Modular Neutrinos

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ROMA TRE UNIVERSITÀ DEGLI STUDI

NOW 2024
Neutrino Oscillation Workshop

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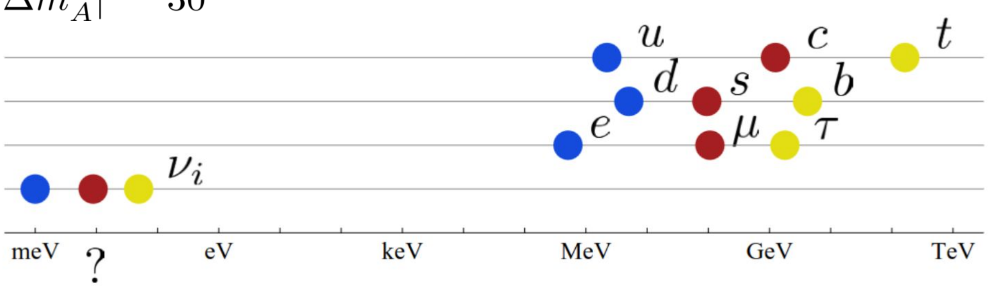
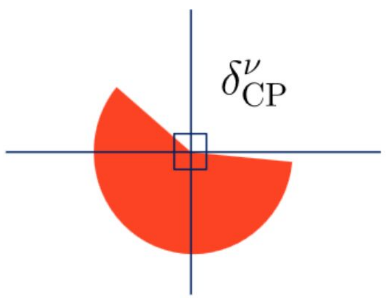
The flavour puzzle

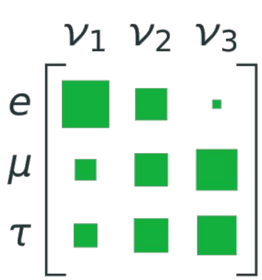
In search of an organising principle...

$$\frac{\Delta m^2_\odot}{|\Delta m^2_A|} \sim \frac{1}{30}$$

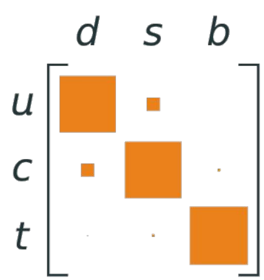


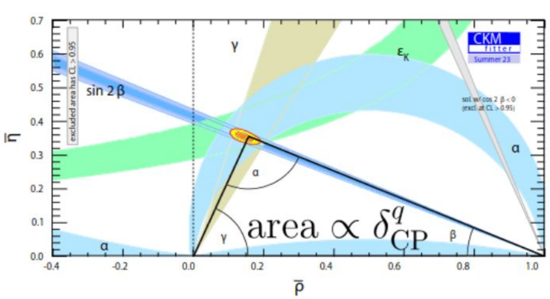


$U_{PMNS} \sim$



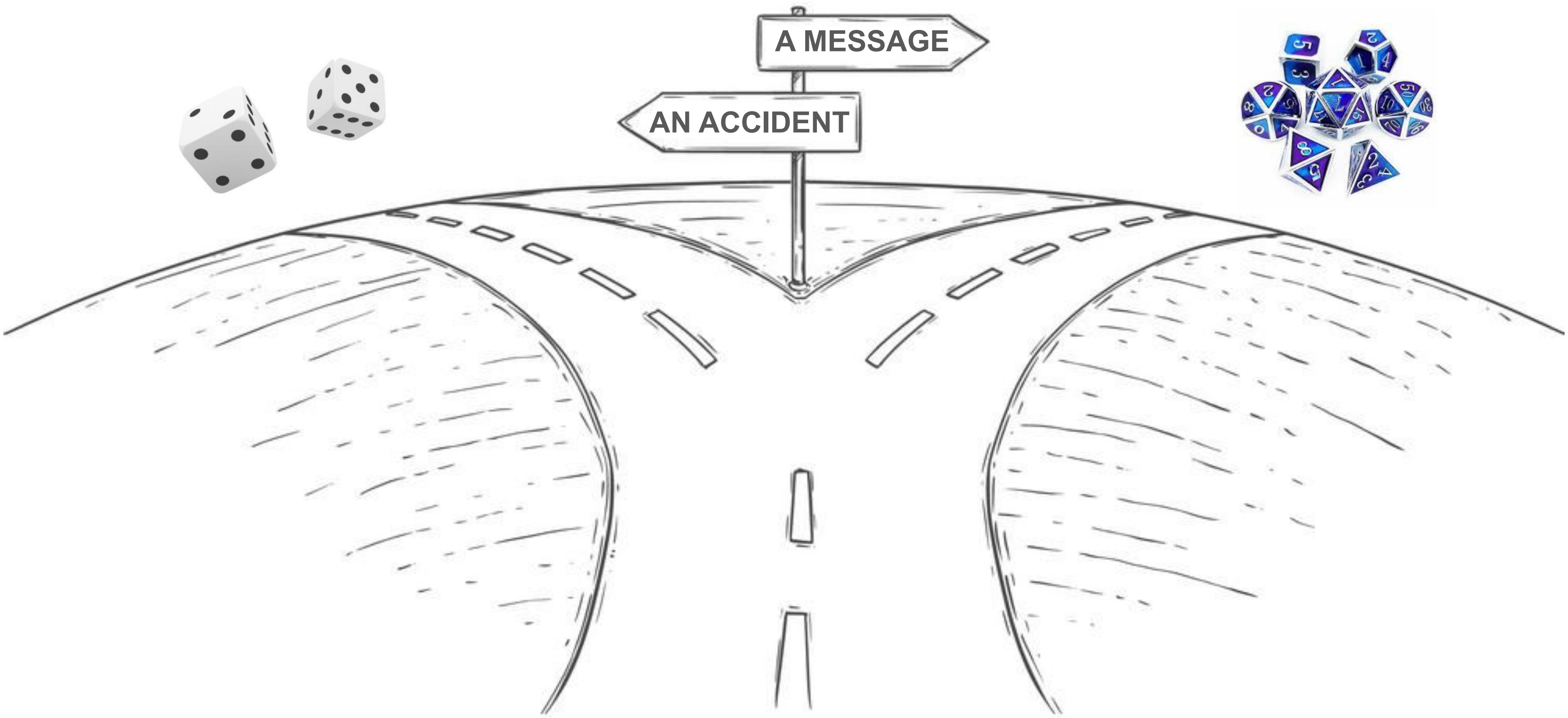
$U_{CKM} \sim$



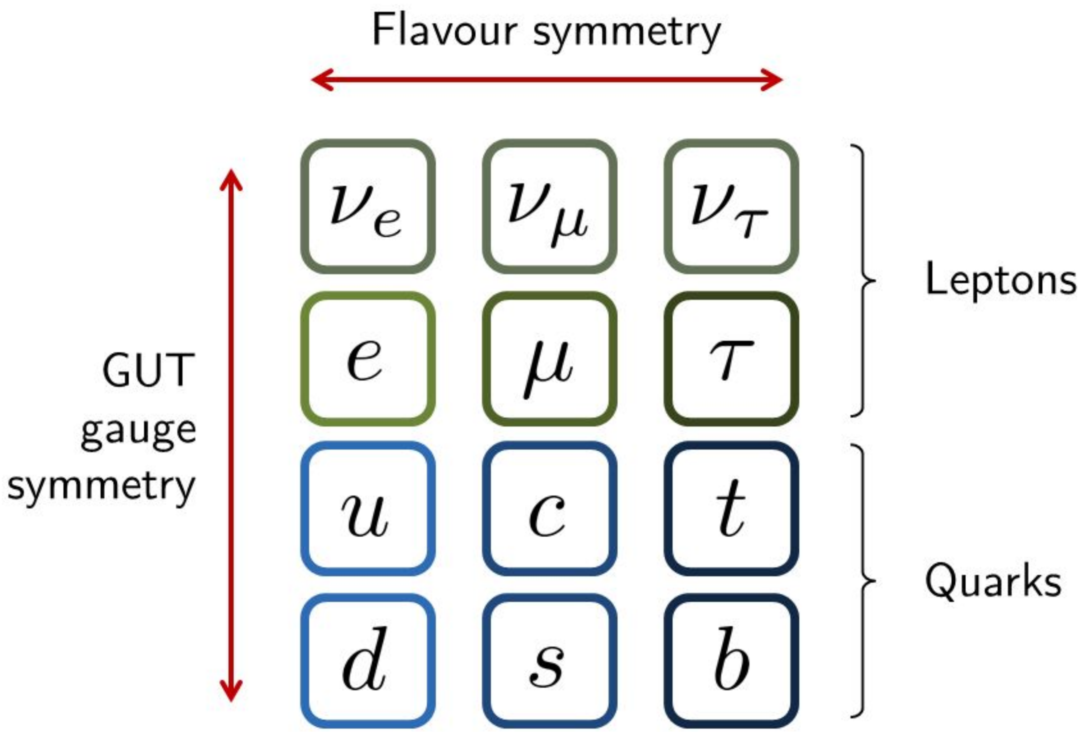


The flavour puzzle

..*is there* an organising principle?



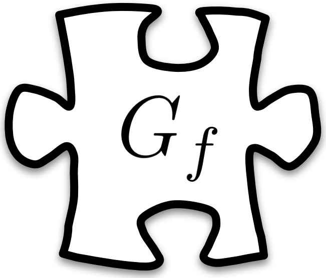
Flavour symmetries



For reviews, see: Altarelli and Feruglio (2010), Ishimori et al. (2010),
King and Luhn (2013), Petcov (2017), Feruglio and Romanino (2019), Ding and Valle (2024)

Flavour symmetries

Non-Abelian discrete flavour symmetries

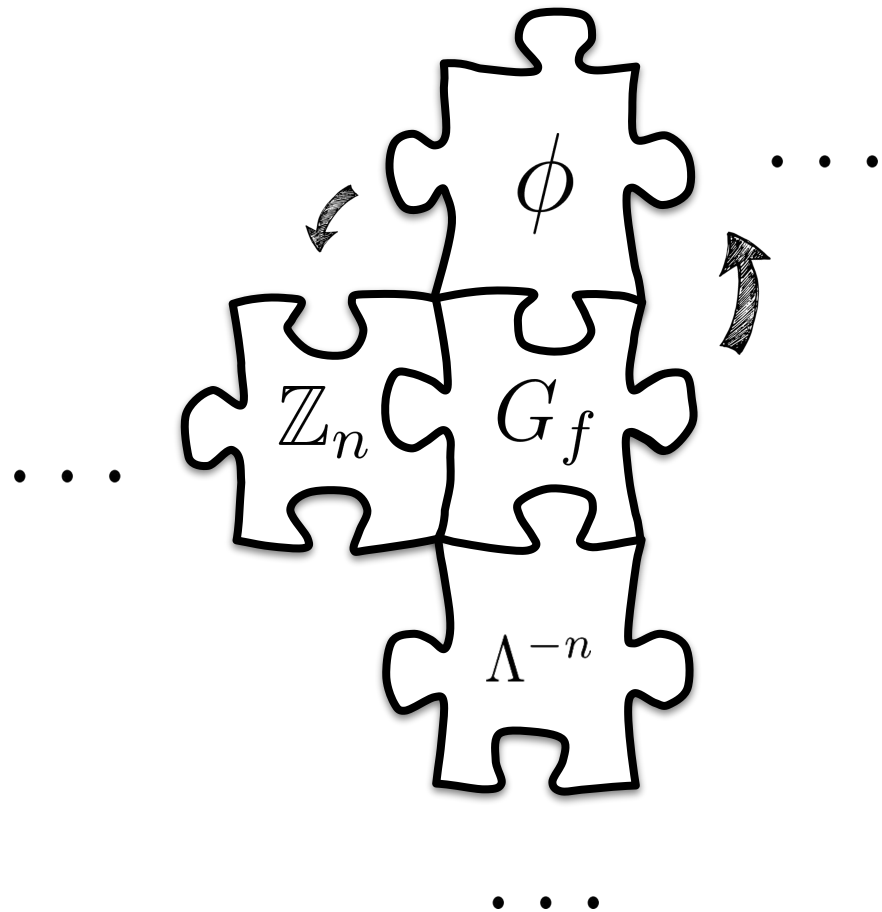


model-independent approaches relying on residual symmetries

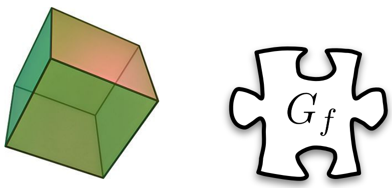
constrain mixing and the Dirac phase

[recall talk by C. Hagedorn on wed.]

Problems with model building

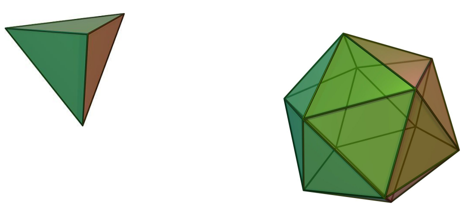


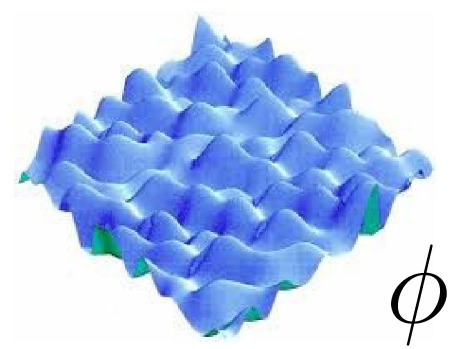
A reversal of the usual logic



Symmetry group
and representations

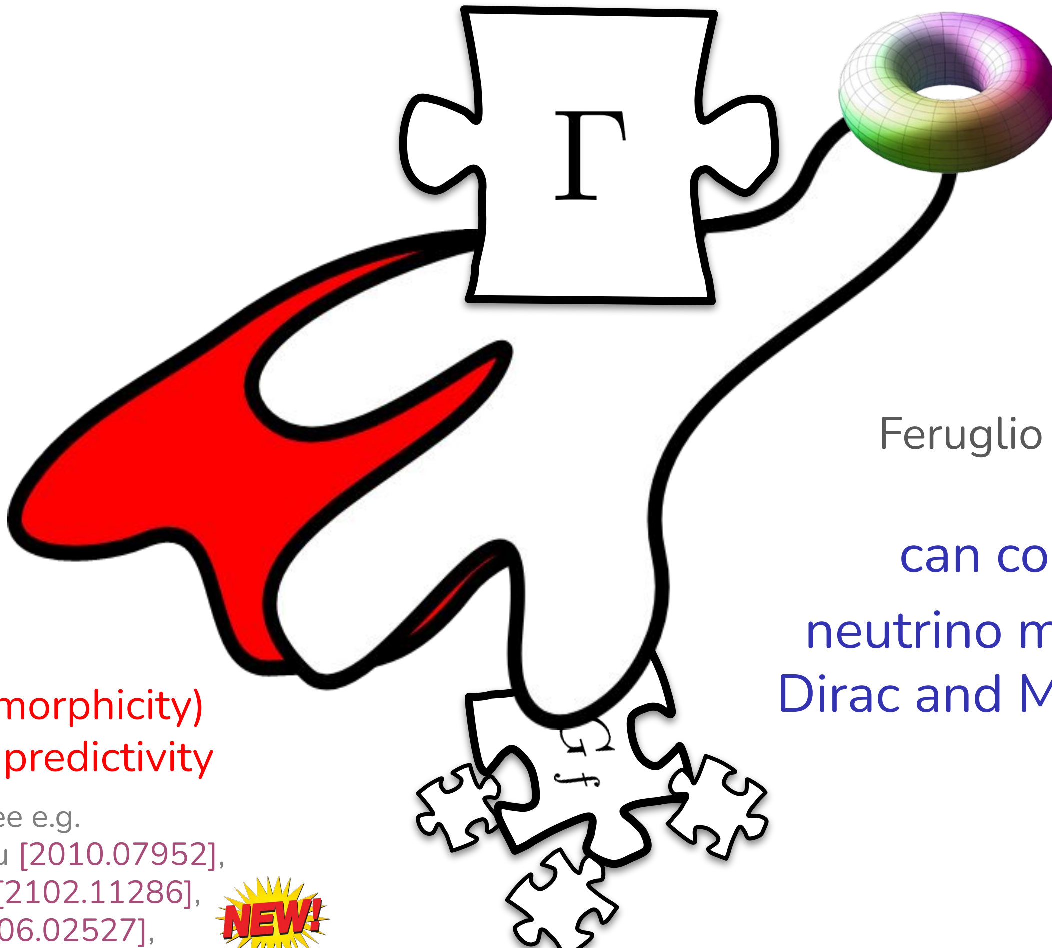
Symmetry-breaking
sector





see Feruglio, PoS DISCRETE 2020-2021, 007

Modular symmetry to the rescue!



