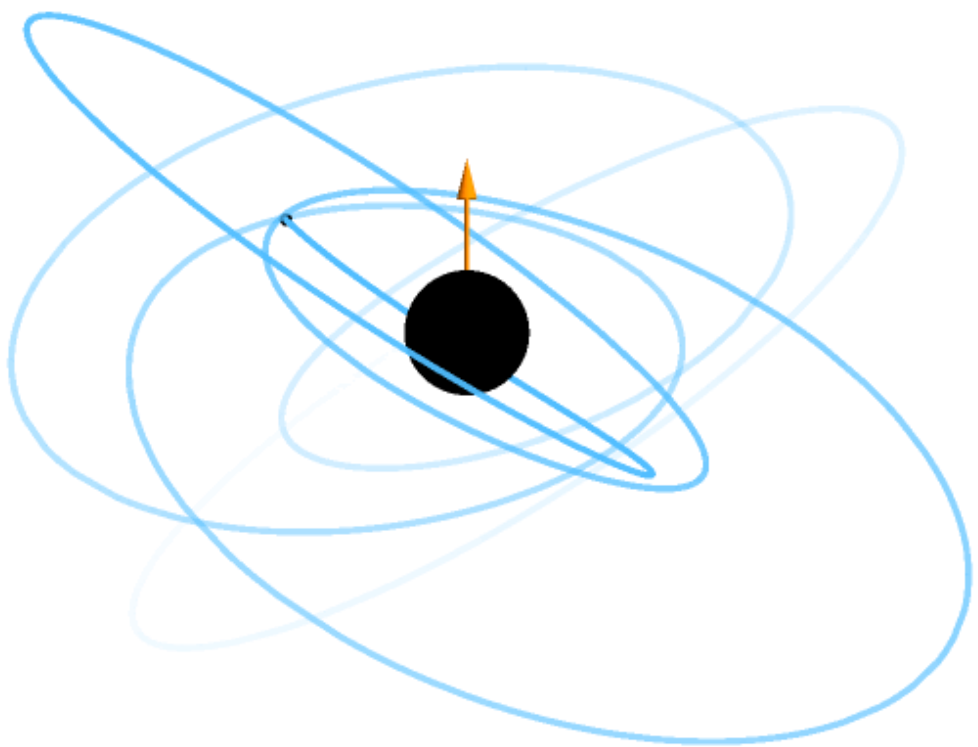
Waveform modelling for I/EMRIs using a multiscale self-force approach



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University College Dublin

1st TEONGRAV international workshop
Sapienza University of Rome
18th September 2024

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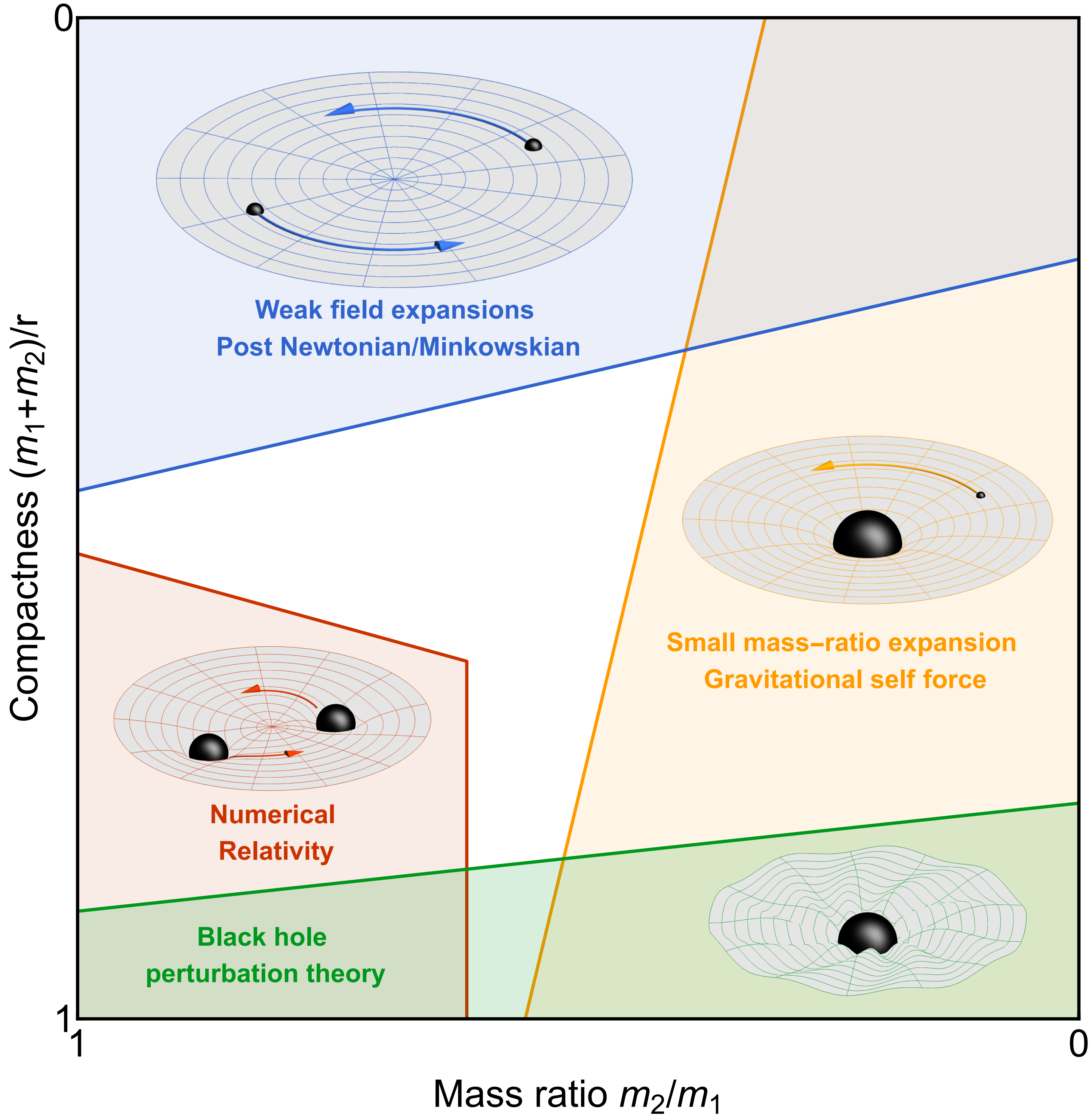
Compact binary parameter space



[Image credit: LISA Consortium Waveform Working Group]

Modelling goals

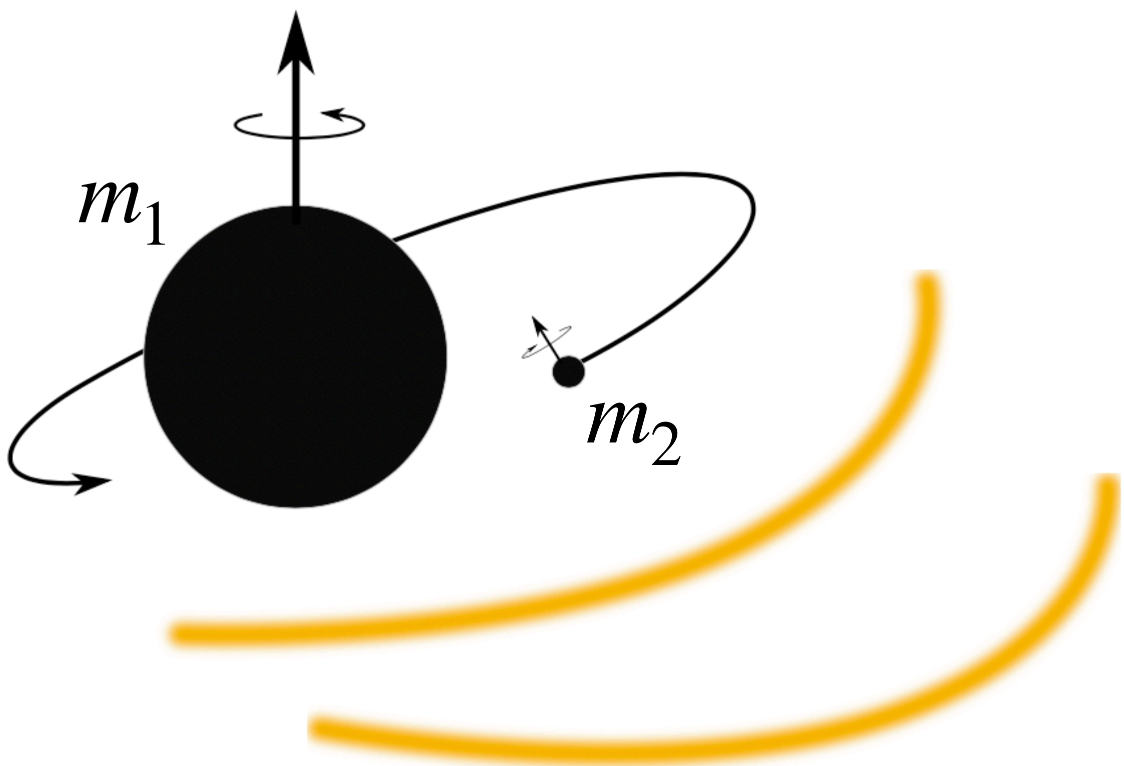
Our EMRI waveform models must be accurate, efficient and extensive

We need the phase error in the waveform to be 'small' $\implies$ we must include adiabatic and post-adiabatic corrections

Search and parameter estimation requires millions of templates $\implies$ each waveform must be computed in less than 1 second

EMRIs will be very generic $\implies$ our models must span the full parameter space of eccentric, precessing systems

Gravitational self-force approach



[Image credit: Adam Pound]

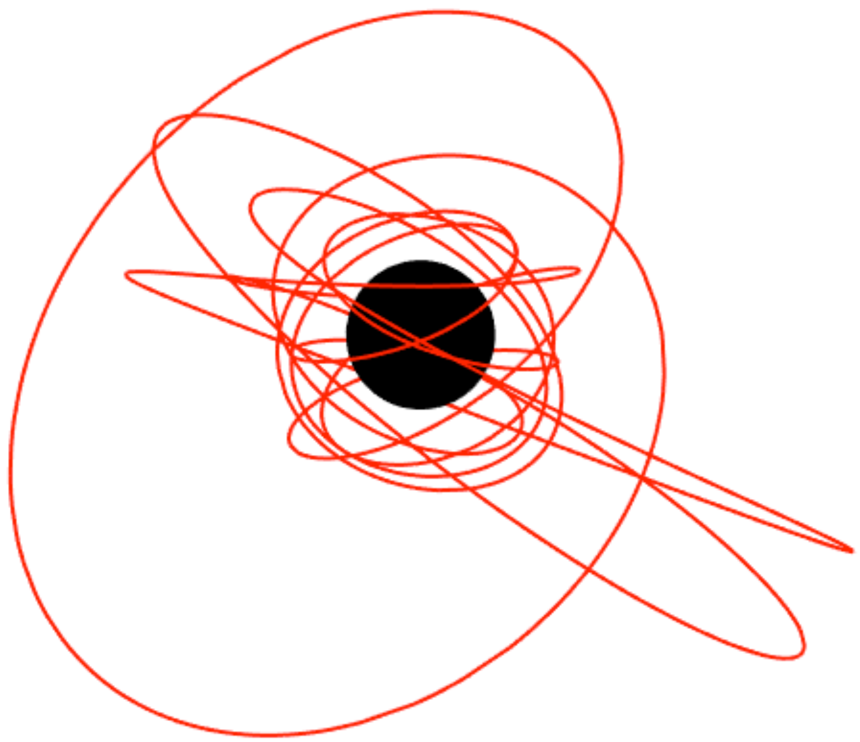
- $\epsilon = 1/q = m_2/m_1 \ll 1$
- Small body perturbs spacetime:

$$g_{\alpha\beta} = g^{\rm Kerr}_{\alpha\beta} + \epsilon h^{(1)}_{\alpha\beta} + \epsilon^2 h^{(2)}_{\alpha\beta} + \ldots$$

