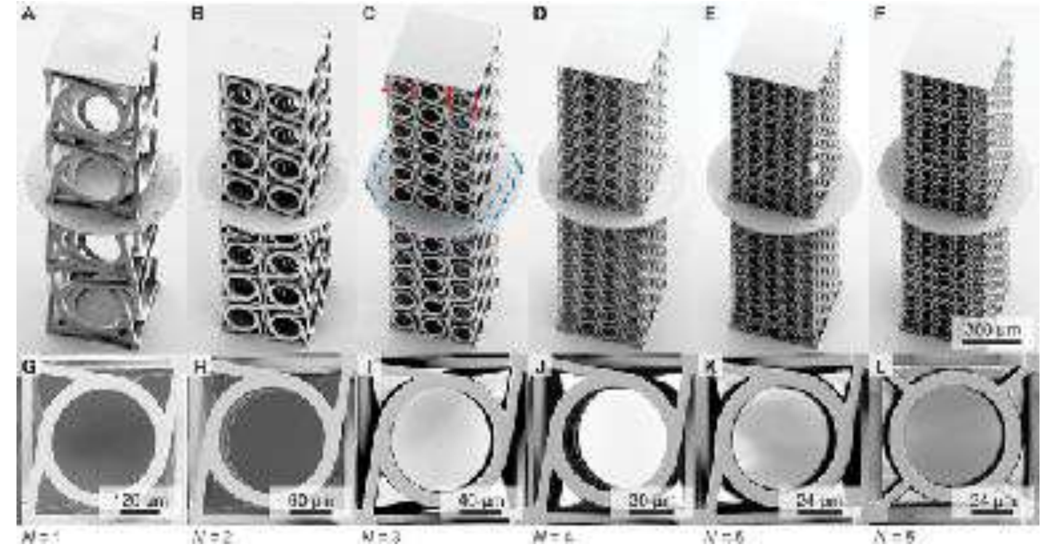


$$\vec{x}'(t) = \vec{x}(t) + \vec{u}(\vec{x},t)$$

The distance vector between two material points at $\vec{x}$ and $\vec{y}$ is changed from $d\vec{x} = \vec{x} - \vec{y}$ to $dx'_a = dx_a + \partial_b u^a dx_a$, and its length from $dr = \sqrt{d\vec{x}^2}$ to $dr' = \sqrt{d\vec{x}^2 + 2u_{ab}dx_a dx_b}$ where the strain tensor u_{ab} is in linear approximation,

$$u_{ab} = \frac{1}{2}\left(\partial_b u^a + \partial_a u^b\right).$$

Cosserat elasticity



Frenzel et al.

The Cosserat theory of elasticity, also known as micropolar elasticity, endows classical elasticity with local rotations. Physically it means that the elastic body is not considered as a collection of points at the microscopic level but rather of extended objects the can rotate in space. In two dimensions the displacement vector u_i is supplemented with an orientation angle θ. In the second step we require that the effective action/free energy is invariant under translations and rotations. Translations require that under the transformation $u_i \to u_i + b_i$, where b_i is a constant vector the action remains invariant. Rotations by a constant angle θ_0 are implemented by two simultaneous transformations $\theta \to \theta + \theta_0$, $u_i \to u_i + \epsilon_{ij} x_j \theta_0$. Gradients of the displacement field are invariant under translations but not under rotations. It is, however, possible to construct a combination

$$\gamma_{ij} = \partial_i u_j - \epsilon_{ij}\theta$$

The free energy reads

$$F = \int dt d^2x \left[\dot{\theta}\dot{\theta} + \dot{u}^i \dot{u}^i - C^{ijkl}\gamma_{ij}\gamma_{kl} + \zeta \tau_i \tau^i \right]$$

Thermoelasticity

We assume that it is a function of the energy, strain, and heat current

$$s = s(\varepsilon, u_{ij}, q_i).$$

This is analogous to viscoelasticity, where the stress or momentum current also contributes to entropy. Taking the divergence one gets

$$ds = \frac{d\varepsilon}{T} + \frac{\partial s}{\partial u_{ij}} du_{ij} + \frac{\partial s}{\partial q_k} dq_k.$$

We impose the second law of thermodynamics

$$\Delta_s = \dot{s} + \nabla_i J_i^s \geq 0,$$

where J^s is the entropy current, and supplement the system with conservation laws

$$\begin{aligned} &\rho \ddot{u}_i + \partial_j t_{ij} = 0 \quad , \\ &\dot{\varepsilon} + \partial_j q_j + \dot{u}_{ij} t_{ij} = 0 \quad , \\ &\dot{\rho} = 0 \quad . \end{aligned}$$