Feruglio [1706.08749]

can constrain all:

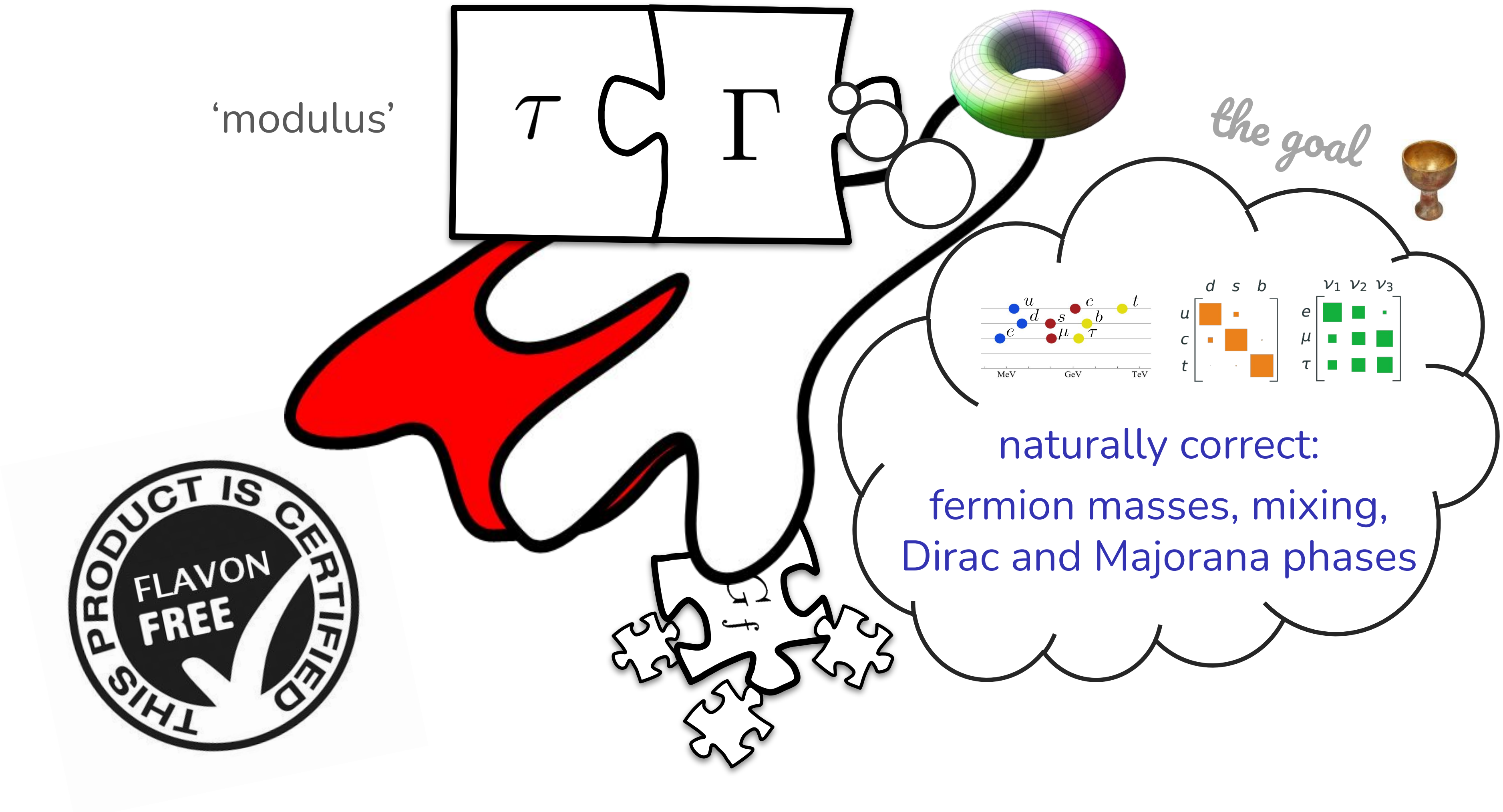
neutrino masses, mixing,
Dirac and Majorana phases

SUSY (holomorphicity)
required for predictivity

...but see e.g.
Ding, Feruglio, Liu [2010.07952],
Almumin et al. [2102.11286],
Ding, Qu[2406.02527],
Ding, Lu, Petcov, Qu [2408.15988]

NEW!

Modular symmetry to the rescue!



Modular symmetries can…

(the outline)

…offer a **predictive framework** for flavour

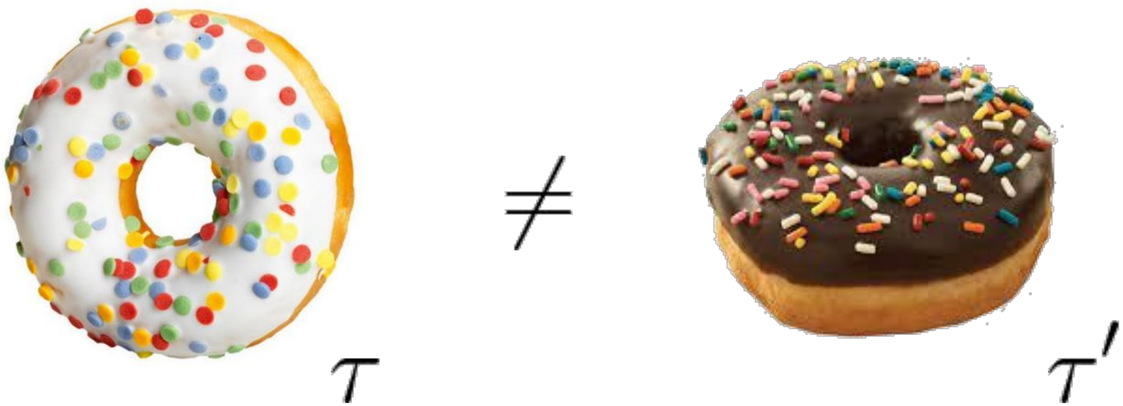
…provide an origin for **CP violation** (CPV)

…explain fermion **mass hierarchies**

The predictive framework

Image credit: sallysbakingaddiction.com

The modulus



τ may describe a torus compactification

(we assume only 1 unfrozen modulus)

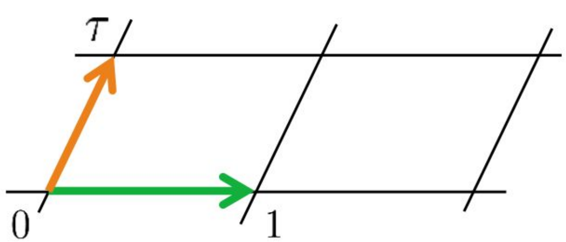
In the **bottom-up** modular approach τ is a dimensionless **spurion**

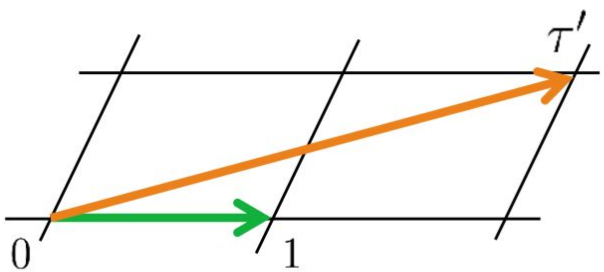
The modulus

τ

$=$

$$\tau' = \frac{a\tau + b}{c\tau + d}$$





$\mathrm{Im}\,\tau > 0$

$ad - bc = 1 \qquad a, b, c, d \in \mathbb{Z}$

The modulus

$=$

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\tau' = \frac{a\tau+b}{c\tau+d}$$

$$\det \gamma = 1$$

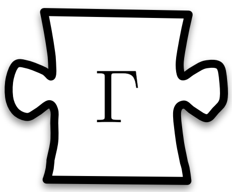
$$a, b, c, d \in \mathbb{Z}$$

the ***modular group***

$$\Gamma \equiv SL(2,\mathbb{Z}) = \{\gamma\}$$

The modular group

$$\tau \;\rightarrow\; \frac{a\tau + b}{c\tau + d}$$



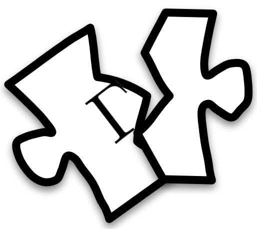
$$\Gamma \equiv SL(2,\mathbb{Z}) = \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle|\; a, b, c, d \in \mathbb{Z},\; \det\gamma = 1 \right\}$$

Presentation in terms of generators S, T, R:

$$S^2 = R, \quad (ST)^3 = R^2 = \mathbb{1}, \quad RT = TR$$

The modular group

$$\langle \tau \rangle \nrightarrow \frac{a\tau + b}{c\tau + d}$$



$$\equiv SL(2,\mathbb{Z}) = \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, a,b,c,d \in \mathbb{Z},\ \det\gamma = 1 \right\}$$

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}:$$

$$\tau \to -1/\tau$$

inver**S**ion

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}:$$

$$\tau \to \tau + 1$$

Translation

$$R = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}:$$

$$\tau \to \tau$$

Redundant

but can affect fields...

The field transformations

$$\psi \;\to\; (c\tau + d)^{-k}\,\rho(\gamma)\,\psi$$

The field transformations

* not necessarily:
rare from top-down!

automorphy factor

$$\psi \;\rightarrow\; \boxed{(c\tau + d)^{-k}}\;\rho(\gamma)\,\psi$$

Weight $k \in \mathbb{Z}$*

* not necessarily: rare from top-down!

The field transformations

NEW!

automorphy factor

$$\psi \to (c\tau + d)^{-k} \, \rho(\gamma) \, \psi$$

Weight $k \in \mathbb{Z}^{*}$

"Almost trivial" representation of the modular group

$$\rho\big(\Gamma(N)\big) = \mathbb{1}$$

$$\rho\big(T\,\Gamma(N)\big) = \rho(T)$$

$$\rho\big(S\,\Gamma(N)\big) = \rho(S)$$

$\cdots$ Feruglio [1706.08749]

* not necessarily: rare from top-down!

The field transformations

automorphy factor

$$\psi \rightarrow \boxed{(c\tau + d)^{-k}}\,\boxed{\rho(\gamma)}\,\psi$$

Weight $k \in \mathbb{Z}^*$

"Almost trivial" representation of the modular group

$$\Gamma(N) \subset \Gamma$$

Principal congruence subgroup of level *N*

$$\Gamma(N) \equiv \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}$$

$$\rho\big(\Gamma(N)\big) = \mathbb{1}$$

$$\rho\big(T\,\Gamma(N)\big) = \rho(T)$$

$$\rho\big(S\,\Gamma(N)\big) = \rho(S)$$

$$\cdots$$

Feruglio [1706.08749]

$\rho(\gamma)$ **is effectively a representation of** $\Gamma'_N \equiv \Gamma/\Gamma(N)$

other choices are possible: in general, ***vector-valued modular forms***, see e.g. [2112.14761, 2311.10136]

The finite modular groups

$\Gamma'_N \equiv \Gamma/\Gamma(N)$ *behave like flavour groups*

N	2	3	4	5
Γ_N	S_3	A_4	S_4	A_5
Γ'_N	S_3	$A'_4 \equiv T'$	$S'_4 \equiv SL(2,\mathbb{Z}_4)$	$A'_5 \equiv SL(2,\mathbb{Z}_5)$

← drop the **R** generator

(in general there in TD)

Presentation in terms of generators S, T, R:

$$S^2 = R, \quad (ST)^3 = R^2 = \mathbb{1}, \quad RT = TR, \quad T^N = \mathbb{1}$$

The finite modular groups

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Γ'_N	S_3	$A'_4 \equiv T'$	$S'_4 \equiv SL(2,\mathbb{Z}_4)$	$A'_5 \equiv SL(2,\mathbb{Z}_5)$

← drop the **R** generator

(in general there in TD)

$\Gamma_2 \simeq S_3$
Kobayashi et al. [1803.10391]
Meloni, Parriciatu [2306.09028]

$\Gamma_3 \simeq A_4$
Feruglio [1706.08749]

$\Gamma_4 \simeq S_4$
JP, Petcov [1806.11040]

$\Gamma_5 \simeq A_5$
Novichkov et al. [1812.02158]

summary in Appendices of Novichkov, JP, Petcov, Titov [1905.11970]

$\Gamma'_3 \simeq A'_4$
Liu, Ding [1907.01488]

$\Gamma'_4 \simeq S'_4$
Novichkov, JP, Petcov [2006.03058]

$\Gamma'_5 \simeq A'_5$
Wang, Yu, Zhou [2010.10159]

For early top-down, see e.g.:

Kobayashi et al. [1804.06644]; Kobayashi, Tamba [1811.11384]; de Anda et al. [1812.05620]; Baur et al. [1901.03251, 1908.00805]; Kariyazono et al. [1904.07546]; Nilles et al. [2001.01736, 2004.05200, 2006.03059]; Kobayashi, Otsuka [2001.07972, 2004.04518]; Abe et al. [2003.03512]; Ohki et al. [2003.04174]; Kikuchi et al. [2005.12642]

by now, O(200) papers...

A lot of model building...

- models based on finite modular groups of higher N
  [2004.12662, 2108.02181, 2307.01419]
- modular models of unification (also without GUTs)
  [1906.10341, 2012.01397, 2101.02266, 2101.12724, 2103.02633, 2103.16311, 2108.09655, 2206.14675, 2312.09255]
- modular models of leptogenesis
  [1909.06520, 2007.00545, 2103.07207, 2201.10429, 2204.08338, 2205.08269, 2206.14675, 2402.18547, 2405.09363]
- models with multiple moduli
  [1811.04933, 1812.11289, 1906.02208, 1908.02770, 2304.05958]

$$\tau \to \begin{pmatrix} \tau_1 & \tau_3 \\ \tau_3 & \tau_2 \end{pmatrix}$$

  based on symplectic modular invariance (**Siegel modular group**)
  and automorphic forms: Ding, Feruglio, Liu [2010.07952, 2402.14915]
  From TD, see e.g.: Nilles et al. [2105.08078], Baur et al. [2012.09586], Kikuchi et al. [2305.16709]

- models relating modular flavour symmetries and inflation
  [2208.10086, 2303.02947, 2405.06497, 2405.08924]
- models exploring the interplay of modular and gCP symmetries
  [1901.03251, 1905.11970, 1910.11553, 2006.03058, 2012.01688, 2012.13390, 2102.06716, 2106.11659]

But how does it work?

$$\psi \sim (\mathbf{r}, k)$$

$$W \sim g(\psi_1 \dots \psi_n)_{\mathbf{1}}$$

$$\psi \to (c\tau + d)^{-k} \rho_{\mathbf{r}}(\gamma)\, \psi$$

Assuming rigid SUSY (*W* not invariant in SUGRA)

Need modular forms

$$\psi \sim (\mathbf{r}, k)$$

$$W \sim g\big(Y(\tau)\, \psi_1 \dots \psi_n\big)_{\mathbf{1}}$$

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$$\psi \to (c\tau + d)^{-k}\,\rho_{\mathbf{r}}(\gamma)\,\psi$$

$$Y(\tau) \to (c\tau + d)^{k_Y}\,\rho_Y(\gamma)\,Y(\tau)$$

Assuming rigid SUSY (W not invariant in SUGRA)

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$$\psi \to (c\tau + d)^{-k}\, \rho_{\mathbf{r}}(\gamma)\, \psi$$

$$Y(\tau) \to (c\tau + d)^{k_Y}\, \rho_Y(\gamma)\, Y(\tau)$$

$$\begin{cases} k_Y = k_1 + \ldots + k_n \\ \rho_Y \otimes \rho_1 \otimes \ldots \otimes \rho_n \supset \mathbf{1} \end{cases}$$

Assuming rigid SUSY (*W* not invariant in SUGRA)

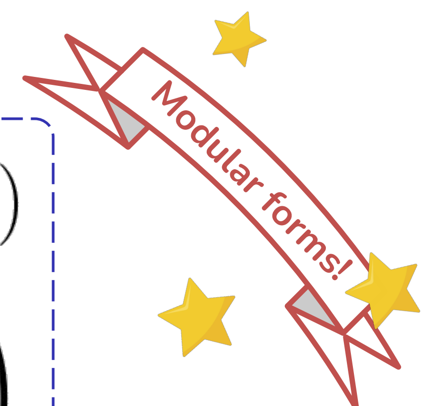
Need modular forms

$$\psi \sim (\mathbf{r}, k)$$

$$W \sim g(Y(\tau)\,\psi_1 \ldots \psi_n)_{\mathbf{1}}$$

$$\psi \to (c\tau + d)^{-k}\,\rho_{\mathbf{r}}(\gamma)\,\psi$$

$$Y(\tau) \to (c\tau + d)^{k_Y}\,\rho_Y(\gamma)\,Y(\tau) = Y\left(\frac{a\tau + b}{c\tau + d}\right)$$



Assuming rigid SUSY (*W* not invariant in SUGRA)

The modular forms

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Γ_N	S_3	A_4	S_4	A_5
Γ'_N	S_3	$A'_4 \equiv T'$	$S'_4 \equiv SL(2,\mathbb{Z}_4)$	$A'_5 \equiv SL(2,\mathbb{Z}_5)$
$\dim \mathcal{M}_k(\Gamma(N))$	$k/2+1$	$k+1$	$2k+1$	$5k+1$

Not so many available!

