Line shape of soft photon radiation generated at zero angle in an undulator with a dispersive medium

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1. The frequency-angular distribution of the number of photons produced in a medium, regardless of the type of radiation

The system of four interrelated Maxwell equations of the electromagnetic field forms the foundation of electrodynamics. The microscopic form of these first-order partial differential equations, when using scalar and vector potentials, is reduced to two second-order equations. By utilizing the freedom in defining the potentials, it is possible to choose a system of potentials that satisfies the Lorenz condition. In this case, the potentials satisfy individual inhomogeneous wave equations, which have the same structure. If the gauge scalar function satisfies the homogeneous wave equation, then the Lorenz condition remains valid for the potentials satisfying this condition before transformation. The Lorenz gauge is independent of the choice of the coordinate system. A particular solution of the inhomogeneous wave equation for scalar and vector potentials, for a confined distribution of charges and currents in the absence of boundaries, is determined through the retarded Green's function in accordance with the causality condition. The electromagnetic field of a point charge, determined by the Liénard–Wiechert potentials, includes not only the static field but also the transverse radiation field, which depends on acceleration. The total energy radiated per unit solid angle is determined by integrating the power over time, which is defined by the radiation field intensity. The Fourier amplitude of the radiation field is expressed as an integral along the charge trajectory. It is assumed that the observation point is sufficiently distant from the region of space where the radiation field is formed, such that it is observed under a small solid angle. Integration is carried out over the time interval during which the acceleration of the charged particle is non-zero. In the frequency-angular distribution of the radiation energy, obtained based on the microscopic form of the Maxwell equations, by replacing the speed of light in a vacuum c with $c/\sqrt{\epsilon(\omega)}$ and the charge e with $e/\sqrt{\epsilon(\omega)}$, where ω is the radiation frequency; we obtain the energy distribution of the radiation produced in the medium.
Dividing the energy distribution of the radiation by $\hbar\omega$, where $\hbar = h/(2\pi)$ and h is Planck's constant, yields the following distribution for the number of radiated photons.

$$\frac{d^3N}{d\omega d(\cos\vartheta)d\varphi} = \frac{\alpha\omega\sqrt{\epsilon(\omega)}}{4\pi^2}|\boldsymbol{I}(\omega,\vartheta)|^2, \tag{1}$$

where $d(\cos\vartheta)d\varphi$ represents the solid angle of radiation: the polar angle ϑ is measured from the unit vector in the direction of radiation $\boldsymbol{n} = -\sin\vartheta\cos\varphi\,\boldsymbol{i} - \sin\vartheta\sin\varphi\,\boldsymbol{j} + \cos\vartheta\,\boldsymbol{k}$, and φ is the azimuthal angle. The constant $\alpha = 1/137$ is the fine-structure constant. The electromagnetic field of the radiation is defined by the following time integral:

$$\begin{aligned}\boldsymbol{I}(\omega,\vartheta) &= \int \boldsymbol{a}(t)\exp\left\{i\omega\left(t - \frac{\sqrt{\epsilon(\omega)}}{c}\boldsymbol{n}\cdot\boldsymbol{r}(t)\right)\right\}dt,\\ \boldsymbol{a}(t) &= [\boldsymbol{n}\times[\boldsymbol{n}\times\boldsymbol{\beta}]],\end{aligned} \tag{2}$$

where $\boldsymbol{\beta}(t)$ is the particle's velocity in units of c, and $\boldsymbol{r}(t)$ is the particle's trajectory. The integration in (2) is performed only over the time interval during which the charged particle interacts with the medium and/or external fields.
When the velocity of the charged particle is constant, radiation occurs when the argument of the exponent goes to zero. This condition corresponds to the law of conservation of energy-momentum and relates the photon energy to the direction of radiation. It is satisfied in the case of Vavilov-Čerenkov radiation $\beta\sqrt{\epsilon(\omega)} > 1$, as well as in transition radiation due to the difference $\Delta\epsilon(\omega)$ at the boundary of two media. When the velocity of the charged particle changes, the law of conservation of energy-momentum holds, taking into account the integrand factor $\boldsymbol{a}(t)$.

