Neutrino oscillations in the interaction picture*

Massimo Blasone

Università di Salerno and INFN, Salerno, Italy

*M. B., F. Giacosa, L. Smaldone and G. Torrieri, EPJC (2023)
V.Bittencourt, M.B. and G.Zanfardino, arXiv:2408.16742[hep-ph]

Summary

1. Quantum Field Theory of neutrino mixing and oscillations

2. Mixing in the interaction picture: QM toy model, boson field, neutrinos

3. Chiral oscillations

Motivations

• CKM quark mixing, meson mixing, massive neutrino mixing (and oscillations) play a crucial role in phenomenology;

• Theoretical interest: origin of mixing in the Standard Model;

• Bargmann superselection rule†: coherent superposition of states with different masses is not allowed in non-relativistic QM;

• Necessity of a QFT treatment: problems in defining Hilbert space for mixed particles‡; oscillation formulas§.

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†V. Bargmann, Ann. Math. (1954); D.M. Greenberger, Phys. Rev. Lett. (2001).

‡C.W. Kim and A. Pevsner, *Neutrinos in Physics and Astrophysics*, (Harwood, 1993). C. Giunti, J. Phys. G (2007).

§M. Beuthe, Phys. Rept. (2003).

Neutrino oscillations in QFT: a brief (early) history

- Pontecorvo theory[*]
- Vacuum-condensate structure and neutrino oscillations[†]
- First attempts to define flavor Fock space[‡§]
- External wavepackets[¶]
- Flavor Fock space approach[‖].

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[*] V. Gribov and B. Pontecorvo, Phys. Lett. B (1969).
[†] L.N. Chang and N.P. Chang, Phys. Rev. Lett. (1980).
[‡] P.T. Mannheim, Phys. Rev. D **37**, 1935 (1988).
[§] C. Giunti, C.W. Kim and U.W. Lee, Phys. Rev. D (1992).
[¶] C. Giunti, C.W. Kim, J.A. Lee and U.W. Lee, Phys. Rev. D (1993).
[‖] M.B. and G.Vitiello, Ann. Phys. (1995).
For a review see M.Beuthe, Phys. Rept. (2003).

Lagrangian (flavor basis)

Free Lagrangian:

$$\mathcal{L}_0 = \sum_{\sigma,\rho=e,\mu} \left[\overline{\nu}_\sigma \left(i\gamma_\mu \partial^\mu - M_\nu^{\sigma\rho}\right) \nu_\rho + \bar{l}_\sigma \left(i\gamma_\mu \partial^\mu - M_l^{\sigma\rho}\right) l_\rho\right]$$

where $l_e \equiv e$, $l_\mu \equiv \mu$, and:

$$M_\nu = \begin{bmatrix} m_e & m_{e\mu} \\ m_{e\mu} & m_\mu \end{bmatrix} \quad ; \quad M_l = \begin{bmatrix} \tilde{m}_e & 0 \\ 0 & \tilde{m}_\mu \end{bmatrix}$$

Interaction term (charged current):

$$\mathcal{L}_{int} = \frac{g}{2\sqrt{2}} \sum_{\sigma=e,\mu} \left[W_\mu^+(x)\, \overline{\nu}_\sigma\, \gamma^\mu \left(1-\gamma^5\right) l_\sigma + h.c.\right]$$

Mixing transformation

Kinetic part diagonalized by mixing transformation

$$\nu_\sigma(x) \ = \ \sum_j U_{\sigma\, j} \nu_j(x)$$

between flavor fields ν_σ and mass fields ν_j. U is the mixing matrix

$$U \ = \ \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

with

$$\tan 2\theta \ = \ 2m_{e\mu}/(m_e - m_\mu)$$

Lagrangian (mass basis)

In the mass basis

$$\mathcal{L}_0 \;=\; \sum_{j=1,2} \overline{\nu}_j \,(i\gamma_\mu \partial^\mu - m_j)\, \nu_j \;+\; \sum_{\sigma=e,\mu} \bar{l}_\sigma \,(i\gamma_\mu \partial^\mu - \tilde{m}_\sigma)\, l_\sigma$$

where

$$\begin{bmatrix} m_1 \\ m_2 \end{bmatrix} = \begin{bmatrix} \cos^2\theta & \sin^2\theta \\ \sin^2\theta & \cos^2\theta \end{bmatrix} \begin{bmatrix} m_e \\ m_\mu \end{bmatrix}$$

Interaction term is no-more diagonal

$$\mathcal{L}_{int} \;=\; \frac{g}{2\sqrt{2}} \sum_{\sigma=e,\mu} \sum_{j=1,2} \left[W^+_\mu(x)\, \overline{\nu}_j\, U^*_{j\sigma}\, \gamma^\mu\, (1-\gamma^5)\, l_\sigma + h.c. \right]$$

In computing an amplitude $\langle \nu_\sigma l^+_\sigma P_F | S | P_I \rangle$, what is definition of $|\nu_\sigma\rangle$?

Pontecorvo flavor states

Pontecorvo states*:

$$|\nu^r_{\mathbf{k},e}\rangle_P = \cos\theta\,|\nu^r_{\mathbf{k},1}\rangle + \sin\theta\,|\nu^r_{\mathbf{k},2}\rangle$$

$$|\nu^r_{\mathbf{k},\mu}\rangle_P = -\sin\theta\,|\nu^r_{\mathbf{k},1}\rangle + \cos\theta\,|\nu^r_{\mathbf{k},2}\rangle$$

Consider the amplitude of the neutrino detection process $\nu_\sigma + X_i \to e^- + X_f$:

$$\langle e^s_{\mathbf{q},-}|\bar{e}(x)\,\gamma^\mu\,(1-\gamma^5)\,\nu_e(x)|\nu^r_{\mathbf{k},\sigma}\rangle_P\,h_\mu(x) \;\not\propto\; \delta_{\sigma e}$$

where h_μ are the matrix elements of the X part.

Problem: since neutrino flavor is defined by the charged-lepton, the above amplitude should be proportional to $\delta_{\sigma e}$.

*S.M. Bilenky and B. Pontecorvo, Phys. Rep. **41**, 225 (1978)

Weak process states

Other proposal: Production (detection) states[†]:

$$|\nu_\sigma^r\rangle_{P(D)} \equiv \Big(\sum_j |\mathcal{A}_{\sigma j}^{P(D)}|^2\Big)^{-\frac{1}{2}} \sum_j \mathcal{A}_{\sigma j}^{P(D)} \, |\nu_j^r\rangle$$

where $\mathcal{A}_{\sigma j}^P = \langle \nu_j l_\sigma^+ P_F|S|P_I\rangle$ and $\mathcal{A}_{\sigma j}^D = \langle l_\sigma^+ X_F|S|X_I \nu_j\rangle$. Flavor states definition depends on the process.

Oscillation probability

$$P_{\sigma\to\rho}(L) = N \sum_{j,k} \mathcal{A}_{\sigma j}^P \mathcal{A}_{\rho j}^{D*} \mathcal{A}_{\sigma k}^{P*} \mathcal{A}_{\rho k}^D e^{-i\frac{\delta m_{kj}^2 L}{2E}}$$

with $\delta m_{kj}^2 \equiv m_k^2 - m_j^2$ and

$$N \equiv \Big(\sum_j |\mathcal{A}_{\sigma j}^P|^2\Big)^{-\frac{1}{2}} \Big(\sum_k |\mathcal{A}_{\rho k}^D|^2\Big)^{-\frac{1}{2}}$$

[†] C. Giunti and C.W. Kim, *Fundamentals of Neutrino Physics and Astrophysics* (Oxford Univ. Press, 2007)

Quantum Field Theory of neutrino mixing and oscillations

Neutrino mixing in QFT

- Mixing relations for two Dirac fields

$$\nu_e(x) \;=\; \cos\theta\,\nu_1(x) + \sin\theta\,\nu_2(x)$$

$$\nu_\mu(x) \;=\; -\sin\theta\,\nu_1(x) + \cos\theta\,\nu_2(x)$$

can be written as*

$$\nu_e^\alpha(x) \;=\; G_\theta^{-1}(t)\,\nu_1^\alpha(x)\,G_\theta(t)$$

$$\nu_\mu^\alpha(x) \;=\; G_\theta^{-1}(t)\,\nu_2^\alpha(x)\,G_\theta(t)$$

– Mixing generator:

$$G_\theta(t) = \exp\left[\theta \int d^3\mathbf{x}\,\left(\nu_1^\dagger(x)\nu_2(x) - \nu_2^\dagger(x)\nu_1(x)\right)\right]$$

For ν_e, we get $\frac{d^2}{d\theta^2}\,\nu_e^\alpha = -\nu_e^\alpha$ with i.c. $\nu_e^\alpha\big|_{\theta=0} = \nu_1^\alpha$, $\frac{d}{d\theta}\nu_e^\alpha\big|_{\theta=0} = \nu_2^\alpha$.

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*M.B. and G.Vitiello, Annals Phys. (1995)

• The vacuum $|0\rangle_{1,2}$ is not invariant under the action of $G_\theta(t)$:

$$|0(t)\rangle_{e,\mu} \equiv G_\theta^{-1}(t)\;|0\rangle_{1,2}$$

• Relation between $|0\rangle_{1,2}$ and $|0(t)\rangle_{e,\mu}$: orthogonality! (for $V \to \infty$)

$$\lim_{V\to\infty}\ {}_{1,2}\langle 0|0(t)\rangle_{e,\mu} = \lim_{V\to\infty} e^{V\int \frac{d^3\mathbf{k}}{(2\pi)^3}\ \ln\left(1-\sin^2\theta\,|V_\mathbf{k}|^2\right)^2} = 0$$

with

$$|V_\mathbf{k}|^2 \;\;\equiv\;\; \sum_{r,s} |\; v^{r\dagger}_{-\mathbf{k},1} u^s_{\mathbf{k},2}\;|^2 \neq 0 \quad for \quad m_1 \neq m_2$$

.

Quantum Field Theory vs. Quantum Mechanics

- Quantum Mechanics:

- finite ♯ of degrees of freedom.

- unitary equivalence of the representations of the canonical commutation relations (von Neumann theorem).

- Quantum Field Theory:

- infinite ♯ of degrees of freedom.

- ∞ many unitarily inequivalent representations of the field algebra $\Leftrightarrow$ many vacua .

- The mapping between interacting and free fields is "weak", i.e. representation dependent (LSZ formalism)*. Example: theories with spontaneous symmetry breaking.

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*F. Strocchi, *Elements of Quantum Mechanics of Infinite Systems* (W. Sc., 1985).