- Perturbation affects m_2's motion:

$$\frac{D^2 z^\mu}{d\tau^2} = \epsilon f^\mu_{(1)} + \epsilon^2 f^\mu_{(2)} + \ldots$$

Zeroth order: geodesics in Kerr



[Image created using the BHPToolkit KerrGeodesics package]

- constants $J_a = (m_1, \chi_1, E, L_z, Q)$
  - Energy E
  - angular momentum L_z
  - Carter constant Q
- phases $\varphi_A = (\varphi_r, \varphi_\theta, \varphi_\phi)$ with frequencies $\Omega_A(J_B)$

- Simple ODEs:

$$\frac{d\varphi_A}{dt} = \Omega_A(J_B)$$

$$\frac{dJ_A}{dt} = 0$$

Post-adiabatic (multi-scale) expansion

Hinderer, Flanagan, Miller, Pound, Moxon, Grant

- evolution due to the self-force:

$$\frac{d\tilde{\varphi}_A}{dt} = \Omega_A(\tilde{J}_B)$$

$$\frac{d\tilde{J}_A}{dt} = \epsilon\left[F_A^{(0)}(\tilde{J}_B) + \epsilon F_A^{(1)}(\tilde{J}_B) + \mathcal{O}(\epsilon^2)\right]$$

- waveform:

$$h_{\ell m} = \sum_{k^i}\left[\epsilon h^{(1)}_{\ell m k^i}(\tilde{J}_A) + \epsilon^2 h^{(2)}_{\ell m k^i}(\tilde{J}_A) + \mathcal{O}(\epsilon^3)\right] e^{-i(m\tilde{\varphi}_\phi + k^i\tilde{\varphi}_i)}$$

- Secondary spin: (i) new slow parameters
  (ii) new precession phase

Post-adiabatic (multi-scale) expansion

Hinderer, Flanagan, Miller, Pound, Moxon, Grant

- Write metric as product of slowly evolving amplitudes and a rapidly evolving phase:

$$h^{(n)}_{\alpha\beta} = \sum_{k^A} h^{(n,k^A)}_{\alpha\beta}(J_A; x^i) e^{-ik^A\varphi_A} \qquad x^i = \{r, \theta, \phi\}$$

- Substitute this into the Einstein field equations. By treating t as a function of (J_A, φ_B) time derivatives can be computed via:

$$\partial_t = \sum_A \dot{\varphi}_A \partial_{\varphi_A} + \dot{J}_A \partial_{J_A}$$

- Lots of extra detail… (gauge choice, regularisation via matched expansions, numerical methods, etc). The first calculation of $h^{(2)}_{\alpha\beta}$ took ~10 years to work through all the details.

Near-identity averaging transformations

Lynch, van de Meent, NW, Witzany

- What are those $\tilde{J}_A$ variables? $$\frac{d\tilde{J}_A}{dt} = \epsilon\left[F_A^{(0)}(\tilde{J}_B) + \epsilon F_A^{(1)}(\tilde{J}_B) + \mathcal{O}(\epsilon^2)\right]$$

- Evolution of J_A depends on phases φ_C:

$$\frac{dJ_A}{dt} = \epsilon\left[F_A^{(0)}(J_B,\varphi_C) + \epsilon F_A^{(1)}(J_B,\varphi_C) + \mathcal{O}(\epsilon^2)\right]$$

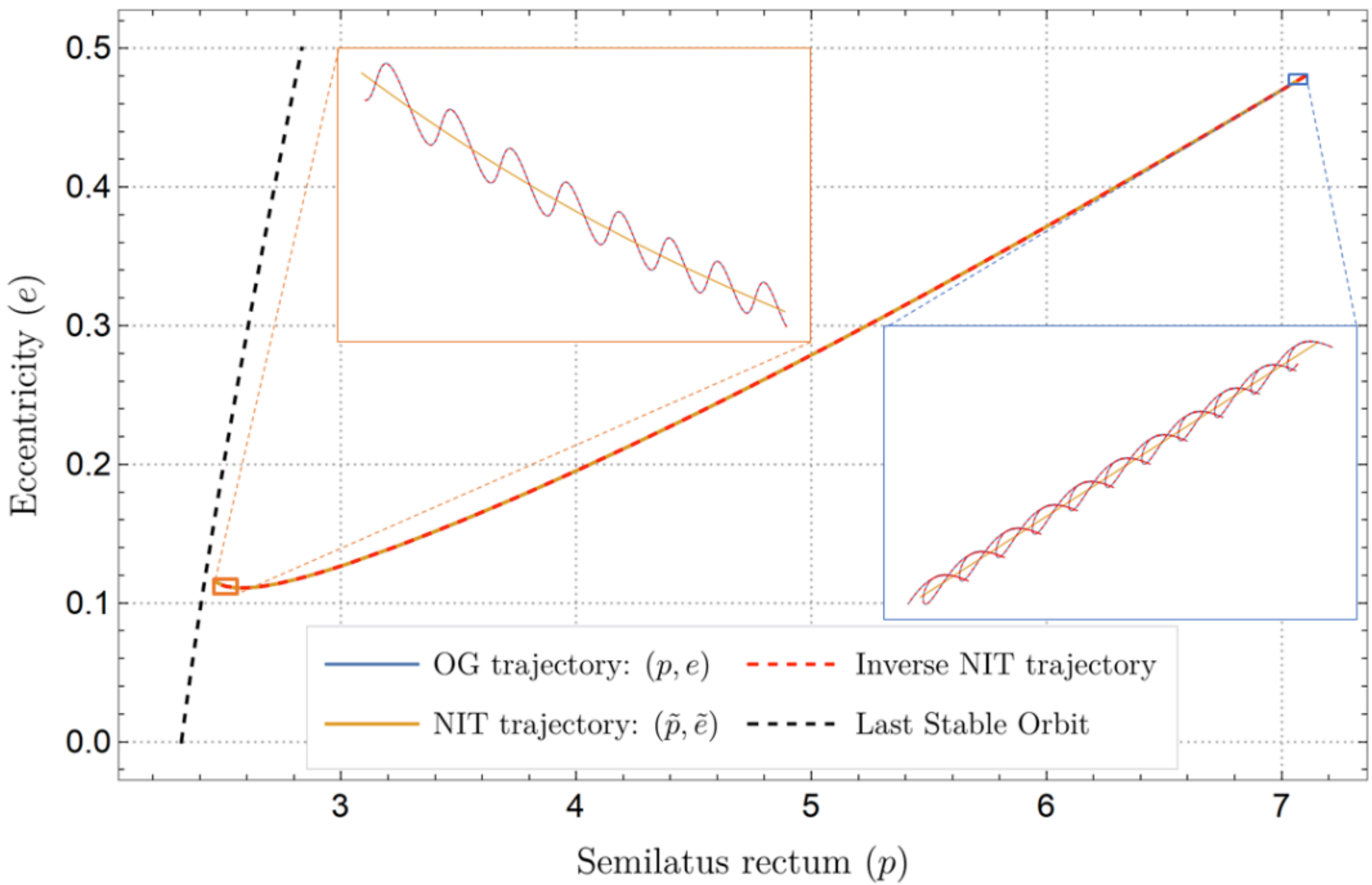
- Introduce a near-identity (averaging) transformation (NIT):

$$\tilde{J}_A = J_A + \epsilon Y_A^{(1)}(J_B,\varphi_C) + \epsilon^2 Y_A^{(2)}(J_B,\varphi_C) + \mathcal{O}(\epsilon^3)$$

- Can choose $Y_A^{(n)}$ to remove dependency on the phase in the equations for $\tilde{J}_A$

Near-identity averaging transformations

Lynch, van de Meent, NW, Witzany



Phase space trajectory computation goes from taking hours to to taking milliseconds

Native rapid waveform generation

Offline step

- solve field equations for amplitudes $h^{(n)}_{lmk^i}$ and forcing functions $F^{(n-1)}_A$ on a grid of $\tilde{J}_A$ values

Online step

- solve ODEs for $\tilde{\varphi}_A$ and $\tilde{J}_A$
- Add up the mode amplitudes $h^{(n)}_{\ell mk^i}$ at each sample time
- `FastEMRIWaveforms` (**FEW**) software package can compute a 2-year long waveform in ~ 10 - 100ms

