Strings In and Out of Equilibrium
&
Dark Radiation (GW Waves)

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Introduction

* An early Universe where high temperature strings dominate the energy density !
* String thermodynamics subject with a long history, yet

  many important open question

  of both formal and phenomenological character remain
* Let me start by listing two recent (of many) reviews:
  * Hagedorn String Thermodynamics in Curved Spacetimes and near Black Hole Horizons

    T. G. Mertens (PhD Thesis), 2015
  * Superstring Cosmology - A Complementary Review R Brandenberger, 2023
* Our focus: Boltzmann Equations, extracting Equilibration rates & connect to some cosmological observables dark radiation, high frequency gravitational waves

This talk is going to be based on

* A Frey, R Mahanta and AM

  (Phys.Rev.D 105 (2022) 6, 066007)

* A Frey, R Mahanta, AM, F Muia, F Quevedo, G Villa

  (JHEP 03 (2024). 112)

* A Frey, R Mahanta, AM, F Quevedo, G Villa

  (2408.13803)

Outline

* Boltzmann equation approach to string thermodynamics (in flat space)

  (To build up towards the cosmology)

* A Stochastic Background of High Frequency Gravitational Waves from High Temperature Strings

1. Boltzmann Equations for Thermal Strings in flat space

* There are various approaches to study to thermal strings, the Boltzmann equation approach was pioneered in:
D Lowe and L Thorlacius '94; S Lee and L Thorlacius '97
E. J. Copeland, T. W. B. Kibble, and D. A. Steer '98

* Here, the basic idea is to write a rate equation for $n(\ell)$
$n(\ell)d\ell$: the number of strings of length $(\ell, \ell + d\ell)$
(ℓ length of strings, defined as $\ell \equiv 2\pi\alpha' M$)

$$\boxed{\frac{\partial n(\ell, t)}{\partial t} = \text{Interaction Rates}; \ell >> 1}$$

* Note that are only keeping track of the number of strings at length ℓ, so a coarse grained description.

We will touch upon three areas:

* The form of the interaction rates (& provide strong evidence that they admit a simple interpretation)
* The structure of detailed balance & equilibrium solutions
* And finally: Non-equilibrium Dynamics

with effectively Non-compact directions, with cosmological applications in mind.

J Manes '02

* Interaction rates
  * The rates can be determined by string perturbation theory. Here, we are working in the limit of $\ell >> 1$.
  * We have found strong evidence that they can be obtain by a random walk picture of string interactions.
* To exhibit this, I will focus on the decay of a highly excited closed string to two closed strings (other cases in the paper)

* The analytic form of the decay rate of a high excited closed string in a specific state $|S\rangle$ to two other such strings

$$|S\rangle \to |s'\rangle + |s''\rangle$$

is unknown.

* On the other hand, the decay rate for the inclusive process where one averages over all initial states of the same mass (length) and sums over final states with the same mass is available.
* These are exactly the kind of decay rates that one is interested if one wants to write a rate equation for $n(\ell)$.

* The decay rate per unit length of the outgoing string ℓ' is given by

$$\frac{d\Gamma}{d\ell'} \sim g_s^2 \ell \left(\frac{\ell}{\ell'(\ell - \ell')} \right)^{d/2}$$

in "d" non-compact directions.

* An important feature of these averaged processes is that the total string length is conserved

$$\ell = \ell' + \ell''$$

Next, let us give this decay rate a random walk interpretation

$$\frac{d\Gamma}{d\ell'} \sim g_s^2 \ell \left(\frac{\ell}{\ell'(\ell - \ell')} \right)^{d/2}$$

So, *What is the random walk interpretation ?* It is has four postulates

* String interactions rates are proportional to their length
* Long strings are in a random walk configuration.
* Interactions take place when strings intersect.
* Interaction rates are weighted by the probabilities for intersection.

Let us analyse our decay rate in this light

$$\frac{d\Gamma}{d\ell'} \sim g_s^2 \ell \left(\frac{\ell}{\ell'(\ell - \ell')} \right)^{d/2}$$

* There is the overall factor of ℓ as string interactions proportional to their length.

* What about the term in the brackets ?

  * For this, Recall: A random walk of length ℓ in “d" dimensions fills in a volume $\ell^{d/2}$

  * The probability that it closes on to itself is proportional to $\frac{1}{\ell^{d/2}}$.