Odd Thermoelasticity

Ostoja-Starzewski, Surówka

Using the conservation laws, after some algebra, we find

$$\Delta_s = q_i \nabla_i (1/T) + \lambda_i \dot{q}_i \geq 0,$$

We now assume, in the linear regime, that

$$\lambda_i = -\alpha_{ij} q_j,$$

for some phenomenological tensor α_{ij}. When α_{ij} is positive the system is bounded from below with a well-defined equilibrium state. Similarly, we impose $\nabla_i(1/T) + \alpha_{ij}\dot{q}_j = \gamma_{ij} q_j$, from which relation we obtain $\tau_{ij}\dot{q}_j = -k_{ij}\nabla_j T - q_i$, where $\tau_{ij} = \alpha_{ik}\gamma^{-1}_{kj}$, $k_{ij} = \gamma^{-1}_{ij}/T^2$. For even passive materials ($\alpha_{ij} = \alpha\delta_{ij}$, $\gamma_{ij} = \gamma_{ji} = \gamma\delta_{ij}$) this reduces to the isotropic, even Maxwell-Cattaneo relation

$$\tau \dot{q}_i = -\lambda \nabla_i T - q_i,$$

where $\tau = \alpha\gamma$ and $\lambda = \gamma/T^2$. However, for odd active thermoelastic materials, we can have a generalized relation, with odd components τ_{odd} and k_{odd}. k_{odd} is an active component of heat conductivity in odd materials.

Hydrodynamics - a theory of conserved quantities

This conservation of particle number is expressed in hydrodynamics as conservation of mass, by the continuity equation

$$\partial_t \rho + \partial_i (\rho u_i) = 0$$

Another equation is the equation of motion of a fluid element. This equation can be written as a momentum conservation equation.

$$\partial_t (\rho u_i) + \partial_j T_{ij} = 0 \qquad\qquad T_{ij} = p\delta_{ij} + \rho u_i u_j$$

We are still one equation short to have a complete system. We add entropy conservation equation, which can be expressed as energy conservation using thermodynamics

$$\partial_t \left(\varepsilon + \frac{\rho u^2}{2} \right) + \partial_i \left[\left(w + \frac{\rho u^2}{2} \right) u_i \right]$$

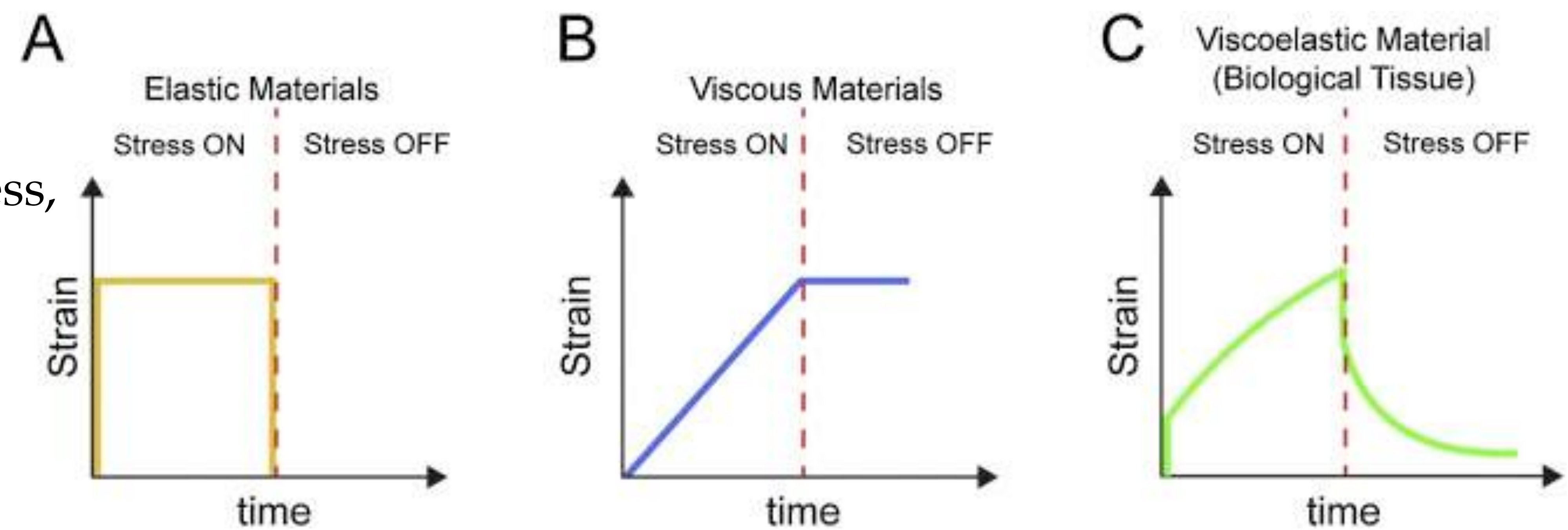
Rewriting we get the Euler's equation

$$\rho \frac{\partial \vec{u}}{\partial t} + \rho \vec{u} \cdot \nabla \vec{u} = -\nabla p$$

Viscoelasticity

Viscoelasticity is the property of materials that exhibit both viscous and elastic characteristics when undergoing deformation.