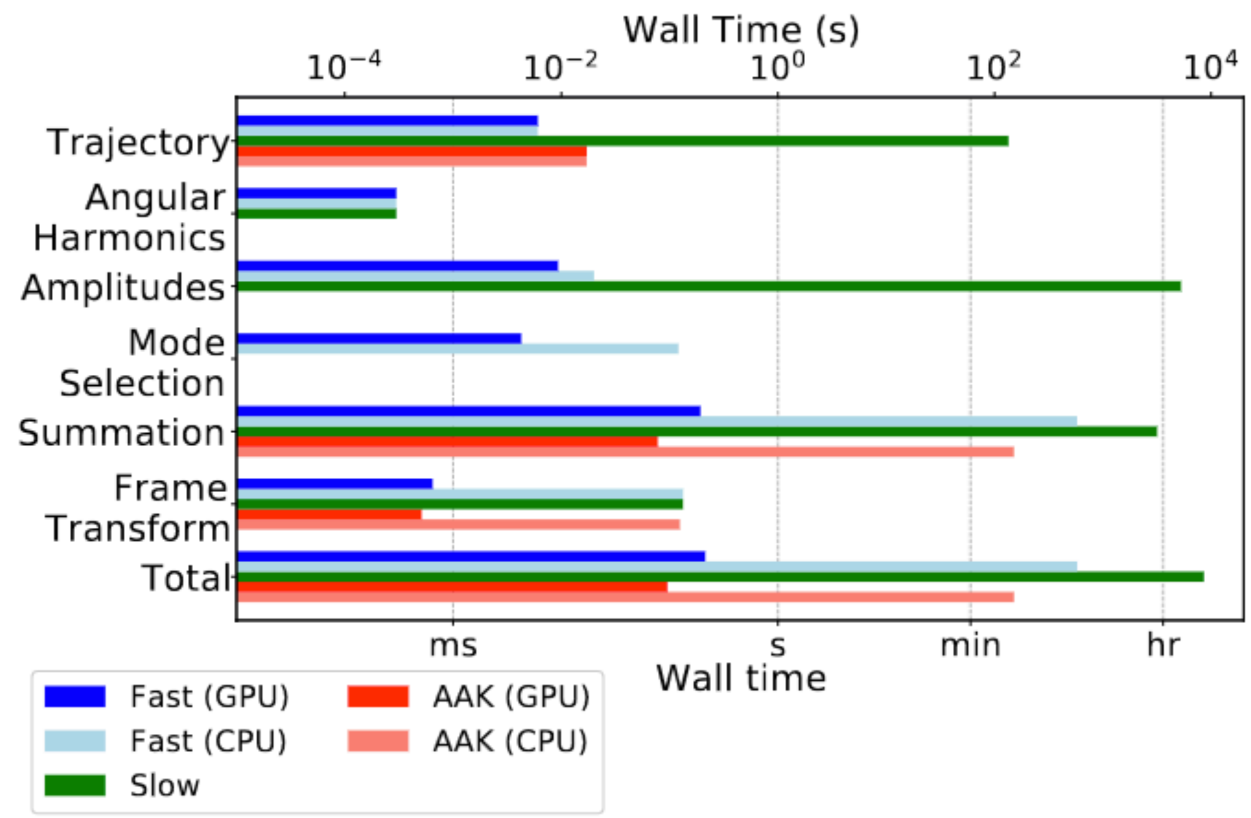
$$h_{\ell m} = \sum_{k^i} \left[\epsilon h^{(1)}_{\ell m k^i}(\tilde{J}_A) + \epsilon^2 h^{(2)}_{\ell m k^i}(\tilde{J}_A) + \mathcal{O}(\epsilon^3) \right] e^{-i(m\tilde{\varphi}_\phi + k^i \tilde{\varphi}_i)}$$

- The number of $h^{(n)}_{\ell m k^i}$ that need to be summed at each time step can be in the thousands.

- The waveform amplitudes vary slowly. These amplitudes are sampled on a sparse set of points, summed, and then upsampled

- GPU acceleration takes generation time down from minutes to milliseconds

- Relativistic adiabatic (0PA) Kerr equatorial model will be publicly available soon



Accuracy and post-adiabatic counting

phases: $$\tilde{\varphi}_A = \epsilon^{-1}\varphi_A^{(0)}(\Omega_B) + \epsilon^{0}\varphi_A^{(1)}(\Omega_B) + \mathcal{O}(\epsilon)$$

Adiabatic (0PA)

From the orbit averaged piece of first-order self-force $\langle f^{\alpha}_{(1)} \rangle$

$\langle f^{\alpha}_{(1)} \rangle$ can be related to the fluxes, thus avoiding a local calculation of the self-force

0PA is sufficient for detection and rough parameter estimation for astrophysics of EMRIs of bright sources

Post-adiabatic order (1PA)

Two contributions:

- oscillatory pieces of the first order self-force $\breve{f}^{\alpha}_{(1)}$
- second-order orbit averaged self-force $\langle f^{\alpha}_{(2)} \rangle$

Needed to extract all sources

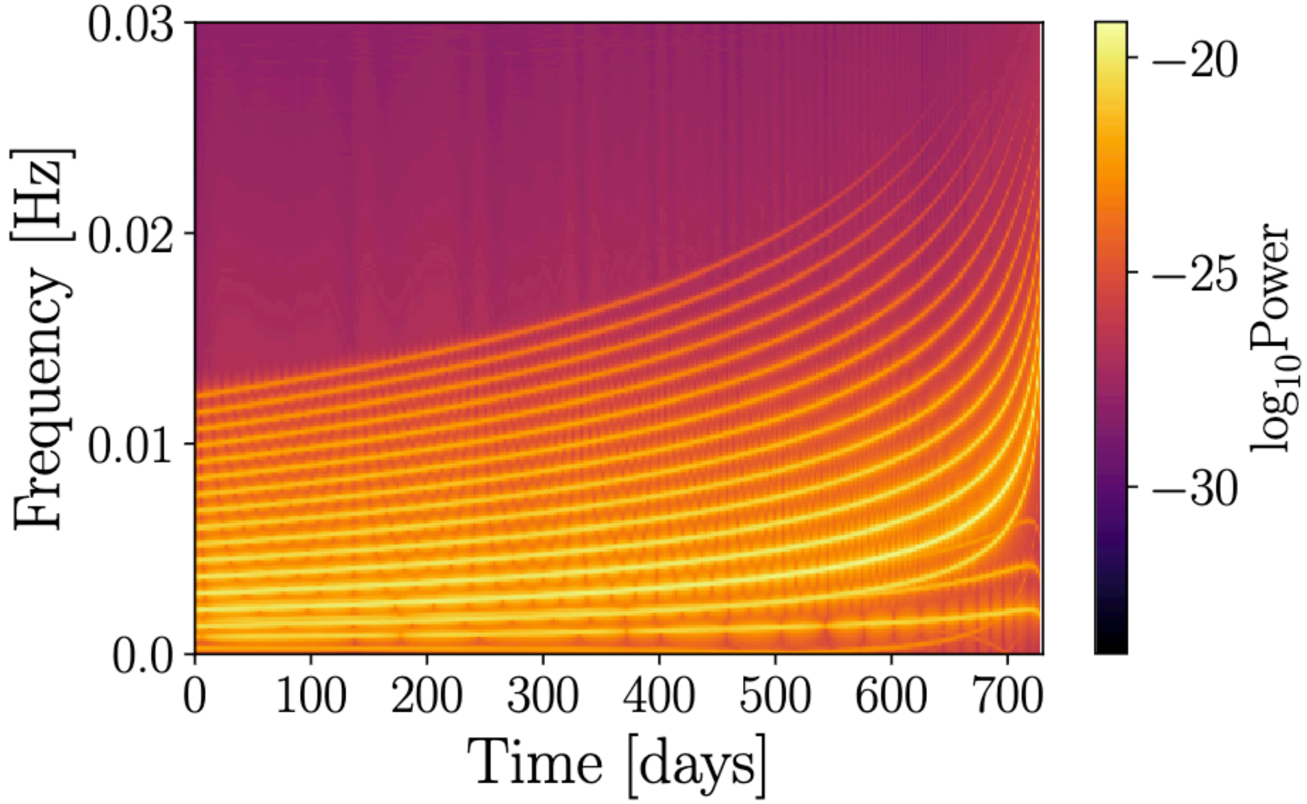
Needed for precision tests of GR

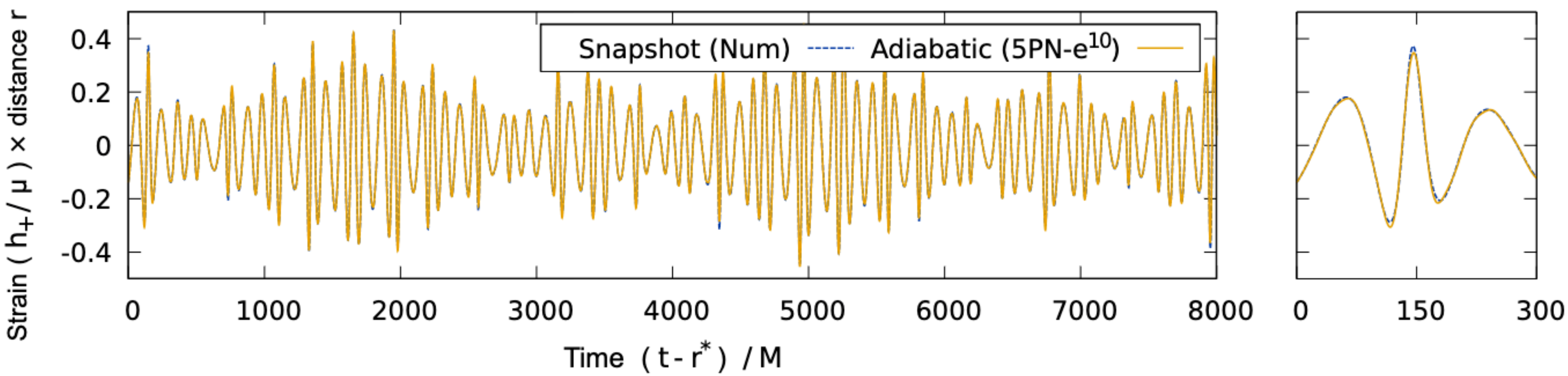
Potential application to IMRIs

Parameter space coverage at 0PA

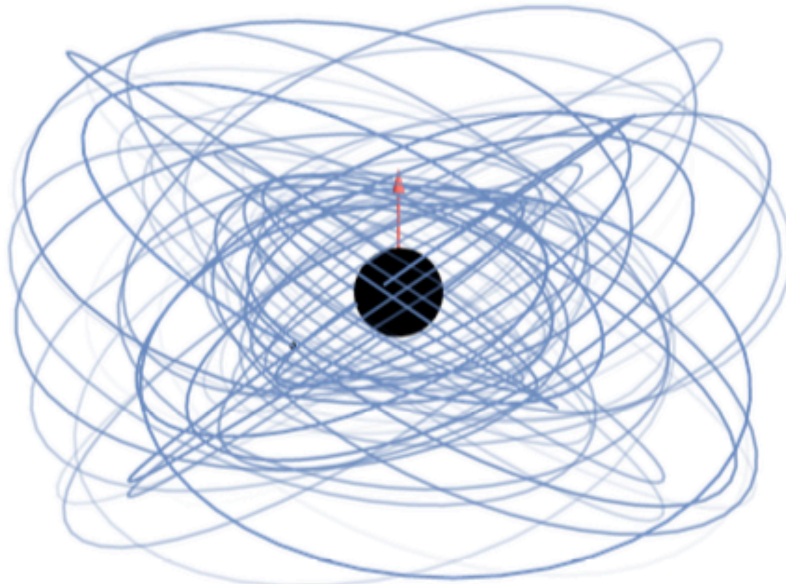
In FEW:

- generic orbits in Kerr: 5.5PN-e^{10} approximation
- equatorial orbits in Kerr: full relativistic waveforms in time or frequency domain
- kludge models

