Theoretical upper bounds on the DM-e^- scattering rate in the generalized susceptibility formalism

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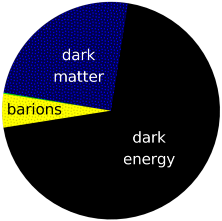
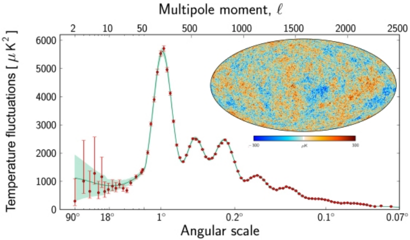
L'Aquila, 9 July 2024

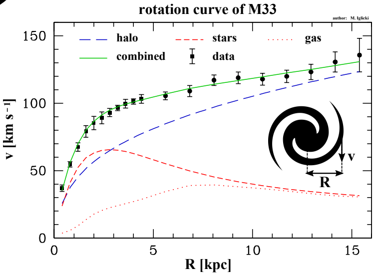
Dark matter

https://apod.nasa.gov/apod

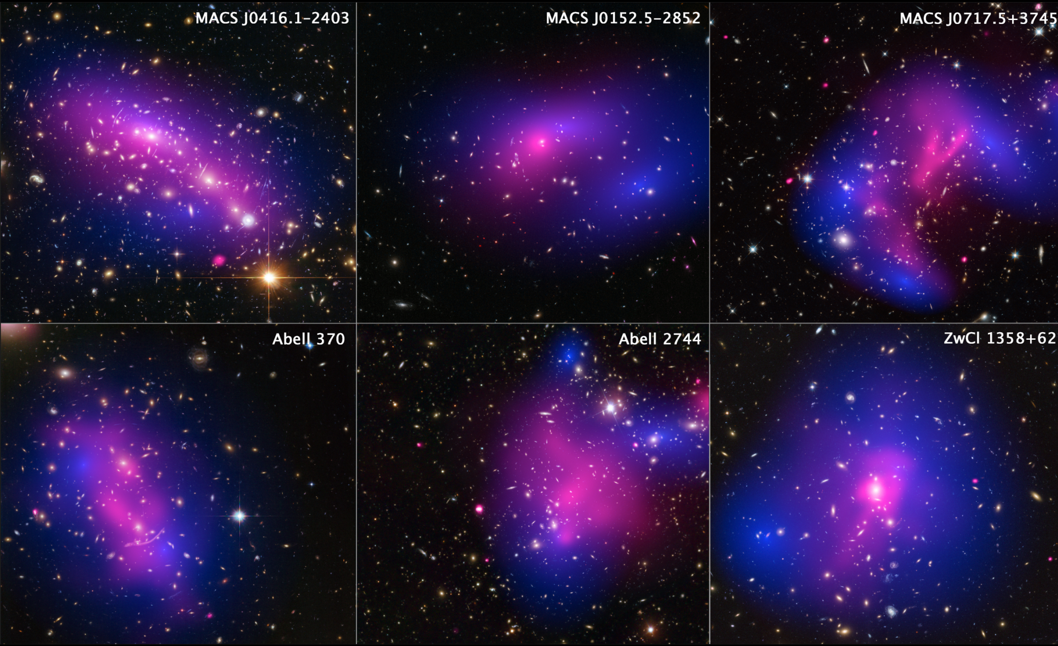




http://sci.esa.int/planck ,
https://wiki.cosmos.esa.int/planck-legacy-archive

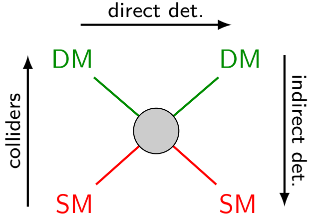


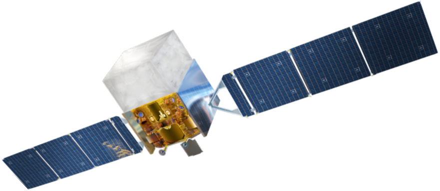
data from: arXiv:astro-ph/9909252



https://hubblesite.org

How to detect particle dark matter?