* The probability that a string closes on to itself is proportional to $\frac{1}{\ell^{d/2}}$.

* With this, we can write the term is the brackets

$$\left(\frac{\ell}{\ell'(\ell-\ell')}\right)^{d/2} = \frac{\left(\frac{1}{\ell'}\right)^{d/2}\left(\frac{1}{\ell''}\right)^{d/2}}{\left(\frac{1}{\ell}\right)^{d/2}}$$

in terms of the closure probabilities of the three strings.

* Thus, the term in brackets is
$$\frac{\text{the probability that closed mother string self intersects such}}{\text{that there are daughter strings of length } \ell' \text{ and } \ell''}$$

* So, the rate is in agreement with the random walk interpretation.

* This was one example, we have found that all interaction rates available in the literature are consistent with the random walk picture

* The next step, is to write the Boltzmann equation. In some situations, some of the interaction rates needed are not available from the string perturbation theory literature. Given the evidence for the random walk picture we use have used it to determine these interactions

* We plan verify these by explicit string perturbation theory computations. Making this connection more concrete is an interesting direction.

Boltzmann Equations

Open and Closed Strings in the presence of space-filling branes

$$\frac{\partial n_c(l)}{\partial t} = +\frac{b}{2N} V_\perp \frac{n_o(l)}{l^{d/2}} - a\frac{N}{V_\perp} l n_c(l) + \frac{1}{2}\int_{l_c}^{l-l_c} dl' \left(\kappa_a \frac{n_c(l')l'\, n_c(l-l')(l-l')}{V} - \kappa_b l n_c(l) \left(\frac{l}{l'(l-l')}\right)^{d/2} \right)$$
$$+ \int_{l+l_c}^{\infty} dl' \left(\kappa_b l' n_c(l') \left(\frac{l'}{l(l'-l)}\right)^{d/2} - \kappa_a \frac{l n_c(l)(l'-l) n_c(l'-l)}{V} \right)$$
$$+ \int_{l+l_c}^{\infty} dl' \left(\kappa_c \frac{(l'-l) n_o(l')}{l^{d/2}} - \kappa_d \frac{l n_c(l)(l'-l) n_o(l'-l)}{V} \right).$$

$$\frac{\partial n_o(l)}{\partial t} = +a\frac{N}{V_\perp} l n_c(l) - \frac{b}{2N} V_\perp \frac{n_o(l)}{l^{d/2}} + \int_{l_c}^{l-l_c} dl' \left(\frac{b}{2NV_\parallel}\, n_o(l') n_o(l-l') - a\frac{N}{V_\perp} n_o(l) \right)$$
$$+ \int_{l+l_c}^{\infty} dl' \left(2a\frac{N}{V_\perp} n_o(l') - \frac{b}{NV_\parallel} n_o(l) n_o(l'-l) \right) + \int_{l_c}^{l-l_c} dl' \left(\kappa_d \frac{l'(l-l') n_c(l') n_o(l-l')}{V} - \kappa_c n_o(l) \frac{l-l'}{l'^{d/2}} \right)$$
$$+ \int_{l+l_c}^{\infty} dl' \left(\kappa_c \frac{n_o(l') l}{(l'-l)^{d/2}} - \kappa_d \frac{l n_o(l)(l'-l) n_c(l'-l)}{V} \right) + \text{(2-2 interactions)},$$

Detailed Balance, Equilibrium Solutions Boltzmann Equations

* Detailed balance: Equilibrium Solutions to Boltzmann equations can be obtained by setting the net rate along <u>every</u> reaction channel to be zero.

* Equilibrium solutions in our case of interest (four dimesnsions, open string on branes)

$$n_c(l) \simeq M_s^4 \frac{e^{-l/L}}{(M_s l)^{5/2}}\,, \qquad n_o(l) \simeq N_D^2 M_s^4 e^{-l/L}\,, \qquad (1)$$

* L length of typical long string
* Also

$$L^{-1} = M_s^2(\beta - \beta_H)$$

Energy Density is dominated by nonrelativistic open strings

* Boltzmann Equations allow us to probe non-equilibrium dynamics Various cosmological applications ...