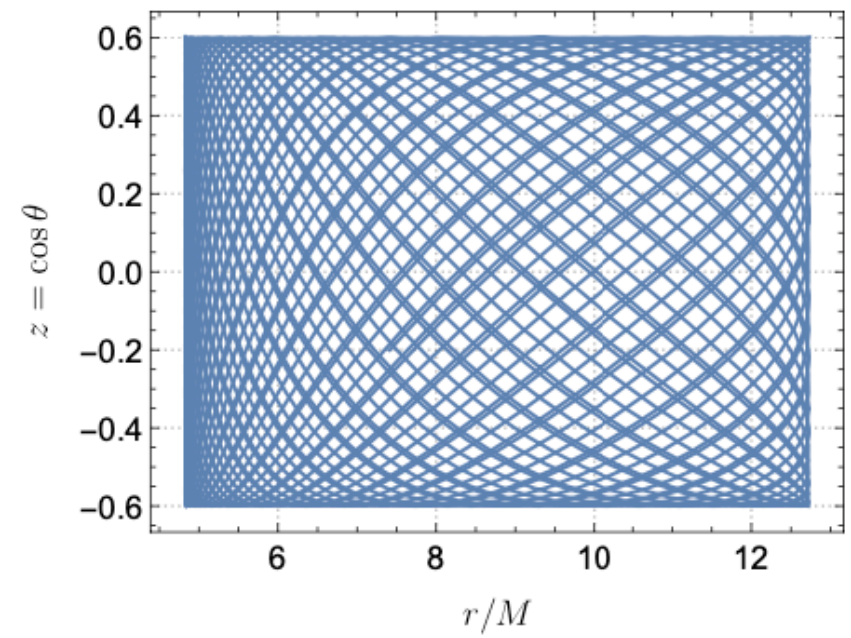




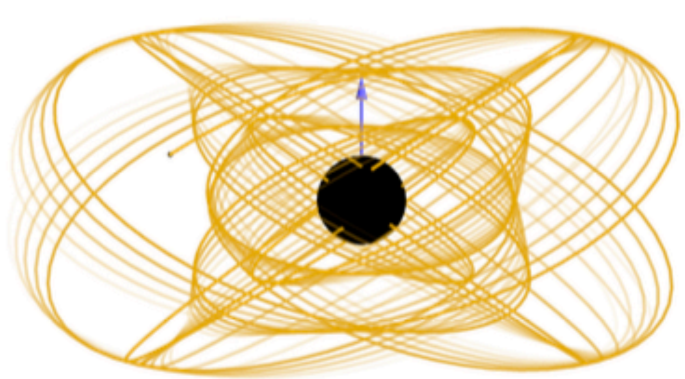
Resonances (0.5PA)



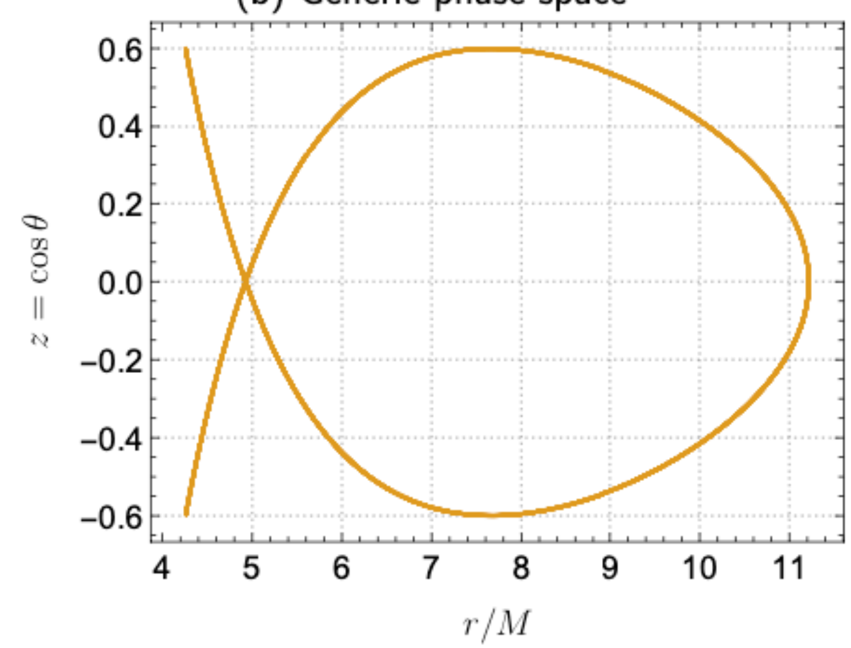
(a) Generic orbit



(b) Generic phase space



(c) Resonant orbit: $q_{r,0} = 0, q_{z,0} = 0$



(d) Resonant phase space: $q_{r,0} = 0, q_{z,0} = 0$

[Image credit: Philip Lynch]

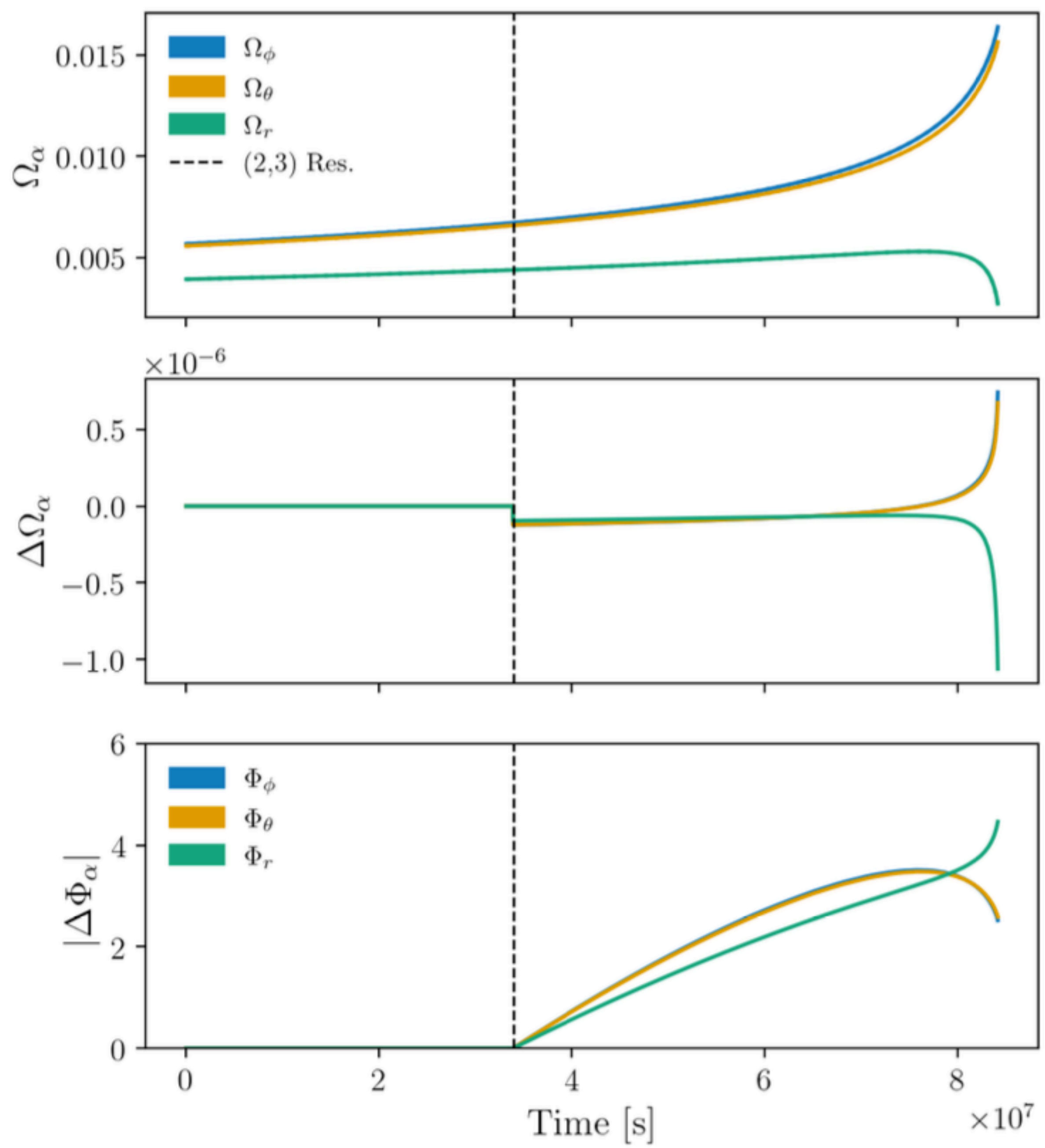
Resonances (0.5PA) in FEW

Chapman-Bird, NW

Ω_r/Ω_θ becomes momentarily rational

Ω_A "jumps" slightly across the resonance

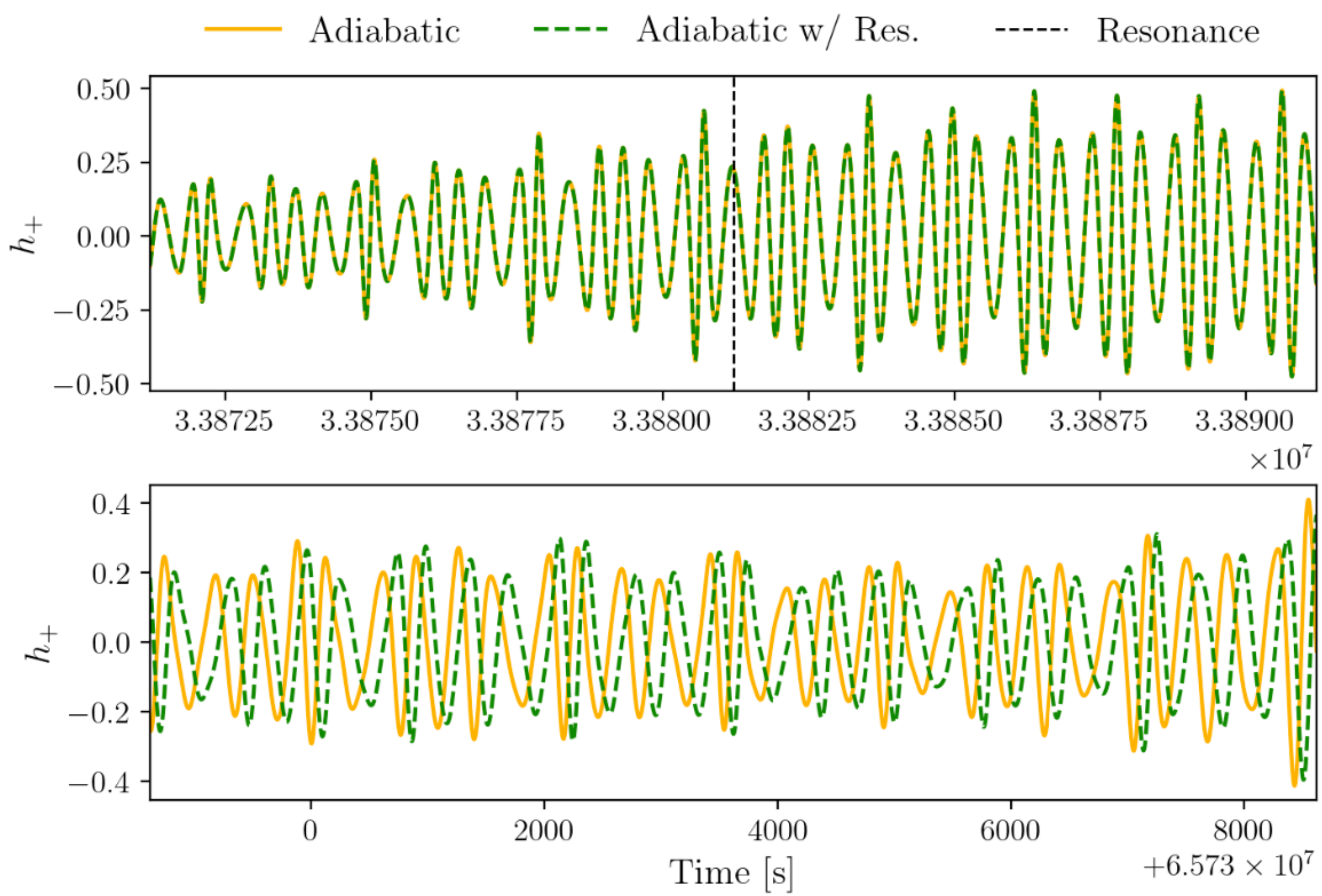
Leads to a significant phase corrections



$$\tilde{\varphi}_A = \epsilon^{-1}\varphi_A^{(0)}(\Omega_B) + \epsilon^{-1/2}\varphi_A^{(1/2)}(\Omega_B) + \epsilon^0\varphi_A^{(1)}(\Omega_B) + \mathcal{O}(\epsilon^{1/2})$$