The creep-recovery test involves loading a material at constant stress, holding that stress for some length of time and then removing the load.



Modelled by a spring

Modelled by a dashpot/ damper

Modelled by a combination of springs and dashpots

Linear spring

The constitutive equation for a material which responds as a linear elastic spring of stiffness E is

$$\varepsilon = \frac{1}{E}\sigma$$

An elastic material undergoes an instantaneous elastic strain upon loading, maintains that strain so long as the load is applied, and instantaneously goes back to the initial position upon removal of the load.

Linear viscous dashpot

A dashpot responds with a strain rate proportional to the applied stress

$$\dot{\varepsilon} = \frac{1}{\eta}\sigma$$

η is the viscosity of the material. This is the characteristic response of Newtonian fluids. The larger is the stress, the faster is the straining.

Creep recovery response

The creep response follows immediately from the solution of constitutive equations

Dashpot

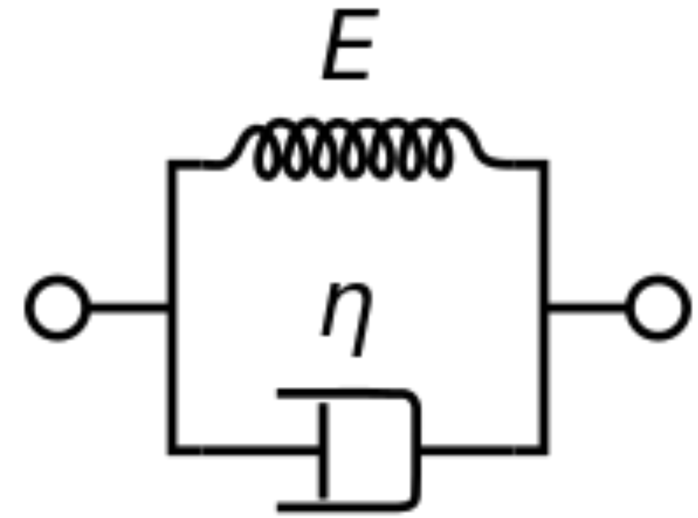
$$\varepsilon(t) = \sigma_0 J(t) \qquad\qquad J(t) = \frac{t}{\eta}$$

$J(t)$ is called the creep (compliance) function.

Linear spring

$$J(t) = \frac{1}{E}$$

Kelvin Voigt solid



We first consider a two-element model, the Kelvin-Voigt model, which consists of a spring and dashpot in parallel. We assume no bending moment.

$$\varepsilon = \frac{1}{E}\sigma_1 \qquad \dot{\varepsilon} = \frac{1}{\eta}\sigma_2 \qquad \sigma = \sigma_1 + \sigma_2$$

Total stress is a sum of the individual stresses in the dashpot and the spring. Responses are controlled by elastic and viscous transport coefficients. Eliminating individual components of the stress one gets the constitutive equation:

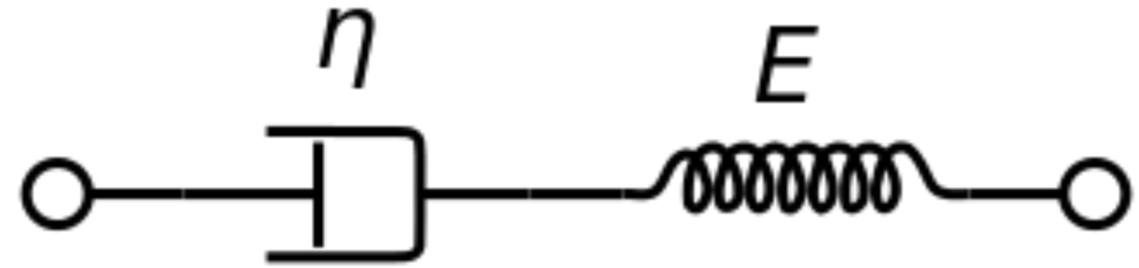
$$\sigma = E\varepsilon + \eta\dot{\varepsilon}$$

Solving the first order non-homogeneous differential with vanishing initial strain gives

$$\varepsilon(t) = \frac{\sigma_0}{E}\left[1 - e^{-(E/\eta)t}\right] \qquad \frac{\eta}{E} \equiv \tau_R$$

Retardation time is a measure of the time taken for the creep strain to accumulate.