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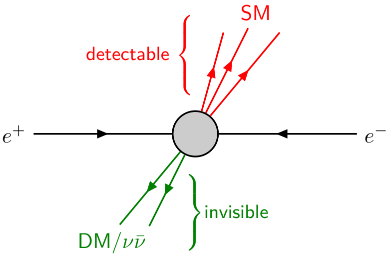
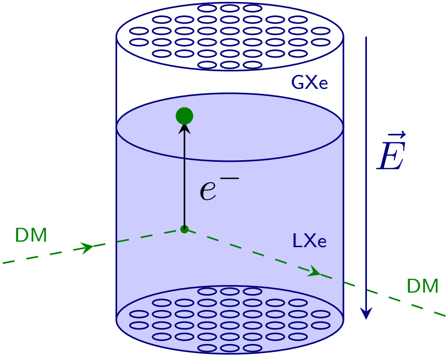
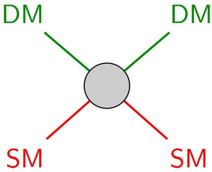




Direct detection experiments



- GeV+ range of masses (WIMPs):
  no success so far
  $\Rightarrow$ sub-GeV DM?

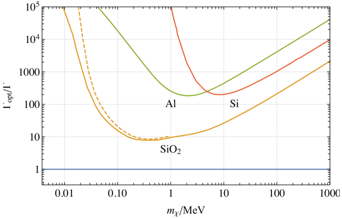
- nuclear vs. electronic recoil of non-relativistic DM

$$\Delta E_{\text{SM}} \leq \frac{4\,\mu}{(1+\mu)^2}\, E_{\text{DM}}^{\text{in}} \qquad \leftarrow \text{ maximized for } \mu \equiv m_{\text{SM}}/m_{\text{DM}} = 1$$

$\Rightarrow$ m_{SM} should be as close to m_{DM} as possible!

$\Rightarrow$ **electrons** preferable for light DM

- what material to use?



Lasenby & Prabhu, 2110.01587

Outline

- effective approach to non-relativistic DM-e^- interactions
- linear response theory

$$[\text{interaction rate}] = \int [\text{DM model}] \times [\text{material properties of the detector}]$$

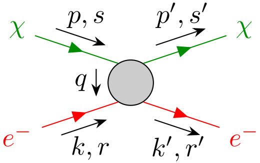
- generalized susceptibilities $\chi_{a^\dagger b}(\omega, q)$
- Kramers-Kronig relations $\Rightarrow$ theoretical upper bound on the interaction rate
- application of Kramers-Kronig relations to generalized susceptibilities calculated within the linear response theory **[new!]**

Effective non-relativistic theory for spin-1/2 DM

Catena et al., 2105.02233

$$\boldsymbol{v}^{\perp} \equiv \frac{\boldsymbol{p}+\boldsymbol{p}'}{2\,m_\chi} - \frac{\boldsymbol{k}+\boldsymbol{k}'}{2\,m_e}$$

$$\boldsymbol{q}\cdot\boldsymbol{v}^{\perp} \xrightarrow{\text{en. cons.}} 0$$



Assumptions:

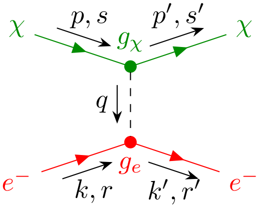
- non-relativistic limit
- Lorentz (Galilean) invariance

$$\Rightarrow \quad \mathcal{M} = \sum_i c_i\,\mathcal{O}_i \qquad i = 1, \cancel{2}, 3, 4, \ldots, 15$$

Possible operators for a spin-1/2 dark particle

Catena et al., 2105.02233

$$\mathcal{M} = \sum_i c_i\,\mathcal{O}_i \qquad i = 1, \cancel{2}, 3, 4, \dots, 15$$



- simple example: scalar coupling

$$\mathcal{M} = g_\chi\,\bar{u}_\chi^{s'}(p')\,u_\chi^{s}(p)\,\frac{i}{q^\mu q_\mu - M^2}\,g_e\,\bar{u}_e^{r'}(k')\,u_e^{r}(k)$$

$$\simeq \underbrace{-i\,\frac{g_\chi\,g_e}{q^2 + M^2}\,4\,m_\chi m_e}_{c_1}\;\underbrace{\delta^{ss'}\delta^{rr'}}_{\mathcal{O}_1}$$

- other examples:

$$\mathcal{O}_1^{rr'ss'} = \delta^{rr'}\delta^{ss'}\,, \qquad \mathcal{O}_4^{rr'ss'} = \frac{\boldsymbol{\sigma}^{rr'}}{2}\cdot\frac{\boldsymbol{\sigma}^{ss'}}{2}$$

$$\mathcal{O}_9^{rr'ss'} = i\left(\frac{\boldsymbol{\sigma}^{rr'}}{2}\times\frac{\boldsymbol{q}}{m_e}\right)\cdot\frac{\boldsymbol{\sigma}^{ss'}}{2}\,, \qquad \mathcal{O}_{12}^{rr'ss'} = \left(\frac{\boldsymbol{\sigma}^{rr'}}{2}\times\boldsymbol{v}^\perp\right)\cdot\frac{\boldsymbol{\sigma}^{ss'}}{2}$$

$$\mathcal{O}_{15}^{rr'ss'} = \left[\left(\frac{\boldsymbol{\sigma}^{rr'}}{2}\times\frac{\boldsymbol{q}}{m_e}\right)\cdot\boldsymbol{v}^\perp\right]\left(\frac{\boldsymbol{\sigma}^{ss'}}{2}\cdot\frac{\boldsymbol{q}}{m_e}\right)$$