*A **finite set** of functions for each k_Y*

Lowest-weight *k* modular forms for each group:

$\Gamma_N^{(\prime)}$	$Y_{\mathbf{r}}^{(k)}$
$\Gamma_2 \simeq S_3$	$Y_{\mathbf{2}}^{(2)}$
$\Gamma'_3 \simeq A'_4$	$Y_{\hat{\mathbf{2}}}^{(1)}$
$\Gamma_3 \simeq A_4$	$Y_{\mathbf{3}}^{(2)}$
$\Gamma'_4 \simeq S'_4$	$Y_{\hat{\mathbf{3}}}^{(1)}$
$\Gamma_4 \simeq S_4$	$Y_{\mathbf{2}}^{(2)}, Y_{\mathbf{3'}}^{(2)}$
$\Gamma'_5 \simeq A'_5$	$Y_{\hat{\mathbf{6}}}^{(1)}$
$\Gamma_5 \simeq A_5$	$Y_{\mathbf{3}}^{(2)}, Y_{\mathbf{3'}}^{(2)}, Y_{\mathbf{5}}^{(2)}$

non-singular, unlike modular *functions*. Can still have an interpretation, see Feruglio, Strumia, Titov [2305.08908]

can generalize modular group to e.g. the larger metaplectic group and get half-integer weight forms, see Liu et al. [2007.13706]

Example

Let's build a modular-invariant term!

$$W \supset N N$$

Example

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Let's build a modular-invariant term!

$$\Gamma_3 \simeq A_4$$

$$N \sim (\mathbf{3}, 1)$$

$$W \supset \Lambda \left(N \otimes N \otimes Y_{\mathbf{3}}^{(2)} \right)_{\mathbf{1}}$$

Example

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$$\Gamma_3 \simeq A_4$$

$$N \sim (\mathbf{3}, 1)$$

$$W \supset \Lambda \left(N \otimes N \otimes Y_{\mathbf{3}}^{(2)} \right)_{\mathbf{1}}$$

$$Y_{\mathbf{3}}^{(2)}(\tau) = \begin{pmatrix} Y_1(\tau) \\ Y_2(\tau) \\ Y_3(\tau) \end{pmatrix}$$

$$M_N = \Lambda \begin{pmatrix} 2Y_1(\tau) & -Y_3(\tau) & -Y_2(\tau) \\ -Y_3(\tau) & 2Y_2(\tau) & -Y_1(\tau) \\ -Y_2(\tau) & -Y_1(\tau) & 2Y_3(\tau) \end{pmatrix}$$

Example

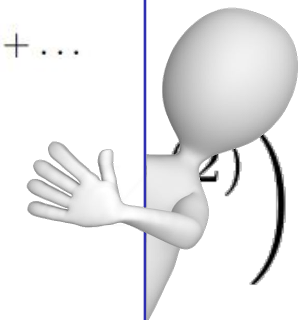
$$W \supset NN$$

Let's build a modular-invariant term!

$$Y_1(\tau) = \frac{i}{2\pi}\left[\frac{\eta'\left(\frac{\tau}{3}\right)}{\eta\left(\frac{\tau}{3}\right)} + \frac{\eta'\left(\frac{\tau+1}{3}\right)}{\eta\left(\frac{\tau+1}{3}\right)} + \frac{\eta'\left(\frac{\tau+2}{3}\right)}{\eta\left(\frac{\tau+2}{3}\right)} - \frac{27\eta'(3\tau)}{\eta(3\tau)}\right] = 1 + 12q + 36q^2 + 12q^3 + \dots$$

$$Y_2(\tau) = \frac{-i}{\pi}\left[\frac{\eta'\left(\frac{\tau}{3}\right)}{\eta\left(\frac{\tau}{3}\right)} + \omega^2\frac{\eta'\left(\frac{\tau+1}{3}\right)}{\eta\left(\frac{\tau+1}{3}\right)} + \omega\frac{\eta'\left(\frac{\tau+2}{3}\right)}{\eta\left(\frac{\tau+2}{3}\right)}\right] = -6q^{1/3}(1 + 7q + 8q^2 + \dots)$$

$$Y_3(\tau) = \frac{-i}{\pi}\left[\frac{\eta'\left(\frac{\tau}{3}\right)}{\eta\left(\frac{\tau}{3}\right)} + \omega\frac{\eta'\left(\frac{\tau+1}{3}\right)}{\eta\left(\frac{\tau+1}{3}\right)} + \omega^2\frac{\eta'\left(\frac{\tau+2}{3}\right)}{\eta\left(\frac{\tau+2}{3}\right)}\right] = -18q^{2/3}(1 + 2q + 5q^2 + \dots)$$



$$\Big)_{\mathbf{1}}$$

$$= \begin{pmatrix} Y_1(\tau) \\ Y_2(\tau) \\ Y_3(\tau) \end{pmatrix}$$

with

$$\omega = e^{2\pi i/3} \qquad \eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n) \qquad q \equiv e^{i2\pi\tau}$$

obey $Y_2^2 + 2Y_1Y_3 = 0$

Feruglio [1706.08749]

$$M_N = \Lambda \begin{pmatrix} 2Y_1(\tau) & -Y_3(\tau) & -Y_2(\tau) \\ -Y_3(\tau) & 2Y_2(\tau) & -Y_1(\tau) \\ -Y_2(\tau) & -Y_1(\tau) & 2Y_3(\tau) \end{pmatrix}$$

Example

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Let's build a modular-invariant term!

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$$W \supset \Lambda \left(N \otimes N \otimes Y_{\mathbf{3}}^{(2)} \right)_{\mathbf{1}}$$

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$$M_N = \Lambda \begin{pmatrix} 2Y_1(\tau) & -Y_3(\tau) & -Y_2(\tau) \\ -Y_3(\tau) & 2Y_2(\tau) & -Y_1(\tau) \\ -Y_2(\tau) & -Y_1(\tau) & 2Y_3(\tau) \end{pmatrix}$$

so now we can build models...

Example: an S_4 lepton model

Novichkov, JP, Petcov, Titov [1811.04933]

Ingredients: **Choose group, field content** $\psi \sim (\mathbf{r}, k)$

Example: an S4 lepton model

Novichkov, JP, Petcov, Titov [1811.04933]

$$N^c \sim (\mathbf{3}', 0)\,, \quad L \sim (\mathbf{3}, 2)$$
$$E^c \sim (\mathbf{1}', 0) \oplus (\mathbf{1}, 2) \oplus (\mathbf{1}', 2)$$

Ingredients: **Choose group, field content**

$$\psi \sim (\mathbf{r}, k)$$

$$W = \alpha \left(E_1^c L\, Y_{\mathbf{3}'}^{(2)}\right)_{\mathbf{1}} H_d + \beta \left(E_2^c L\, Y_{\mathbf{3}}^{(4)}\right)_{\mathbf{1}} H_d + \gamma \left(E_3^c L\, Y_{\mathbf{3}'}^{(4)}\right)_{\mathbf{1}} H_d$$
$$+ g \left(N^c L\, Y_{\mathbf{2}}^{(2)}\right)_{\mathbf{1}} H_u + g' \left(N^c L\, Y_{\mathbf{3}'}^{(2)}\right)_{\mathbf{1}} H_u + \Lambda \left(N^c N^c\right)_{\mathbf{1}}\,,$$

$\in \mathbb{C}$ only physical phase

Procedure: **Fit couplings and τ**

$$\min \chi^2(\tau,\, g'/g,\, g^2/\Lambda,\, \alpha, \beta, \gamma)$$

Example: an S4 lepton model

$N^c \sim (3', 0) \qquad L \sim (3, 2)$

$$\alpha \begin{pmatrix} Y_1(\tau) & Y_3(\tau) & Y_2(\tau) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{Y^{(2)}_{\mathbf{3'}}} \quad \beta \begin{pmatrix} 0 & 0 & 0 \\ Y_1(\tau) & Y_3(\tau) & Y_2(\tau) \\ 0 & 0 & 0 \end{pmatrix}_{Y^{(4)}_{\mathbf{3}}} \quad \gamma \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ Y_1(\tau) & Y_3(\tau) & Y_2(\tau) \end{pmatrix}_{Y^{(4)}_{\mathbf{3'}}}$$

$$W = \alpha \left(E_1^c L\, Y^{(2)}_{\mathbf{3'}}\right)_{\mathbf{1}} H_d + \beta \left(E_2^c L\, Y^{(4)}_{\mathbf{3}}\right)_{\mathbf{1}} H_d + \gamma \left(E_3^c L\, Y^{(4)}_{\mathbf{3'}}\right)_{\mathbf{1}} H_d$$

$$+\, g \left(N^c L\, Y^{(2)}_{\mathbf{2}}\right)_{\mathbf{1}} H_u + g' \left(N^c L\, Y^{(2)}_{\mathbf{3'}}\right)_{\mathbf{1}} H_u + \Lambda \left(N^c N^c\right)_{\mathbf{1}} ,$$

$\in \mathbb{C}$ only physical phase

P

$\min \chi^2(\tau, g'/g$

$$g \begin{pmatrix} 0 & Y_1(\tau) & Y_2(\tau) \\ Y_1(\tau) & Y_2(\tau) & 0 \\ Y_2(\tau) & 0 & Y_1(\tau) \end{pmatrix}_{Y^{(2)}_{\mathbf{2}}}$$

$$g' \begin{pmatrix} 2Y_1(\tau) & -Y_3(\tau) & -Y_2(\tau) \\ -Y_3(\tau) & 2Y_2(\tau) & -Y_1(\tau) \\ -Y_2(\tau) & -Y_1(\tau) & 2Y_3(\tau) \end{pmatrix}_{Y^{(2)}_{\mathbf{3'}}}$$

$$\Lambda \begin{pmatrix} 1 & & \\ & & 1 \\ & 1 & \end{pmatrix}$$

Example: an S4 lepton model

Novichkov, JP, Petcov, Titov [1811.04933]

$$N^c \sim (\mathbf{3}', 0)\,, \quad L \sim (\mathbf{3}, 2)$$
$$E^c \sim (\mathbf{1}', 0) \oplus (\mathbf{1}, 2) \oplus (\mathbf{1}', 2)$$

Ingredients: **Choose group, field content**

$$\psi \sim (\mathbf{r}, k)$$

$$W = \alpha \left(E_1^c L\, Y_{\mathbf{3}'}^{(2)}\right)_{\mathbf{1}} H_d + \beta \left(E_2^c L\, Y_{\mathbf{3}}^{(4)}\right)_{\mathbf{1}} H_d + \gamma \left(E_3^c L\, Y_{\mathbf{3}'}^{(4)}\right)_{\mathbf{1}} H_d + g \left(N^c L\, Y_{\mathbf{2}}^{(2)}\right)_{\mathbf{1}} H_u + g' \left(N^c L\, Y_{\mathbf{3}'}^{(2)}\right)_{\mathbf{1}} H_u + \Lambda\, (N^c N^c)_{\mathbf{1}}\,,$$

$g' \in \mathbb{C}$ only physical phase

Procedure: **Fit couplings and τ**

$$\min \chi^2(\tau,\, g'/g,\, g^2/\Lambda,\, \alpha, \beta, \gamma)$$

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$$+ g \left(N^c L\, Y_{\mathbf{2}}^{(2)}\right)_{\mathbf{1}} H_u + g' \left(N^c L\, Y_{\mathbf{3}'}^{(2)}\right)_{\mathbf{1}} H_u + \Lambda\, (N^c N^c)_{\mathbf{1}}\,,$$

Procedure: **Fit couplings and τ**

$$\min \chi^2(\tau,\, g'/g,\, g^2/\Lambda,\, \alpha, \beta, \gamma)$$

$$\text{gCP} \Rightarrow g' \in \mathbb{R}$$

Novichkov, JP, Petcov, Titov [1905.11970]

τ can be the only source of CPV

Example: an S4 lepton model (results)

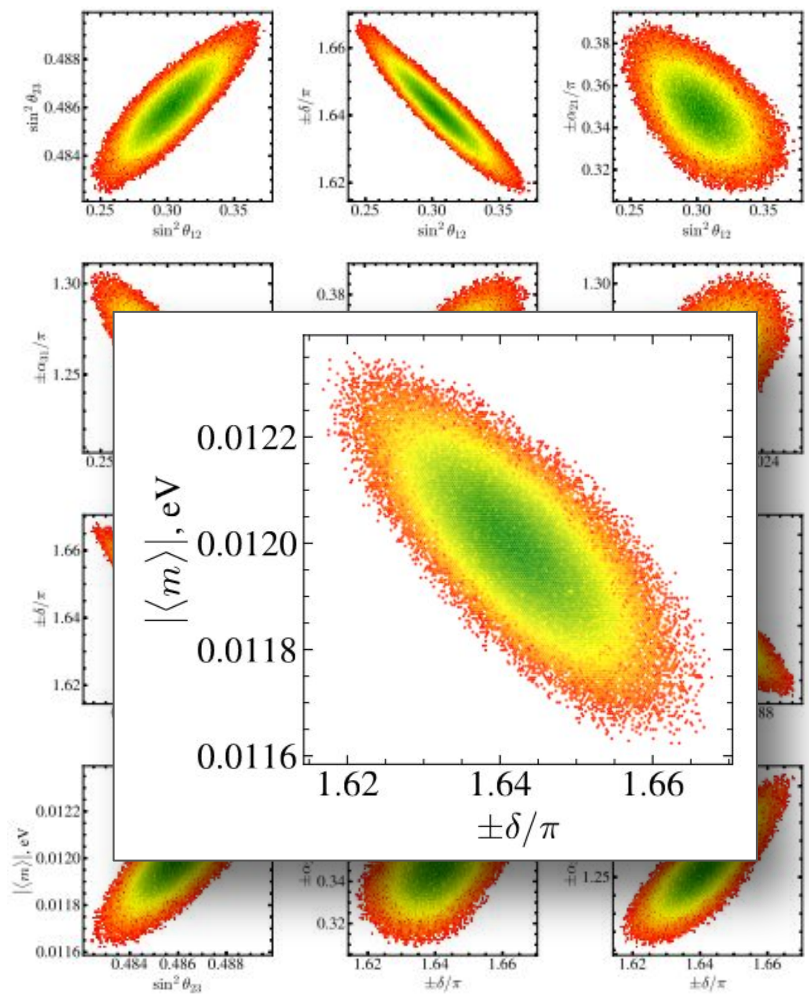
Novichkov, JP, Petcov, Titov
[1811.04933, 1905.11970]

$$\sin^2\theta_{23} \sim 0.49$$
$$\delta \sim 1.6\pi$$
$$\alpha_{21} \sim 0.3\pi$$
$$\alpha_{31} \sim 1.3\pi$$

$$|\langle m \rangle|_{\beta\beta} \sim 12 \text{ meV}$$
$$\sum_i m_i \sim 0.08 \text{ eV}$$