Resonances (0.5PA) in FEW

Chapman-Bird, NW

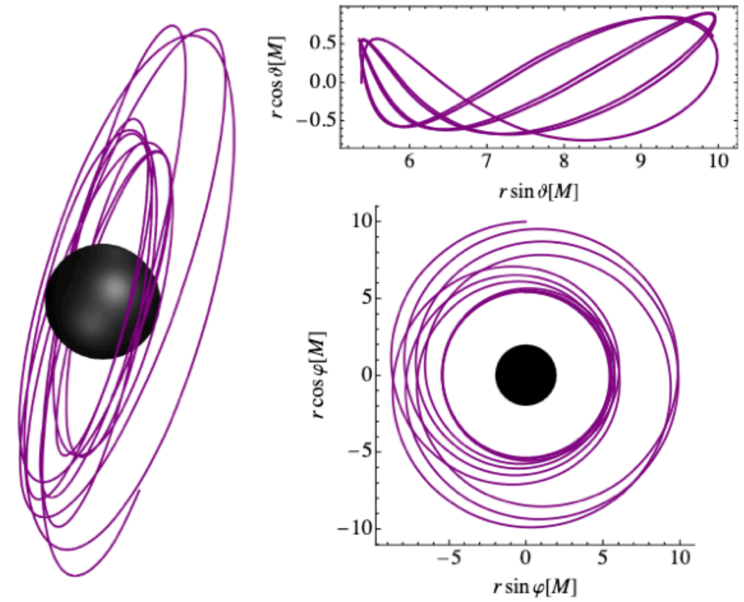


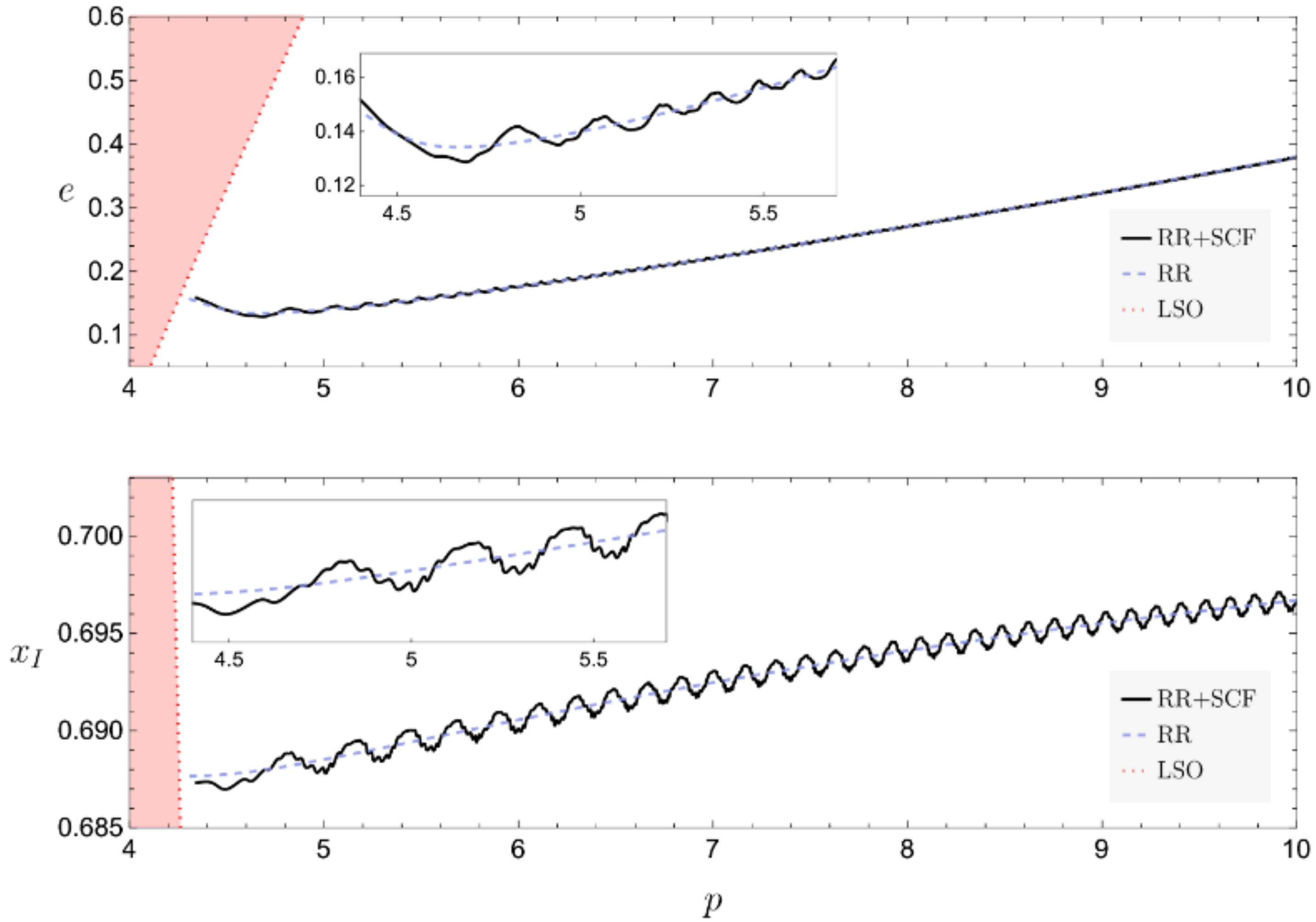
Goal: modular framework in FEW. Given a resonance surface and jump conditions FEW can efficiently model any resonant phenomena.

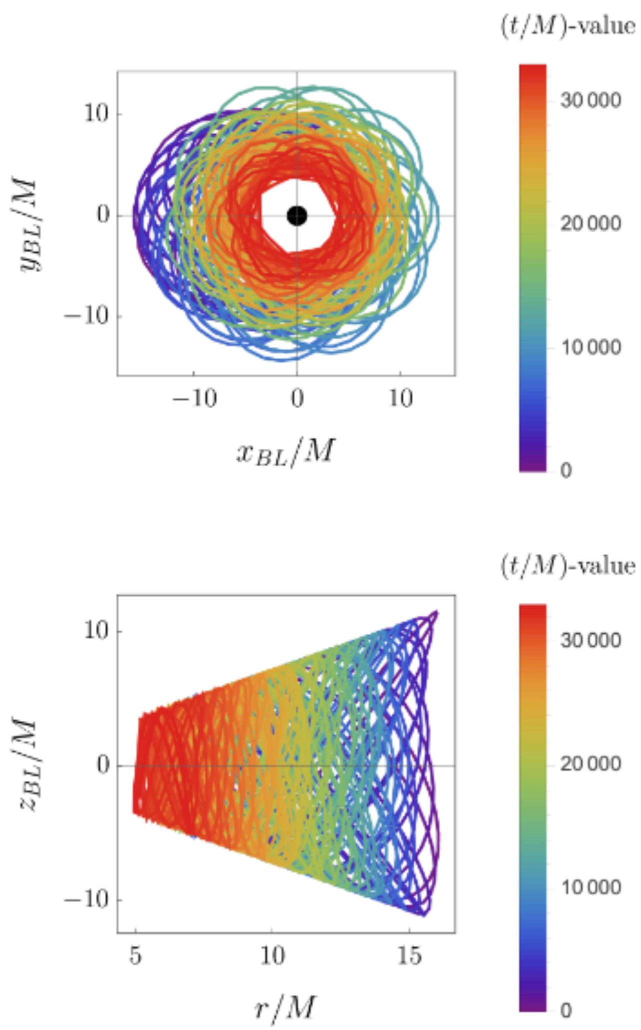
1PA secondary spin effects

Piovano, Witzany
Drummond, Hughes, Lynch et al
Skoupý et al.

- Inspiral trajectory including 1PA conservative effects has been NIT'ed (fast to compute)
- Dissipative correction is computable but still need to tile parameter space and build into FEW