Linear response theory

Catena & Spaldin, 2402.06817

- the electronic part of each operator can be factored out, giving

$$\mathcal{M} = \sum_i c_i\,\mathcal{O}_i = \sum_a F_a^{ss'}(\boldsymbol{q},\boldsymbol{v})\,J_a^{rr'}(\boldsymbol{v}_e^\perp)\,, \qquad \boldsymbol{v} \equiv \frac{\boldsymbol{p}}{m_\chi}\,, \qquad \boldsymbol{v}_e^\perp \equiv \frac{\boldsymbol{k}+\boldsymbol{k}'}{2m_e}$$

where

$$J_0^{rr'} \equiv \delta^{rr'}\,, \quad J_A^{rr'} \equiv \boldsymbol{v}_e^\perp\cdot\sigma^{rr'}\,,$$
$$\boldsymbol{J}_5^{rr'} \equiv \boldsymbol{\sigma}^{rr'}\,, \quad \boldsymbol{J}_M^{rr'} \equiv \boldsymbol{v}_e^\perp\,\delta^{rr'}\,, \quad \boldsymbol{J}_E^{rr'} \equiv -i\,\boldsymbol{v}_e^\perp\times\boldsymbol{\sigma}^{rr'}\,,$$

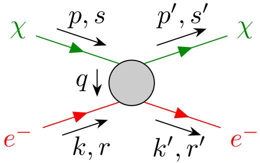
and $F_a^{ss'}(\boldsymbol{q},\boldsymbol{v})$ contain $c_1,\ldots,c_{15}$.

- scalar coupling:

$$\mathcal{M} \simeq \overbrace{\underbrace{-i\,\frac{g_\chi\,g_e}{q^2+M^2}\,4\,m_\chi m_e}_{c_1}\,\delta^{ss'}}^{F_0}\;\underbrace{\overbrace{\delta^{rr'}}^{J_0}}_{\mathcal{O}_1}\,.$$

Interaction rate for bounded electrons

Catena et al., 1912.08204



- electron-averaged matrix element

$$|\widetilde{\mathcal{M}}_{i\boldsymbol{k}\to i'\boldsymbol{k}'}|^2 \equiv \frac{1}{4}\sum_{\text{sp.}}\left|\int \frac{d^3l}{(2\pi)^3}\psi^*_{i'\boldsymbol{k}'}(\boldsymbol{l}+\boldsymbol{q})\mathcal{M}(\boldsymbol{l},\boldsymbol{p},\boldsymbol{q})\psi_{i\boldsymbol{k}}(\boldsymbol{l})\right|^2$$

i – energy band, $\boldsymbol{k}$ – momentum in the 1^{st} Brillouin zone

- interaction rate per dark particle

$$\Gamma(\boldsymbol{v}) \sim \int \frac{d^3q}{(2\pi)^3}\sum_{ii'}\int \frac{d^3k}{(2\pi)^3}\frac{d^3k'}{(2\pi)^3}|\widetilde{\mathcal{M}}_{i\boldsymbol{k}\to i'\boldsymbol{k}'}|^2\,\delta(\text{cons.})$$

- total interaction rate

$$\begin{aligned}\Gamma &\equiv \frac{1}{n_\chi V}\frac{dN}{dt}\\ &= \int d^3v\,\rho(\boldsymbol{v})\,\Gamma(\boldsymbol{v})\end{aligned}$$

Linear response theory, cont.

Catena & Spaldin, 2402.06817

- we have

$$|\widetilde{\mathcal{M}}_{\boldsymbol{i}\boldsymbol{k}\to i'\boldsymbol{k}'}|^2 \equiv \frac{1}{4}\sum_{\text{sp.}}\left|\int \frac{d^3l}{(2\pi)^3}\psi^*_{i'\boldsymbol{k}'}(\boldsymbol{l}+\boldsymbol{q})\mathcal{M}(\boldsymbol{l},\boldsymbol{p},\boldsymbol{q})\psi_{i\boldsymbol{k}}(\boldsymbol{l})\right|^2$$