Maxwell fluid



Another possibility for a two element representation of a viscoelasticity is a spring and dashpot in series, known as the Maxwell model.

$$\varepsilon_1 = \frac{1}{E}\sigma \qquad \dot{\varepsilon}_2 = \frac{1}{\eta}\sigma \qquad \varepsilon = \varepsilon_1 + \varepsilon_2$$

Total strain is a sum of the individual strains in the dashpot and the spring. One can eliminate the individual strains to get the constitutive equation

$$\sigma + \frac{\eta}{E}\dot{\sigma} = \eta\dot{\varepsilon}$$

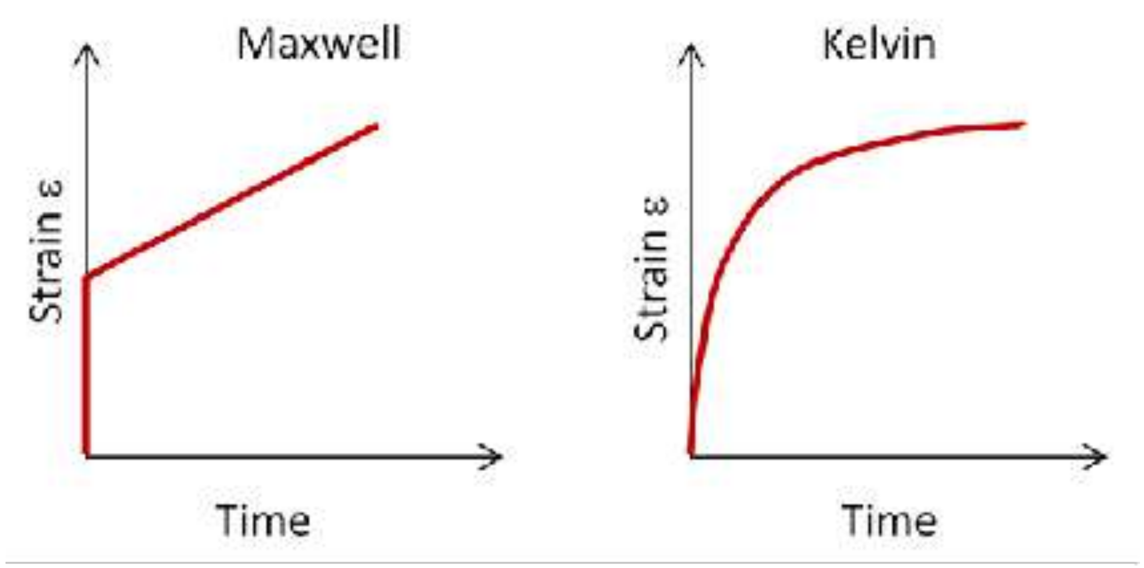
When the Maxwell model is subjected to a stress, the spring will stretch immediately and the dashpot will take time to react. Using this as the initial condition gives

$$\varepsilon(t) = \sigma_0\left(\frac{1}{\eta}t + \frac{1}{E}\right)$$

A new feature is the stress relaxation.

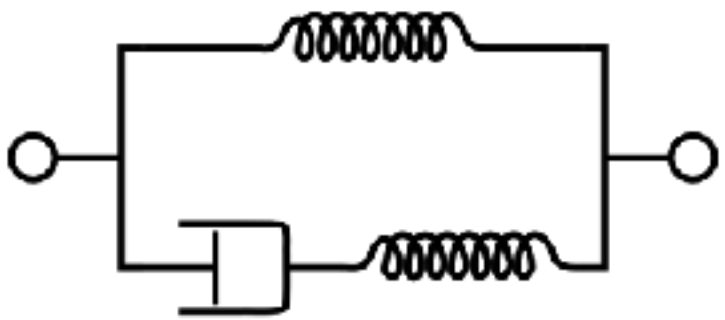
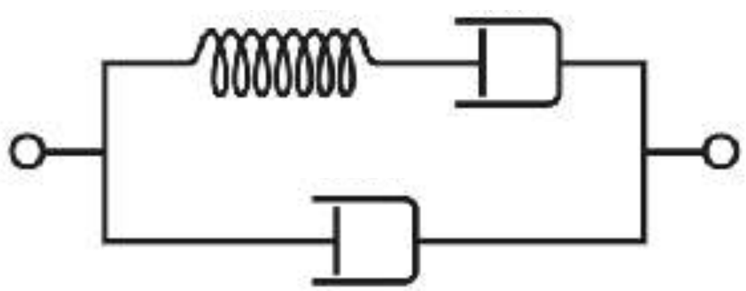
Limitations

- The Maxwell model predicts creep, but it does not decrease with time. There is no anelastic recovery (strain recovers over time).
- No stress relaxation in the Kelvin-Voigt model
- Not covariant (frame indifferent)
- Not applicable to chiral systems
- No plasticity



Three element models

The usual procedure to get more realistic models of viscoelasticity is to increase the number of elements. the simplest extension is to add one spring or one dashpot. Again one distinguishes two classes of models: standard linear solids (Zener models) or standard linear fluids (Jeffreys models)

Constitutive relation	Representation
$\sigma + \tau\dot{\sigma} = \mathcal{E}_1\varepsilon + \tau\mathcal{E}_2\dot{\varepsilon}$	
$\sigma + \tau\dot{\sigma} = \eta_1\dot{\varepsilon} + \tau\eta_2\ddot{\varepsilon}$	

In order to derive the constitutive equations we need to solve equations for individual elements. The can be done in one dimension but becomes not practical in higher dimensions. Therefore we would like to follow the symmetry approach.