* I will discuss the case of closed strings in all compact directions in detail in the talk. Consider a perturbation $\delta n(\ell, t)$ about the equilibrium solution.

$$n(\ell, t) = \frac{e^{-\ell/L}}{\ell} + \delta n(\ell, t)$$

* This satisfies an integro-differential equation

$$\frac{V}{\kappa}\frac{\partial \delta n(\ell, t)}{\partial t} = -\left(\frac{\ell^2}{2} + \ell L\right)\delta n(\ell, t) + \int_0^l dl'\, \ell' \delta n(\ell', t)\left(e^{\frac{-(\ell-\ell')}{L}} - 1\right) - \delta E\left(e^{-\ell/L} - 1\right),$$

where δE is the energy of the perturbation,

$$\delta E \equiv \int_0^\infty d\ell'\, \ell' \delta n(\ell', t)\,.$$

The integro-differential equation:

$$\frac{V}{\kappa}\frac{\partial \delta n(\ell,t)}{\partial t} = -\left(\frac{\ell^2}{2}+\ell L\right)\delta n(\ell,t) + \int_0^l d\ell'\,\ell'\delta n(\ell',t)\left(e^{\frac{-(\ell-\ell')}{L}}-1\right) - \delta E\left(e^{-\ell/L}-1\right)\,,$$

* Interestingly, it is possible to find explicit solutions. By taking derivatives, we show that $\delta n(\ell,t)$ needs to satisfy the differential equation

$$\left[2(\ell+L)+\left(\frac{\ell^2}{2}+\ell L+\frac{V}{\kappa}\frac{\partial}{\partial t}\right)\frac{\partial}{\partial \ell}\right]\left(\delta n(\ell,t)+L\frac{\partial \delta n}{\partial \ell}\right)=0\,.$$

* And thus the problem can be divided into two simpler problems:
  a) Find the kernel of the operator

$$\mathcal{L}\equiv 2(\ell+L)+\left(\frac{\ell^2}{2}+\ell L+\frac{V}{\kappa}\frac{\partial}{\partial t}\right)\frac{\partial}{\partial l}\,,$$

  b) Translate the functions in the kernel, denoted $K(\ell,t)$, into fluctuations through a first order inhomogeneous ODE:

$$\delta n(\ell,t)+L\frac{\partial \delta n(\ell,t)}{\partial \ell}=K(\ell,t)\,,$$

These have the form:

*
$$\delta n(\ell, t) = e^{-\ell/L}$$

*
$$\delta n_c(\ell, t) = \sqrt{\frac{\pi(c + tL^2)}{2}} \frac{e^{-\frac{\ell}{L} + A(t)^2}}{L} \mathrm{Erf}\left(A(t), A(t) + \sqrt{\frac{c + tL^2}{2}} \frac{\ell}{L} \right)$$

where $\mathrm{Erf}(z_1, z_2) = \frac{2}{\sqrt{\pi}} \int_{z_1}^{z_2} e^{-t^2} dt$ is the incomplete error function, and

$$A(t) = \sqrt{\frac{c + tL^2}{2}} \left(1 - \frac{1}{c + tL^2} \right) .$$

* Only zero energy perturbations

$$\delta E \equiv \int_0^\infty d\ell' \, \ell' \delta n(\ell', t) \ = 0$$

settle to the background equilibrium solutions.

* These can be obtained by considering linear combination of the two solutions
* They have a length dependent equilibration rates

$$\Gamma(\ell) = \frac{\kappa}{V} \left(\frac{\ell^2}{2} + \ell L \right)$$

result in keeping with the basic estimates of D Lowe and L Thorlacius '94

* A similar approach can be used to study the dynamics in the presence of non-compact directions case of interest (admixture of open and closed strings, d=3)

$$\begin{aligned}
\Gamma_{o,o}(l) &\simeq g_s N_D M_s^2 \left(L + \frac{l}{2} \right) \gtrsim g_s N_D M_s^2 L, \\
\Gamma_{o,c}(l) &\simeq g_s N_D M_s^2 l + \frac{g_s M_s}{2 N_D (M_s l)^{3/2}} \gtrsim g_s N_D M_s^2 l \sim g_s N_D M_s^2 L\,, \\
\Gamma_{c,c}(l) &\simeq g_s^2 l \left(\frac{\rho_c}{M_s^2} + \frac{M_s^2}{(M_s l_c)^{1/2}} \right) \simeq g_s^2 M_s^2 l \sim g_s^2 M_s^2 L.
\end{aligned} \tag{2}$$