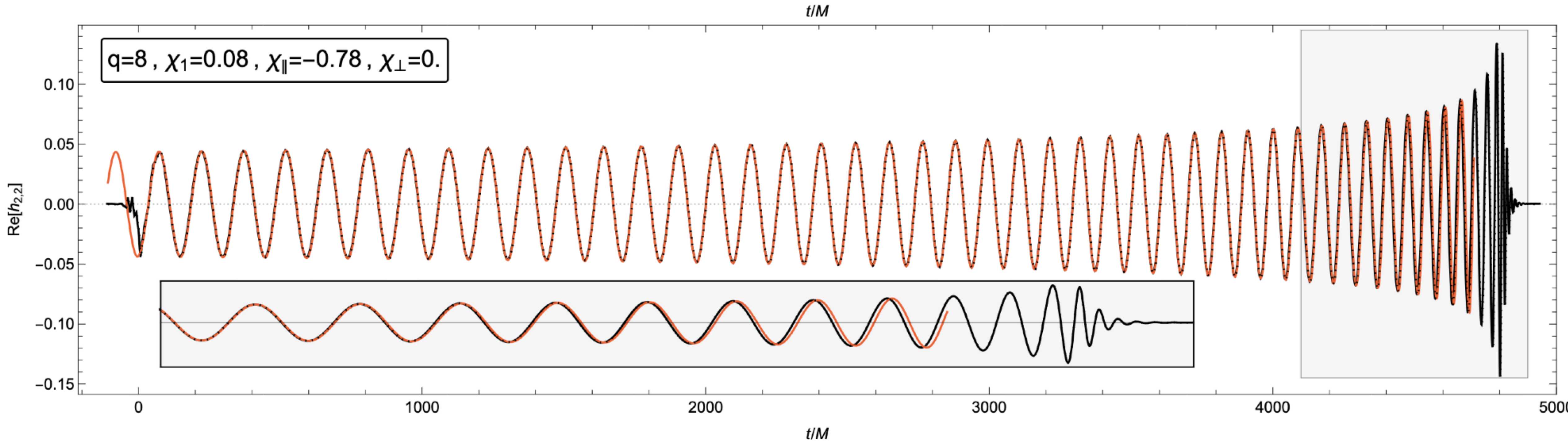




Complete 1PA inspiral waveforms

Pound, NW, Wardell, Durkan, Miller, Mathews

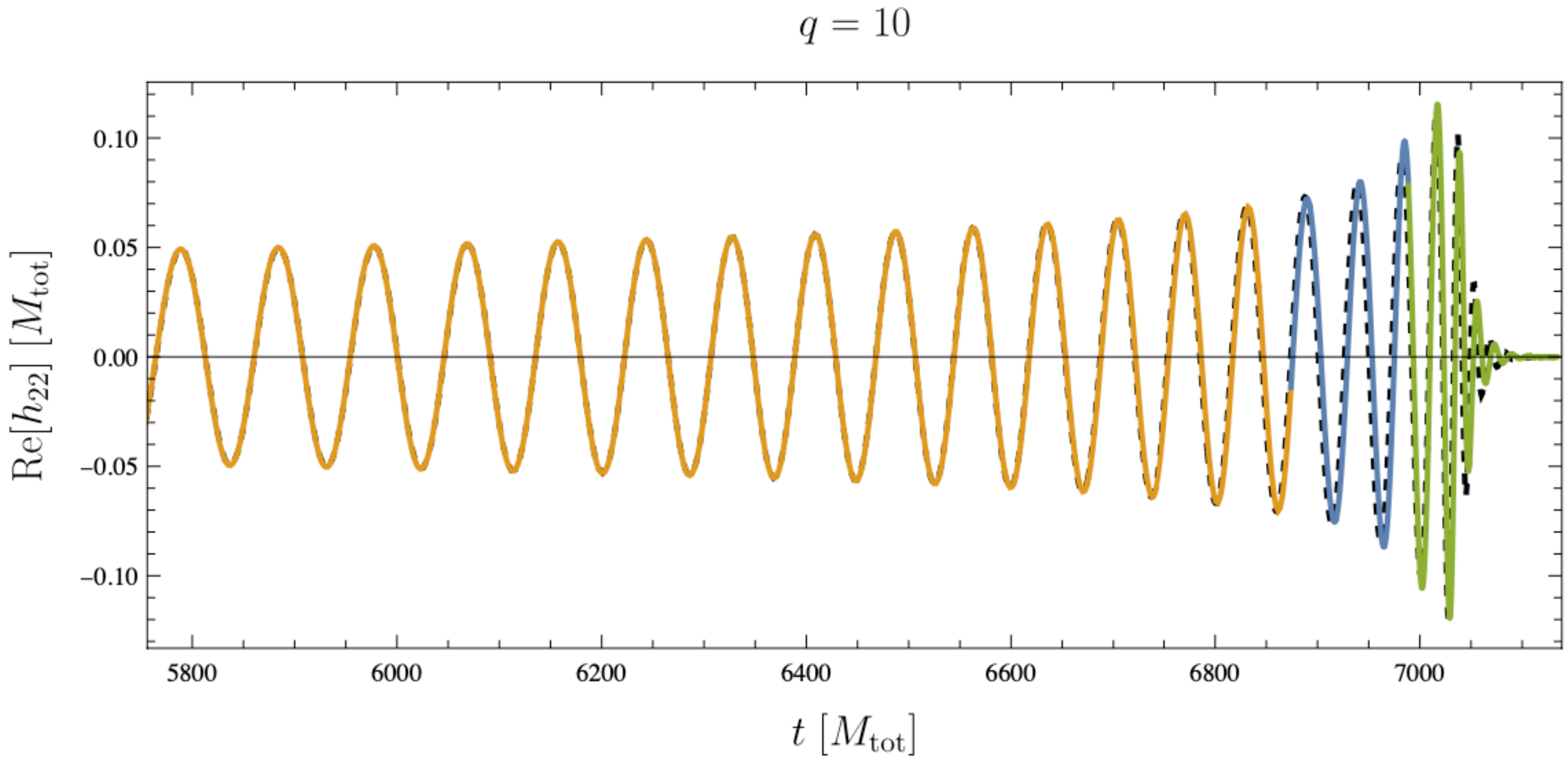
Comparison with NR waveform from SXS collaboration



- Complete quasi-circular 1PA inspiral model with generic (precessing) secondary spin, linear primary spin and evolving m_1 and χ_1
- Implementation in FEW will be public soon

Complete inspiral-merger-ringdown models

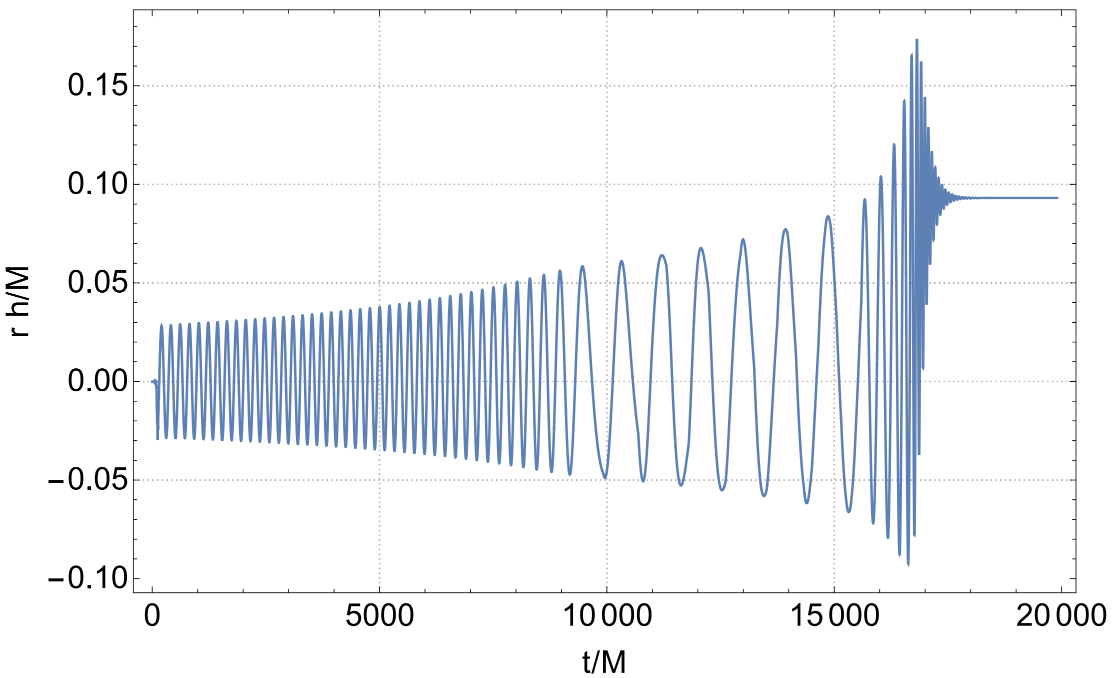
Küchler, Compère, Durkan, Pound



- The multiscale expansion used in the inspiral breaks down at the ISCO
- Implement new expansions for the transition-to-plunge and plunge region
- First results appearing. Fast waveform generation speed maintained.

Gravitational wave memory

Cunningham, Kavanagh, Trestini, Pound, NW

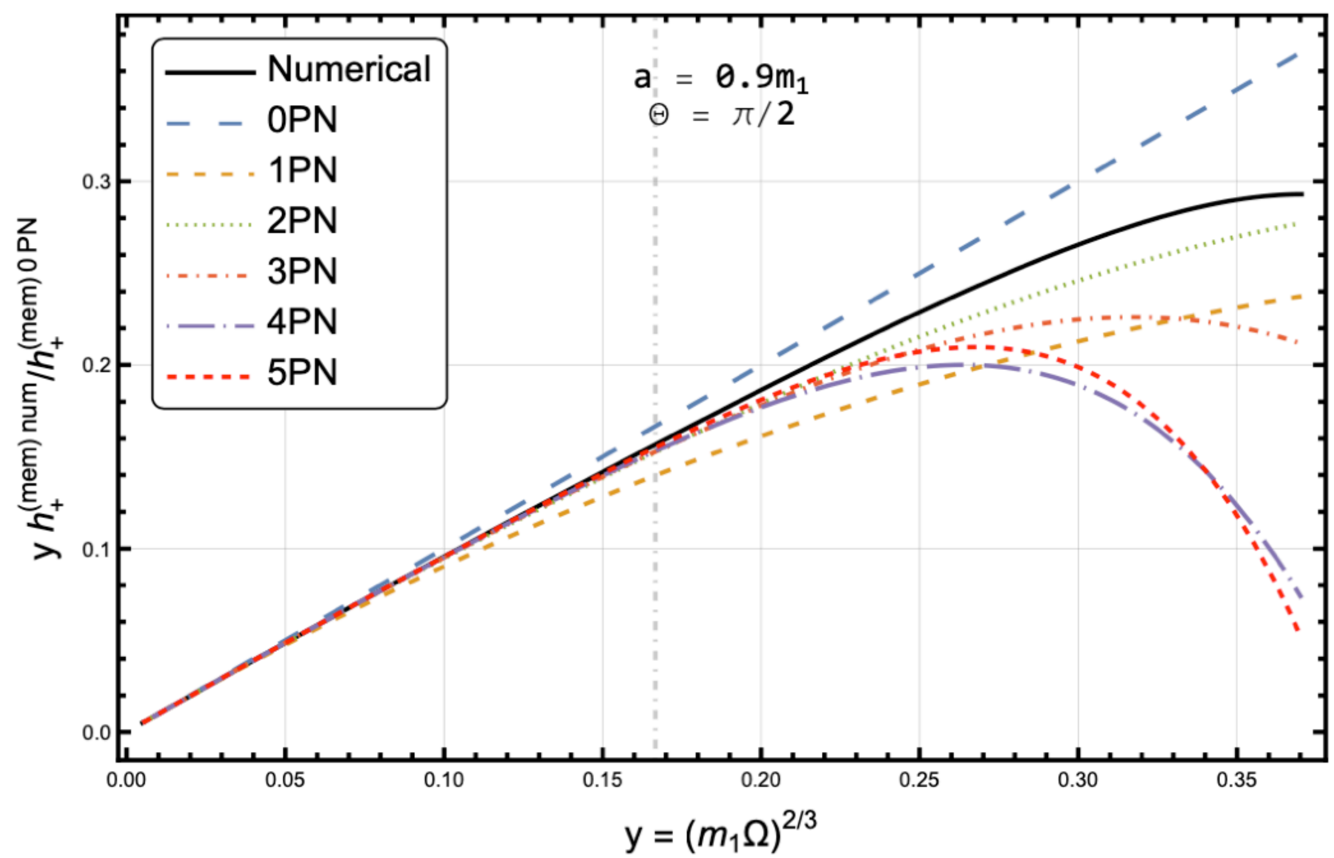


Memory in SXS:BBH:1124 (q=1)

- GW memory leads to a permanent displacement of the test masses after the GW has passed