- non-relativistic limit

$$\mathcal{M}(\boldsymbol{l},\boldsymbol{p},\boldsymbol{q}) = \sum_a F_a(\boldsymbol{p},\boldsymbol{q})\, J_a(\boldsymbol{v}_e^\perp)$$

$$\simeq \sum_a F_a(\boldsymbol{q},\boldsymbol{v})\left(J_a^0(\boldsymbol{q}) + \frac{\boldsymbol{l}}{m_e}\cdot\boldsymbol{J}_a^1(\boldsymbol{q})\right)$$

$$\Rightarrow \quad |\widetilde{\mathcal{M}}_{i\boldsymbol{k}\to i'\boldsymbol{k}'}|^2 \simeq \sum_{ab}\overbrace{\mathcal{F}_{ab}(\boldsymbol{q},\boldsymbol{v})}^{\text{DM physics}}\underbrace{\Big[\;|f_{i\boldsymbol{k}\to i'\boldsymbol{k}'}(\boldsymbol{q})|^2\,\mathcal{J}_{ab}(\boldsymbol{q}) + f_{i\boldsymbol{k}\to i'\boldsymbol{k}'}(\boldsymbol{q})\,\boldsymbol{f}^*_{i\boldsymbol{k}\to i'\boldsymbol{k}'}(\boldsymbol{q})\cdot\boldsymbol{\mathcal{J}}_{ab}(\boldsymbol{q}) + \text{c.c.} + \boldsymbol{f}_{i\boldsymbol{k}\to i'\boldsymbol{k}'}(\boldsymbol{q})^*\cdot\hat{\mathcal{J}}_{ab}(\boldsymbol{q})\cdot\boldsymbol{f}_{i\boldsymbol{k}\to i'\boldsymbol{k}'}(\boldsymbol{q})\Big]}_{\text{material response}}$$

Generalized susceptibilities

Catena & Spaldin, 2402.06817

- total interaction rate

$$\Gamma \equiv \int d^3v\, \rho(\boldsymbol{v})\, \Gamma(\boldsymbol{v})$$

$$\Gamma(\boldsymbol{v}) \sim \int \frac{d^3q}{(2\pi)^3} \sum_{ii'} \int \frac{d^3k}{(2\pi)^3} \frac{d^3k'}{(2\pi)^3} |\widetilde{\mathcal{M}}_{i\boldsymbol{k}\to i'\boldsymbol{k}'}|^2\, \delta(\text{cons.})$$

$$|\widetilde{\mathcal{M}}_{i\boldsymbol{k}\to i'\boldsymbol{k}'}|^2 \simeq \sum_{ab} \overbrace{\mathcal{F}_{ab}(\boldsymbol{q},\boldsymbol{v})}^{\text{DM physics}} \times [\text{material response}]$$

- the **material response** part gives the generalized susceptibilities

$$\Gamma(\boldsymbol{v}) \sim \int \frac{d^3q}{(2\pi)^3} \sum_{ab} \mathcal{F}_{ab}(\boldsymbol{q},\boldsymbol{v})\, (\chi_{a^\dagger b} - \chi^*_{b^\dagger a})(\boldsymbol{q}, \omega_{\boldsymbol{v},\boldsymbol{q}})\ ,$$

- for $a = b = n_0$,

$$\chi_{00} = \chi = 1 - \frac{4\pi\alpha}{q^2}\varepsilon^{-1} \qquad \leftarrow \text{„standard'' susceptibility}$$

Kramers-Kronig relations and the dielectric function

Lasenby & Prabhu, 2110.01587

- if function $f:\mathbb{C}\to\mathbb{C}$
  - is analytic in the upper half-plane
  - satisfies $f(z)\xrightarrow{|z|\to\infty} 0$,

  then

$$\begin{cases} \Re f(0) = \frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty}\frac{\Im f(x)}{x} \\ \Im f(0) = -\frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty}\frac{\Re f(x)}{x} \end{cases}$$

- the dielectric function ε^{-1}
  - is causal, so $\varepsilon^{-1}(\omega)$ – well-defined for $\Im\omega \geq 0$;
  - is real in the time-domain, so $\Re\varepsilon^{-1}(\omega)$ – even, $\Im\varepsilon^{-1}(\omega)$ – odd;
  - satisfies $\varepsilon^{-1}(\omega)\xrightarrow{\omega\to\infty} 1$.