- The "flavor vacuum" $|0(t)\rangle_{e,\mu}$ is a $SU(2)$ generalized coherent state[†]:

$$|0\rangle_{e,\mu} = \prod_{\mathbf{k},r} \Big[(1-\sin^2\theta\,|V_{\mathbf{k}}|^2) - \epsilon^r \sin\theta\,\cos\theta\,|V_{\mathbf{k}}|\,(\alpha^{r\dagger}_{\mathbf{k},1}\beta^{r\dagger}_{-\mathbf{k},2} + \alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},1})$$

$$+\,\epsilon^r \sin^2\theta\,|V_{\mathbf{k}}||U_{\mathbf{k}}|\,(\alpha^{r\dagger}_{\mathbf{k},1}\beta^{r\dagger}_{-\mathbf{k},1} - \alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},2}) + \sin^2\theta\,|V_{\mathbf{k}}|^2\,\alpha^{r\dagger}_{\mathbf{k},1}\beta^{r\dagger}_{-\mathbf{k},2}\alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},1}\Big]\,|0\rangle_{1,2}$$

- Condensation density:

$$_{e,\mu}\langle 0(t)|\alpha^{r\dagger}_{\mathbf{k},i}\alpha^{r}_{\mathbf{k},i}|0(t)\rangle_{e,\mu} \;=\; _{e,\mu}\langle 0(t)|\beta^{r\dagger}_{\mathbf{k},i}\beta^{r}_{\mathbf{k},i}|0(t)\rangle_{e,\mu} \;=\; \sin^2\theta\,|V_{\mathbf{k}}|^2$$

vanishing for $m_1 = m_2$ and/or $\theta = 0$ (in both cases no mixing).

– Condensate structure as in systems with SSB (e.g. superconductors)

– Exotic condensate: mixed pairs

– Note that $|0\rangle_{e\,\mu} \neq |a\rangle_1 \otimes |b\rangle_2 \Rightarrow$ entanglement.

[†] A. Perelomov, *Generalized Coherent States,* (Springer V., 1986)

• Structure of the annihilation operators for $|0(t)\rangle_{e,\mu}$:

$$\alpha^r_{\mathbf{k},e}(t) = \cos\theta\,\alpha^r_{\mathbf{k},1} + \sin\theta\left(U^*_{\mathbf{k}}(t)\,\alpha^r_{\mathbf{k},2} + \epsilon^r V_{\mathbf{k}}(t)\,\beta^{r\dagger}_{-\mathbf{k},2}\right)$$

$$\alpha^r_{\mathbf{k},\mu}(t) = \cos\theta\,\alpha^r_{\mathbf{k},2} - \sin\theta\left(U_{\mathbf{k}}(t)\,\alpha^r_{\mathbf{k},1} - \epsilon^r V_{\mathbf{k}}(t)\,\beta^{r\dagger}_{-\mathbf{k},1}\right)$$

$$\beta^r_{-\mathbf{k},e}(t) = \cos\theta\,\beta^r_{-\mathbf{k},1} + \sin\theta\left(U^*_{\mathbf{k}}(t)\,\beta^r_{-\mathbf{k},2} - \epsilon^r V_{\mathbf{k}}(t)\,\alpha^{r\dagger}_{\mathbf{k},2}\right)$$

$$\beta^r_{-\mathbf{k},\mu}(t) = \cos\theta\,\beta^r_{-\mathbf{k},2} - \sin\theta\left(U_{\mathbf{k}}(t)\,\beta^r_{-\mathbf{k},1} + \epsilon^r V_{\mathbf{k}}(t)\,\alpha^{r\dagger}_{\mathbf{k},1}\right)$$

• Mixing transformation = Rotation + Bogoliubov transformation .

– Bogoliubov coefficients:

$$U_{\mathbf{k}}(t) = u^{r\dagger}_{\mathbf{k},2} u^r_{\mathbf{k},1}\, e^{i(\omega_{k,2}-\omega_{k,1})t} \quad ; \qquad V_{\mathbf{k}}(t) = \epsilon^r\, u^{r\dagger}_{\mathbf{k},1} v^r_{-\mathbf{k},2}\, e^{i(\omega_{k,2}+\omega_{k,1})t}$$

$$|U_{\mathbf{k}}|^2 + |V_{\mathbf{k}}|^2 = 1$$

Decomposition of mixing generator *

The mixing generator can be expressed in terms of a rotation and a Bogoliubov transformation. Define:

$$R(\theta) \equiv \exp\left\{\theta \sum_{\mathbf{k},r}\left[\left(\alpha^{r\dagger}_{\mathbf{k},1}\alpha^{r}_{\mathbf{k},2} + \beta^{r\dagger}_{\mathbf{k},1}\beta^{r}_{\mathbf{k},2}\right)e^{i\psi_k} - \left(\alpha^{r\dagger}_{\mathbf{k},2}\alpha^{r}_{\mathbf{k},1} + \beta^{r\dagger}_{\mathbf{k},2}\beta^{r}_{\mathbf{k},1}\right)e^{-i\psi_k}\right]\right\},$$

$$B_i(\Theta_i) \equiv \exp\left\{\sum_{\mathbf{k},r}\Theta_{\mathbf{k},i}\;\epsilon^r\left[\alpha^{r}_{\mathbf{k},i}\beta^{r}_{-\mathbf{k},i}e^{-i\phi_{k,i}} - \beta^{r\dagger}_{-\mathbf{k},i}\alpha^{r\dagger}_{\mathbf{k},i}e^{i\phi_{k,i}}\right]\right\}, \qquad i = 1,2$$

Since $[B_1, B_2] = 0$ we put $B(\Theta_1, \Theta_2) \equiv B_1(\Theta_1)\, B_2(\Theta_2)$.

- We find:

$$G_\theta \;=\; B(\Theta_1, \Theta_2)\;\; R(\theta)\;\; B^{-1}(\Theta_1, \Theta_2)$$

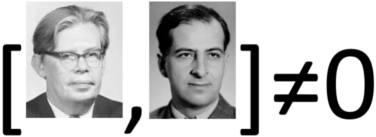
which is realized when the $\Theta_{\mathbf{k},i}$ are chosen as:

$$U_{\mathbf{k}} = e^{-i\psi_k}\;\cos(\Theta_{\mathbf{k},1} - \Theta_{\mathbf{k},2})\,; \qquad V_{\mathbf{k}} = e^{\frac{(\phi_{k,1}+\phi_{k,2})}{2}}\;\sin(\Theta_{\mathbf{k},1} - \Theta_{\mathbf{k},2})$$

* M. B., M.V. Gargiulo and G. Vitiello, Phys. Lett. B (2017)

Bogoliubov vs Pontecorvo

- Bogoliubov and Pontecorvo do not commute!



As a result, flavor vacuum gets a non-trivial term:

$$|0\rangle_{e,\mu} \equiv G_\theta^{-1}|0\rangle_{1,2} \;=\; |0\rangle_{1,2} \;+\; \left[B(m_1,m_2)\;,\; R^{-1}(\theta)\right]\;|\widetilde{0}\rangle_{1,2}$$

with $|\widetilde{0}\rangle_{1,2} \equiv B^{-1}(\Theta_1,\Theta_2)|0\rangle_{1,2}$.

- Non-diagonal Bogoliubov transformation

$$|0\rangle_{e,\mu} \cong \left[\mathbb{1} + \theta\, a \int \frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}}\;\tilde{V}_{\mathbf{k}}\sum_r \epsilon^r\left(\alpha^{r\dagger}_{\mathbf{k},1}\beta^{r\dagger}_{-\mathbf{k},2} + \alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},1}\right)\right]|0\rangle_{1,2}\,,$$

with $a \equiv \frac{(m_2-m_1)^2}{m_1 m_2}$.

Currents and charges for mixed fermions *

– Lagrangian in the mass basis:

$$\mathcal{L} \,=\, \bar{\nu}_m \,(i \not\partial - M_d)\, \nu_m$$

where $\nu_m^T = (\nu_1, \nu_2)$ and $M_d = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}$.

- $\mathcal{L}$ invariant under global $U(1)$ with conserved charge Q= total charge.

  – Consider now the $SU(2)$ transformation:

$$\nu'_m \,=\, e^{i\alpha_j \tau_j}\, \nu_m \qquad ; \qquad j \,=\, 1, 2, 3.$$

with $\tau_j = \sigma_j/2$ and σ_j being the Pauli matrices.

* M. B., P. Jizba and G. Vitiello, Phys. Lett. B (2001)

The associated currents are:

$$\delta\mathcal{L} = i\alpha_j\,\bar{\nu}_m\,[\tau_j, M_d]\,\nu_m = -\alpha_j\,\partial_\mu J^\mu_{m,j}$$

$$J^\mu_{m,j} = \bar{\nu}_m\,\gamma^\mu\,\tau_j\,\nu_m$$

– The charges $Q_{m,j}(t) \equiv \int d^3\mathbf{x}\,J^0_{m,j}(x)$, satisfy the $su(2)$ algebra:

$$[Q_{m,j}(t), Q_{m,k}(t)] = i\,\epsilon_{jkl}\,Q_{m,l}(t)\,.$$

– Casimir operator proportional to the total charge: $C_m = \frac{1}{2}Q$.

- $Q_{m,3}$ is conserved $\Rightarrow$ charge conserved separately for ν_1 and ν_2:

$$Q_1 = \frac{1}{2}Q + Q_{m,3} = \int d^3\mathbf{x}\,\nu_1^\dagger(x)\,\nu_1(x)$$

$$Q_2 = \frac{1}{2}Q - Q_{m,3} = \int d^3\mathbf{x}\,\nu_2^\dagger(x)\,\nu_2(x).$$

These are the flavor charges in the absence of mixing.

The currents in the flavor basis

– Lagrangian in the flavor basis:

$$\mathcal{L} = \bar{\nu}_f \left(i \not\partial - M \right) \nu_f$$

where $\nu_f^T = (\nu_e, \nu_\mu)$ and $M = \begin{pmatrix} m_e & m_{e\mu} \\ m_{e\mu} & m_\mu \end{pmatrix}$.

– Consider the $SU(2)$ transformation:

$$\nu'_f = e^{i\alpha_j \tau_j} \, \nu_f \qquad ; \qquad j = 1, 2, 3.$$

with $\tau_j = \sigma_j/2$ and σ_j being the Pauli matrices.

– The charges $Q_{f,j} \equiv \int d^3\mathbf{x} \, J^0_{f,j}$ satisfy the $su(2)$ algebra:

$$[Q_{f,j}(t), Q_{f,k}(t)] = i \, \epsilon_{jkl} \, Q_{f,l}(t).$$

– Casimir operator proportional to the total charge $C_f = C_m = \frac{1}{2} Q$.

- $Q_{f,3}$ is not conserved $\Rightarrow$ exchange of charge between ν_e and ν_μ.

Define the flavor charges as:

$$Q_e(t) \;\equiv\; \frac{1}{2}Q + Q_{f,3}(t) = \int d^3\mathbf{x}\, \nu_e^\dagger(x)\, \nu_e(x)$$

$$Q_\mu(t) \;\equiv\; \frac{1}{2}Q - Q_{f,3}(t) = \int d^3\mathbf{x}\, \nu_\mu^\dagger(x)\, \nu_\mu(x)$$

where $Q_e(t) + Q_\mu(t) = Q$.

– We have:

$$Q_e(t) \;=\; \cos^2\theta\, Q_1 + \sin^2\theta\, Q_2 + \sin\theta\cos\theta \int d^3\mathbf{x} \left[\nu_1^\dagger \nu_2 + \nu_2^\dagger \nu_1\right]$$

$$Q_\mu(t) \;=\; \sin^2\theta\, Q_1 + \cos^2\theta\, Q_2 - \sin\theta\cos\theta \int d^3\mathbf{x} \left[\nu_1^\dagger \nu_2 + \nu_2^\dagger \nu_1\right]$$

In conclusion:

– In presence of mixing, neutrino flavor charges are defined as

$$Q_e(t) \equiv \int d^3\mathbf{x}\, \nu_e^\dagger(x)\, \nu_e(x) \quad ; \qquad Q_\mu(t) \equiv \int d^3\mathbf{x}\, \nu_\mu^\dagger(x)\, \nu_\mu(x)$$

– They are not conserved charges ⇒ flavor oscillations.