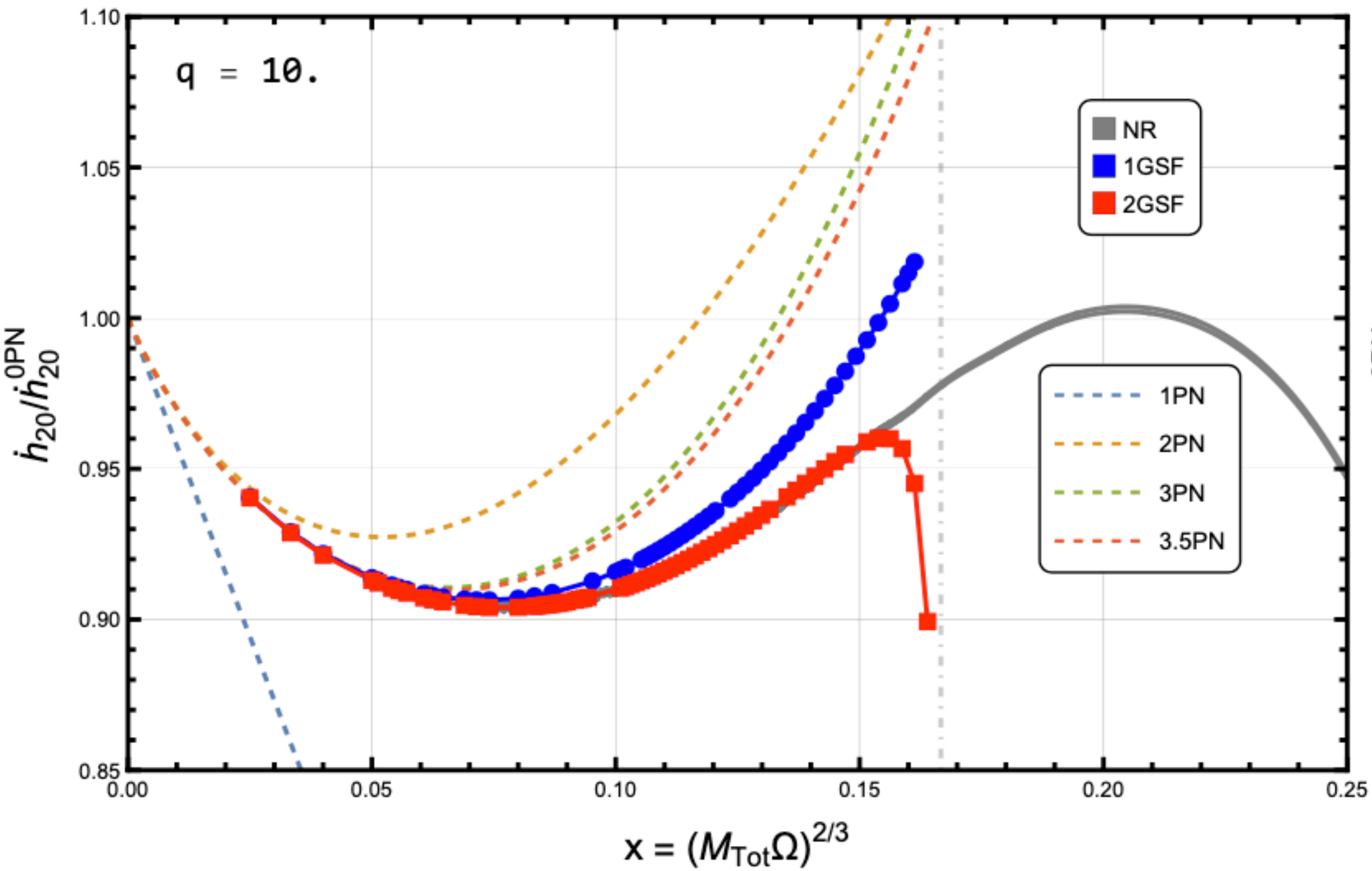
In a forthcoming paper we:

- Calculate the memory from $h^{(1)}_{\alpha\beta}$ during inspiral for a quasi-circular orbit into a Kerr BH
- Numerical and 5PN results



Gravitational wave memory

Cunningham, Kavanagh, Trestini, Pound, NW



We also make the computation including $h^{(2)}_{\alpha\beta}$ and find good agreement with NR at, e.g., $q = 10$

Modelling IMRIs

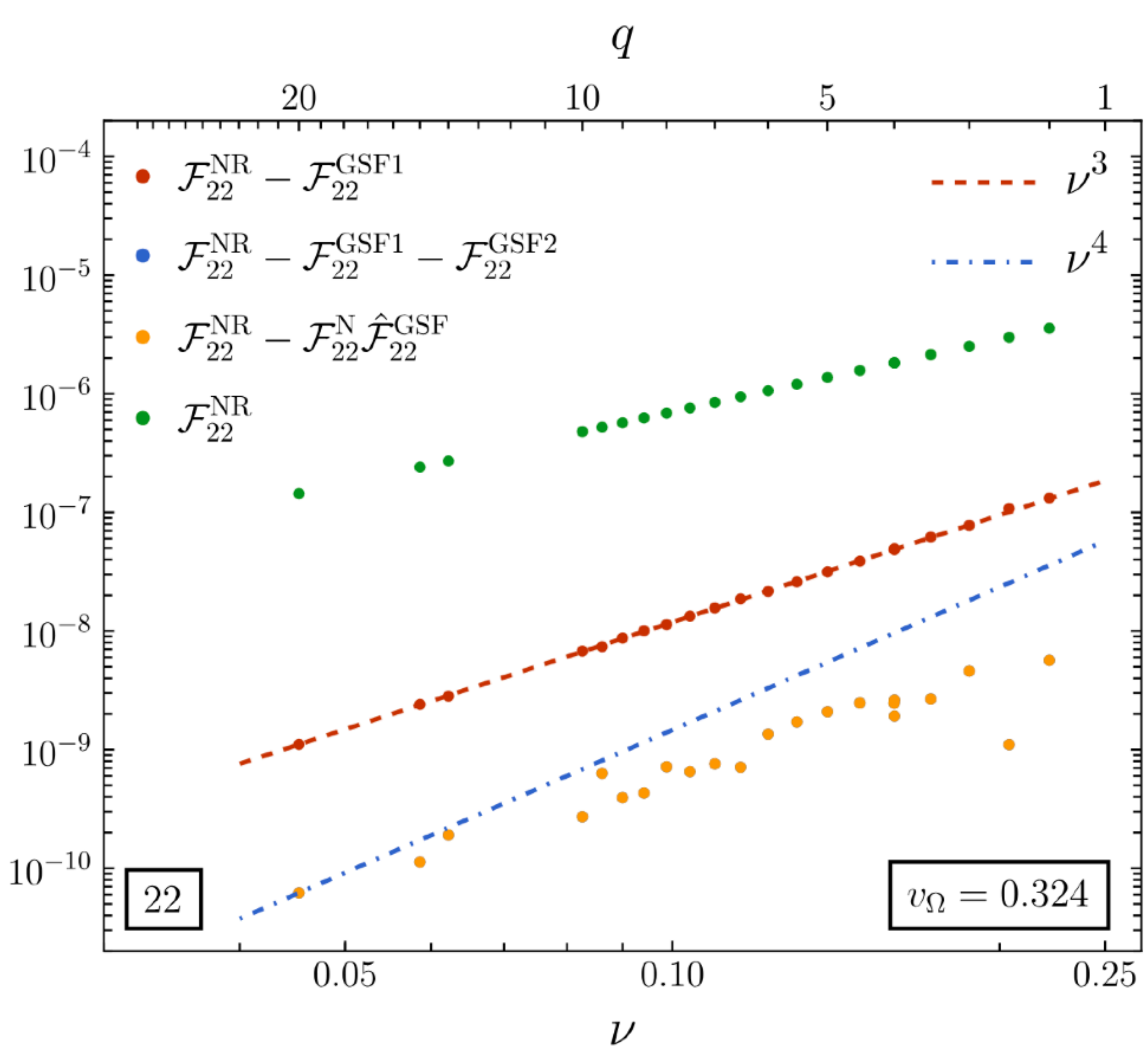
- IMRIs span mass ratio range from $q \simeq 10$ to $q \simeq 10^4$
- NR simulations are extremely expensive at $q \gtrsim 20$
- Q: How will we test our self-force models at, e.g., $q = \{100, 1000\}$?
- A: our perturbative expansive gives us precise control of our phase error, so long as we can estimate the coefficient of the unknown term

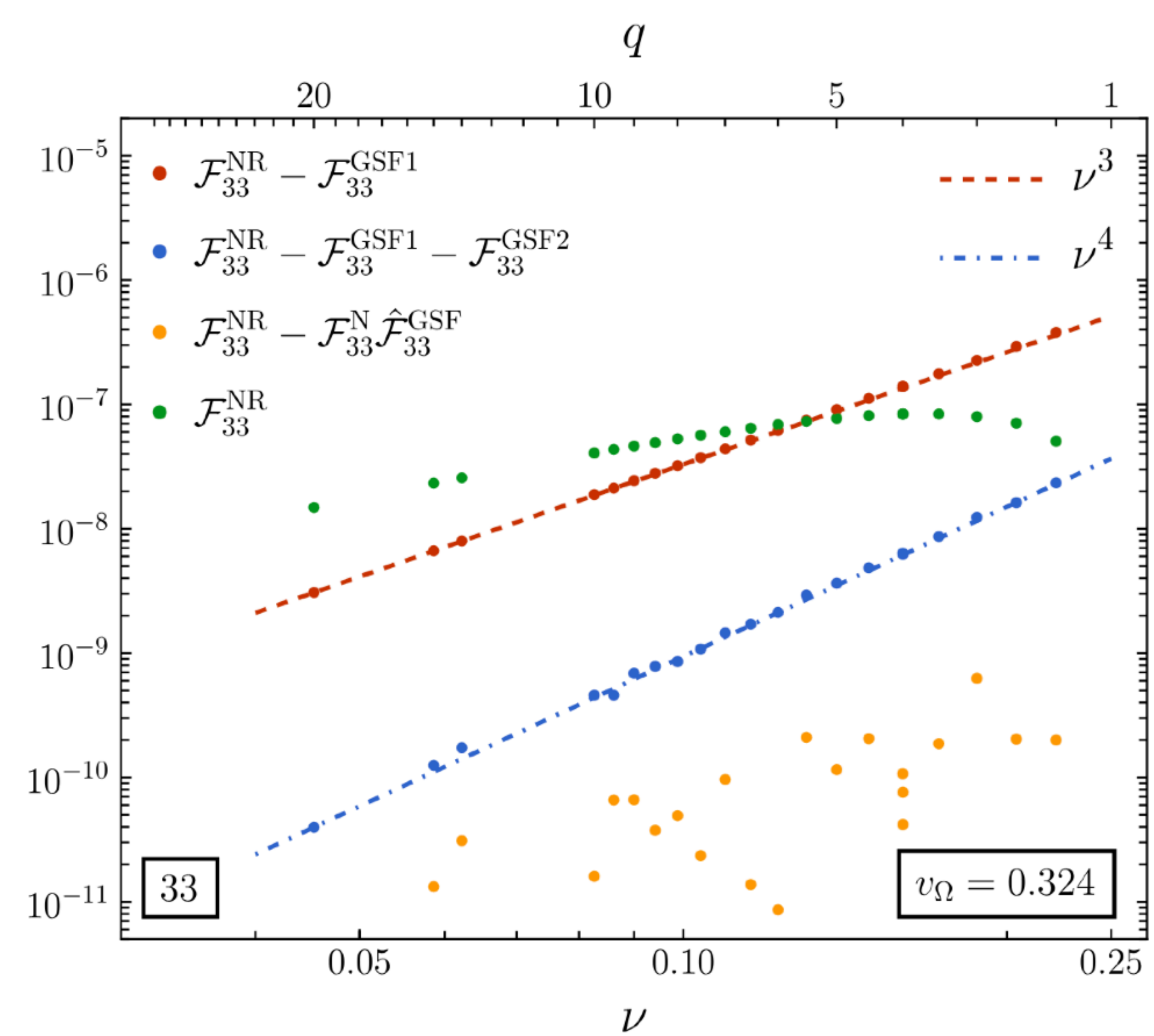
$$\tilde{\varphi}_A = \epsilon^{-1}\varphi_A^{(0)}(\Omega_B) + \epsilon^0\varphi_A^{(1)}(\Omega_B) + \mathcal{O}(\epsilon)$$