  Hence,

$$\int_0^{\infty}\frac{d\omega}{\omega}\Im\left[1-\varepsilon^{-1}(\omega,k)\right] = \frac{\pi}{2}\left[1-\varepsilon^{-1}(0,k)\right]$$

- our generalization for generalized susceptibilities

$$\int_0^{\infty}\frac{d\omega}{\omega}\Im\left[\frac{4\pi\alpha}{q^2}\chi_{a^\dagger a}(\omega,\boldsymbol{q})\right] = \frac{\pi}{2}\left[\frac{4\pi\alpha}{q^2}\chi_{a^\dagger a}(0,\boldsymbol{q})\right]$$

Kramers-Kronig relations and the generalized susceptibilities

- KK relation for $\chi_{a^\dagger a}$

$$\int_0^\infty \frac{d\omega}{\omega} \Im\left[\frac{4\pi\alpha}{q^2}\,\chi_{a^\dagger a}(\omega,\boldsymbol{q})\right] = \frac{\pi}{2}\left[\frac{4\pi\alpha}{q^2}\,\chi_{a^\dagger a}(0,\boldsymbol{q})\right]$$

- interaction rate

$$\Gamma \equiv \int d^3v\,\rho(\boldsymbol{v}) \int \frac{d^3q}{(2\pi)^3} \sum_{ab} \mathcal{F}_{ab}(\boldsymbol{q},\boldsymbol{v})\,(\chi_{a^\dagger b} - \chi^*_{b^\dagger a})(\boldsymbol{q},\omega_{\boldsymbol{v},\boldsymbol{q}})$$

- diagonal terms ($a = b$) for $\mathcal{F}_{aa}(\boldsymbol{q},\boldsymbol{v}) = \mathcal{F}^{(0)}_{aa}(\omega,q) + \mathcal{F}^{(2)}_{aa}(\omega,q)\,v^2$

$$\Gamma_{aa} \propto \int q^4 dq \int_0^\infty d\omega\, f_{aa}(\omega,q)\,\Im\frac{4\pi\alpha}{q^2}\,\chi_{a^\dagger a}(\boldsymbol{q},\omega_{\boldsymbol{v},\boldsymbol{q}})$$

$$< \frac{\pi}{2}\underbrace{\max_q\left[\frac{4\pi\alpha}{q^2}\,\chi_{a^\dagger a}(0,\boldsymbol{q})\right]}_{\text{material response } \lesssim 1}\underbrace{\int q^4 dq \max_\omega\left[\omega f_{aa}(\omega,q)\right]}_{\text{calculable for a given DM model}}$$

$$f_{aa}(\omega,q) \equiv \rho^{(0)}_\omega(\omega;q)\,\mathcal{F}^{(0)}_{aa}(\omega,q) + \rho^{(2)}_\omega(\omega;q)\,\mathcal{F}^{(2)}_{aa}(\omega,q)$$

Preliminary results

material-dependent exact value $\rightarrow$
material-independent upper bound $\rightarrow$

$$\boxed{\begin{array}{c}\Gamma_{a^\dagger a} = \int q^4 dq \int_0^\infty d\omega\, f_{aa}(\omega, q)\, \Im \frac{4\pi\alpha}{q^2}\, \chi_{a^\dagger a}(\boldsymbol{q}, \omega_{\boldsymbol{v},\boldsymbol{q}}) \\ \Gamma^{\text{opt}}_{a^\dagger a} = \frac{\pi}{2} \int q^4 dq\, \max_\omega \left[\omega f_{aa}(\omega, q)\right]\end{array}}$$

$$f_{aa}(\omega, q) \equiv \rho^{(0)}_\omega(\omega; q)\, \mathcal{F}^{(0)}_{aa}(\omega, q) + \rho^{(2)}_\omega(\omega; q)\, \mathcal{F}^{(2)}_{aa}(\omega, q)$$

- truncated thermal local distribution of DM $\Rightarrow$ $\rho^{(0)}_\omega(\omega; q)$, $\rho^{(2)}_\omega(\omega; q)$
- effective models of photon-mediated DM $\Rightarrow$ $\mathcal{F}^{(0)}_{aa}(\omega, q)$, $\mathcal{F}^{(2)}_{aa}(\omega, q)$
  - dark photon
  - anapole
  - magnetic dipole
  - electric dipole
- material science $\Rightarrow$ $\Im \frac{4\pi\alpha}{q^2}\, \chi_{a^\dagger a}(\boldsymbol{q}, \omega_{\boldsymbol{v},\boldsymbol{q}})$
  - numerical data for $\Im\chi$ based on Catena et al., 2105.02233, 2210.07305