– They are still (approximately) conserved in the vertex ⇒ define flavor neutrinos as their eigenstates

- Problem: find the eigenstates of the above charges.

- Flavor charge operators are diagonal in the flavor ladder operators:

$$:: Q_\sigma(t) :: \;\equiv\; \int d^3\mathbf{x} :: \nu_\sigma^\dagger(x)\, \nu_\sigma(x) ::$$

$$= \sum_r \int d^3\mathbf{k} \left(\alpha^{r\dagger}_{\mathbf{k},\sigma}(t) \alpha^{r}_{\mathbf{k},\sigma}(t) - \beta^{r\dagger}_{-\mathbf{k},\sigma}(t) \beta^{r}_{-\mathbf{k},\sigma}(t) \right), \quad \sigma = e, \mu.$$

Here $:: \ldots ::$ denotes normal ordering w.r.t. flavor vacuum:

$$:: A :: \equiv A - {}_{e,\mu}\langle 0|A|0\rangle_{e,\mu}$$

- Define flavor neutrino states with definite momentum and helicity:

$$|\nu^r_{\mathbf{k},\sigma}\rangle \equiv \alpha^{r\dagger}_{\mathbf{k},\sigma}(0)\, |0\rangle_{e,\mu}$$

– Such states are eigenstates of the flavor charges (at t=0):

$$:: Q_\sigma :: |\nu^r_{\mathbf{k},\sigma}\rangle = |\nu^r_{\mathbf{k},\sigma}\rangle$$

Neutrino oscillation formula (QFT)

– We have, for an electron neutrino state:

$$\mathcal{Q}_{\mathbf{k},\sigma}(t) \equiv \langle \nu^r_{\mathbf{k},e}| :: Q_\sigma(t) :: |\nu^r_{\mathbf{k},e}\rangle$$
$$= \left|\left\{\alpha^r_{\mathbf{k},\sigma}(t), \alpha^{r\dagger}_{\mathbf{k},e}(0)\right\}\right|^2 + \left|\left\{\beta^{r\dagger}_{-\mathbf{k},\sigma}(t), \alpha^{r\dagger}_{\mathbf{k},e}(0)\right\}\right|^2$$

with $Q_\sigma(t) \equiv \int d^3\mathbf{x}\, \nu^\dagger_\sigma(x)\, \nu_\sigma(x)$.

• Neutrino oscillation formula (exact result)*:

$$\mathcal{Q}_{\mathbf{k},e}(t) = 1 - |U_\mathbf{k}|^2 \sin^2(2\theta) \sin^2\left(\frac{\omega_{k,2} - \omega_{k,1}}{2}\, t\right) - |V_\mathbf{k}|^2 \sin^2(2\theta) \sin^2\left(\frac{\omega_{k,2} + \omega_{k,1}}{2}\, t\right)$$

- For $k \gg \sqrt{m_1 m_2}$, $|U_\mathbf{k}|^2 \to 1$ and $|V_\mathbf{k}|^2 \to 0 \Rightarrow$ Pontecorvo formula is recovered.

*M.B., P.Henning and G.Vitiello, Phys. Lett. **B** (1999).

Lepton charge violation for Pontecorvo states[†]

– Pontecorvo states:

$$|\nu^r_{\mathbf{k},e}\rangle_P \;=\; \cos\theta\, |\nu^r_{\mathbf{k},1}\rangle \;+\; \sin\theta\, |\nu^r_{\mathbf{k},2}\rangle$$

$$|\nu^r_{\mathbf{k},\mu}\rangle_P \;=\; -\sin\theta\, |\nu^r_{\mathbf{k},1}\rangle \;+\; \cos\theta\, |\nu^r_{\mathbf{k},2}\rangle \;,$$

are *not* eigenstates of the flavor charges.

⇒ *violation of lepton charge conservation* in the production/detection vertices, at tree level:

$${}_P\langle\nu^r_{\mathbf{k},e}| :Q_e(0): |\nu^r_{\mathbf{k},e}\rangle_P \;=\; \cos^4\theta + \sin^4\theta + 2|U_{\mathbf{k}}|\, \sin^2\theta\cos^2\theta \;<\; 1,$$

for any $\theta \neq 0$, $\mathbf{k} \neq 0$ and for $m_1 \neq m_2$.

[†] M. B., A. Capolupo, F. Terranova and G. Vitiello, Phys. Rev. **D** (2005)
C. C. Nishi, Phys. Rev. **D** (2008).

Neutrino ontology: flavor or mass?

- In view of the unitary inequivalence of mass and flavor representations, we have the problem of the fundamental (ontological) nature of neutrino.

Flavor or mass, that is the question...

Neutrino ontology: research directions

- How to verify the fundamental nature of neutrino states?

Two directions:

- Investigate the phenomenology of flavor neutrinos, with corrections expected in the non-relativistic regime: oscillations, beta decay endpoint, quantum correlations, ...
- Use the formal consistency of QFT, by comparing neutrino processes in two different frames (inertial and comoving) for accelerated particle: Unruh effect.*

*M. B., G. Lambiase, G. Luciano and L.Petruzziello, Phys. Rev. D (2018); G.Cozzella, S.Fulling, A.Landulfo, G.Matsas and D.Vanzella, Phys.Rev.(2018) M. B., G.Lambiase, G. Luciano and L.Petruzziello, Phys. Lett. B (2020)

Flavor neutrino as unstable particles

- Time-energy uncertainty relations (TEUR) in the Mandelstam–Tamm form, furnish lower-bounds on neutrino energy uncertainty compatible with flavor oscillations*.
- QFT formulation of neutrino oscillations suggests that these bounds can be read as flavor-energy uncertainty relations (FEUR)†. Energy uncertainty is connected with the intrinsic unstable nature of flavor neutrinos.

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*S.M Bilenky, F. von Feilitzsch and W. Potzel, J. Phys. G (2008)

†M. B., P. Jizba and L. Smaldone, Phys. Rev. D (2019)

Time-energy uncertainty relations

Mandelstam–Tamm TEUR is*:

$$\Delta E \, \Delta t \, \geq \frac{1}{2}$$

where

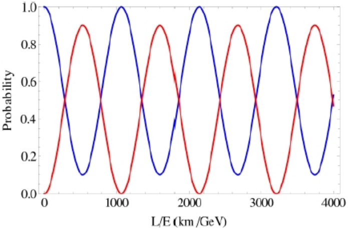
$$\Delta E \equiv \sigma_H \qquad \Delta t \equiv \sigma_O / \left| \frac{\mathrm{d}\langle O(t)\rangle}{\mathrm{d}t} \right|$$

Here $\langle \ldots \rangle \equiv \langle \psi | \ldots | \psi \rangle$ and $O(t)$ represents the "clock observable" whose dynamics quantifies temporal changes in a system.

– The above inequality is obtained by means of the Cauchy-Schwarz inequality and using the fact that $[\hat{O}, \hat{H}] \neq 0$.

*L. Mandelstam and I.G. Tamm, J. Phys. USSR (1945)

Clock observables



Flavor-energy uncertainty relations

Choose flavor charges as clock observables. Then $[Q_{\nu_\sigma}(t)\,,\,H] \neq 0 \Rightarrow$ flavor-energy uncertainty relation†:

$$\langle \Delta H \rangle \, \langle \Delta Q_{\nu_\sigma}(t) \rangle \;\geq\; \frac{1}{2} \left| \frac{\mathrm{d}\langle Q_{\nu_\sigma}(t) \rangle}{\mathrm{d}t} \right|$$

Taking the state $|\psi\rangle \;=\; |\nu^r_{\mathbf{k},\sigma}\rangle$ we have $\langle Q_{\nu_\sigma}(t)\rangle = \mathcal{Q}_{\sigma\to\sigma}(t)$ and

$$\langle \Delta Q_{\nu_\sigma}(t) \rangle \;=\; \sqrt{\mathcal{Q}_{\sigma\to\sigma}(t)(1-\mathcal{Q}_{\sigma\to\sigma}(t))} \leq \frac{1}{2}.$$

Integrating over time from 0 to T, and using the triangular inequality, we obtain:

$$\Delta E \, T \;\geq\; \mathcal{Q}_{\sigma\to\rho}(T)\,, \quad \sigma \neq \rho$$

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†M. B., P. Jizba and L.Smaldone, Phys. Rev. D (2019)

Neutrino oscillation condition

When $m_i/|\mathbf{k}| \to 0$:

$$\Delta E \ \geq \ \frac{2\,\sin^2 2\theta}{L_{osc}}$$

This relation is usually interpreted as neutrino oscillation condition[‡].

The situation is similar to that of unstable particles:

$$\Delta E \ \approx \ \frac{1}{2\tau}$$

where the τ is the particle life-time.

– As for unstable particles only energy distribution are meaningful.
The width of the distribution is related to the oscillation length.

[‡]S.M Bilenky, F. von Feilitzsch and W. Potzel, J. Phys. G (2008)

Mixing in the interaction picture: QM toy model, boson field, neutrinos

Motivations

- Both Pontecorvo neutrino states and weak process states are not eigenstates of flavor charge;
- Exact flavor (lepton) charge eigenstates require the introduction of the flavor vacuum which breaks Poincaré invariance[§].

Consider another approach: treat the mixing term of the Lagrangian as a perturbation and compute oscillation formula from QFT at finite time[¶].

[§] M. B., P. Jizba, N.E. Mavromatos and L. Smaldone, Phys. Rev. D (2020);
[¶] M. B., F. Giacosa, L. Smaldone and G.Torrieri, EPJC (2023)

Neutrino mixing and time-evolution operator

- Decompose neutrino Lagrangian as $(g = 0)$

$$\mathcal{L} \ = \ \mathcal{L}_0 + \mathcal{L}_{int}$$

with

$$\mathcal{L}_0 \ = \ \sum_{\sigma=e,\mu} \overline{\nu}_\sigma \left(i\partial\!\!\!/ - m_\sigma\right) \nu_\sigma$$

$$\mathcal{L}_{int} \ = \ -m_{e\mu} \left(\overline{\nu}_e \nu_\mu + \overline{\nu}_\mu \nu_e\right)$$

Time-evolution operator

$$U(t_i, t_f) \ = \ \mathcal{T} \exp\left[-i \int_{t_i}^{t_f} \mathrm{d}^4 x \ :\mathcal{H}_{int}(x):\right]$$

$\mathcal{H}_{int}(x) = -\mathcal{L}_{int}(x)$.

• Why finite time? Analogy with unstable particles[‖].

Flavor-energy uncertainty relation[**]

$$\Delta E\, T \;\geq\; \mathcal{Q}_{\sigma\to\rho}(T)\,, \quad \sigma \neq \rho$$

It follows (at T_h oscillation probability$=\frac{1}{2}$)

$$\Delta E\, T_h \geq \frac{1}{2}$$

For unstable particles:

$$\Delta E \;\approx\; \frac{1}{2\tau}$$

where the τ is the particle life-time.