7 (4) parameters
vs.
12 (9) observables

Minimal model found with one less parameter:
Ding, Liu, Yao [2211.04546]



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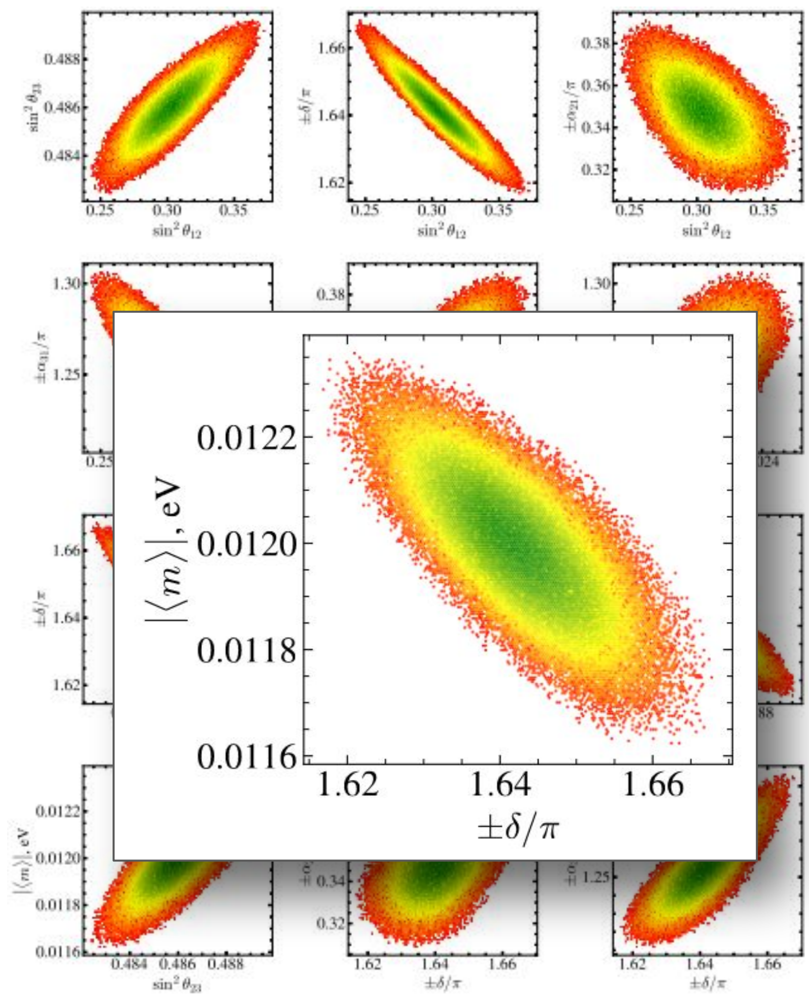
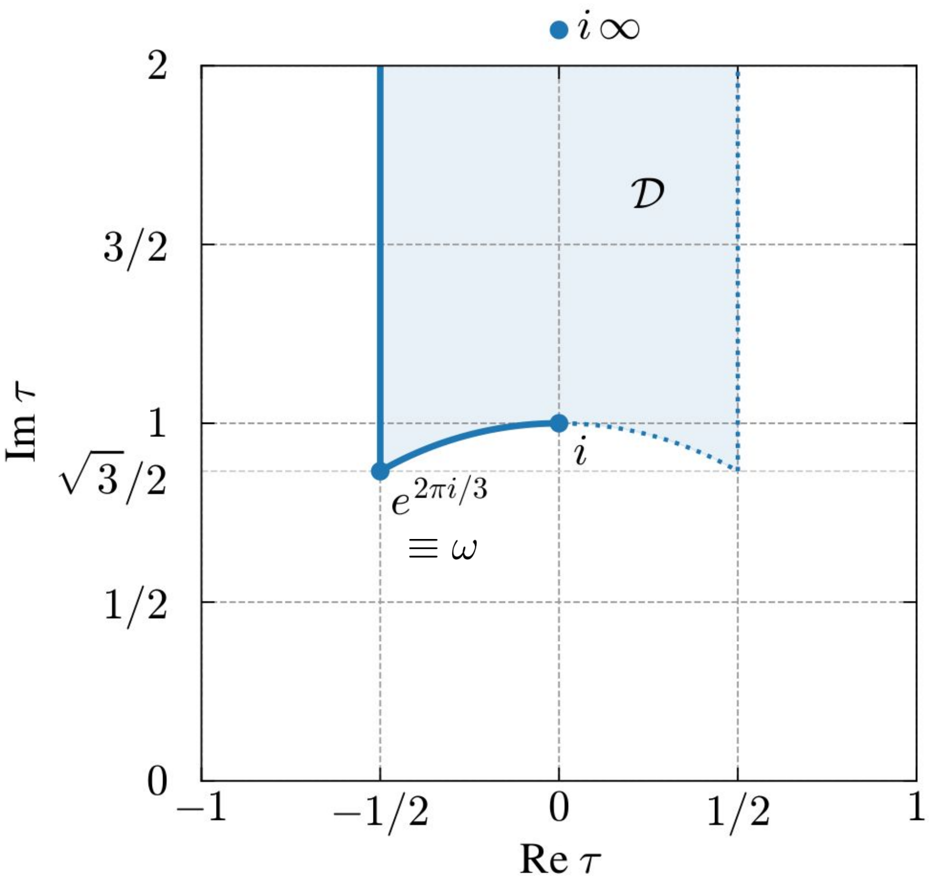


...and one does not need to consider the whole ½ plane

Intermezzo: The fundamental domain

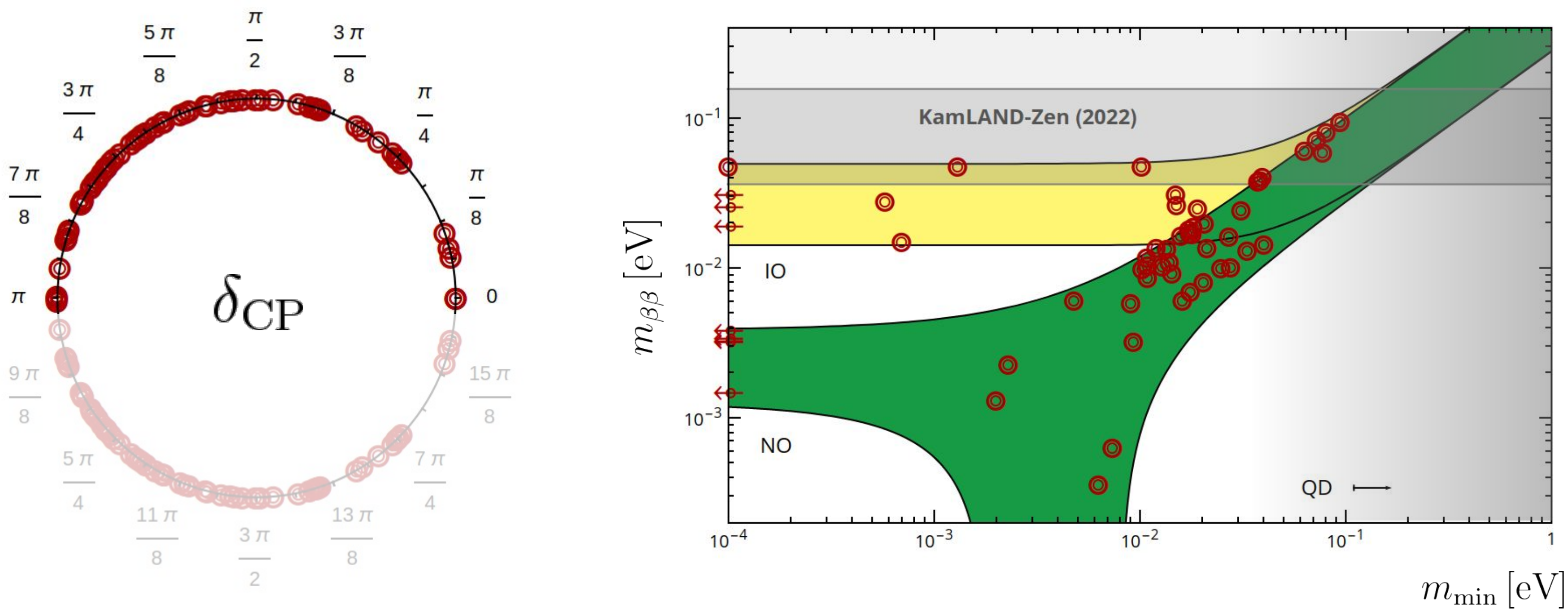


- **Any τ** breaks the full modular symmetry
- To **fit** a model which is invariant under the full modular group, it is enough to scan τ in the **fundamental domain**

In some cases, can avoid fit by looking at invariants see Chen et al. [2211.04546]

…and one does not need to consider the whole ½ plane

Benchmarks from the literature



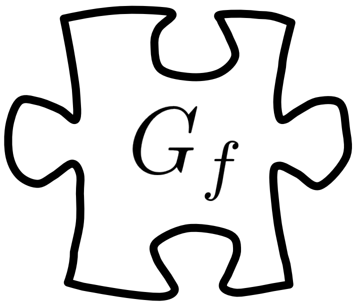
Produced by surveying the models in Table 14 of the Ding, King 2023 review
“Neutrino Mass and Mixing with Modular Symmetry” (2311.09282)

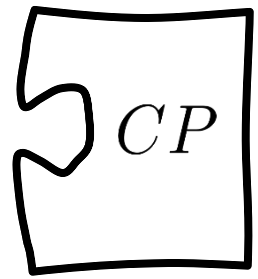
only b.f. points are shown

CP violation

Image credit: www.davidharber.co.uk

Flavour symmetries + gCP (general~~ized~~ CP)





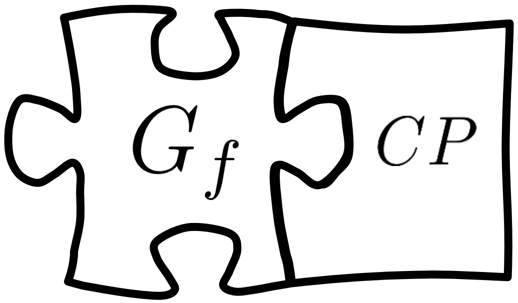
$$\psi(x) \to \rho_{\mathbf{r}}(g)\,\psi(x)$$

$$\psi(x) \to X_{\mathbf{r}}^{\mathrm{CP}}\,\overline{\psi}(x_{\mathrm{P}})$$

Branco, Lavoura, Rebelo (1986)
Harrison, Scott (2002)
Grimus, Lavoura (2003)
Farzan, Smirnov (2006)
Ferreira et al. (2012)
...

Flavour symmetries + gCP (genera~~lized~~ CP)

$\psi(x) \to \rho_{\mathbf{r}}(g)\,\psi(x)$



$\psi(x) \to X_{\mathbf{r}}^{\mathrm{CP}}\,\overline{\psi}(x_{\mathrm{P}})$

can constrain mixing, the Dirac **and the Majorana phases**

[recall talk by C. Hagedorn]

Consistency condition [Feruglio et al. (2012), Holthausen et al. (2012)]

$$X_{\mathbf{r}}^{\mathrm{CP}}\,\rho_{\mathbf{r}}^{*}(g)\left(X_{\mathbf{r}}^{\mathrm{CP}}\right)^{-1} = \rho_{\mathbf{r}}(u(g))$$

u is a class-inverting outer automorphism [Chen et al. (2014)]

Modular symmetry to the rescue!

‘modulus’

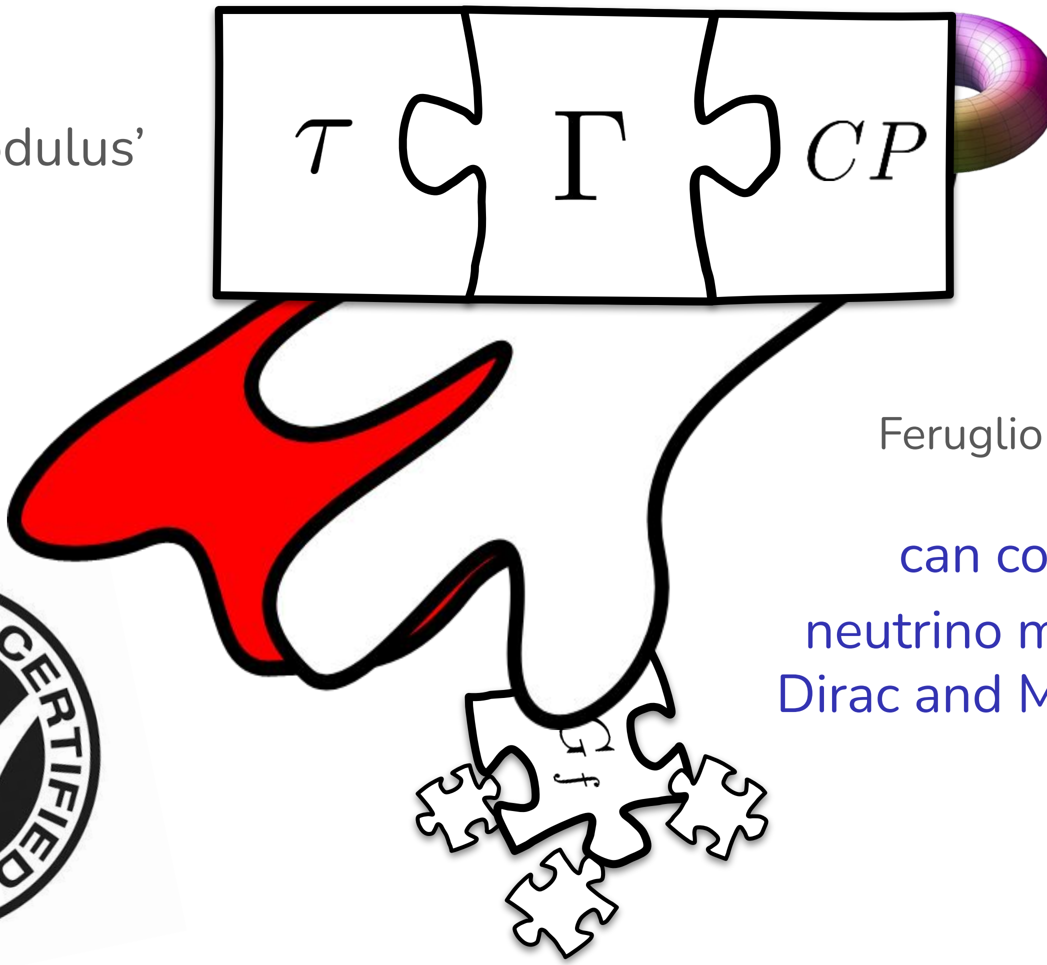
Feruglio [1706.08749]

can constrain all:

neutrino masses, mixing,
Dirac and Majorana phases

Modular symmetry + gCP

'modulus'