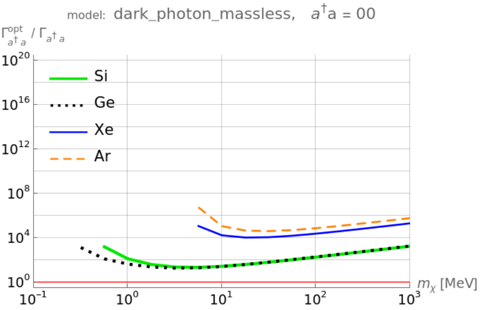
Preliminary results

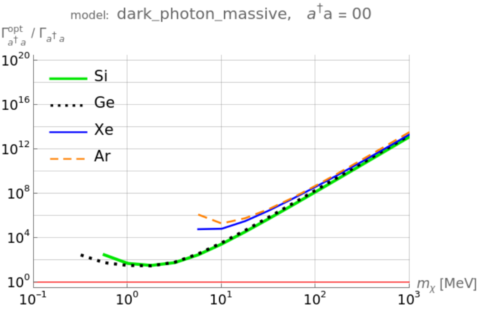
dark photon model

$$\mathcal{L}_{\text{int}} = g_x\, \bar{\chi}\gamma^\mu\chi\, A'_\mu\,, \qquad A'_\mu \text{ mixes kinetically with } A_\mu$$

$$\Rightarrow \qquad \mathcal{F}_{00} \propto (m_{A'}^2 + q^2)^{-2} \propto \begin{cases} q^{-4} & \text{light mediator} \\ 1 & \text{heavy mediator} \end{cases} \qquad \text{(others 0)}$$

(note: the same applies to the scalar-mediator model)



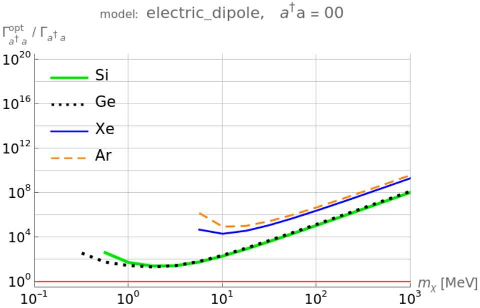


Preliminary results

electric dipole model

$$\mathcal{L}_{\text{int}} = \frac{g}{\Lambda}\, i\, \bar{\chi}\sigma^{\mu\nu}\gamma_5\chi\, F_{\mu\nu}$$

$$\Rightarrow \qquad \mathcal{F}_{00} \propto q^{-2} \qquad \text{(others 0)}$$



Preliminary results

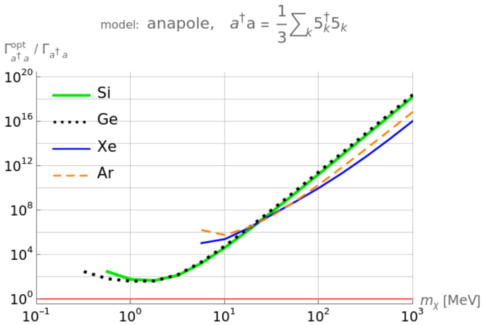
anapole model

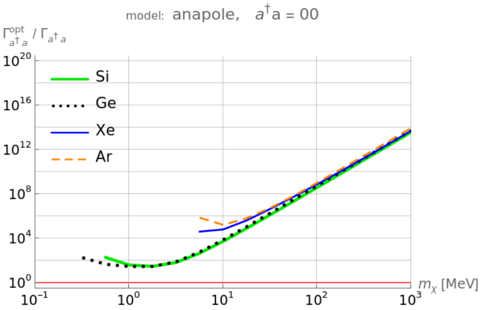
$$\mathcal{L}_{\text{int}} = \frac{g}{2\Lambda^2}\,\bar{\chi}\gamma^\mu\gamma_5\chi\,\partial^\nu F_{\mu\nu}$$

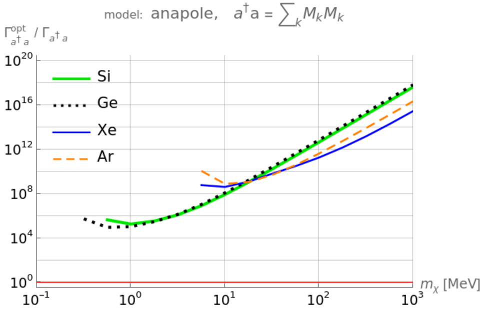
$$\Rightarrow \quad \mathcal{F}_{00} \propto \frac{q}{4\,m_\chi}\left(v_q - \frac{q}{4\,m_\chi}\right) + \frac{1}{4}v^2\,,$$

$$\text{Tr}\mathcal{F}_{MM} \propto \frac{3}{4}\,, \qquad \frac{1}{3}\text{Tr}\mathcal{F}_{55} \propto \frac{q^2}{24\,m_e^2}$$

(others vanish)