Navier-Stokes equation from symmetries

What terms we can write to describe a Galilean invariant fluid? We postulate that in very reference frame the physics is the same

$$t' = t, \qquad \vec{x'} = \vec{x} + \vec{u}t, \qquad \vec{v'}(t', \vec{x'}) = \vec{v}(t, \vec{x}) + \vec{u}$$

We check how the derivatives transform: $\vec{\nabla}' = \vec{\nabla} \qquad \nabla'^2 \vec{v}' = \nabla^2 \vec{v}$

$$\frac{\partial}{\partial t'} = \frac{\partial}{\partial t} - \vec{u} \cdot \vec{\nabla} \qquad\qquad \frac{\partial \vec{v}'}{\partial t'} = \frac{\partial \vec{v}}{\partial t} - (\vec{u} \cdot \vec{\nabla})\vec{v}$$

There is a leftover term. We check how it transforms

$$(\vec{v}' \cdot \vec{\nabla}')\vec{v}' = (\vec{v} \cdot \vec{\nabla})\vec{v} + (\vec{u} \cdot \vec{\nabla})\vec{v}$$

We can construct an invariant equation

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{v} = \eta \nabla^2 \vec{v} - \frac{1}{\rho} \vec{\nabla} P$$

Oldroyd models

Oldroyd in 1950 formulated the first systematic attempt to provide constitutive models for viscoelastic fluids in a way that respects material frame indifference. In other words stresses in a continuous medium should arise from deformations only and not from rotations. We saw that in the context of NS equations

$$d/dt \to \partial/\partial t + (\mathbf{v} \cdot \nabla)$$

This simple substitution does not work if we act on tensors. Oldroyd proposed several derivatives that transform covariantly w.r.t. rotations. In the modern language such a corresponds to a covariance under diffeomorphisms of the fluid manifold. From differential geometry we know that the derivative that generates the diffeomorphism is the Lie derivative.

$$\frac{D}{Dt} A^{i\ldots m}{}_{j\ldots n} = \dot{A}^{i\ldots m}{}_{j\ldots n} + \mathcal{L}_N A^{i\ldots m}{}_{j\ldots n}$$

N^i describes the movement of a fluid particle w.r.t. the coordinate (frame) choice in the fluid space.

Transport and elastic coefficients

A small deformation parametrized by a displacement vector u_i, $i = 1, \ldots, d$ produces a stress that depends on the strain $u_{ij} = \partial_i u_j + \partial_j u_i$ and the strain rate $\dot{u}_{ij} \equiv \partial_t u_{ij}$ through the elastic modulus (K) and viscosity (η) tensors

$$T_{ij} = p\delta_{ij} - K_{ijkl}u_{kl} - \eta_{ijkl}\dot{u}_{kl}.$$

As a warm-up exercise let us consider fluids. The first term is the pressure. When time reversal invariance is not broken, the viscosity tensor satisfies Onsager's relations

$$\eta_{ijkl} = \eta_{klij}.$$

For a rotationally invariant system the above relation allows one only two possible transport coefficients, the shear and bulk viscosities

$$\eta_{ijkl} = \eta(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \left(\zeta - \frac{2}{d}\eta\right)\delta^{ij}\delta_{kl}.$$

“Odd” transport in two dim.

When time reversal invariance is broken, as for instance if a background magnetic field is turned on, the conditions Onsager are relaxed and it is possible to have an `odd' contribution to the viscosity

$$\eta^{(A)}_{ijkl} = -\eta^{(A)}_{klij}.$$

A peculiarity of the odd viscosity is that can be dissipationless. The variation of the energy density under a strain is

$$\delta\varepsilon = -T_{ij}\delta u_{ij}.$$

Using the first law of thermodynamics $\delta\varepsilon = T\delta s - p\delta V$, with s the entropy density, T the temperature and V the volume, the change of entropy with time becomes