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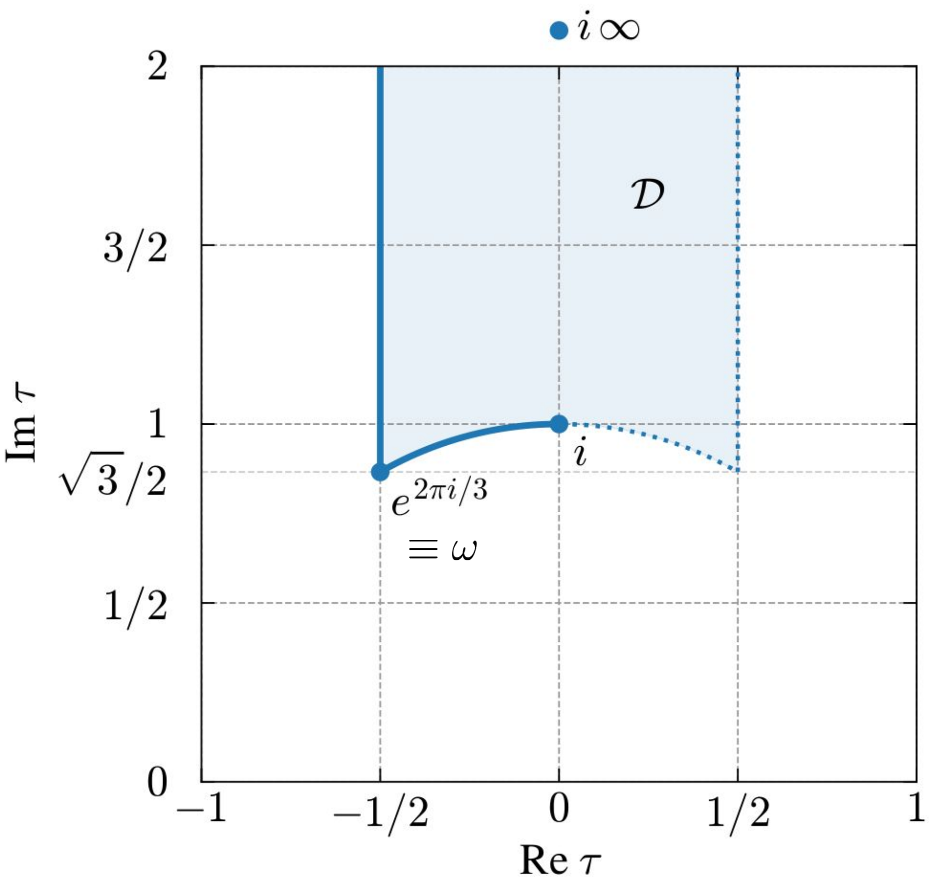
Modular symmetry + gCP (back to the fundamental domain)



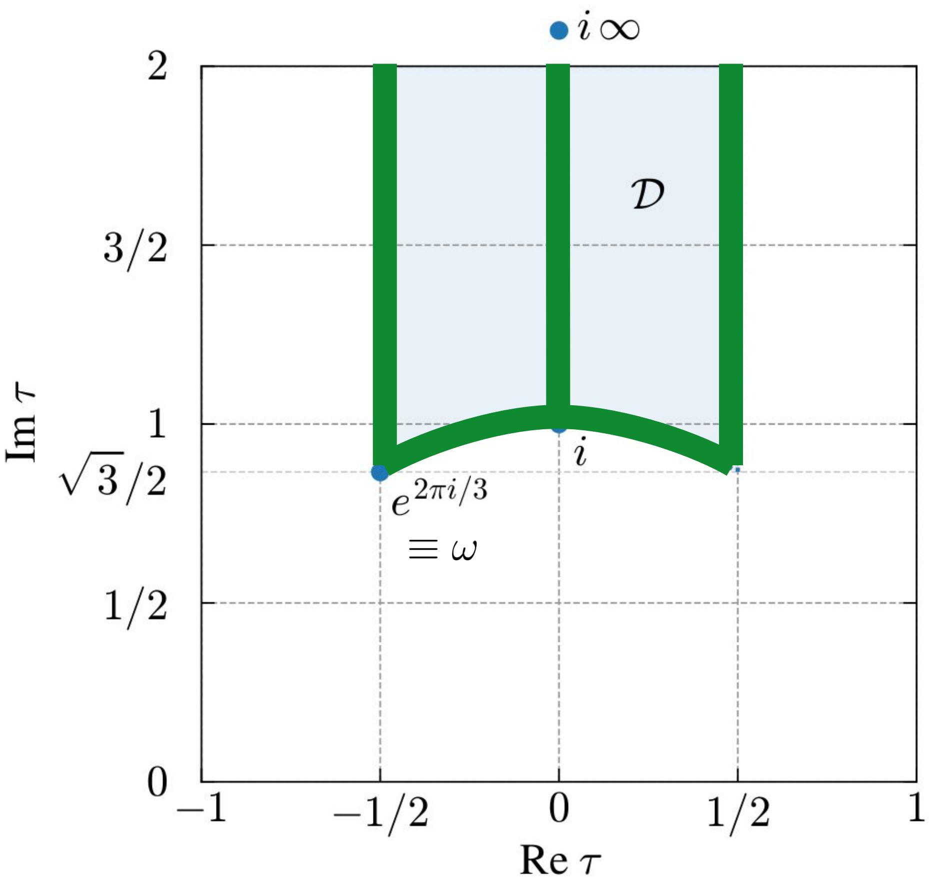
- **Any τ** breaks the full modular symmetry
- Special values of τ preserve the CP symmetry
- The modulus can be the **only source of CP violation**! (recall the example S4 model...)

- CP is violated by the modulus unless

$$-\tau^* = \gamma\tau$$

special regions of the fundamental domain

Modular symmetry + gCP (back to the fundamental domain)



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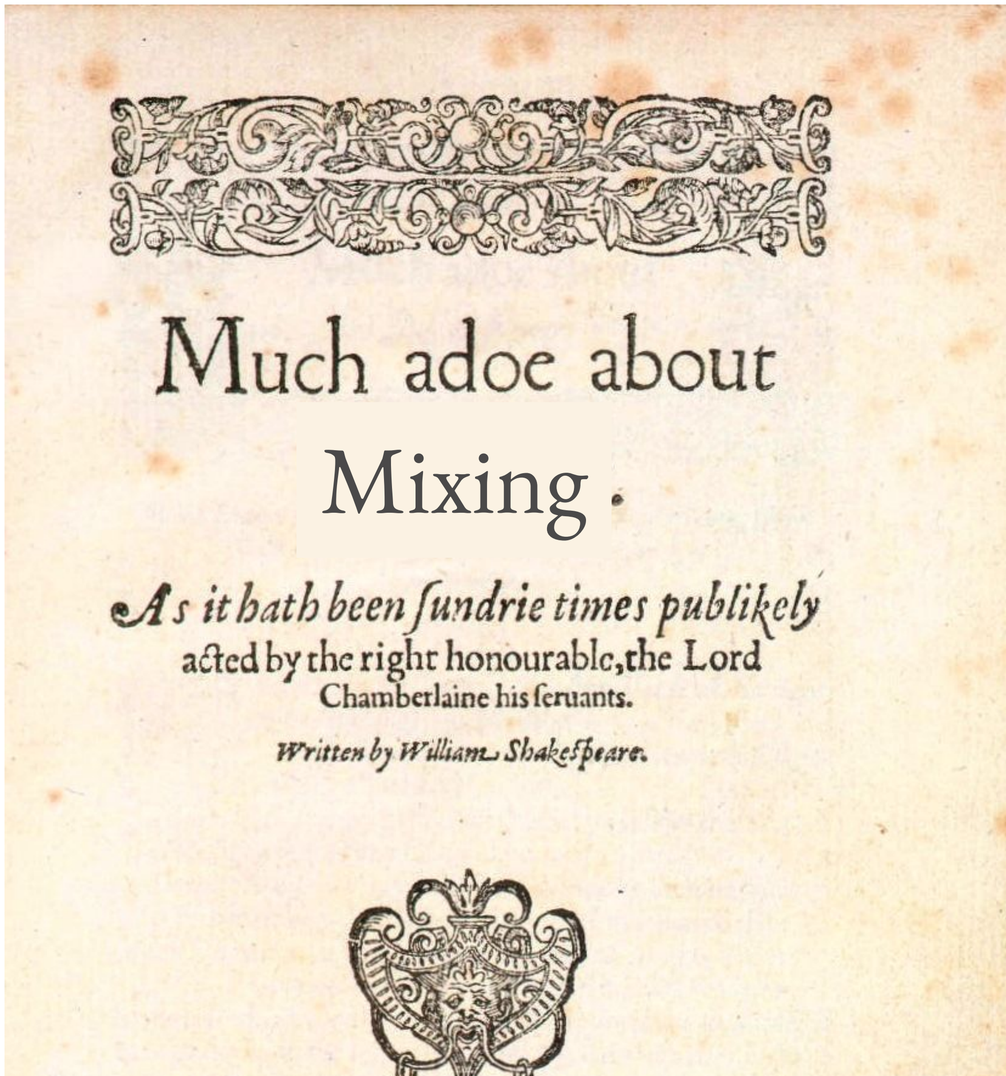
special regions of the fundamental domain

Fermion mass hierarchies

Image credit: prompted StabilityAI's Stable Diffusion 2.1

Mass hierarchies from modular symmetry?



Mass hierarchies from modular symmetry?

- Usually fermion mass hierarchies are put in **by hand**: hierarchies (or cancellations) between superpotential parameters

e.g. in the previously shown S4 model, $\gamma \ll \alpha \ll \beta$

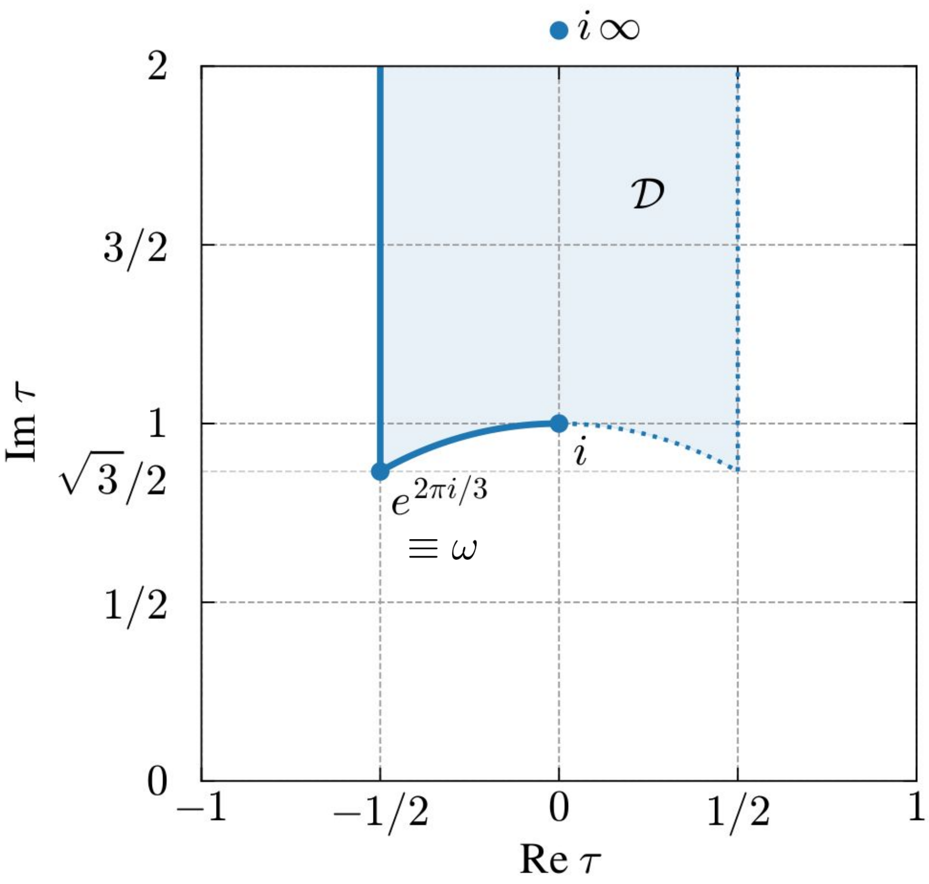
- **Other approaches** - New (weighted) scalars enter mass matrices a la Froggatt-Nielsen. Weights are analogous to FN charges Criado, Feruglio, King [1908.11867]; King^2 [2002.00969]

- **This approach** - No new scalars, mechanism uses **only** $\boldsymbol{\tau}$, with common weights across generations (unlike FN charges)

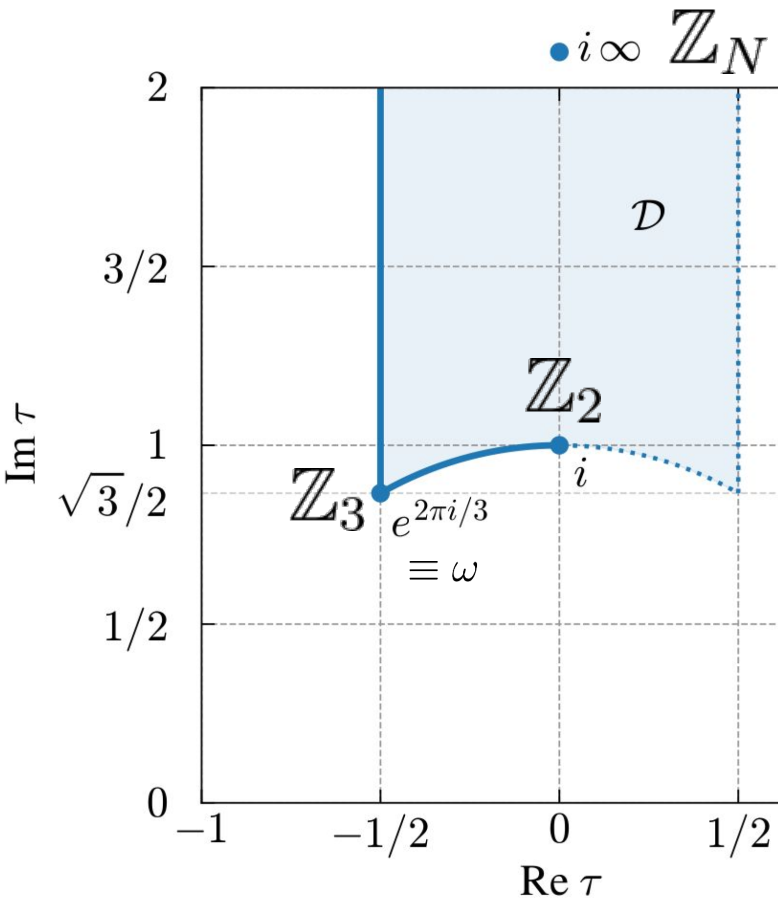
Novichkov, JP, Petcov [2102.07488]

Idea by now applied in several works, see:
Petcov, Tanimoto [2212.13336]; Abe, Higaki, Kawamura, Kobayashi [2301.07439, 2302.11183]; Kikuchi, Kobayashi, Nasu, Takada, Uchida [2301.03737, 2302.03326]

Residual modular symmetries



Residual modular symmetries



- At special values of τ, some **residual symmetry** remains

  see e.g. Novichkov et al. [1811.04933]; Novichkov et al.' [1812.11289]

- Near them, these symmetries are linearly realized

  see e.g. Feruglio [2302.11580]

Can be used for texture zeros, see e.g. [2207.04609]

Key idea for hierarchies:

some couplings vanish as we approach a symmetric point

Novichkov, JP, Petcov [2102.07488]

Corrections to vanishing couplings

$$\tau = \tau_{\text{sym}}$$

$$M \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\psi^c \, M \, \psi$$

Key idea for hierarchies:

some couplings vanish as we approach a symmetric point

Novichkov, JP, Petcov [2102.07488]

Corrections to vanishing couplings

$$\tau = \tau_{\text{sym}}$$

$$\epsilon \sim |\tau - \tau_{\text{sym}}| > 0$$

$$M \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \rightarrow \quad M \sim \begin{pmatrix} 1 & \epsilon^{\cdots} & \epsilon^{\cdots} \\ \epsilon^{\cdots} & \epsilon^{\cdots} & \epsilon^{\cdots} \\ \epsilon^{\cdots} & \epsilon^{\cdots} & \epsilon^{\cdots} \end{pmatrix}$$

$$\psi^c \, M \, \psi$$

In the vicinity of the sym. point, the couplings are $\mathcal{O}(\epsilon^l)$

Key idea for hierarchies:

some couplings vanish as we approach a symmetric point

Novichkov, JP, Petcov [2102.07488]

Decompositions under residual groups (determine $\mathcal{O}(\epsilon^l)$)

$\tau_{\rm sym}$	Residual sym.	Possible powers ϵ^l
i	$\mathbb{Z}_2$	$l = 0, 1$
ω	$\mathbb{Z}_3$	$l = 0, 1, 2$
$i\infty$	$\mathbb{Z}_N$	$l = 0, 1, \ldots, N$