Preliminary results

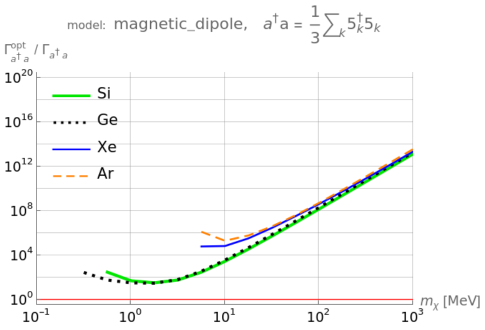
magnetic dipole model

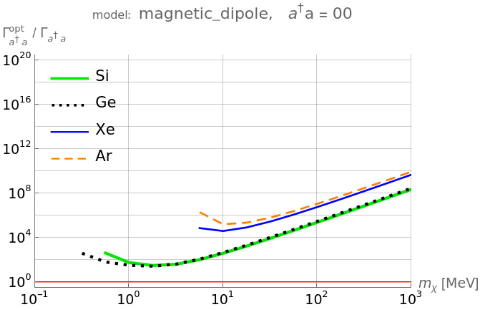
$$\mathcal{L}_{\text{int}} = \frac{g}{\Lambda}\,\bar{\chi}\sigma^{\mu\nu}\chi\,F_{\mu\nu}$$

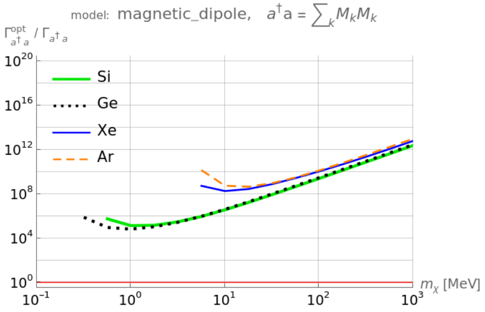
$$\Rightarrow \qquad \mathcal{F}_{00} \propto 1 - 4\,v_q^2\,\frac{m_\chi^2}{q^2} + 4\,v^2\,\frac{m_\chi^2}{q^2}\,,$$

$$\text{Tr}\mathcal{F}_{MM} \propto 8\,\frac{m_\chi^2}{q^2}\,, \qquad \frac{1}{3}\text{Tr}\mathcal{F}_{55} \propto 2\,\frac{m_\chi^2}{m_e^2}$$

(others vanish)







Summary

- effective approach to non-relativistic DM-e^- interactions
  - 14 operators in the leading order
- linear response theory

$$[\text{interaction rate}] = \int [\text{DM model}] \times [\text{material response of the detector}]$$

- material response $\rightarrow$ generalized susceptibilities $\chi_{a^\dagger b}(\omega, q)$
- Kramers-Kronig relations

$$f: \text{ causal, analytic} \qquad \Rightarrow \qquad \int_0^\infty \frac{d\omega}{\omega} \Im f(\omega) = \frac{\pi}{2} f(0)$$

- material-independent theoretical upper bound on the interaction rate

$$\begin{aligned} \Gamma_{a^\dagger a} &= \int q^4 dq \int_0^\infty d\omega\, f_{aa}(\omega, q)\, \Im \frac{4\pi\alpha}{q^2} \chi_{a^\dagger a}(\boldsymbol{q}, \omega_{\boldsymbol{v},\boldsymbol{q}}) \\ &\leq \quad \Gamma^{\text{opt}}_{a^\dagger a} \equiv \frac{\pi}{2} \int q^4 dq \max_\omega [\omega f_{aa}(\omega, q)] \end{aligned}$$

- application of Kramers-Kronig relations to generalized susceptibilities calculated within the linear response theory **[new]**
  - solids usually better than nobles (excl. $m_\chi \gtrsim 20$ MeV in the anapole model)

Summary

thank you!

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BACKUP SLIDES

Truncated thermal distribution

Baxter et al., 2105.00599

$$\rho(\boldsymbol{v}) = \mathcal{N}\, \exp\left[-\frac{(\boldsymbol{v}+\boldsymbol{v}_\oplus)^2}{v_0^2}\right]\theta\left(v_{\mathsf{esc}} - |\boldsymbol{v}+\boldsymbol{v}_\oplus|\right)$$

$$\rho_\omega^{(0)}(\omega;q) = \frac{\pi}{q}\int_{v_q}^{\infty} v\,dv \int_{-1}^{1} d\cos(\boldsymbol{v},\boldsymbol{v}_\oplus)\,\rho(\boldsymbol{v})$$

$$\rho_\omega^{(2)}(\omega;q) = \frac{\pi}{q}\int_{v_q}^{\infty} v^3\,dv \int_{-1}^{1} d\cos(\boldsymbol{v},\boldsymbol{v}_\oplus)\,\rho(\boldsymbol{v})$$

$$v_{\mathsf{esc}} = 544\,\mathsf{km/s}$$
$$v_\oplus = 250.5\,\mathsf{km/s}$$
$$v_0 = 238\,\mathsf{km/s}$$

$$\mathcal{N} = \frac{1}{2\pi\, v_0^3}\left(\frac{\sqrt{\pi}}{2}\,\mathsf{erf}\left[\frac{v_{\mathsf{esc}}}{v_0}\right] - \frac{v_{\mathsf{esc}}}{v_0}\,\exp\left[-\frac{v_{\mathsf{esc}}^2}{v_0^2}\right]\right)^{-1}$$