‖ C.Bernardini, L.Maiani and M.Testa, Phys. Rev. Lett. (1993).
P. Facchi and S. Pascazio, La regola d'oro di Fermi, (Bibliopolis, 1999).
** M. B., P. Jizba and L. Smaldone, Phys. Rev. D (2019)

A toy model

$0 + 1D$ field theory (QM)

$$L = \frac{1}{2}\left(\frac{dx_A}{dt}\right)^2 - \frac{\omega_A^2}{2}x_A^2 + \frac{1}{2}\left(\frac{dx_B}{dt}\right)^2 - \frac{\omega_B^2}{2}y^2 - \omega_{AB}^2 x_A x_B$$

In the interaction picture

$$x_\sigma(t) \;\; = \;\; \frac{1}{\sqrt{2\omega_A}}\left(a_\sigma e^{-i\omega_\sigma t} + a_\sigma^\dagger e^{i\omega_\sigma t}\right) , \qquad \sigma = A, B$$

"Flavor" states

$$|A\rangle = a_A^\dagger|0\rangle , \qquad |B\rangle = a_B^\dagger|0\rangle$$

Interaction term

$$H_{int} = \omega_{AB}^2 x_A(t) x_B(t) = \frac{\omega_{AB}^2}{2\sqrt{\omega_A\omega_B}}\left[a_B^\dagger a_A e^{i(\omega_B-\omega_A)t} + a_B^\dagger a_A^\dagger e^{i(\omega_A+\omega_B)t} + h.c.\right]$$

Decay probability

- Consider the process

$$|A\rangle \;\rightarrow\; |B\rangle$$

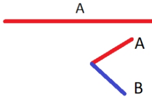
Amplitude at first order in ω_{AB}^2

$$\langle B|U(t_f,t_i)|A\rangle \;=\; \frac{\omega_{AB}^2}{\sqrt{2\omega_A}\sqrt{2\omega_B}}\frac{e^{-i(\omega_A-\omega_B)t_f}-e^{-i(\omega_A-\omega_B)t_i}}{(\omega_A-\omega_B)}$$

Transition probability

$$\mathcal{P}_{A\rightarrow B}(\Delta t)=\frac{\omega_{AB}^4}{\omega_A\omega_B}\frac{\sin^2\left[\frac{(\omega_A-\omega_B)\Delta t}{2}\right]}{(\omega_A-\omega_B)^2} \qquad \Delta t = t_f - t_i$$

Decay probability



- The other non-trivial process

$$|A\rangle \;\rightarrow\; |A\rangle|A\rangle|B\rangle$$

Normalized amplitude (first order in ω_{AB}^2)

$$\frac{1}{\sqrt{2}}\langle 0|a_B a_A^2\, U(t_f,t_i)\, a_A^\dagger|0\rangle \;=\; \frac{\sqrt{2}\omega_{AB}^2}{\sqrt{2\omega_A}\sqrt{2\omega_B}}\frac{e^{-i(\omega_A+\omega_B)t_f} - e^{-i(\omega_A+\omega_B)t_i}}{(\omega_A+\omega_B)}$$

Hence

$$\mathcal{P}_{A\to AAB}(\Delta t) = \frac{2\omega_{AB}^4}{\omega_A\omega_B}\frac{\sin^2\left[\frac{(\omega_A+\omega_B)\Delta t}{2}\right]}{(\omega_A+\omega_B)^2}$$

Decay probability

Flavor transition (decay) probability $\mathcal{P}_{A\to B}(\Delta t) + \mathcal{P}_{A\to AAB}(\Delta t)$

$$\mathcal{P}^A_D(\Delta t) \;=\; \frac{\omega^4_{AB}}{\omega_A\omega_B}\left[\frac{\sin^2\left[\frac{(\omega_A-\omega_B)\Delta t}{2}\right]}{(\omega_A-\omega_B)^2}+2\frac{\sin^2\left[\frac{(\omega_A+\omega_B)\Delta t}{2}\right]}{(\omega_A+\omega_B)^2}\right]$$

Survival probability

Survival amplitude

$$\langle A|U(t_f,t_i)|A\rangle = 1 - i\,\mathcal{T}\langle 0|a_A \int_{t_i}^{t_f} dt_1 H_{int}(t_1)\int_{t_i}^{t_1} dt_2 H_{int}(t_2) a_A^\dagger|0\rangle$$

one gets the survival probability of the state $|A\rangle$ as:

$$\begin{aligned}\mathcal{P}_{A\to A}(\Delta t) &= \Bigg|1 - \frac{\omega_{AB}^4}{4\omega_A\omega_B}\Bigg[2\frac{t}{i(\omega_A+\omega_B)} - 2\frac{e^{-i(\omega_A+\omega_B)\Delta t}-1}{(\omega_A+\omega_B)^2} \\ &\quad + \frac{t}{-i(\omega_A-\omega_B)} - \frac{e^{i(\omega_A-\omega_B)\Delta t}-1}{(\omega_A-\omega_B)^2}\Bigg]\Bigg|^2 \\ &= |1-R-iI|^2 = 1-2R+\ldots\,,\end{aligned}$$

with

$$R = \frac{\omega_{AB}^4}{2\omega_A\omega_B}\left(\frac{\sin^2\left[\frac{(\omega_A-\omega_B)\Delta t}{2}\right]}{(\omega_A-\omega_B)^2} + 2\frac{\sin^2\left[\frac{(\omega_A+\omega_B)\Delta t}{2}\right]}{(\omega_A+\omega_B)^2}\right)\,,$$

Survival probability

$$\mathcal{P}_S^A(\Delta t) \quad = \quad 1 - \frac{\omega_{AB}^4}{\omega_A \omega_B} \left(\frac{\sin^2\left[\frac{(\omega_A - \omega_B)\Delta t}{2}\right]}{(\omega_A - \omega_B)^2} + 2\frac{\sin^2\left[\frac{(\omega_A + \omega_B)\Delta t}{2}\right]}{(\omega_A + \omega_B)^2} \right)$$

Unitarity

$$\mathcal{P}_S^A(\Delta t) + \mathcal{P}_D^A(\Delta t) \ = \ 1$$

Diagonalization

Of course, the problem can be also solved by introducing the rotation

$$\begin{pmatrix} x_A \\ x_B \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

with

$$\begin{aligned} \tan 2\theta &= \frac{2\omega_{AB}^2}{\omega_B^2 - \omega_A^2} \,, \\ \omega_A^2 &= \omega_1^2 \cos^2\theta + \omega_2^2 \sin^2\theta \ , \\ \omega_B^2 &= \omega_1^2 \sin^2\theta + \omega_2^2 \cos^2\theta \ . \end{aligned}$$

Denoting $|\Omega\rangle$ the vacuum of the full Hamiltonian ($a_i \left|\Omega\right\rangle = 0$, $i = 1, 2$), one may also consider the state

$$|a\rangle = \cos\theta a_1^\dagger \left|\Omega\right\rangle + \sin\theta a_2^\dagger \left|\Omega\right\rangle \ ,$$

yet it is clear that $|a\rangle \neq |A\rangle = a_A^\dagger \left|a\right\rangle$,

Diagonalization

In terms of $|a\rangle$, the survival probability takes the form:

$$\mathcal{P}^a_S(\Delta t) \;=\; 1 - \sin^2 2\theta \sin^2\left[\frac{(\omega_1 - \omega_2)\Delta t}{2}\right] .$$

In the limit of small θ, the previous expression is approximated by:

$$\begin{aligned}\mathcal{P}^a_S(\Delta t) \quad &\simeq \quad 1 - \frac{4\omega_{AB}^4}{\left(\omega_B^2 - \omega_A^2\right)^2} \sin^2\left[\frac{(\omega_A - \omega_B)\Delta t}{2}\right] \\ &= \quad 1 - \frac{4\omega_{AB}^4}{\left(\omega_B + \omega_A\right)^2} \frac{\sin^2\left[\frac{(\omega_A - \omega_B)\Delta t}{2}\right]}{\left(\omega_B - \omega_A\right)^2} \quad .\end{aligned}$$

which is different from the perturbative formula both for the absence of the fast oscillating term and for the amplitude of the standard oscillating term.

Scalar field mixing

Lagrangian

$$\mathcal{L} = \frac{1}{2}\left(\partial_\alpha \phi_A\right)^2 - \frac{m_A^2}{2}\phi_A^2 + \frac{1}{2}\left(\partial_\alpha \phi_B\right)^2 - \frac{m_B^2}{2}\phi_B^2 - m_{AB}^2 \phi_A \phi_B$$

In the interaction picture (in a volume V box)

$$\phi_A(x) \ = \frac{1}{\sqrt{V}} \sum_{\mathbf{k}=2\pi\mathbf{n}/L} \frac{1}{\sqrt{2\omega_{\mathbf{k},A}}} \left(a_{\mathbf{k},A} e^{-ikx} + a^\dagger_{\mathbf{k},A} e^{ikx} \right)$$

One-particle state

$$|A, \mathbf{p}\rangle = a^\dagger_{\mathbf{p},A} \, |0\rangle$$

The same for ϕ_B.

Decay probability

- Process $|A, \mathbf{p}\rangle \to |B, \mathbf{k}\rangle$

$$\mathcal{A}_{A\to B}\left(\mathbf{p},\mathbf{k};t_i,t_f\right) \; = \; \frac{m_{AB}^2}{\sqrt{2\omega_{\mathbf{p},A}}\sqrt{2\omega_{\mathbf{k},B}}}\delta_{\mathbf{k},\mathbf{p}}\frac{e^{-i\left(\omega_{\mathbf{p},A}-\omega_{\mathbf{k},B}\right)t_f}-e^{-i\left(\omega_{\mathbf{p},A}-\omega_{\mathbf{k},B}\right)t_i}}{\omega_{\mathbf{p},A}-\omega_{\mathbf{k},B}}$$

Probability

$$\mathcal{P}_{A\to B}(\mathbf{p};\Delta t) = \sum_{\mathbf{k}} \left|\mathcal{A}_{A\to B}\left(\mathbf{p},\mathbf{k};t_i,t_f\right)\right|^2$$

Then

$$\mathcal{P}_{A\to B}(\mathbf{p};\Delta t) = \frac{m_{AB}^4}{\omega_{\mathbf{p},A}\omega_{\mathbf{p},B}}\frac{\sin^2\left[\frac{(\omega_{\mathbf{p},A}-\omega_{\mathbf{p},B})\Delta t}{2}\right]}{\left(\omega_{\mathbf{p},A}-\omega_{\mathbf{p},B}\right)^2}$$

Decay probability

- The other non-trivial process

$$|A, \mathbf{p}\rangle \to |A, \mathbf{k}_1\rangle \, |A, \mathbf{k}_2\rangle \, |B, \mathbf{k}_3\rangle$$