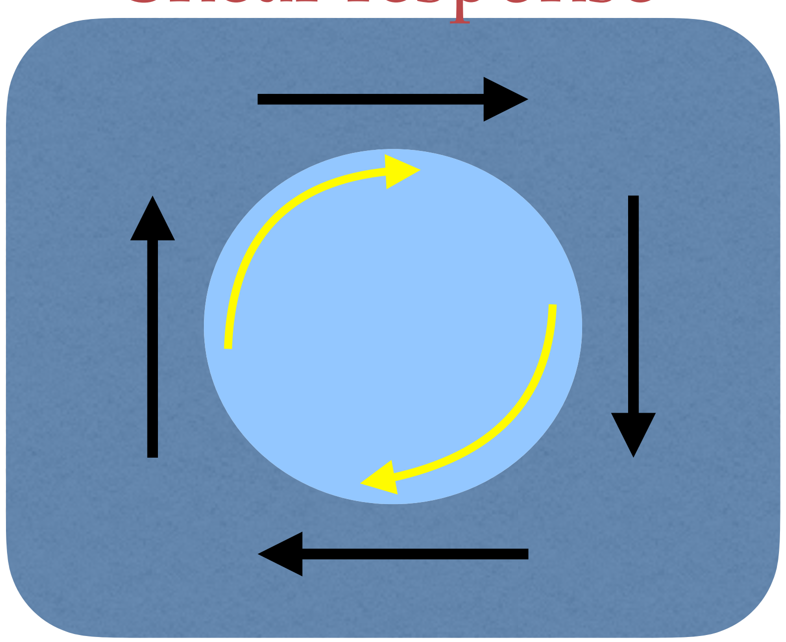
$$T\dot{s} = \eta_{ijkl}\dot{u}_{ij}\dot{u}_{kl}.$$

Odd viscosity

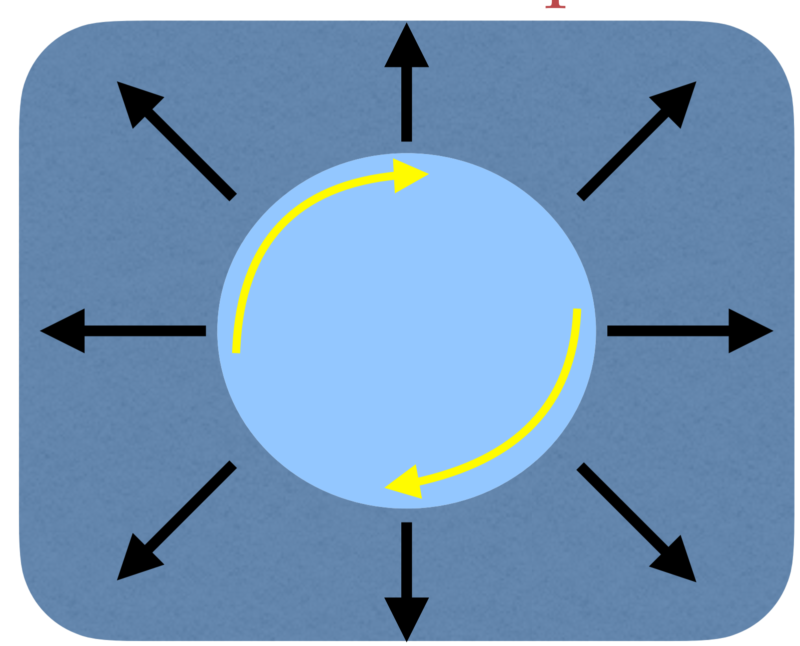
In general, $\eta^{(A)} = 0$ if rotational invariance is not broken. However, for $d = 2$ spatial dimensions an odd viscosity is allowed if parity is also broken

$$\eta^{(A)}_{ijkl} = -\frac{\eta_H}{2}(\epsilon^{ik}\delta^{jl} + \epsilon^{jk}\delta^{il} + \epsilon^{il}\delta^{jk} + \epsilon^{jl}\delta^{ik}).$$

Shear response



Odd/Hall response



Is it possible to have an analogous expression for elasticity?

"Everyone knew it was impossible, until a fool who didn't know came along and did it."— Albert Einstein

Odd Elasticity

Free energy for elasticity reads

$$F = \frac{1}{2} \int dt d^2x \Big[\dot{u}^i \dot{u}^i - K^{ijkl} u_{ij} u_{kl} \Big]$$

Odd terms vanish identically

$$K^{[ij]kl} = 0$$ left minor symmetry

$$K^{ij[kl]} = 0$$ right minor symmetry

$$K^{ijkl} = K^{klij}$$ major symmetry

Odd elasticity implies a violation of major symmetries. Differs from Cosserat elasticity.

Elasticity in two dimensions

$$u^0(\mathbf{x}) = \tau^0_{ij} u_{ij}(\mathbf{x}) \quad \text{Dilation}$$
$$u^1(\mathbf{x}) = \tau^1_{ij} u_{ij}(\mathbf{x}) \quad \text{Rotation}$$
$$u^2(\mathbf{x}) = \tau^2_{ij} u_{ij}(\mathbf{x}) \quad \text{Shear strain 1}$$
$$u^3(\mathbf{x}) = \tau^3_{ij} u_{ij}(\mathbf{x}) \quad \text{Shear strain 2}$$