Electronic couplings

— scalar —

symmetry:

$$\mathcal{J}_{ba} = \mathcal{J}^*_{ab}$$

values:

$$\mathcal{J}_{00} = 1$$

$$\mathcal{J}_{AA} = \frac{q^2}{4m_e}$$

$$\mathcal{J}_{5_k 5_l} = \delta_{kl}$$

$$\mathcal{J}_{M_k M_l} = \frac{q_k q_l}{4m_e^2}$$

$$\mathcal{J}_{E_k E_l} = \frac{\delta_{kl} q^2 - q_k q_l}{2m_e^2}$$

$$\mathcal{J}_{0M_k} = \mathcal{J}_{A5_k} = \frac{q_k}{2m_e}$$

$$\mathcal{J}_{5_k E_l} = -i\varepsilon_{klm} \frac{q_m}{2m_e}$$

others 0
(up to the symmetry)

— vector —

symmetry:

$$\boldsymbol{\mathcal{J}}_{ab} + \boldsymbol{\mathcal{J}}^*_{ba} = \boldsymbol{C}_{ab}$$

$$\equiv \frac{1}{2} \sum_{rr'} \nabla_{\boldsymbol{v}_e^\perp} \left(J_a^{rr'} J_b^{rr'} \right) \Big|_{\boldsymbol{v}_e^\perp = \frac{\boldsymbol{q}}{2m_e}}$$

values:

$$\boldsymbol{\mathcal{J}}_{0\bullet} = \boldsymbol{\mathcal{J}}_{5_k \bullet}$$

$$= \boldsymbol{\mathcal{J}}_{AM_k} = \boldsymbol{\mathcal{J}}_{M_k E_k} = 0$$

$$\boldsymbol{\mathcal{J}}_{AA} = \frac{\boldsymbol{q}}{2m_e}$$

$$\boldsymbol{\mathcal{J}}_{M_k M_l} = \frac{q_l}{2m_e} \boldsymbol{e}_k$$

$$\boldsymbol{\mathcal{J}}_{E_k E_l} = \frac{\delta_{kl} \boldsymbol{q} - q_k \boldsymbol{e}_l}{2m_e}$$

$$\boldsymbol{\mathcal{J}}_{AE_k} = -i \boldsymbol{e}_k \times \frac{\boldsymbol{q}}{2m_e}$$

others 0
(up to the symmetry)

$$\boldsymbol{v} \cdot \boldsymbol{e}_i \equiv v_i$$

— tensor —

symmetry:

$$\hat{\mathcal{J}}_{ba} = \hat{\mathcal{J}}^\dagger_{ab}$$

values:

$$\hat{\mathcal{J}}_{AA} = \mathbb{1}$$

$$\hat{\mathcal{J}}_{M_k M_l} = \hat{e}_{kl}$$

$$\hat{\mathcal{J}}_{E_k E_l} = \delta_{kl} \mathbb{1} - \hat{e}_{lk}$$

$$\hat{\mathcal{J}}_{AE_k} = i\varepsilon_{kij} \hat{e}_{ij}$$

$$\boldsymbol{v} \cdot \hat{e}_{ij} \cdot \boldsymbol{u} \equiv v_i u_j$$

others 0
(up to the symmetry)

$$\boldsymbol{C}_{AA} = \frac{\boldsymbol{q}}{m_e}$$

$$\boldsymbol{C}_{M_k M_l} = \frac{q_k \boldsymbol{e}_l + q_l \boldsymbol{e}_k}{2m_e}$$

$$\boldsymbol{C}_{E_k E_l} = \frac{2\delta_{kl} \boldsymbol{q} - q_k \boldsymbol{e}_l - q_l \boldsymbol{e}_k}{2m_e}$$

$$\boldsymbol{C}_{0M_k} = C_{a5_k} = \boldsymbol{e}_k$$

$$\boldsymbol{C}_{5_k E_l} = -i\varepsilon_{klm} \boldsymbol{e}_m$$

others 0 (up to $\boldsymbol{C}_{ba} = \boldsymbol{C}^*_{ab}$)