– When $\mathbf{k}_1 \neq \mathbf{k}_2$

$$\mathcal{P}^{\mathbf{k}_1 \neq \mathbf{k}_2}_{A \to AAB}(\mathbf{p}; \Delta t) = \sum_{\mathbf{k}_3} \frac{m^4_{AB}}{\omega_{\mathbf{k}_3,A}\,\omega_{\mathbf{k}_3,B}} \frac{\sin^2\left[\frac{\left(\omega_{\mathbf{k}_3,A}+\omega_{\mathbf{k}_3,B}\right)\Delta t}{2}\right]}{\left(\omega_{\mathbf{k}_3,A}+\omega_{\mathbf{k}_3,B}\right)^2} - \frac{m^4_{AB}}{\omega_{\mathbf{p},A}\,\omega_{\mathbf{p},B}} \frac{\sin^2\left[\frac{\left(\omega_{\mathbf{p},A}+\omega_{\mathbf{p},B}\right)\Delta t}{2}\right]}{\left(\omega_{\mathbf{p},A}+\omega_{\mathbf{p},B}\right)^2}$$

Large V

$$\mathcal{P}^{\mathbf{k}_1 \neq \mathbf{k}_2}_{A \to AAB}(\mathbf{p}; \Delta t) = V \int \frac{\mathrm{d}^3 \mathbf{k}_3}{(2\pi)^3} \frac{m^4_{AB}}{\omega_{\mathbf{k}_3,A}\,\omega_{\mathbf{k}_3,B}} \frac{\sin^2\left[\frac{\left(\omega_{\mathbf{k}_3,A}+\omega_{\mathbf{k}_3,B}\right)\Delta t}{2}\right]}{\left(\omega_{\mathbf{k}_3,A}+\omega_{\mathbf{k}_3,B}\right)^2} - \frac{m^4_{AB}}{\omega_{\mathbf{p},A}\,\omega_{\mathbf{p},B}} \frac{\sin^2\left[\frac{\left(\omega_{\mathbf{p},A}+\omega_{\mathbf{p},B}\right)\Delta t}{2}\right]}{\left(\omega_{\mathbf{p},A}+\omega_{\mathbf{p},B}\right)^2}$$

First piece on the r.h.s. IR divergent vacuum contribution

Total decay probability

– When $\mathbf{k}_1 = \mathbf{k}_2$

$$\mathcal{P}^{\mathbf{k}_1=\mathbf{k}_2}_{A\to AAB}(\mathbf{p},\Delta t) = 2\frac{m^4_{AB}}{\omega_{\mathbf{p},A}\omega_{\mathbf{p},B}}\frac{\sin^2\left[\frac{(\omega_{\mathbf{p},A}+\omega_{\mathbf{p},B})\Delta t}{2}\right]}{\left(\omega_{\mathbf{p},A}+\omega_{\mathbf{p},B}\right)^2}\ .$$

- Total decay probability

$$\mathcal{P}^A_D(\mathbf{p};\Delta t) \ = \ \mathcal{P}_{A\to B}(\mathbf{p};\Delta t) + \mathcal{P}^{\mathbf{k}_1\neq\mathbf{k}_2}_{A\to AAB}(\mathbf{p};\Delta t) + \mathcal{P}^{\mathbf{k}_1=\mathbf{k}_2}_{A\to AAB}(\mathbf{p};\Delta t)$$

Subtracting the divergent term

$$\mathcal{P}^A_D(\mathbf{p};\Delta t) = \frac{m^4_{AB}}{\omega_{\mathbf{p},A}\omega_{\mathbf{p},B}}\left(\frac{\sin^2\left[\frac{(\omega_{\mathbf{p},A}-\omega_{\mathbf{p},B})\Delta t}{2}\right]}{\left(\omega_{\mathbf{p},A}-\omega_{\mathbf{p},B}\right)^2}+\frac{\sin^2\left[\frac{(\omega_{\mathbf{p},A}+\omega_{\mathbf{p},B})\Delta t}{2}\right]}{\left(\omega_{\mathbf{p},A}+\omega_{\mathbf{p},B}\right)^2}\right)$$

Survival probability

- Survival process

$$|A, \mathbf{p}\rangle \rightarrow |A, \mathbf{k}\rangle$$

Amplitude

$$\mathcal{A}_{A\rightarrow A}(\mathbf{p}, \mathbf{k}; t_i, t_f) = \delta_{\mathbf{k},\mathbf{p}} + (-i)^2 \int_{t_i}^{t_f} dt_1 \int_{t_i}^{t_1} dt_2 \langle 0|a_{\mathbf{k},A} H_{int}(t_1) H_{int}(t_2) a^{\dagger}_{\mathbf{p},A}|0\rangle$$

Unitarity

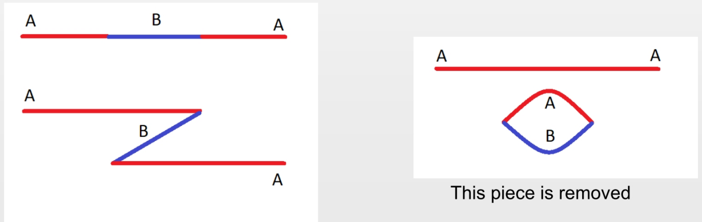
$$\mathcal{P}^A_D(\mathbf{p}; \Delta t) + \mathcal{P}^A_S(\mathbf{p}; \Delta t) \;=\; 1$$

- Survival probability

$$\mathcal{P}_S^A(\mathbf{p};\Delta t) = \sum_{\mathbf{k}} \left|\mathcal{A}_{A\to A}(\mathbf{p},\mathbf{k};t_i,t_f)\right|^2$$

$$= 1 - \frac{m_{AB}^4}{\omega_{\mathbf{p},A}\omega_{\mathbf{p},B}} \frac{\sin^2\left[\frac{\left(\omega_{\mathbf{p},A}-\omega_{\mathbf{p},B}\right)\Delta t}{2}\right]}{\left(\omega_{\mathbf{p},A}-\omega_{\mathbf{p},B}\right)^2} - \frac{m_{AB}^4}{\omega_{\mathbf{p},A}\omega_{\mathbf{p},B}} \frac{\sin^2\left[\frac{\left(\omega_{\mathbf{p},A}+\omega_{\mathbf{p},B}\right)\Delta t}{2}\right]}{\left(\omega_{\mathbf{p},A}+\omega_{\mathbf{p},B}\right)^2}$$

$$-V\int \frac{\mathrm{d}^3\mathbf{q}_1}{(2\pi)^3} \frac{m_{AB}^4}{\omega_{\mathbf{q}_1,A}2\omega_{\mathbf{q}_1,B}} \frac{\sin^2\left[\frac{\left(\omega_{\mathbf{q}_1,A}+\omega_{\mathbf{q}_1,B}\right)\Delta t}{2}\right]}{\left(\omega_{\mathbf{q}_1,A}+\omega_{\mathbf{q}_1,B}\right)^2},$$



Neutrino oscillations in the interaction picture

Neutrino fields

$$\nu_\sigma(x) = \frac{1}{\sqrt{V}} \sum_{\mathbf{k},r} \left[u^r_{\mathbf{k},\sigma}(t)\, \alpha^r_{\mathbf{k},\sigma} + v^r_{-\mathbf{k},\sigma}(t)\, \beta^{r\dagger}_{-\mathbf{k},\sigma} \right] e^{i\mathbf{k}\cdot\mathbf{x}}$$

with $\sigma = e, \mu$. Spinor normalization

$$u^{r\dagger}_{\mathbf{k},\rho} u^s_{\mathbf{k},\rho} = v^{r\dagger}_{\mathbf{k},\rho} v^s_{\mathbf{k},\rho} \;=\; \delta_{rs} \quad , \quad u^{r\dagger}_{\mathbf{k},\rho} v^s_{-\mathbf{k},\rho} = 0$$

Neutrino flavor state

$$|\nu^r_{\mathbf{p},\sigma}\rangle \equiv \alpha^{r\dagger}_{\mathbf{p},\sigma} |0\rangle$$

Interaction Hamiltonian

$$H_{int}(t) = m_{e\mu} \sum_{s,s'=1,2} \sum_{\mathbf{p}} \Big[\beta^{s}_{\mathbf{p},\mu} \beta^{s\dagger}_{\mathbf{p},e} \delta_{ss'} W^{*}_{\mathbf{p}}(t) + \alpha^{r\dagger}_{\mathbf{p},\mu} \alpha^{r}_{\mathbf{p},e} \delta_{ss'} W_{\mathbf{p}}(t)$$

$$+ \quad \beta^{s}_{-\mathbf{p},\mu} \alpha^{s'}_{e,\mathbf{p}} \left(Y^{ss'}_{\mathbf{p}}(t) \right)^{*} + \alpha^{s\dagger}_{\mathbf{p},\mu} \beta^{s'\dagger}_{-\mathbf{p},e} Y^{ss'}_{\mathbf{p}}(t) \, + \, e \leftrightarrow \mu \Big]$$

where

$$W_{\mathbf{p}}(t) \;=\; \overline{u}^{s}_{\mathbf{p},\mu} u^{s}_{\mathbf{p},e} e^{i(\omega_{\mathbf{k},\mu} - \omega_{\mathbf{k},e})t} \;=\; W_{\mathbf{p}} \, e^{i(\omega_{\mathbf{p},\mu} - \omega_{\mathbf{p},e})t}$$

$$Y^{ss'}_{\mathbf{p}}(t) \;=\; \overline{u}^{s}_{\mathbf{p},\mu} v^{s'}_{-\mathbf{p},e} e^{i(\omega_{\mathbf{k},\mu} + \omega_{\mathbf{k},e})t} \;=\; Y^{ss'}_{\mathbf{p}} \, e^{i(\omega_{\mathbf{p},\mu} + \omega_{\mathbf{p},e})t}$$

Explicit form of coefficients:

$$W_{\mathbf{p}} = \sqrt{\frac{(\omega_{\mathbf{p},e}+m_e)(\omega_{\mathbf{p},\mu}+m_\mu)}{4\omega_{\mathbf{p},e}\omega_{\mathbf{p},\mu}}}\left(1-\frac{|\mathbf{p}|^2}{(\omega_{\mathbf{p},e}+m_e)(\omega_{\mathbf{p},\mu}+m_\mu)}\right)$$

$$Y^{22}_{\mathbf{p}} = -Y^{11}_{\mathbf{p}} = \frac{p_3}{\sqrt{4\omega_{\mathbf{p},e}\omega_{\mathbf{p},\mu}}}\left(\sqrt{\frac{\omega_{\mathbf{p},\mu}+m_\mu}{\omega_{\mathbf{p},e}+m_e}}+\sqrt{\frac{\omega_{\mathbf{p},e}+m_e}{\omega_{\mathbf{p},\mu}+m_\mu}}\right)$$

$$Y^{12}_{\mathbf{p}} = \left(Y^{21}_{\mathbf{p}}\right)^* = -\frac{p_1-ip_2}{\sqrt{4\omega_{\mathbf{p},e}\omega_{\mathbf{p},\mu}}}\left(\sqrt{\frac{\omega_{\mathbf{p},\mu}+m_\mu}{\omega_{\mathbf{p},e}+m_e}}+\sqrt{\frac{\omega_{\mathbf{p},e}+m_e}{\omega_{\mathbf{p},\mu}+m_\mu}}\right)$$