2. Line shape of soft and hard photons emitted at zero angle

To the extent that small energy losses due to radiation justify a classical approach to the problem, the energy of the particle is conserved. As the transverse component of the velocity of a particle with energy γmc^2 changes; where γ is the Lorentz factor and mc^2 is the rest energy; the longitudinal component also changes. However, radiation caused by changes in the longitudinal component of velocity is smaller by a factor of γ^2 than the radiation caused by changes in the transverse component. Therefore, we will assume that the particle moves in the longitudinal direction of the undulator with a root-mean-square velocity $\beta_\parallel = \sqrt{\langle\beta_\parallel(t)^2\rangle}$. Then, due to energy conservation, $\beta^2 = \langle\beta_\perp(t)^2\rangle + \langle\beta_\parallel(t)^2\rangle = \langle\beta_\perp(t)^2\rangle + \beta_\parallel^2$, it follows that the energy of the longitudinal motion of the particle $\gamma_\parallel = \frac{1}{\sqrt{1-\beta_\parallel^2}}$ is less than the total energy γ by a factor of $\sqrt{Q} = \sqrt{1+\gamma^2\langle\beta_\perp(t)^2\rangle}$. The transverse velocity of the particle in the magnetic field of a planar undulator (a one-dimensional oscillator) varies according to the law $\boldsymbol{\beta}_\perp(t) = -\beta_\perp\sin(\Omega t)\,\boldsymbol{j}$, where $\beta_\perp$ is the undulator parameter, and Ω is the oscillation frequency. Thus, the trajectory of the particle is $\boldsymbol{r}(t) = (\beta_\perp c/\Omega)\cos(\Omega t)\,\boldsymbol{j} + \beta_\parallel c\,\boldsymbol{k}$. Note that for a one-dimensional oscillator, $Q = 1 + q^2/2$, where $q = \beta_\perp\gamma$ is the radiation parameter, unlike a two-dimensional oscillator, for which $Q = 1 + q^2$.
If the particle in the planar undulator undergoes n_0 oscillations with frequency Ω, it is convenient to integrate (2) with respect to the variable $\tau = \Omega t$ over the range $[-\pi n_0, \pi n_0]$. Since we are interested in the line shape of the frequency distribution of radiation at $\vartheta \to 0$, it is easy to show that $\boldsymbol{a}(t) = \boldsymbol{n}\left(-\vartheta\beta_\perp\cos\varphi\sin\tau + \beta_\parallel\cos\vartheta\right) - \boldsymbol{\beta}(t) \approx \boldsymbol{\beta}_\perp(t) = \frac{\boldsymbol{j}\beta_\perp}{2i}\left(e^{-i\tau} - e^{i\tau}\right)$, and $\frac{\sqrt{\epsilon(\omega)}}{c}\boldsymbol{n}\cdot\boldsymbol{r}(t)$ approaches $\beta_\parallel\sqrt{\epsilon(\omega)}\cos\vartheta$.

Using the dimensionless frequency $\xi = \omega/(\Omega\gamma_\parallel^2)$, for the frequency-angular distribution of the number of emitted photons, we obtain:

$$\frac{d^3N}{d\omega d(\cos\vartheta)d\varphi} = \frac{\alpha(\Omega\gamma_\parallel^2)^2\xi\sqrt{\epsilon(\omega)}}{4\pi^2}|\boldsymbol{I}(\omega,\vartheta)|^2, \tag{3a}$$

$$\boldsymbol{I} = \boldsymbol{j}\frac{\beta_\perp}{2i\Omega}\int_{-\pi n_0}^{\pi n_0}\left(e^{-i\tau} - e^{i\tau}\right)e^{i\xi\gamma_\parallel^2\left(1-\beta_\perp\sqrt{\epsilon(\xi)}\cos\vartheta\right)\tau}d\tau, \tag{3b}$$

where $\sqrt{\epsilon(\xi)} = 1 - \frac{1}{2\gamma_\parallel^2}\left(\frac{r}{\xi}\right)^2$, $r = \frac{\gamma_{\text{th}}}{\gamma_\parallel}$, $\gamma_{\text{th}} = \frac{\omega_p}{\Omega}$.
Considering the expansions $\beta_\parallel \approx 1 - \gamma_\parallel^{-2}/2$, $\cos\vartheta \approx 1 - \vartheta^2/2$, the argument of the exponent of the first integrand term, which corresponds to the law of conservation of energy-momentum during radiation, is:

$$\phi(\xi,\theta) = \frac{1}{2}\left((1+\theta^2)\xi - 2 + \frac{r^2}{\xi}\right), \tag{4}$$

where $\theta = \gamma_\parallel\vartheta$. This argument vanishes at the values:

$$\xi_{1,\,2} = \frac{1}{1+\theta^2}\left(1 \mp \sqrt{1 - r^2(1+\theta^2)}\right). \tag{5}$$