← Feruglio, Gherardi, Romanino, Titov [2101.08718] (for A_4, m_e=0)

$$\psi^c \, M \, \psi$$

$$\psi \xrightarrow{\gamma} (c\tau + d)^{-k} \rho(\gamma)\, \psi$$

$$\psi^c \xrightarrow{\gamma} (c\tau + d)^{-k^c} \rho^c(\gamma)\, \psi^c$$

$$M(\tau) \xrightarrow{\gamma} M(\gamma\tau) = (c\tau + d)^K \rho^c(\gamma)^* M(\tau) \rho(\gamma)^\dagger$$

$$\psi \rightsquigarrow \mathbf{1}_{\ldots} \oplus \mathbf{1}_{\ldots} \oplus \mathbf{1}_{\ldots}$$

$$\psi^c \rightsquigarrow \mathbf{1}_{\ldots} \oplus \mathbf{1}_{\ldots} \oplus \mathbf{1}_{\ldots}$$

In general, depend on weights

Determined for all $N \leqslant 5$

Example: hierarchical mass matrix (A5)

$$\psi \sim (\mathbf{3}, k)$$
$$\psi^c \sim (\mathbf{3}', k^c)$$

$$\Rightarrow$$

Under the residual group of

$$\tau_{\text{sym}} = i\infty$$

$$\psi \rightsquigarrow 1_0 \oplus \mathbf{1}_1 \oplus \mathbf{1}_4$$
$$\psi^c \rightsquigarrow 1_0 \oplus \mathbf{1}_2 \oplus \mathbf{1}_3$$

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$$\psi \rightsquigarrow 1_0 \oplus \mathbf{1}_1 \oplus \mathbf{1}_4$$
$$\psi^c \rightsquigarrow 1_0 \oplus \mathbf{1}_2 \oplus \mathbf{1}_3$$

For $\psi^c M \psi$, we expect:

$$M \sim \begin{pmatrix} 1 & \epsilon^4 & \epsilon \\ \epsilon^3 & \epsilon^2 & \epsilon^4 \\ \epsilon^2 & \epsilon & \epsilon^3 \end{pmatrix} \Rightarrow$$

with $\epsilon = e^{-2\pi \operatorname{Im} \tau/5}$

fermion spectrum

$$\sim (1, \epsilon, \epsilon^4)$$ ✔

Indeed the case, provided enough modular forms contribute to *M*

(otherwise, $m_e = 0$)

Example: lepton model close to ω

Only S₄' model from a scan requiring minimal # params., $m_e > 0$, and Dirac phase within 2σ range (otherwise unconstrained):

$$L \sim (\hat{\mathbf{1}} \oplus \hat{\mathbf{1}} \oplus \hat{\mathbf{1}}', 2),\; E^c \sim (\hat{\mathbf{3}}, 4),\; N^c \sim (\mathbf{3}', 1)$$

Superpotential:

$$\begin{aligned} W = & \left[\alpha_1 \left(Y^{(4,6)}_{\mathbf{3}',1} E^c L_1\right)_{\mathbf{1}} + \alpha_3 \left(Y^{(4,6)}_{\mathbf{3}',1} E^c L_2\right)_{\mathbf{1}} + \alpha_4 \left(Y^{(4,6)}_{\mathbf{3}',2} E^c L_2\right)_{\mathbf{1}} + \alpha_5 \left(Y^{(4,6)}_{\mathbf{3}} E^c L_3\right)_{\mathbf{1}}\right] H_d \\ & + \left[g_1 \left(Y^{(4,3)}_{\hat{\mathbf{3}}} N^c L_1\right)_{\mathbf{1}} + g_2 \left(Y^{(4,3)}_{\hat{\mathbf{3}}} N^c L_2\right)_{\mathbf{1}} + g_3 \left(Y^{(4,3)}_{\hat{\mathbf{3}}'} N^c L_3\right)_{\mathbf{1}}\right] H_u \\ & + \Lambda \left(Y^{(4,2)}_{\mathbf{2}} (N^c)^2\right)_{\mathbf{1}} . \end{aligned}$$

with gCP imposed

Example: lepton model close to ω

Only S_4' model from a scan requiring minimal # params., $m_e > 0$, and Dirac phase within 2σ range (otherwise unconstrained):

$$L \sim (\hat{\mathbf{1}} \oplus \hat{\mathbf{1}} \oplus \hat{\mathbf{1}}', 2),\ E^c \sim (\hat{\mathbf{3}}, 4),\ N^c \sim (\mathbf{3}', 1)$$

$$M_e \propto \begin{pmatrix} 1 & \alpha - 2\beta & 2\sqrt{3}i\gamma \\ \sqrt{3}\epsilon & \sqrt{3}(\alpha + 2\beta)\epsilon & 2i\gamma\epsilon \\ \frac{5}{2}\epsilon^2 & \left(\frac{5}{2}\alpha - \beta\right)\epsilon^2 & -\frac{5}{\sqrt{3}}i\gamma\epsilon^2 \end{pmatrix}$$

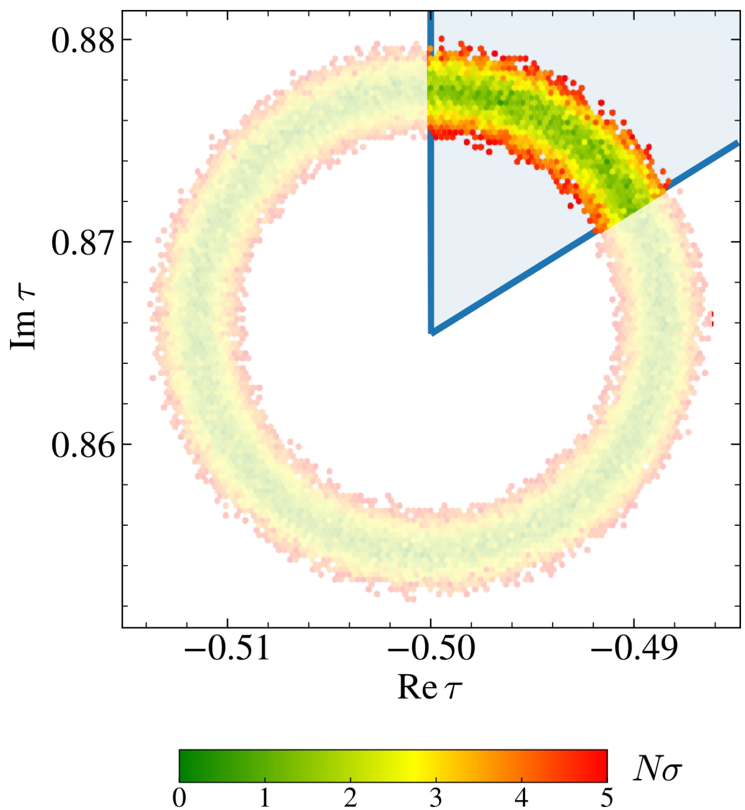
$$|\epsilon| \simeq 2.8 \left|\frac{\tau - \omega}{\tau - \omega^2}\right| \sim \left|\tau - e^{2\pi i/3}\right|$$

$$M_\nu \propto \epsilon \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & a \\ 1 & a & 2i\sqrt{\frac{2}{3}}b \end{pmatrix}$$

$\|\epsilon\| \simeq 0.02$	$\alpha = 2.45 \pm 0.44$
$a = 1.5 \pm 0.15$	$\beta = 2.14 \pm 0.32$
$b = 2.22 \pm 0.17$	$\gamma = 0.91 \pm 0.05$

Example: lepton model close to ω

$|\epsilon| \simeq 0.02$



$$m_e = \mathcal{O}(\epsilon^2)$$
$$m_\mu = \mathcal{O}(\epsilon)$$
$$m_\tau = \mathcal{O}(1)$$

$$\text{NO}, \quad m_{\nu_1} = 0 \qquad \delta \simeq \pi$$
$$m_{\beta\beta} = (1.44 \pm 0.33)\ \text{meV}$$

Naturally allows for **hierarchies**, **large mixing**, and some **predictivity** to conclude…

Modular symmetries can...

(in lieu of conclusions)

...offer a **predictive framework** for flavour

in principle

...provide an origin for **CP violation** (CPV)

?? XOR ??

...explain fermion **mass hierarchies**

Parting words (i.e. what next?)

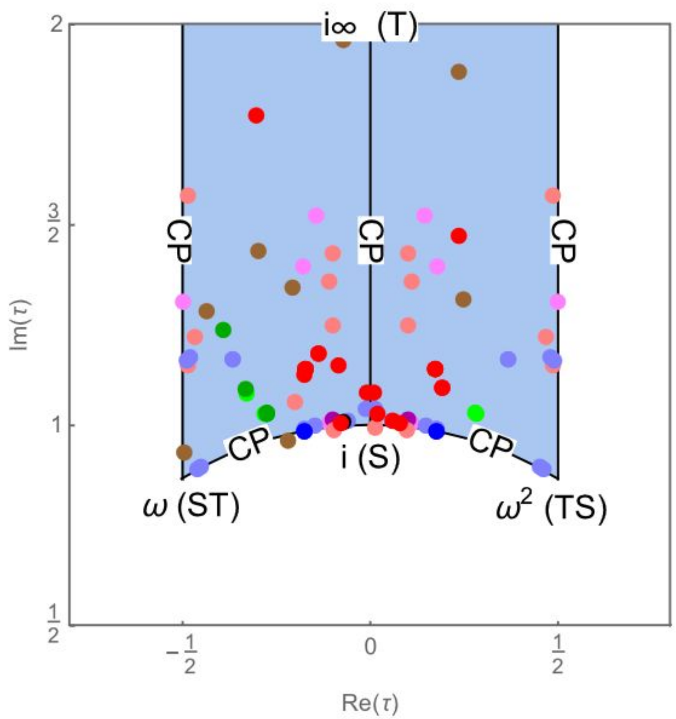
- modular symmetry as the origin of CPV **and** mass hierarchies?
- do away with SUSY?
- hints of universality?
- use TD to fix Kähler, irreps, weights!
- pheno beyond masses and mixing?

Grazie mille!

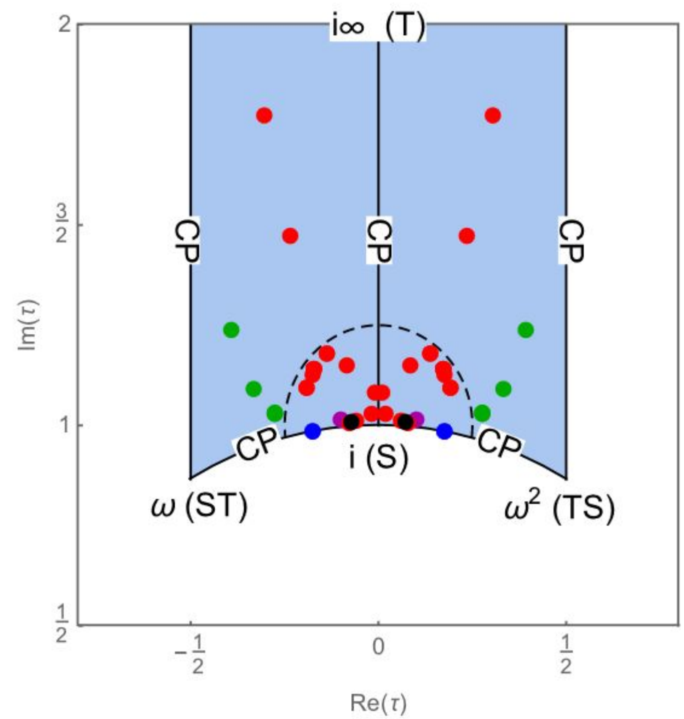
Backup slides

Parting words (i.e. what next?)

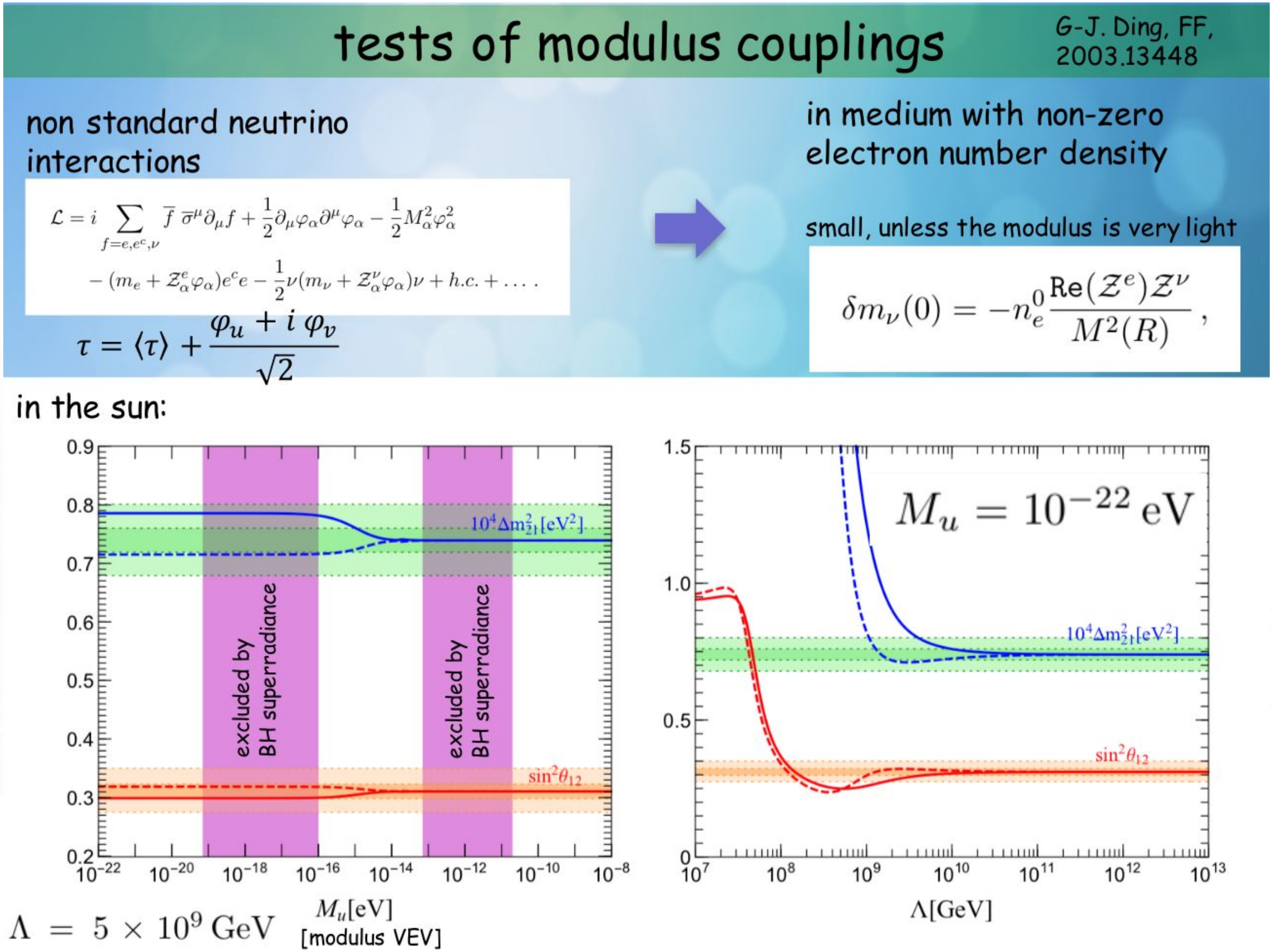
Hints of universality?