* N_D: Number of D-branes
* g_s string coupling

A consistent cosmology

Putting all the elements together, one can have a cosmology with

* With equilibrium maintained by the splitting and joining process of long strings
* Temperature
$$L(t) = L_* \left(\frac{a_*}{a(t)} \right)^{3/2} , \tag{3}$$
* Hubble during the epoch
$$H \simeq \frac{\sqrt{\rho}}{M_p} = N_D L M_s^2 \frac{M_s}{M_p} , \tag{4}$$
One can check that such an epoch can take place after inflation if there is a hierarchy between the Planck M_p and String scale M_s.
* Epoch comes to an end with the decay of the long strings to massless ones

* Gravitions are produced during the epoch from the decay of the long open string.
* This process is Planck suppressed and takes the "greybody" from

$$\frac{d\Gamma_{o,g}}{d\omega} = A\left(\frac{M_s}{M_p}\right)^2 M_s \ell (\omega/T_H)^2 \sigma(\omega/T_H) \frac{e^{-\omega/T_H}}{1-e^{-\omega/T_H}},$$

Here, ℓ : length of the string, T_H Hagedorn temperature.
$\sigma(\omega/T_H)$: grey body factor.

* This channel is not in equilibrium. There is a constant production of gravitions from the direct decay of long strings.

The Stochastic gravitational background

* The stochastic gravitational wave background today can be computed:

$$
\begin{aligned}
h^2\Omega_{GW} &= 5\cdot 10^{-7}\left(\frac{N_D}{5}\right)\left(\frac{L_{\rm end}M_s}{5}\right)\left(\frac{A}{1}\right)\left(\frac{1}{\Upsilon}\right)^2\left(\frac{G}{0.32}\frac{X}{1}\right)^4\left(\frac{M_s}{10^{15}\,\mathrm{GeV}}\right)\left(\frac{Y}{1}\right)^{5/2} \\
X &\quad I\left(Y, B, \frac{L_s}{L_{\rm end}}\right) \qquad Y\equiv \omega_0/T_0 GX = 2\pi f_0/T_0 GX
\end{aligned} \tag{5}
$$

Many quantities in the formulae, I which I do not define. It was two very interesting features:

* Peaks in the 50 GHz region

* Amplitude is directly proportional to the string scale.

* The Amplitude is significantly larger than the amplitudes of GW waves obtained from the reheating epoch of the Standard Model or its field theoretic extensions.

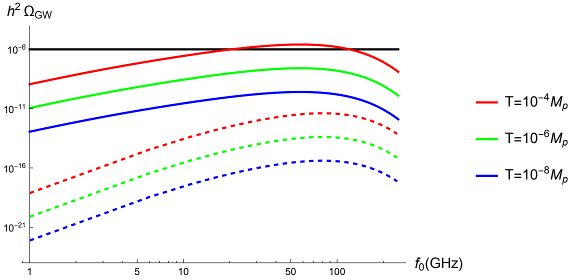

Figure 1: Comparison of the GW wave spectrum with that of the the Standard Model

WHY ?

* Direct decay channels

$$\ell \to \ell' + g$$

* Exponential density of states (at high mass) implies that states with $m >> T$ can be relevant for the process.

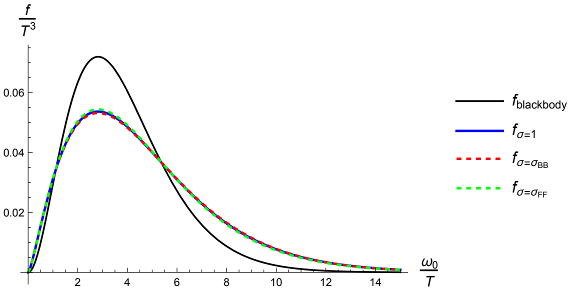

Figure 2: Has a distinct shape

Comparing with Blackbody with the same $\Delta N_{\rm eff}$ and same peak

* Broader than the BB
* Low frequency behaviour: milder fall off than w^3 than of blackbody.

Conclusions

* Boltzmann equations for highly excited strings: random walk picture, equilibration rates

* A high frequency stochastic GW waves, with amplitude significantly larger than that of the SM and its field theoretic extensions