Avron

$$\begin{pmatrix} \sigma^0(\mathbf{x}) \\ \sigma^1(\mathbf{x}) \\ \sigma^2(\mathbf{x}) \\ \sigma^3(\mathbf{x}) \end{pmatrix} = 2 \begin{pmatrix} K^{00} & K^{01} & K^{02} & K^{03} \\ K^{10} & K^{11} & K^{12} & K^{13} \\ K^{20} & K^{21} & K^{22} & K^{23} \\ K^{30} & K^{31} & K^{32} & K^{33} \end{pmatrix} \begin{pmatrix} u^0(\mathbf{x}) \\ u^1(\mathbf{x}) \\ u^2(\mathbf{x}) \\ u^3(\mathbf{x}) \end{pmatrix} .$$

Viscoelastic odd KV solids

Scheibner, Souslov, Banerjee,
Surówka, Irvine, Vitelli

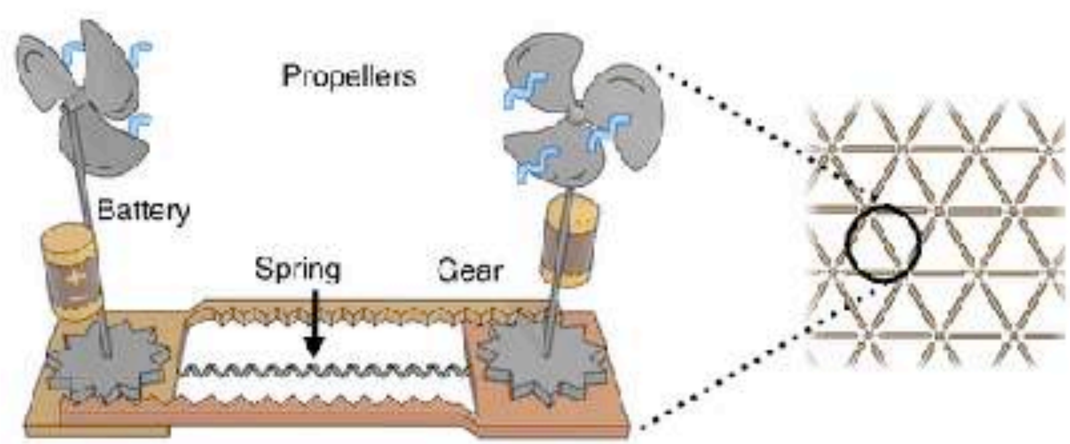
Isotropy and conservation laws fix the form of the elastic tensor. In the usual case two positive elastic moduli.

$$K^{\alpha\beta} = 2 \begin{pmatrix} \lambda+\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & \mu \end{pmatrix}^{\alpha\beta}$$

If one doesn't impose the conservation of energy two new coefficients are allowed. Stability requires adding a relaxation mechanism e.g. viscosity.

$$K^{\alpha\beta} = 2 \begin{pmatrix} \lambda+\mu & 0 & 0 & 0 \\ A & 0 & 0 & 0 \\ 0 & 0 & \mu & -\kappa_o \\ 0 & 0 & \kappa_o & \mu \end{pmatrix}^{\alpha\beta}$$

Possible mechanical realisation



Odd viscoelastic Maxwell fluids

The constitutive equation together with the momentum conservation equation for the Maxwell fluid read

$$v_{kl} = \eta^{-1}_{ijkl}\sigma_{ij} + \kappa^{-1}_{ijkl}\frac{D}{Dt}\sigma_{ij}\,,$$

$$\rho\frac{D}{Dt}v_i = -\partial_i p + \partial_j \sigma_{ij}\,,$$

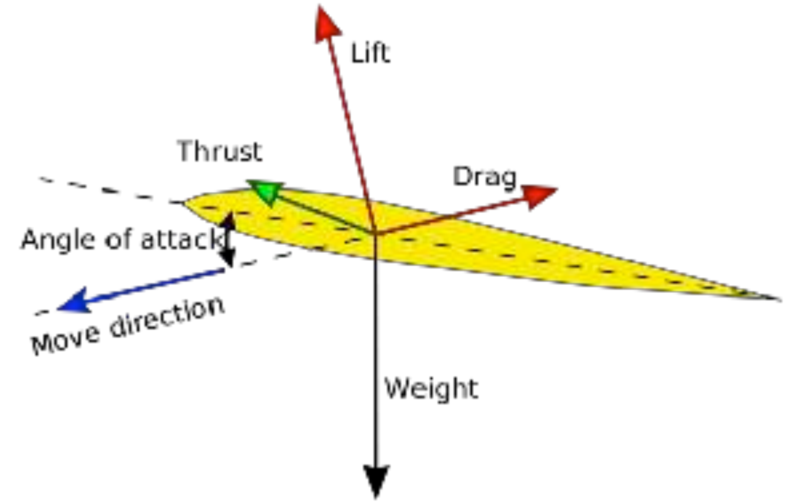
Banerjee, Vitelli, Jülicher, Surówka

These equations are much more complicated to deal with then for solids with a lot of unknown properties. As simple physical example we can look at relaxation times

$$\tilde{\tau} = \frac{\eta + \zeta}{\mu + \lambda} \qquad \tilde{\tau}_{1,2} = \frac{\eta^o\kappa^o + \eta\mu \pm i(\eta^o\mu - \eta\kappa^o)}{\mu^2 + (\kappa^o)^2}$$