only gCP-invariant models



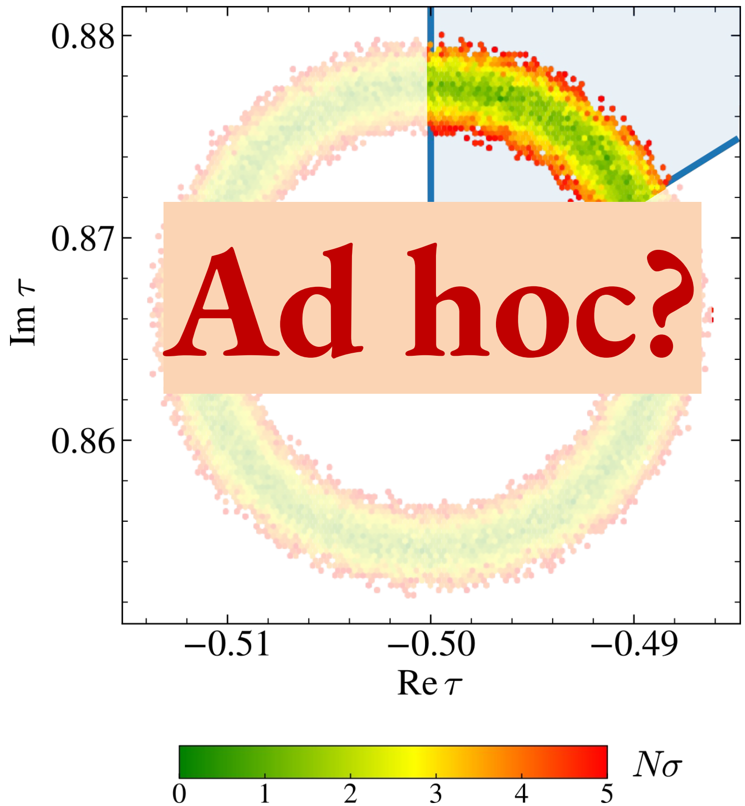
from Feruglio [2211.00659]
see also Feruglio [2302.11580] and Ding, Feruglio, Liu [2402.14915]



from Feruglio's slides at Bethe Workshop, citing [2003.13448]

Example: lepton model close to ω

$|\epsilon| \simeq 0.02$



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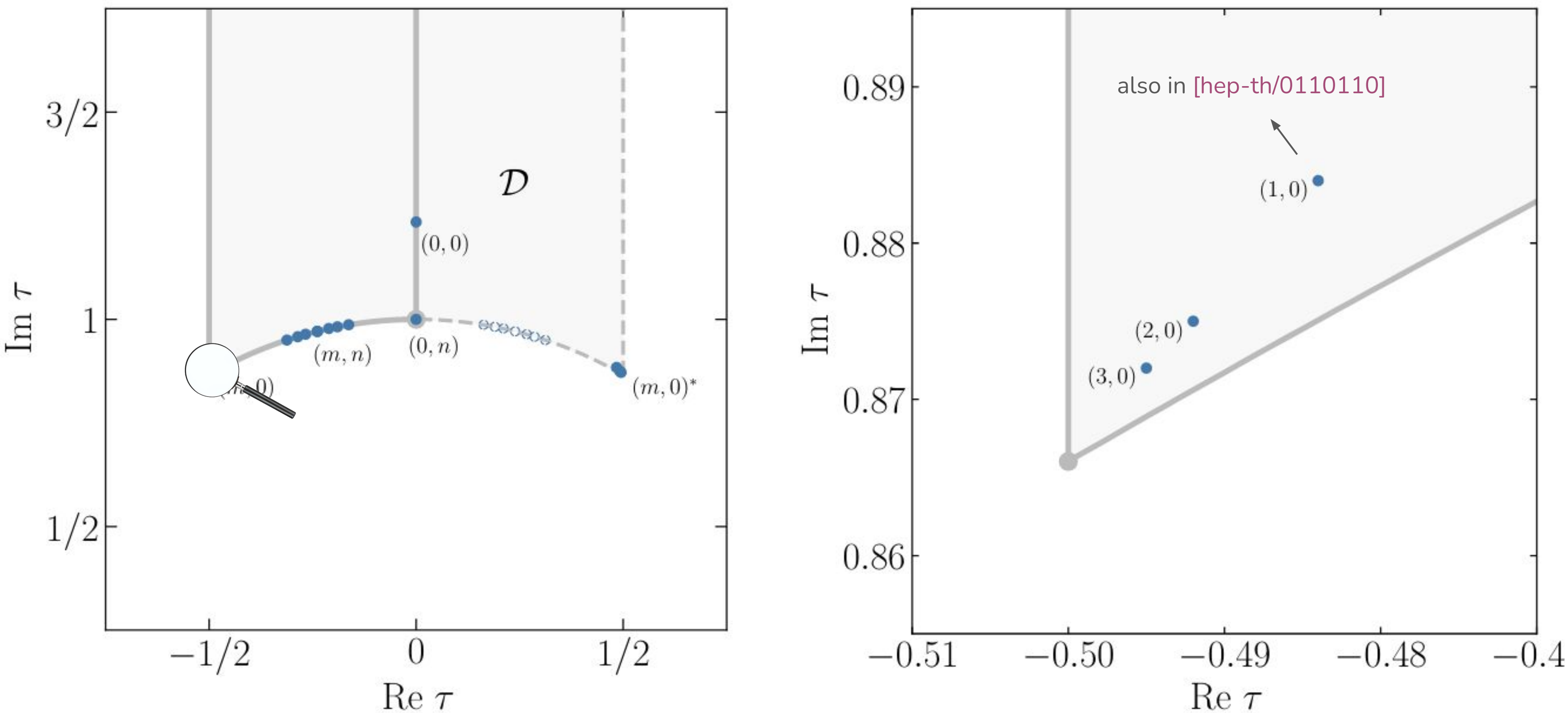
NO, $m_{\nu_1} = 0$ $\delta \simeq \pi$

$m_{\beta\beta} = (1.44 \pm 0.33)$ meV

Naturally allows for **hierarchies**, **large mixing**, and some **predictivity**

Novichkov, JP, Petcov [2201.02020]

Global minima for toy SUGRA potentials

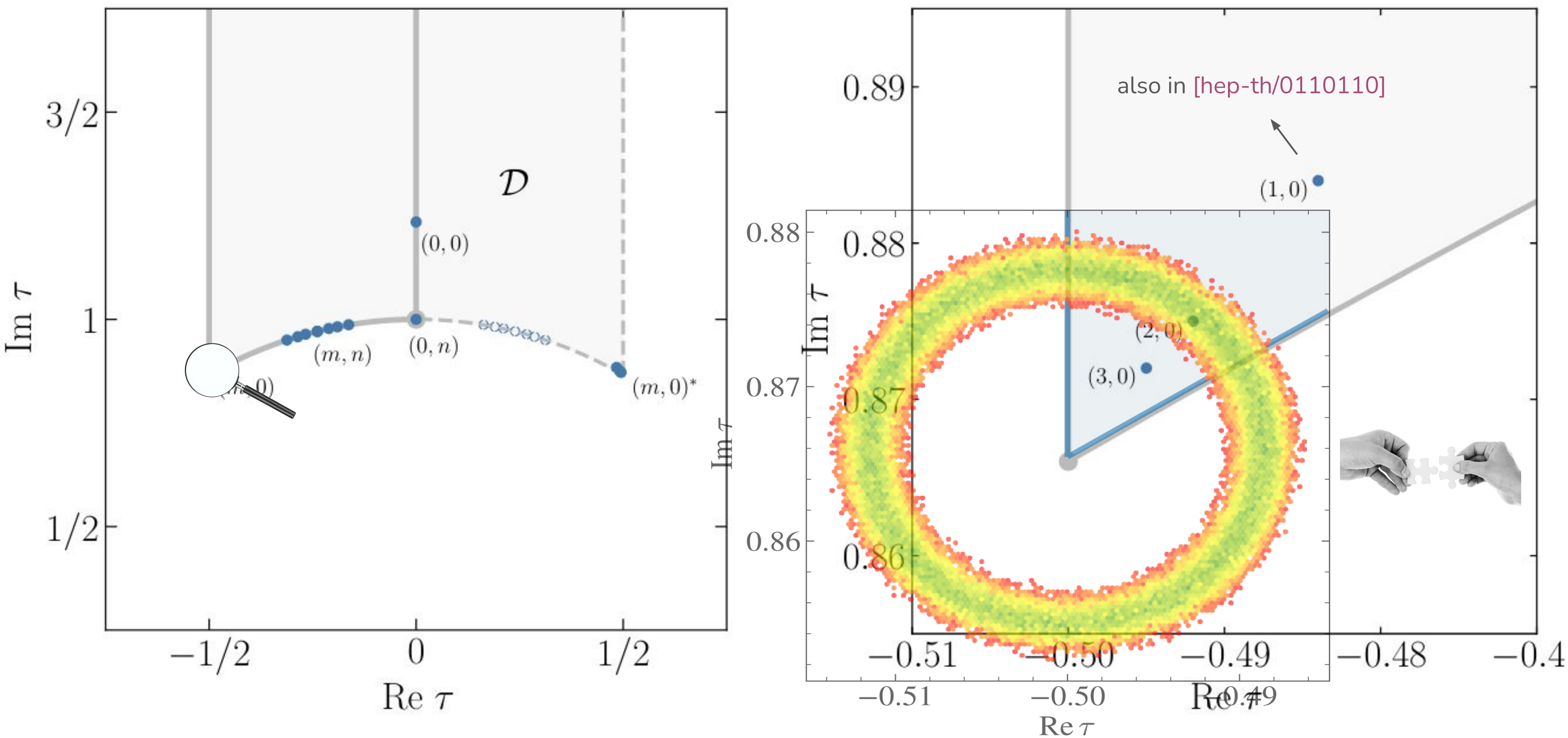


❌ “(…) we conjecture that all extrema of V entirely lie on [the boundary].” — Cvetič et al.

our results later confirmed by Leedom, Righi, Westphal [2212.03876]

Novichkov, JP, Petcov [2201.02020]

Global minima for toy SUGRA potentials



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SUSY breaking effects?

- **RGEs & threshold corrections** need to be considered, depend on tan β and unknown SUSY spectrum
- **SUSY-breaking** corrections can be made negligible via separation of scales (power counting argument)
- Under reasonable conditions, predictions may be unaffected

Feruglio and Criado [1807.01125]

What about... the Kähler?

- **Not holomorphic**: unconstrained by the symmetry!
- Under a modular transformation, invariant up to:

$$K(\chi_i,\overline{\chi}_i;\tau,\overline{\tau}) \;\rightarrow\; K(\chi_i,\overline{\chi}_i;\tau,\overline{\tau}) + f(\chi_i;\tau) + f(\overline{\chi}_i;\overline{\tau})$$

- Minimal choice:

$$K(\chi_i,\overline{\chi}_i;\tau,\overline{\tau}) = -h\,\Lambda_0^2\,\log(-i(\tau-\overline{\tau})) + \sum_i \frac{|\chi_i|^2}{(-i(\tau-\overline{\tau}))^{k_i}}$$

impacts pheno → should be justified from the top-down

Chen, Ramos-Sánchez and Ratz [1909.06910]

- Further constraints may arise from the (unavoidable...) combination of modular + traditional flavour symmetries

Nilles, Ramos-Sanchez, Vaudrevange [2004.05200]

Lessons from eclectic flavor symmetries

see [1901.03251, 1908.00805, 2001.01736, **2004.05200**, 2006.03059, 2112.06940]
and Nilles, Ramos-Sánchez [2404.16933] for a recent summary

- There is no possible scheme with just modular flavor symmetries
- Also discrete R-symmetries seem unavoidable
- A limited type of groups appear (e.g. *T'*)

<table>
<tr><th colspan="2">nature of symmetry</th><th>outer automorphism of Narain space group</th><th colspan="5">flavor groups</th></tr>
<tr><td rowspan="6">eclectic</td><td rowspan="2">modular</td><td>rotation S $\in$ $\mathrm{SL}(2,\mathbb{Z})_T$</td><td>$\mathbb{Z}_4$</td><td colspan="3" rowspan="2">T'</td><td rowspan="6">$\Omega(2)$</td></tr>
<tr><td>rotation T $\in$ $\mathrm{SL}(2,\mathbb{Z})_T$</td><td>$\mathbb{Z}_3$</td></tr>
<tr><td rowspan="4">traditional flavor</td><td>translation A</td><td>$\mathbb{Z}_3$</td><td rowspan="2">$\Delta(27)$</td><td rowspan="3">$\Delta(54)$</td><td rowspan="4">$\Delta'(54,2,1)$</td></tr>
<tr><td>translation B</td><td>$\mathbb{Z}_3$</td></tr>
<tr><td>rotation $\mathrm{C} = \mathrm{S}^2 \in \mathrm{SL}(2,\mathbb{Z})_T$</td><td colspan="2">$\mathbb{Z}_2^R$</td></tr>
<tr><td>rotation $\mathrm{R} = \gamma_{(3)} \in \mathrm{SL}(2,\mathbb{Z})_U$</td><td colspan="3">$\mathbb{Z}_9^R$</td></tr>
</table>

Nilles, Ramos-Sanchez, Vaudrevange [2006.03059]

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see [1901.03251, 1908.00805, 2001.01736, **2004.05200**, 2006.03059, 2112.06940]
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- There is no possible scheme with just modular flavor symmetries
- Also discrete R-symmetries seem unavoidable
- A limited type of groups appear (e.g. *T'*)
- Only some weights and irreps available at low energy
- Weights (typically fractional) correlated with irreps

sector	matter fields Φ_n	eclectic flavor group $\Omega(2)$: modular T' subgroup: irrep $\boldsymbol{s}$	$\rho_{\boldsymbol{s}}(\mathrm{S})$	$\rho_{\boldsymbol{s}}(\mathrm{T})$	n	traditional $\Delta(54)$ subgroup: irrep $\boldsymbol{r}$	$\rho_{\boldsymbol{r}}(\mathrm{A})$	$\rho_{\boldsymbol{r}}(\mathrm{B})$	$\rho_{\boldsymbol{r}}(\mathrm{C})$	$\mathbb{Z}_9^R$: R
bulk	Φ_0	$\mathbf{1}$	1	1	0	$\mathbf{1}$	1	1	$+1$	0
	Φ_{-1}	$\mathbf{1}$	1	1	-1	$\mathbf{1}'$	1	1	-1	3
θ	$\Phi_{-2/3}$	$\mathbf{2}'\oplus\mathbf{1}$	$\rho(\mathrm{S})$	$\rho(\mathrm{T})$	$-2/3$	$\mathbf{3}_2$	$\rho(\mathrm{A})$	$\rho(\mathrm{B})$	$+\rho(\mathrm{C})$	1
	$\Phi_{-5/3}$	$\mathbf{2}'\oplus\mathbf{1}$	$\rho(\mathrm{S})$	$\rho(\mathrm{T})$	$-5/3$	$\mathbf{3}_1$	$\rho(\mathrm{A})$	$\rho(\mathrm{B})$	$-\rho(\mathrm{C})$	-2
θ^2	$\Phi_{-1/3}$	$\mathbf{2}''\oplus\mathbf{1}$	$(\rho(\mathrm{S}))^*$	$(\rho(\mathrm{T}))^*$	$-1/3$	$\bar{\mathbf{3}}_1$	$\rho(\mathrm{A})$	$(\rho(\mathrm{B}))^*$	$-\rho(\mathrm{C})$	2
	$\Phi_{+2/3}$	$\mathbf{2}''\oplus\mathbf{1}$	$(\rho(\mathrm{S}))^*$	$(\rho(\mathrm{T}))^*$	$+2/3$	$\bar{\mathbf{3}}_2$	$\rho(\mathrm{A})$	$(\rho(\mathrm{B}))^*$	$+\rho(\mathrm{C})$	5
super-potential	$\mathcal{W}$	$\mathbf{1}$	1	1	-1	$\mathbf{1}'$	1	1	-1	3

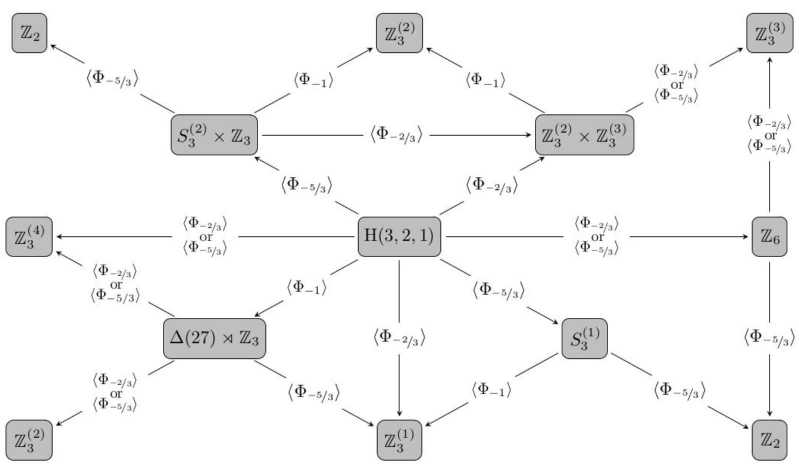
Nilles, Ramos-Sanchez, Vaudrevange [2004.05200]

Lessons from eclectic flavor symmetries

see [1901.03251, 1908.00805, 2001.01736, **2004.05200**, 2006.03059, 2112.06940]
and Nilles, Ramos-Sánchez [2404.16933] for a recent summary

- There is no possible scheme with just modular flavor symmetries
- Also discrete R-symmetries seem unavoidable
- A limited type of groups appear (e.g. T')
- Only some weights and irreps available at low energy
- Weights (typically fractional) correlated with irreps
- Breaking reintroduces flavons :(

Baur et al. [2112.06940]



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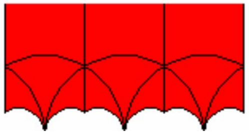
- A limited type of groups appear (e.g. T')
- Only some weights and irreps available at low energy
- Weights (typically fractional) correlated with irreps
- Breaking reintroduces flavons :(
- Both Kähler and superpotential play a crucial role
- Larger fundamental domains ($\Gamma(N)$ instead of Γ?)
- Top-down and bottom-up do **not yet** meet

but there are a few BU attempts:
Chen et al. [2108.02240]; Ding et al. [2303.02071]; Li, Ding [2308.16901]

Larger fundamental domains?