Decay probability

- Amplitude of the $|\nu^r_{\mathbf{p},e}\rangle \;\rightarrow\; |\nu^s_{\mathbf{k},\mu}\rangle$ process

$$\mathcal{A}^{rs}_{e\rightarrow\mu}(\mathbf{p},\mathbf{k},;t_i,t_f) \;=\; \delta_{rs}\delta_{\mathbf{k},\mathbf{p}}\frac{m_{e\mu}\,W_{\mathbf{p}}}{\omega_{\mathbf{p},e}-\omega_{\mathbf{p},\mu}}\;\left(e^{i(\omega_{\mathbf{p},\mu}-\omega_{\mathbf{p},e})t_f}-e^{i(\omega_{\mathbf{p},\mu}-\omega_{\mathbf{p},e})t_i}\right)$$

Probability

$$\mathcal{P}_{e\rightarrow\mu}(\mathbf{p};\Delta t) \;=\; \sum_{\mathbf{k},s}|\mathcal{A}^{rs}_{e\rightarrow\mu}(\mathbf{p},\mathbf{k};t_i,t_f)|^2$$

Explicitly

$$\mathcal{P}_{e\rightarrow\mu}(\mathbf{p};\Delta t) \;=\; W^2_{\mathbf{p}}\,\frac{4m^2_{e\mu}}{(\omega_{\mathbf{p},e}-\omega_{\mathbf{p},\mu})^2}\sin^2\left[\frac{(\omega_{\mathbf{p},\mu}-\omega_{\mathbf{p},e})\,\Delta t}{2}\right]$$

Decay probability

- Consider the process

$$|\nu^r_{\mathbf{p},e}\rangle \;\to\; |\nu^{s_1}_{\mathbf{k}_1,e}\rangle|\nu^{s_2}_{\mathbf{k}_2,e}\rangle|\overline{\nu}^{s_3}_{\mathbf{k}_3,\mu}\rangle\,, \qquad \mathbf{k}_1 \neq \mathbf{k}_2 \;\vee\; s_1 \neq s_2\,.$$

Probability (After non-trivial subtractions!)

$$\mathcal{P}_{e\to ee\overline{\mu}}(\mathbf{p};\Delta t) \;=\; \frac{4m^2_{e\mu}Y^2_{\mathbf{p}}}{\left(\omega_{\mathbf{p},e}+\omega_{\mathbf{p},\mu}\right)^2}\sin^2\left(\frac{\left(\omega_{\mathbf{p},\mu}+\omega_{\mathbf{p},e}\right)\Delta t}{2}\right)$$

where

$$Y^2_{\mathbf{p}} \;=\; \sum_{\mathbf{s}}\left(Y^{rs}_{\mathbf{p}}\right)^* Y^{rs}_{\mathbf{p}}$$

Neutrino oscillation formula

Total flavor transition probability

$$\mathcal{P}^e_D(\mathbf{p};\Delta t) \;=\; 4m^2_{e\mu}\left[\frac{W^2_{\mathbf{p}}}{\left(\omega^-_{\mathbf{p}}\right)^2}\sin^2\left(\frac{\omega^-_{\mathbf{p}}\Delta t}{2}\right)+\frac{Y^2_{\mathbf{p}}}{\left(\omega^+_{\mathbf{p}}\right)^2}\sin^2\left(\frac{\omega^+_{\mathbf{p}}\Delta t}{2}\right)\right]$$

with $\omega^\pm_{\mathbf{p}} \equiv \omega_{\mathbf{p},e} \pm \omega_{\mathbf{p},\mu}$. Note that

$$|U_{\mathbf{p}}| \;=\; W_{\mathbf{p}}\frac{m_\mu - m_e}{\omega^-_{\mathbf{p}}}\,, \qquad |V_{\mathbf{p}}| \;=\; Y_{\mathbf{p}}\frac{m_\mu - m_e}{\omega^+_{\mathbf{p}}}$$

when $m_1 \approx m_e$, $m_2 \approx m_\mu$. Then

$$\mathcal{P}^e_D(\mathbf{p};\Delta t) \;=\; \sin^2(2\theta)\left[|U_{\mathbf{p}}|^2\sin^2\left(\frac{\omega^-_{\mathbf{p}}\Delta t}{2}\right)+|V_{\mathbf{p}}|^2\sin^2\left(\frac{\omega^+_{\mathbf{p}}\Delta t}{2}\right)\right]$$

with $\theta = m_{e\mu}/(m_\mu - m_e) \approx \sin\theta$. Oscillation formula of the flavor Fock-space approach!!

Survival probability

- The amplitude of $|\nu^r_{\mathbf{p},e}\rangle \;\rightarrow\; |\nu^s_{\mathbf{k},e}\rangle$ is decomposed as

$$\mathcal{A}^{rs}_{e\rightarrow e}(\mathbf{p},\mathbf{k};t_i,t_f) = \delta_{\mathbf{k},\mathbf{p}}\delta_{rs} + \frac{1}{2}\mathcal{A}^{(2)rs}_{e\rightarrow e}(\mathbf{p},\mathbf{k};t_i,t_f)$$

Probability

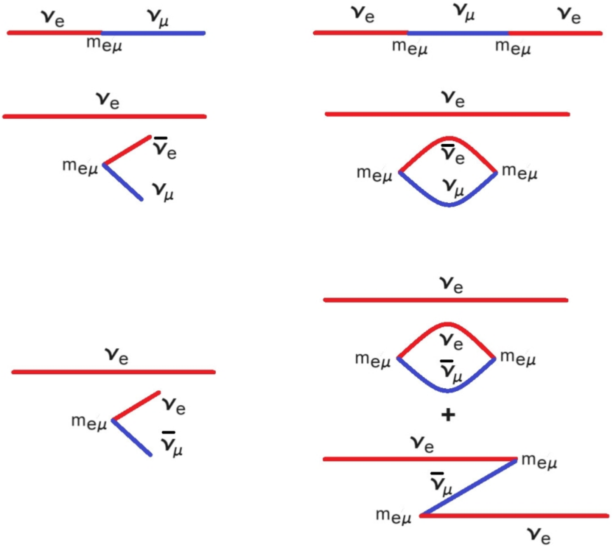
$$\mathcal{P}^e_S(\mathbf{p};\Delta t) = \sum_{\mathbf{k},s}\mathcal{A}^{rs}_{e\rightarrow e}(\mathbf{p},\mathbf{k},t_i,t_f) \approx 1 + 2\,\Re e\left(\tilde{\mathcal{A}}^{(2)}_{e\rightarrow e}(\mathbf{p};t_i,t_f)\right)$$

with

$$\tilde{\mathcal{A}}^{(2)}_{e\rightarrow e}(\mathbf{p};t_i,t_f) \;\equiv\; \sum_{\mathbf{k},s} A^{(2)rs}_{e\rightarrow e}(\mathbf{p},\mathbf{k};t_i,t_f)$$

$$\mathcal{P}^e_D(\mathbf{p};\Delta t) + \mathcal{P}^e_S(\mathbf{p};\Delta t) \;=\; 1.$$

Diagrams for neutrino oscillations



Conclusions and perspectives

• The interaction picture approach* matches results of the flavor Fock space approach, at the lowest order in $m_{e\mu}$

• It should be possible to sum up the perturbative series and recover the flavor space (nonperturbative) result.

• Chiral oscillations should be also accommodated in this scheme.

*M.B., F.Giacosa, L.Smaldone and G.Torrieri, EPJC (2023)

Chiral oscillations

Chiral oscillations

- Taking into account (bi)spinorial nature of neutrinos and chiral nature of weak interaction, one naturally gets chiral oscillations [*]
- Interplay with flavor oscillations in the non-relativistic region[†]
- For CνB, chiral oscillations reduce detection by a factor of 2.[‡]
- Application: lepton-antineutrino entanglement and chiral oscillations in pion decay.[§]

[*] A. Bernardini and S. De Leo, Phys. Rev. D (2005)

[†] V.A.Bittencourt, A.Bernardini and M.B.,Eur.Phys.J.C(2021);EPL Persp.(2022); M. W. Li, Z. L. Huang and X. G. He, Phys. Lett. B (2024); K. Kimura and A. Takamura, arXiv:2101.03555 [hep-ph].

[‡] S.-F. Ge and P.Pasquini, Phys. Lett. B (2020)

[§] V.A.Bittencourt, A.Bernardini and M.B., Universe (2021)

Chiral oscillations

Chiral representation of the Dirac matrices

$$\alpha_i = \begin{bmatrix} \sigma_i & 0 \\ 0 & -\sigma_i \end{bmatrix}, \qquad \beta = \begin{bmatrix} 0 & I_2 \\ I_2 & 0 \end{bmatrix},$$

and $\gamma_5 = (I_2, -I_2)$. Any bispinor $|\xi\rangle$ can be written in this representation as

$$|\xi\rangle = \begin{bmatrix} |\xi_R\rangle \\ |\xi_L\rangle \end{bmatrix},$$

The Dirac equation $H_D\,|\xi\rangle = i|\dot{\xi}\rangle$ can then be written as

$$i\partial_t\,|\xi_R\rangle - \mathbf{p}\cdot\sigma\,|\xi_R\rangle = m\,|\xi_L\rangle\,,$$
$$i\partial_t\,|\xi_L\rangle + \mathbf{p}\cdot\sigma\,|\xi_L\rangle = m\,|\xi_R\rangle\,,$$

• Evolution under the free Dirac Hamiltonian $\hat{H}_D$ induces left-right chiral oscillations.

Take initial state $|\psi(0)\rangle = [0,\ 0,\ 0,\ 1]^T$ which has negative helicity and negative chirality: $\hat{\gamma}_5 \, |\psi(0)\rangle = -\, |\psi(0)\rangle$.

The time evolved state $|\psi_m(t)\rangle = e^{-i\hat{H}_D t} \, |\psi(0)\rangle$ is given by

$$
\begin{aligned}
|\psi_m(t)\rangle \quad = \quad & \sqrt{\frac{E_{p,m}+m}{4E_{p,m}}} \left[\left(1 + \frac{p}{E_{p,m}+m}\right) e^{-iE_{p,m}t} \, |u_-(p,m)\rangle \right. \\
& \left. - \left(1 - \frac{p}{E_{p,m}+m}\right) e^{iE_{p,m}t} \, |v_-(-p,m)\rangle \right],
\end{aligned}
$$

with (for one-dimensional propagation along the $\mathbf{e}_z$ direction)

$$
|u_\pm(p,m)\rangle = \sqrt{\frac{E_{p,m}+m}{4E_{p,m}}} \begin{bmatrix} \left(1 \pm \frac{p}{E_{p,m}+m}\right) |\pm\rangle \\ \left(1 \mp \frac{p}{E_{p,m}+m}\right) |\pm\rangle \end{bmatrix},
$$

$$
|v_\pm(p,m)\rangle = \sqrt{\frac{E_{p,m}+m}{4E_{p,m}}} \begin{bmatrix} \left(1 \pm \frac{p}{E_{p,m}+m}\right) |\pm\rangle \\ -\left(1 \mp \frac{p}{E_{p,m}+m}\right) |\pm\rangle \end{bmatrix},
$$

• Survival probability of initial left-handed state

$$\mathcal{P}(t) = |\,\langle\psi_m(0)|\psi_m(t)\rangle\,|^2 = 1 - \frac{m^2}{E_{p,m}^2}\sin^2\left(E_{p,m}t\right),$$

Average value of the chiral operator $\langle\hat{\gamma}_5\rangle(t)$

$$\langle\hat{\gamma}_5\rangle(t) = \langle\psi_m(t)|\hat{\gamma}_5\,|\psi_m(t)\rangle = -1 + \frac{2m^2}{E_{p,m}^2}\sin^2\left(E_{p,m}t\right).$$

– Chiral oscillation period: $T_{ch} = \frac{2\pi}{E_{p,m}}$

– Chiral oscillation length: $L_{ch} = v\,\frac{2\pi}{E_{p,m}} = \frac{2\pi p}{E_{p,m}^2}$

Chiral and flavor oscillations

- State of a neutrino of flavor α at a given t:

$$|\nu_\alpha(t)\rangle = \sum_i U_{\alpha,i} \, |\psi_{m_i}(t)\rangle \otimes |\nu_i\rangle \, ,$$

where $|\psi_{m_i}(t)\rangle$ are bispinors.