These values are valid (real) if $r\sqrt{1+\theta^2} < 1$. When $r = 1$ ($\gamma_\parallel = \gamma_{\text{th}}$), photons with frequency $\xi_0 = 1$ ($\omega_0 = \Omega\gamma_{\text{th}}^2 = \omega_p^2/\Omega$) are emitted at zero angle. When $\gamma_\parallel > \gamma_{\text{th}}$ two photons with frequencies $\omega_1 < \omega_0$ and $\omega_2 > \omega_0$ are emitted at zero angle. For $\gamma_\parallel \gg \gamma_{\text{th}}$, with accuracy up to small terms of order r^2, we have $\xi_1 = r^2/2$ ($\omega = \omega_p^2/(2\Omega)$) and $\xi_2(\theta) = 2/(1+\theta^2)$ ($\omega_2(\theta) = \frac{2\Omega\gamma_\parallel^2}{1+\theta^2}$). Note that for soft photons, the unit of angle ϑ is 1, while for hard photons ($\theta = \gamma_\parallel\vartheta$): $\gamma_\parallel^{-1}$. Therefore, for soft photons at $\vartheta = 0$, we obtain $\int d(\cos\vartheta)\,d\varphi = 2\pi$, and for hard photons: $\int d(\cos\theta)\,d\varphi = 2\pi$. Considering (3b), $\boldsymbol{I}(\xi,0) = \boldsymbol{j}\frac{\beta_\perp}{i\Omega}\frac{\sin\phi(\xi)}{\phi(\xi)}$. Consequently, $|\boldsymbol{I}|^2 = \frac{\beta_\perp^2}{\Omega^2}F(\xi_{1,\,2})$. The function $F(\xi_{1,\,2}) = \frac{\sin^2(\pi n_0\phi(\xi_{1,\,2}))}{\phi^2(\xi_{1,\,2})}$, for $n_0 \gg 1$ is a delta-like function with a peak value of $\pi^2 n_0^2$ and a width of $1/n_0$, regardless of the values of $\xi_{1,\,2}$. This line shape $F(\xi_{1,\,2})$ of radiation at zero angle is the same for both soft and hard photons.

3. Number of radiated photons

To an accuracy of order $1/n_0$, the formula

$$\begin{aligned}F(\xi_{1,\,2}) &= \pi^2 n_0\delta\left(\phi(\xi_{1,\,2})\right),\\ \delta(\xi_{1,\,2}) &= \frac{\delta(\xi-\xi_{1,\,2})}{|\phi'(\xi)|_{\xi=\xi_{1,\,2}}} = \frac{\delta(\xi-\xi_{1,\,2})}{1/2 - \xi_1/\xi_{1,\,2}^2} = \begin{cases}\xi_1\delta(\xi-\xi_1) & \text{for } \xi_1,\\ 2\delta(\xi-\xi_2) & \text{for } \xi_2.\end{cases},\\ \phi(\xi) &= \frac{\xi}{2} - 1 + \frac{\xi_1}{\xi},\end{aligned} \tag{6}$$

can be used, where it is considered that $\xi_1 \ll 1 < \xi_2 = 2$.
Thus, after integration, for soft and hard photons we have:

$$N(\xi_1) = 2\pi\alpha n_0\beta_\perp^2\left(\frac{\omega_p}{2\Omega}\right)^4, \tag{7a}$$

$$N(\xi_2) = 2\pi\alpha n_0 q_\parallel^2, \quad q_\parallel = \beta_\perp\gamma_\parallel = \frac{q}{\sqrt{Q}}. \tag{7b}$$

It should be noted that, at a specific energy of the charged particle, the number of soft and hard photons emitted at zero angle is the same.

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