- Despite working with representations of the quotients, theories in the BU are typically **fully modular invariant**
- To have canonical kinetic terms,

$$\tau \to \frac{a\tau + b}{c\tau + d} \quad \Rightarrow \quad g_i \to (c\tau + d)^{-k_{Y_i}} g_i$$

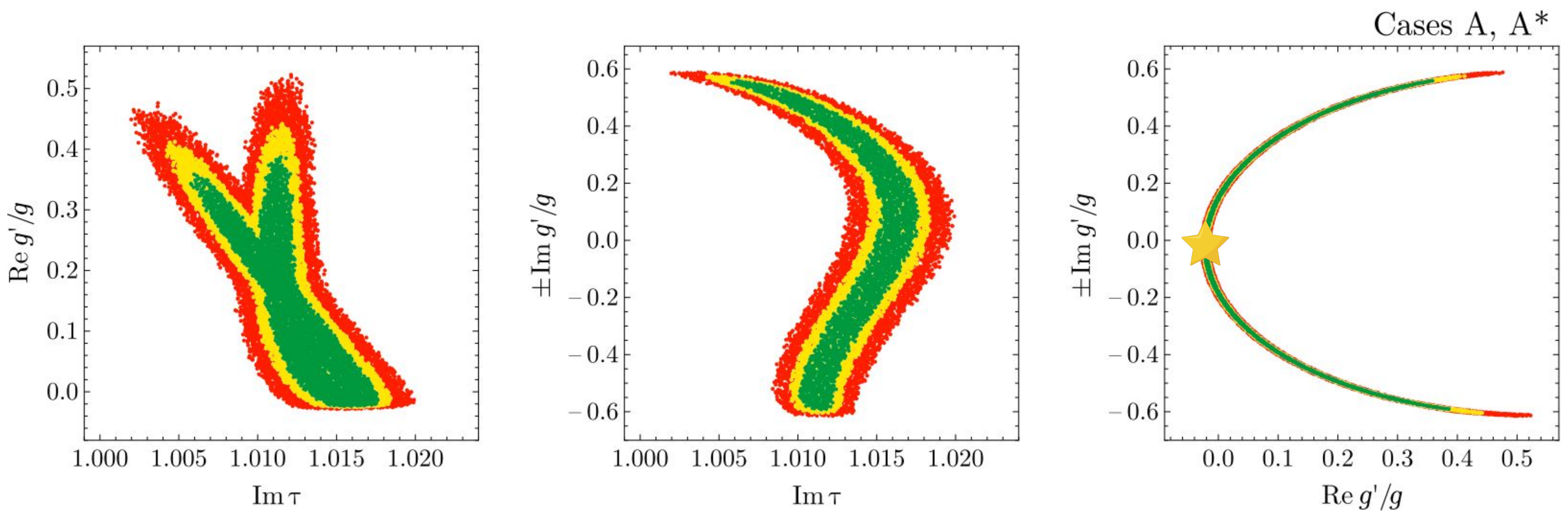
- e.g. in a particular model, see sec. 4 of Novichkov, JP, Petcov, Titov [1811.04933]

$$(\tau,\, \beta/\alpha,\, \gamma/\alpha,\, g'/g,\, \ldots,\, \Lambda'/\Lambda,\, \ldots) \to$$
$$\left(\frac{a\tau + b}{c\tau + d},\, (c\tau + d)^{-2}\,\beta/\alpha,\, (c\tau + d)^{-2}\,\gamma/\alpha,\, g'/g,\, \ldots,\, \Lambda'/\Lambda,\, \ldots\right)$$

these different parameter sets lead to the same observables

- Things may be different if **flavons** are present!

Correlations between parameters in the first S4 example model



Novichkov, JP, Petcov, Titov [1811.04933]

Details of the last model fit

Model	Section 4.2 (S_4')
$\mathrm{Re}\,\tau$	$-0.496^{+0.009}_{-0.016}$
$\mathrm{Im}\,\tau$	$0.877^{+0.0023}_{-0.024}$
α_2/α_1	—
α_3/α_1	$2.45^{+0.44}_{-0.42}$
α_4/α_1	$-2.37^{+0.36}_{-0.3}$
α_5/α_1	$1.01^{+0.06}_{-0.06}$
g_2/g_1	$1.5^{+0.15}_{-0.14}$
g_3/g_1	$2.22^{+0.17}_{-0.15}$
$v_d\,\alpha_1$, GeV	$4.61^{+1.32}_{-1.33}$
$v_u^2\,g_1/\Lambda$, eV	$0.268^{+0.057}_{-0.063}$
$\epsilon(\tau)$	$0.0186^{+0.0028}_{-0.0023}$
CL mass pattern	$(1, \epsilon, \epsilon^2)$
max(BG)	0.848

m_e/m_μ	$0.00475^{+0.00061}_{-0.00052}$
m_μ/m_τ	$0.0556^{+0.0136}_{-0.0116}$
r	$0.0298^{+0.00196}_{-0.0023}$
δm^2, 10^{-5} eV^2	$7.38^{+0.35}_{-0.44}$
$\|\Delta m^2\|$, 10^{-3} eV^2	$2.48^{+0.05}_{-0.04}$
$\sin^2\theta_{12}$	$0.304^{+0.039}_{-0.036}$
$\sin^2\theta_{13}$	$0.0221^{+0.0019}_{-0.002}$
$\sin^2\theta_{23}$	$0.539^{+0.0522}_{-0.099}$
m_1, eV	0
m_2, eV	$0.0086^{+0.0002}_{-0.00026}$
m_3, eV	$0.0502^{+0.00046}_{-0.00043}$
$\Sigma_i m_i$, eV	$0.0588^{+0.0002}_{-0.0002}$
$\|\langle m\rangle\|$, eV	$0.00144^{+0.00035}_{-0.00033}$
δ/π	$1 \pm \mathcal{O}(10^{-6})$
α_{21}/π	0
α_{31}/π	$1 \pm \mathcal{O}(10^{-5})$
$N\sigma$	0.563

Example: hierarchical mass matrix (A5)

$$\psi \sim (\mathbf{3}, k)$$

$$\Rightarrow$$

Under the residual group of

$$\tau_{\rm sym} = i\infty$$

Not like Froggatt-Nielsen. Instead, it is an **improvement**!

Explicit example at weight 2

$$W \supset \sum_s \alpha_s \left(Y_{\mathbf{5}}^{(5,2)}(\tau)\psi^c\psi\right)_{\mathbf{1},s} \quad \Rightarrow \quad M(\tau) = \alpha \begin{pmatrix} \sqrt{3}Y_1 & Y_5 & Y_2 \\ Y_4 & -\sqrt{2}Y_3 & -\sqrt{2}Y_5 \\ Y_3 & -\sqrt{2}Y_2 & -\sqrt{2}Y_4 \end{pmatrix}_{Y_{\mathbf{5}}^{(5,2)}}$$

$$(Y_1, Y_2, Y_3, Y_4, Y_5) \simeq \mathcal{N}\left(-1/\sqrt{6},\, q,\, 3q^2,\, 4q^3,\, 7q^4\right)$$

with $\epsilon = e^{-2\pi\,\mathrm{Im}\,\tau/5}$

Indeed the case, provided enough modular forms contribute to *M*

(otherwise, $m_e = 0$)

Promising hierarchical patterns (leptons)

N	Γ'_N	Pattern	Sym. point	Viable $\mathbf{r}\otimes\mathbf{r}^c$
2	S_3	$(1,\epsilon,\epsilon^2)$	$\tau\simeq\omega$	
3	A'_4	$(1,\epsilon,\epsilon^2)$	$\tau\simeq\omega$ $\tau\simeq i\infty$	
4	S'_4	$(1,\epsilon,\epsilon^2)$	$\tau\simeq\omega$	
		$(1,\epsilon,\epsilon^3)$	$\tau\simeq i\infty$	$\hat{\mathbf{3}}'\otimes(\hat{\mathbf{2}}\oplus\hat{\mathbf{1}})$
5	A'_5	$(1,\epsilon,\epsilon^4)$	$\tau\simeq i\infty$	$\mathbf{3}\otimes\mathbf{3}'$

$\hat{\mathbf{3}}'\otimes(\hat{\mathbf{2}}\oplus\hat{\mathbf{1}})$ → $L\sim(\hat{\mathbf{2}}\oplus\hat{\mathbf{1}},2),\ E^c\sim(\hat{\mathbf{3}}',2),\ N^c\sim(\mathbf{3},1)$

8 parameters

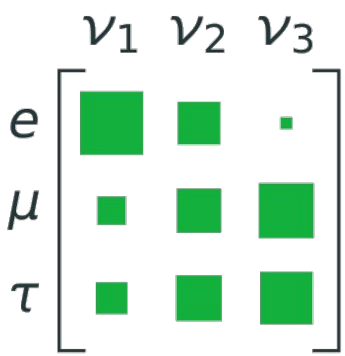
$\mathbf{3}\otimes\mathbf{3}'$ → $L\sim(\mathbf{3},3),\ E^c\sim(\mathbf{3}',1),\ N^c\sim(\hat{\mathbf{2}},2)$

8 parameters

Masses are OK, but mixing is tuned :(

Wrong PMNS in the symmetric limit: parameters are driven into cancellations

How to avoid fine-tuning (in the lepton sector)



$$\xrightarrow{\tau \to \tau_{\text{sym}}} \begin{bmatrix} \star & \star & 0 \\ \star & \star & \star \\ \star & \star & \star \end{bmatrix} \text{ or } \begin{bmatrix} \star & \star & \star \\ \star & \star & \star \\ \star & \star & \star \end{bmatrix}$$

1. $\begin{cases} L \sim 1 \oplus 1 \oplus 1 \\ E^c \sim 1 \oplus \mathbf{r} \not\supset 1 \end{cases}$

2. $\begin{cases} L \sim \mathbf{1} \oplus \mathbf{1} \oplus \bar{\mathbf{1}} \\ E^c \sim \bar{\mathbf{1}} \oplus \mathbf{r} \not\supset \mathbf{1}, \bar{\mathbf{1}} \end{cases}$

3. $m_e = m_\mu = m_\tau = 0$

4. $m_{\nu_1} = m_{\nu_2} = m_{\nu_3} = 0$

Reyimuaji, Romanino [1801.10530]

Simplest modular-invariant potentials?

- Studied by Cvetič, Font, Ibáñez, Lüst and Quevedo (1991)
  $\mathcal{N} = 1$ SUGRA

$$V(\tau,\bar{\tau}) = \frac{\Lambda_V^4}{8(\mathrm{Im}\,\tau)^3|\eta|^{12}}\left[\frac{4}{3}\left|iH' + \frac{3}{2\pi}H\hat{G}_2\right|^2(\mathrm{Im}\,\tau)^2 - 3|H|^2\right]$$

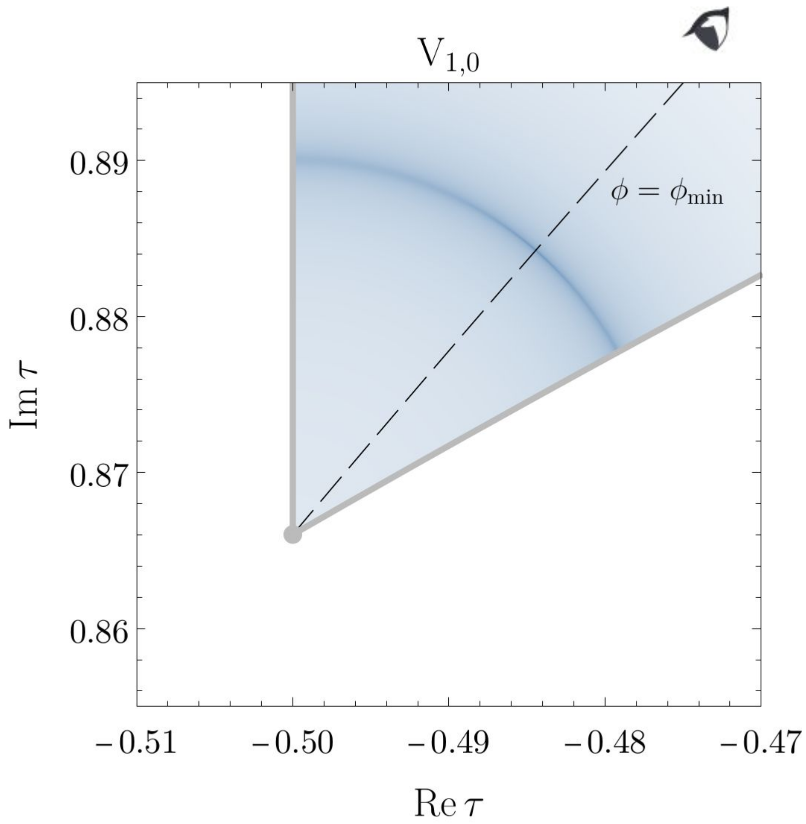
$$H(\tau) = (j(\tau) - 1728)^{m/2}\, j(\tau)^{n/3}$$

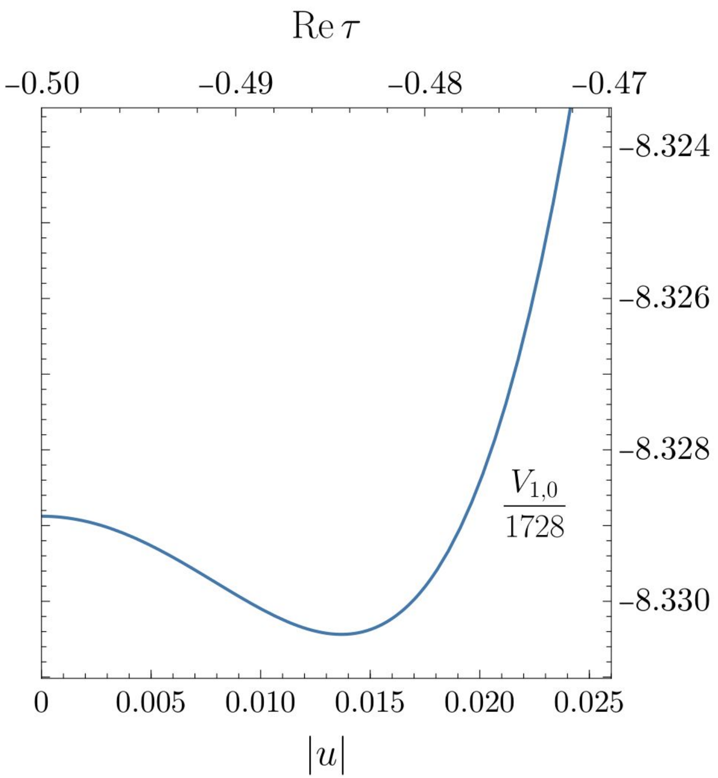
$$W(\tau) = \Lambda_W^3 \frac{H(\tau)}{\eta(\tau)^6}$$

$m, n = 0, 1, 2, \ldots$

- This potential is **modular-** and **CP-invariant**
- Simplified model, independent of the level *N*

The (*m*,*0*) family of potentials (*m* = *1*)





"Mexican"-hat potential