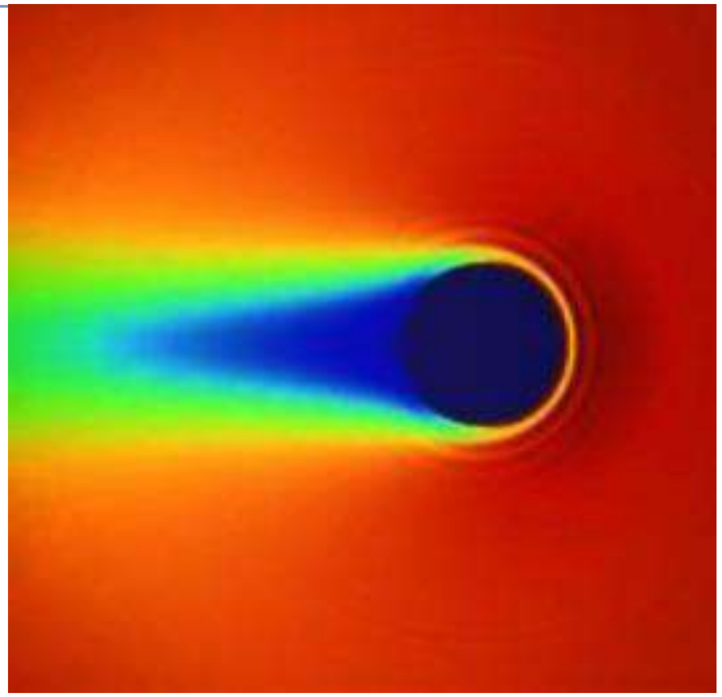
Contrary to even viscoelastic fluids transverse and longitudinal modes cannot be decoupled. The non-dissipative part corresponds to chiral metric hydrodynamics proposed by Son to describe fractional Hall states.

Lift force



New incarnation of an old problem: what are the forces on a cylinder in an odd fluid?

In a fluid without parity breaking the cylinder experiences only drag. Symmetries in odd fluids do not forbid lift, although it was shown by Ganeshan and Abanov that incompressible odd fluid does not experience lift.



$$f_l \propto \tilde{\omega} \ln \tilde{\omega}$$

Lier, Duclut, Bo, Armas, Jülicher, Surówka

What about compressible odd fluids? In the steady case the lift force is present only if mass is not conserved. However, the lift force can be present if the fluid is oscillating. This opens up a possibility to measure it by microrheological experiments.

Elasticity as a gauge theory

The action (or free energy at finite temperature) is

$$S[u^i] = \int dt d^2x \left[\dot{u}_i \dot{u}_i - C^{ijkl} u_{ij} u_{kl} \right] ,$$

where C^{ijkl} is a tensor of elastic moduli.

The equation of motion takes the form of a conservation law for momentum

$$\dot{P}^i + \partial_j T^{ij} = 0 \qquad \Leftrightarrow \qquad \partial_\mu T^{i\mu} = 0 \, .$$

Next we reformulate the partition function in terms of the dual variables by essentially performing a Legendre transformation

$$S[P^i, T^{ij}, u^i] = \int dt d^2x \left[P_i P^i + C^{-1}_{ijkl} T^{ij} T^{kl} + u_i (\partial_\mu T^{i\mu}) \right] .$$

Pretko, Radzihovsky

we split the displacement filed into the smooth and singular part and perform integral over the smooth part.

$$Z = \int DP^i DT^{ij} e^{iS[P^i, T^{ij}]} \delta\left(\partial_\mu T^{i\mu}\right) \ .$$

We are going to resolve the constraint by introducing a tensor gauge field

$$T^{i\mu} = \epsilon^{\mu\nu\rho}\partial_\nu A^i_\rho \, .$$

The formulation contains the gauge redundancy of the stress tensor

$$\delta A^i_\mu = \partial_\mu \alpha^i \, .$$

It is convenient to define the generalized electric and magnetic fields

$$B^i = \epsilon^{kl}\partial_k A^i_l \, , \qquad E^i_j = \epsilon^i{}_k(-\partial_0 A^k_j + \partial_j \Phi^k) \, .$$

The momentum and stress tensor map to the vector magnetic field and tensor electric field

$$P^i = B^i \, , \qquad T^{ij} = \epsilon_i{}^k \epsilon_j{}^l E^{kl} \ .$$

Conclusions

- Odd viscoelasticity is a new phenomenon, which can shed light on aspects of meta-materials and active matter
- Microscopic models allow for analytic control
- New applications
- Playground for physicists and engineers

Thank you!