- The state at $t = 0$ reads

$$|\nu_\alpha(0)\rangle = |\psi(0)\rangle \otimes \sum_i U_{\alpha,i} \, |\nu_i\rangle = |\psi(0)\rangle \otimes |\nu_\alpha\rangle \, ,$$

where $|\psi(0)\rangle$ is a left handed bispinor.

- Survival probability:

$$\mathcal{P}_{\alpha\to\alpha} = |\langle\nu_\alpha(0)|\nu_\alpha(t)\rangle|^2 = \left|\sum_i |U_{\alpha,i}|^2 \, \langle\psi(0)|\psi_{m_i}(t)\rangle\right|^2 .$$

Two flavor mixing:

$$|\nu_e(t)\rangle = \left[\cos^2\theta\,|\psi_{m_1}(t)\rangle + \sin^2\theta\,|\psi_{m_2}(t)\rangle\right] \otimes |\nu_e\rangle$$
$$+ \sin\theta\cos\theta\left[|\psi_{m_1}(t)\rangle - |\psi_{m_2}(t)\rangle\right] \otimes |\nu_\mu\rangle\,,$$

- The survival probability can be decomposed as

$$\mathcal{P}_{e\to e}(t) = \mathcal{P}^S_{e\to e}(t) + \mathcal{A}_e(t) + \mathcal{B}_e(t).$$

$\mathcal{P}^S_{e\to e}(t)$ is the standard flavor oscillation formula

$$\mathcal{P}^S_{e\to e}(t) = 1 - \sin^2 2\theta \sin^2\left(\frac{E_{p,m_2} - E_{p,m_1}}{2}t\right)$$

and

$$\mathcal{A}_e(t) = -\left[\frac{m_1}{E_{p,m_1}}\cos^2\theta\sin\left(E_{p,m_1}t\right) + \frac{m_2}{E_{p,m_2}}\sin^2\theta\sin\left(E_{p,m_2}t\right)\right]^2,$$

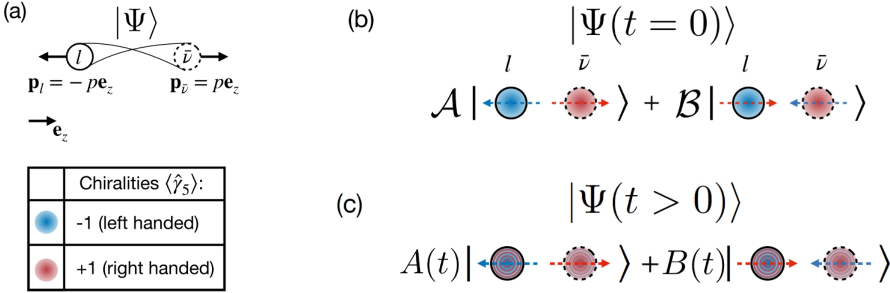
$$\mathcal{B}_e(t) = \frac{1}{2}\sin^2 2\theta\sin(E_{p,m_1}t)\sin(E_{p,m_2}t)\left(\frac{p^2 + m_1 m_2}{E_{p,m_1}E_{p,m_2}} - 1\right),$$

are correction terms due to the bispinorial structure.

- Agreement with the QFT formula.

Lepton-antineutrino entanglement and chiral oscillations[*]

• As an application of chiral oscillations, we consider induced spin correlations in pion decay products ($\pi \to l + \bar{\nu}$)



[*]V.A.Bittencourt, A.Bernardini and M.B., Universe (2021)
V.A.Bittencourt, M.B. and G.Zanfardino, arXiv:2308.14574

Quantum field theory of chiral oscillations[†]

- Dirac Lagrangian density

$$\mathcal{L} = \overline{\psi}\,(i\gamma^\mu\partial_\mu - m)\,\psi$$

Invariance under global phase transformations ⇒ conserved charge

$$Q = \int d^3\mathbf{x}\,\psi^\dagger(x)\psi(x)$$

Dirac field ψ can be split as $\psi = \psi_L + \psi_R$ where

$$\psi_L \equiv P_L\psi(x) = \frac{1-\gamma^5}{2}\psi(x), \qquad \psi_R \equiv P_R\psi(x) = \frac{1+\gamma^5}{2}\psi(x)$$

and hence Dirac Lagrangian can be written as

$$\mathcal{L} = \overline{\psi_L}\,i\gamma^\mu\,\partial_\mu\,\psi_L + \overline{\psi_R}\,i\gamma^\mu\partial_\mu\psi_R - m\left(\overline{\psi_L}\,\psi_R + \overline{\psi_R}\,\psi_L\right)$$

[†]V.Bittencourt, M.B. and G.Zanfardino, arXiv:2408.16742[hep-ph]

• Chiral symmetry is explicitly broken by the Dirac mass term.

• Separate global phase transformations for ψ_L and ψ_R lead to the non-conserved chiral charges

$$Q_L(t) = \int d^3\mathbf{x}\, \psi_L^\dagger(x)\psi_L(x), \qquad Q_R(t) = \int d^3\mathbf{x}\, \psi_R^\dagger(x)\psi_R(x).$$

• The total (conserved) charge is equal to the sum of the (time dependent) chiral charges

$$Q \,=\, Q_L(t) \,+\, Q_R(t).$$

Quantization of Dirac field

$$\{\psi_\alpha(\mathbf{x},t),\psi^\dagger_\beta(\mathbf{y},t)\} = \delta_{\alpha,\beta}\delta^3(\mathbf{x}-\mathbf{y}),$$
$$\{\psi_\alpha(\mathbf{x},t),\psi_\beta(\mathbf{y},t)\} = \{\psi^\dagger_\alpha(\mathbf{x},t),\psi^\dagger_\beta(\mathbf{y},t)\} = 0$$

The expansion of the field in terms of creation and annihilation is

$$\psi(x) = \sum_{r=1,2}\int \frac{d^3\mathbf{k}}{(2\pi)^3}\Big[u^r_{\mathbf{k}}\,\alpha^r_{\mathbf{k}}\,e^{-i\omega_k t} \,+\, v^r_{-\mathbf{k}}\,\beta^{r\dagger}_{-\mathbf{k}}\,e^{i\omega_k t}\Big]e^{i\mathbf{k}\cdot\mathbf{x}}$$

with $\omega_k = \sqrt{k^2+m^2}$ and

$$\Big\{\alpha^r_{\mathbf{k}},\alpha^{s\dagger}_{\mathbf{p}}\Big\} = \delta^3(\mathbf{k}-\mathbf{p})\delta_{rs}\,; \qquad \Big\{\beta^r_{\mathbf{k}},\beta^{s\dagger}_{\mathbf{p}}\Big\} = \delta^3(\mathbf{k}-\mathbf{p})\delta_{rs}$$

and other anticommutators vanishing. We denote by $|0\rangle$ the vacuum state annihilated by the above operators for the Dirac field:

$$\alpha^r_{\mathbf{k}}\,|0\rangle \,=\, \beta^r_{\mathbf{k}}\,|0\rangle \,=\, 0.$$

In terms of ladder operators, we have

$$Q = \sum_r \int d^3\mathbf{k} \left(\alpha^{r\dagger}_{\mathbf{k}} \alpha^r_{\mathbf{k}} - \beta^{r\dagger}_{-\mathbf{k}} \beta^r_{-\mathbf{k}} \right)$$

If we now consider the chiral charges, we find:

$$Q_{L/R}(t) \;\; = \;\; \frac{1}{2} \left(Q \mp \int d^3\mathbf{x}\, \psi^\dagger(x) \gamma^5 \psi(x) \right)$$

Time dependence comes from the second piece in the above equation, which is indeed non-diagonal in the ladder operators:

$$\int d^3\mathbf{x}\, \psi^\dagger(x)\gamma^5\psi(x) \;\; = \;\; \sum_{r=1,2} \int d^3\mathbf{k} \left[(u^{r\dagger}_{\mathbf{k}}\, \gamma^5\, u^s_{\mathbf{k}})\, \alpha^{r\dagger}_{\mathbf{k}} \alpha^r_{\mathbf{k}} + \; (v^{r\dagger}_{-\mathbf{k}}\, \gamma^5\, v^s_{-\mathbf{k}})\, \beta^r_{-\mathbf{k}} \beta^{r\dagger}_{-\mathbf{k}} \right.$$

$$\left. + \;\; e^{-2i\omega_k t}\, (v^{r\dagger}_{-\mathbf{k}}\, \gamma^5\, u^s_{\mathbf{k}})\, \beta^r_{-\mathbf{k}} \alpha^r_{\mathbf{k}} + e^{2i\omega_k t}\, (u^{r\dagger}_{k}\, \gamma^5\, v^s_{-\mathbf{k}})\, \alpha^{r\dagger}_{\mathbf{k}} \beta^{r\dagger}_{-\mathbf{k}} \right].$$

Explicit form for Q_L (similar expression for Q_R):

$$
\begin{aligned}
Q_L \;=\; & \frac{1}{2}\sum_{r=1,2}\int d^3\mathbf{k}\left[\left(1+\epsilon^r\frac{|\mathbf{k}|}{\omega_k}\right)\alpha^{r\dagger}_{\mathbf{k}}\alpha^r_{\mathbf{k}} - \left(1-\epsilon^r\frac{|\mathbf{k}|}{\omega_k}\right)\beta^{r\dagger}_{-\mathbf{k}}\beta^r_{-\mathbf{k}}\right. \\
& \left. - \frac{m}{\omega_k}\left(e^{-2i\omega_k t}\,\beta^r_{-\mathbf{k}}\alpha^r_{\mathbf{k}} + e^{2i\omega_k t}\,\alpha^{r\dagger}_{\mathbf{k}}\beta^{r\dagger}_{-\mathbf{k}}\right)\right]
\end{aligned}
$$

with $\epsilon^r = (-1)^r$.

In the relativistic limit $\omega_k \gg m$:

$$
\begin{aligned}
Q_L(t)|_{m=0} &= \int d^3\mathbf{k}\left(\alpha^{2\dagger}_{\mathbf{k}}\alpha^2_{\mathbf{k}} - \beta^{1\dagger}_{-\mathbf{k}}\beta^1_{-\mathbf{k}}\right) \\
Q_R(t)|_{m=0} &= \int d^3\mathbf{k}\left(\alpha^{1\dagger}_{\mathbf{k}}\alpha^1_{\mathbf{k}} - \beta^{2\dagger}_{-\mathbf{k}}\beta^2_{-\mathbf{k}}\right)
\end{aligned}
$$

conserved (Noether) charges for the Weyl fields ψ_L and ψ_R.

Diagonalization of chiral charges

- Introduce the following canonical (Bogoliubov) transformation:

$$\begin{aligned}
\alpha_{\mathbf{k},L} &= \cos\theta_k\,\alpha^2_{\mathbf{k}} - e^{i\phi_k}\,\sin\theta_k\,\beta^{2\dagger}_{-\mathbf{k}} \\
\beta^\dagger_{-\mathbf{k},L} &= \cos\theta_k\,\beta^{1\dagger}_{-\mathbf{k}} - e^{-i\phi_k}\,\sin\theta_k\,\alpha^1_{\mathbf{k}} \\
\alpha_{\mathbf{k},R} &= \cos\theta_k\,\alpha^1_{\mathbf{k}} + e^{i\phi_k}\,\sin\theta_k\,\beta^{1\dagger}_{-\mathbf{k}} \\
\beta^\dagger_{-\mathbf{k},R} &= \cos\theta_k\,\beta^{2\dagger}_{-\mathbf{k}} + e^{-i\phi_k}\,\sin\theta_k\,\alpha^1_{\mathbf{k}}
\end{aligned}$$

- Condition for diagonalization

$$\begin{aligned}
\cos^2\theta_k &= \frac{1}{2}\left(1+\frac{|\mathbf{k}|}{\omega_k}\right), \quad \sin^2\theta_k = \frac{1}{2}\left(1-\frac{|\mathbf{k}|}{\omega_k}\right), \\
\cos 2\theta_k &= \frac{|\mathbf{k}|}{\omega_k}, \quad \sin 2\theta_k = -\frac{m}{\omega_k}, \quad \phi_k = 2\omega_k t.
\end{aligned}$$

Thus the above defined chiral ladder operators are time-dependent and satisfy (equal time) canonical anticommutation relations (CAR):

$$\left\{\alpha^r_{\mathbf{k},L}(t), \alpha^{s\dagger}_{\mathbf{k},L}(t)\right\} = \delta^3(\mathbf{k}-\mathbf{p})\delta_{rs}\,, \qquad \left\{\beta^r_{\mathbf{k},L}(t), \beta^{s\dagger}_{\mathbf{k},L}(t)\right\} = \delta^3(\mathbf{k}-\mathbf{p})\delta_{rs}$$

- Chiral charges are diagonal in the new operators

$$\begin{aligned} Q_L(t) &= \int d^3\mathbf{k}\, \left(\alpha^\dagger_{\mathbf{k},L}(t)\alpha_{\mathbf{k},L}(t) - \beta^\dagger_{-\mathbf{k},L}(t)\beta_{-\mathbf{k},L}(t)\right), \\ Q_R(t) &= \int d^3\mathbf{k}\, \left(\alpha^\dagger_{\mathbf{k},R}(t)\alpha_{\mathbf{k},R}(t) - \beta^\dagger_{-\mathbf{k},R}(t)\beta_{-\mathbf{k},R}(t)\right). \end{aligned}$$

Dirac field expansion:

$$
\begin{aligned}
\psi(x) &= \int \frac{d^3\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} \Big[u^1_{\mathbf{k}} \left(\cos\theta_k\, \alpha_{\mathbf{k},R} - \; e^{i\phi_k}\, \sin\theta_k\, \beta^\dagger_{-\mathbf{k},L} \right) e^{-i\omega_k t} \\
&+ u^2_{\mathbf{k}} \left(\cos\theta_k\, \alpha_{\mathbf{k},L} + \; e^{i\phi_k}\, \sin\theta_k\, \beta^\dagger_{-\mathbf{k},R} \right) e^{-i\omega_k t} \\
&+ v^1_{-\mathbf{k}} \left(\cos\theta_k\, \beta^\dagger_{-\mathbf{k},L} + \; e^{-i\phi_k}\, \sin\theta_k\, \alpha_{\mathbf{k},R} \right) e^{i\omega_k t} \\
&+ v^2_{-\mathbf{k}} \left(\cos\theta_k\, \beta^\dagger_{-\mathbf{k},R} - \; e^{-i\phi_k}\, \sin\theta_k\, \alpha_{\mathbf{k},L} \right) e^{i\omega_k t} \Big]
\end{aligned}
$$

can be rearranged in the following form (using $\phi_k = 2\omega_k t$)

$$
\begin{aligned}
\psi(x) &= \int \frac{d^3\mathbf{k}}{(2\pi)^3} \Big[u_{\mathbf{k},L}\, \alpha_{\mathbf{k},L}(t)\, e^{-i\omega_k t} + \; v_{-\mathbf{k},L}\, \beta^\dagger_{-\mathbf{k},L}(t)\, e^{i\omega_k t} \Big] e^{i\mathbf{k}\cdot\mathbf{x}} \\
&+ \int \frac{d^3\mathbf{k}}{(2\pi)^3} \Big[u_{\mathbf{k},R}\, \alpha_{\mathbf{k},R}(t)\, e^{-i\omega_k t} + \; v_{-\mathbf{k},R}\, \beta^\dagger_{-\mathbf{k},R}(t)\, e^{i\omega_k t} \Big] e^{i\mathbf{k}\cdot\mathbf{x}} \\
&= \psi_L(x) + \psi_R(x)
\end{aligned}
$$

with

$$
\begin{aligned}
u_{\mathbf{k},L} &\equiv \cos\theta_k\, u^2_{\mathbf{k}} - \sin\theta_k\, v^2_{-\mathbf{k}}, \quad u_{\mathbf{k},R} \equiv \cos\theta_k\, u^1_{\mathbf{k}} + \sin\theta_k\, v^1_{-\mathbf{k}} \\
v_{-\mathbf{k},L} &\equiv \cos\theta_k\, v^1_{-\mathbf{k}} - \sin\theta_k\, u^1_{\mathbf{k}}, \quad v_{-\mathbf{k},R} \equiv \cos\theta_k\, v^2_{-\mathbf{k}} + \sin\theta_k\, u^2_{\mathbf{k}}
\end{aligned}
$$

$$
\begin{aligned}
u^\dagger_{\mathbf{k},L} u_{\mathbf{k},L} = u^\dagger_{\mathbf{k},R} u_{\mathbf{k},R} = 1\,, \quad v^\dagger_{-\mathbf{k},L} v_{-\mathbf{k},L} = v^\dagger_{-\mathbf{k},R} v_{-\mathbf{k},R} = 1 \\
u^\dagger_{\mathbf{k},L} u_{\mathbf{k},R} = v^\dagger_{-\mathbf{k},L} v_{-\mathbf{k},R} = 0\,, \quad u^\dagger_{\mathbf{k},L} v_{-\mathbf{k},L} = u^\dagger_{\mathbf{k},R} v_{-\mathbf{k},L} = 0
\end{aligned}
$$

and the completeness relation:

$$
u_{\mathbf{k},R}\, u^\dagger_{\mathbf{k},R} + u_{\mathbf{k},L}\, u^\dagger_{\mathbf{k},L} + v_{-\mathbf{k},R}\, v^\dagger_{-\mathbf{k},R} + v_{-\mathbf{k},L}\, v^\dagger_{-\mathbf{k},L} = 1\!\!1
$$

Consistency relations:

$$
\begin{aligned}
P_L\, u_{\mathbf{k},L} &= u_{\mathbf{k},L}\,, \qquad P_L\, v_{-\mathbf{k},L} = v_{-\mathbf{k},L} \\
P_R\, u_{\mathbf{k},R} &= u_{\mathbf{k},R}\,, \qquad P_R\, v_{-\mathbf{k},R} = v_{-\mathbf{k},R} \\
P_R\, u_{\mathbf{k},L} &= P_R\, v_{-\mathbf{k},L} = P_L\, u_{\mathbf{k},R} = P_L\, v_{-\mathbf{k},R} = 0
\end{aligned}
$$

The Bogoliubov is written as

$$\alpha_{\mathbf{k},L} = G_t^{-1}\alpha^2_{\mathbf{k}}G_t \quad , \quad \beta_{\mathbf{k},L} = G_t^{-1}\beta^1_{\mathbf{k}}G_t$$

$$\alpha_{\mathbf{k},R} = G_t^{-1}\,\alpha^1_{\mathbf{k}}\,G_t \quad , \quad \beta_{\mathbf{k},R} = G_t^{-1}\,\beta^2_{\mathbf{k}}\,G_t$$

with generator

$$G_t(\theta,\phi) = \exp\left[\sum_r \int d^3\mathbf{k}\,\theta_k\epsilon^r \left(e^{-i\phi_k}\alpha^r_{\mathbf{k}}\beta^r_{-\mathbf{k}} - e^{i\phi_k}\beta^{r\dagger}_{-\mathbf{k}}\alpha^{r\dagger}_{\mathbf{k}}\right)\right]$$

- Explicit form for the *chiral vacuum*:

$$|\tilde{0}(t)\rangle_{LR} = \prod_{\mathbf{k},r}\left[\cos\theta_k + \epsilon^r e^{i\phi_k}\sin\theta_k\alpha^{r\dagger}_{\mathbf{k}}\beta^{r\dagger}_{-\mathbf{k}}\right]|0\rangle$$

- The chiral vacuum $|\tilde{0}(t)\rangle_{LR}$ and the Dirac vacuum $|0\rangle$ are orthogonal in the infinite volume limit:

$$\lim_{V\to\infty}\langle 0|\tilde{0}(t)\rangle_{LR} = 0,$$

generating *unitarily inequivalent representations* of the field algebra.

Chiral oscillation formula

Define the state $|\alpha_L\rangle \equiv \alpha^{\dagger}_{\mathbf{k},L}|\tilde{0}\rangle_{LR}$, with $|\tilde{0}\rangle_{LR} \equiv |\tilde{0}(0)\rangle_{LR}$.

Left chiral operator at time t

$$\alpha_{\mathbf{k},L}(t) = \cos\theta_k \, e^{-i\omega_k t} \, \alpha^2_{\mathbf{k}} \, - \, \sin\theta_k \, e^{i\omega_k t} \, \beta^{2\dagger}_{-\mathbf{k}}$$

- Chiral oscillation formula

$$\langle \alpha_{\mathbf{k},L}|Q_L(t)|\alpha_{\mathbf{k},L}\rangle = |\{\alpha_{\mathbf{k},L}(t), \alpha^{\dagger}_{\mathbf{k},L}(0)\}|^2$$

with

$$\{\alpha_{\mathbf{k},L}(t), \alpha^{\dagger}_{\mathbf{k},L}(0)\} = \cos^2\theta_k e^{-i\omega_k t} + \sin^2\theta_k e^{i\omega_k t}$$

We obtain

$$\langle \alpha_{\mathbf{k},L}|Q_L(t)|\alpha_{\mathbf{k},L}\rangle = 1 - \sin^2(2\theta_k)\sin^2(\omega_k t) = 1 - \frac{m^2}{\omega_k^2}\sin^2(\omega_k t)$$

Conclusions and perspectives

- Consistent treatment of oscillating particles in QFT
- Unified approach for flavor and chiral oscillations
- Weak interactions appear to be non trivial at representation (particle) level.