- Fortunately this can be achieved by comparison with NR simulations in the range $q = 1$ to $q \simeq 15$

Modelling IMRIs

Example: scaling with symmetric mass ratio ν for the flux





[Figures from van de Meent et al. arXiv:2303.18026]

Adding more physics

- So long as your extra physics acts on a longtime scale (or can be NIT'ed), the equations of motion become:

$$\frac{d\tilde{\varphi}_A}{dt} = \Omega_A(\tilde{J}_B)$$

$$\frac{d\tilde{J}_A}{dt} = \epsilon\left[F_A^{(0)}(\tilde{J}_B) + \epsilon F_A^{(1)}(\tilde{J}_B) + \mathcal{O}(\epsilon^2)\right] + \kappa F_A^{(\kappa)}(\tilde{J}_B)$$

- Examples include:
  - accretion disks
  - third-body perturbers (adds new resonances)
  - beyond-GR physics

- Once you have $F_A^{(\kappa)}(\tilde{J}_B)$, the multiscale framework and the modular construction of `FEW` means you can generate waveforms quickly

EMRIs beyond GR

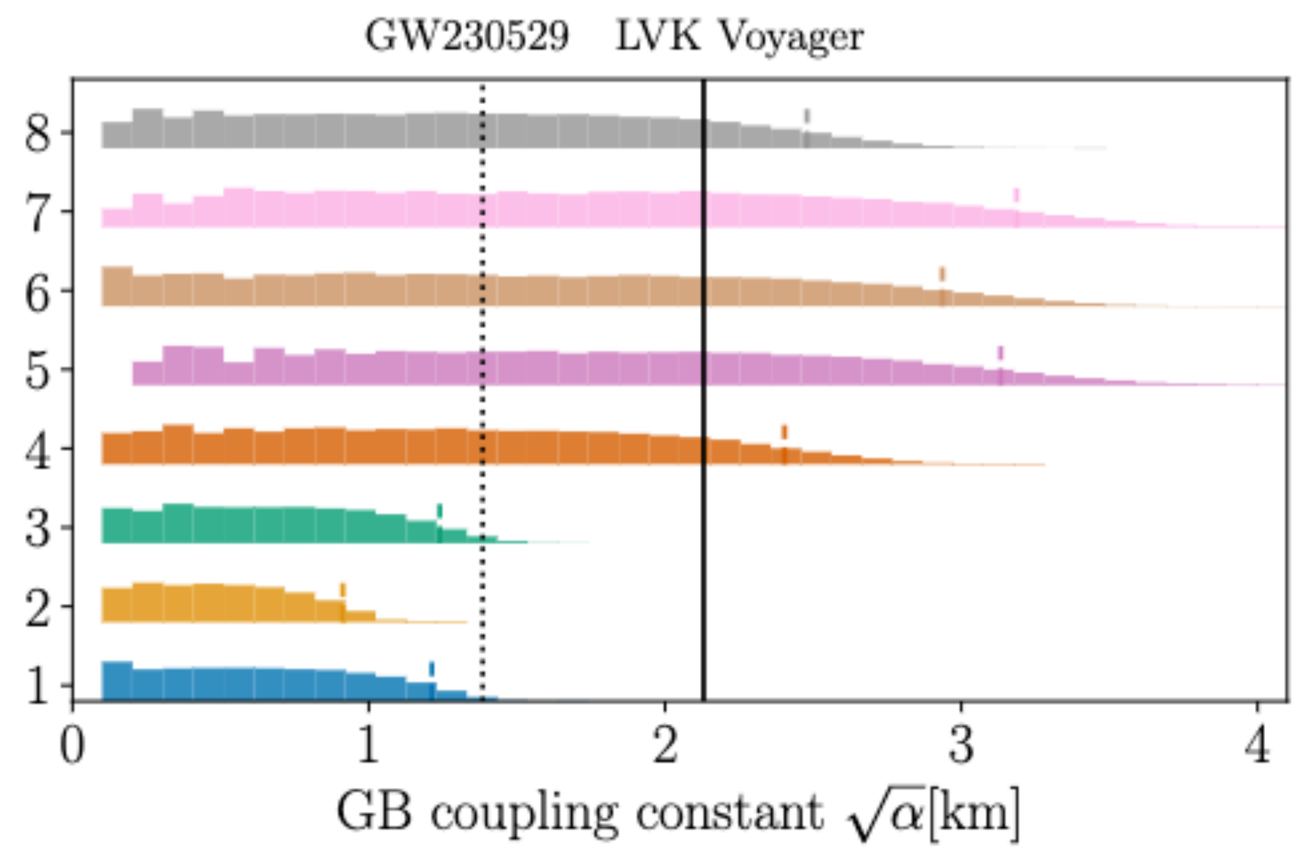
- A wide class of beyond-GR theories, e.g., linear-Gauss-Bonnet gravity, can be model by a secondary carrying a scalar charge

$$G^{\alpha\beta}[h^{(1)}_{\alpha\beta}] = 8\pi m_2 \int \delta^{(4)}(x - x_p(\lambda)) u^\alpha u^\beta g^{-1/2}\, d\lambda$$

$$\Box \phi^{(1)} = -4\pi d m_2 \int \delta^{(4)}(x - x_p(\lambda)) g^{-1/2}\, d\lambda$$

$$F^{(\kappa)}_A \propto \langle \partial_t \phi^{(1)} \rangle$$

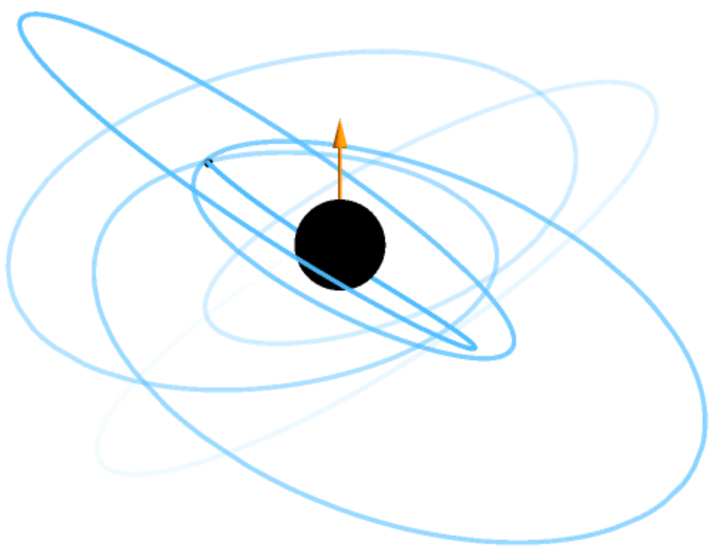
- By solving the scalar wave equation for eccentric orbits in Kerr we could carry out MCMC parameter estimation studies and explore constraints possible with LISA



Conclusions

- Using the post-adiabatic (multiscale) expansion we can compute $h^{(1)}_{\alpha\beta}$ and $h^{(2)}_{\alpha\beta}$

- There is a native, fast waveform generation scheme, which when combined with FEW gives EMRI waveforms in 10s of ms



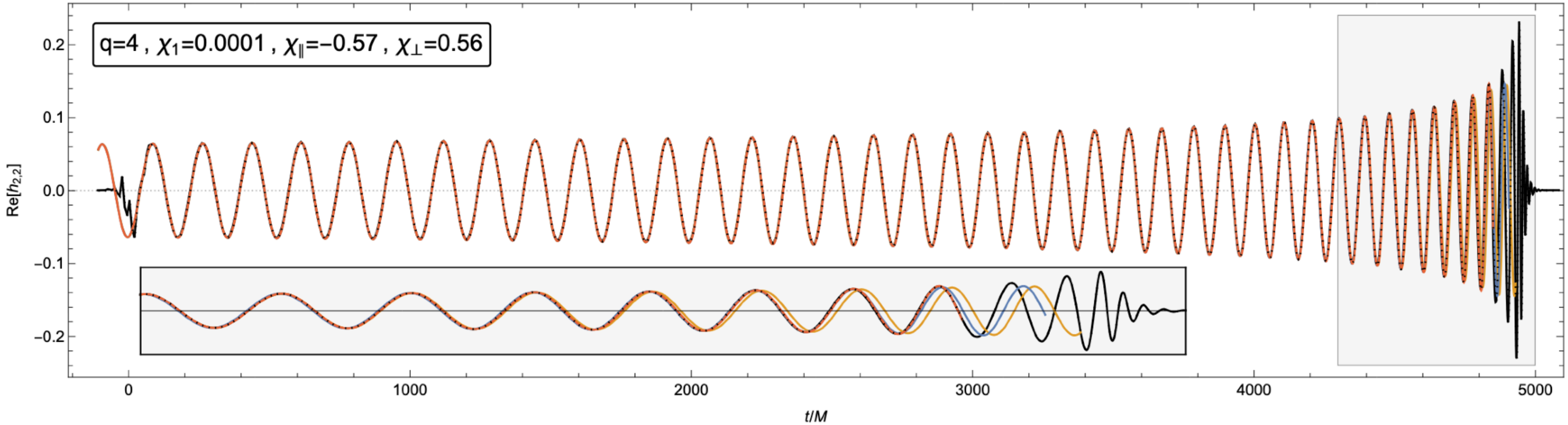
[Movie credit: Philip Lynch]

- Post-adiabatic waveforms agree very remarkably with NR waveforms even for $q \simeq 10$. This suggests we can model IMRIs with 1PA waveforms.

- Much of the Multiscale Self-Force (MSF) collaboration is pushing towards computing $h^{(2)}_{\alpha\beta}$ for a rotating primary

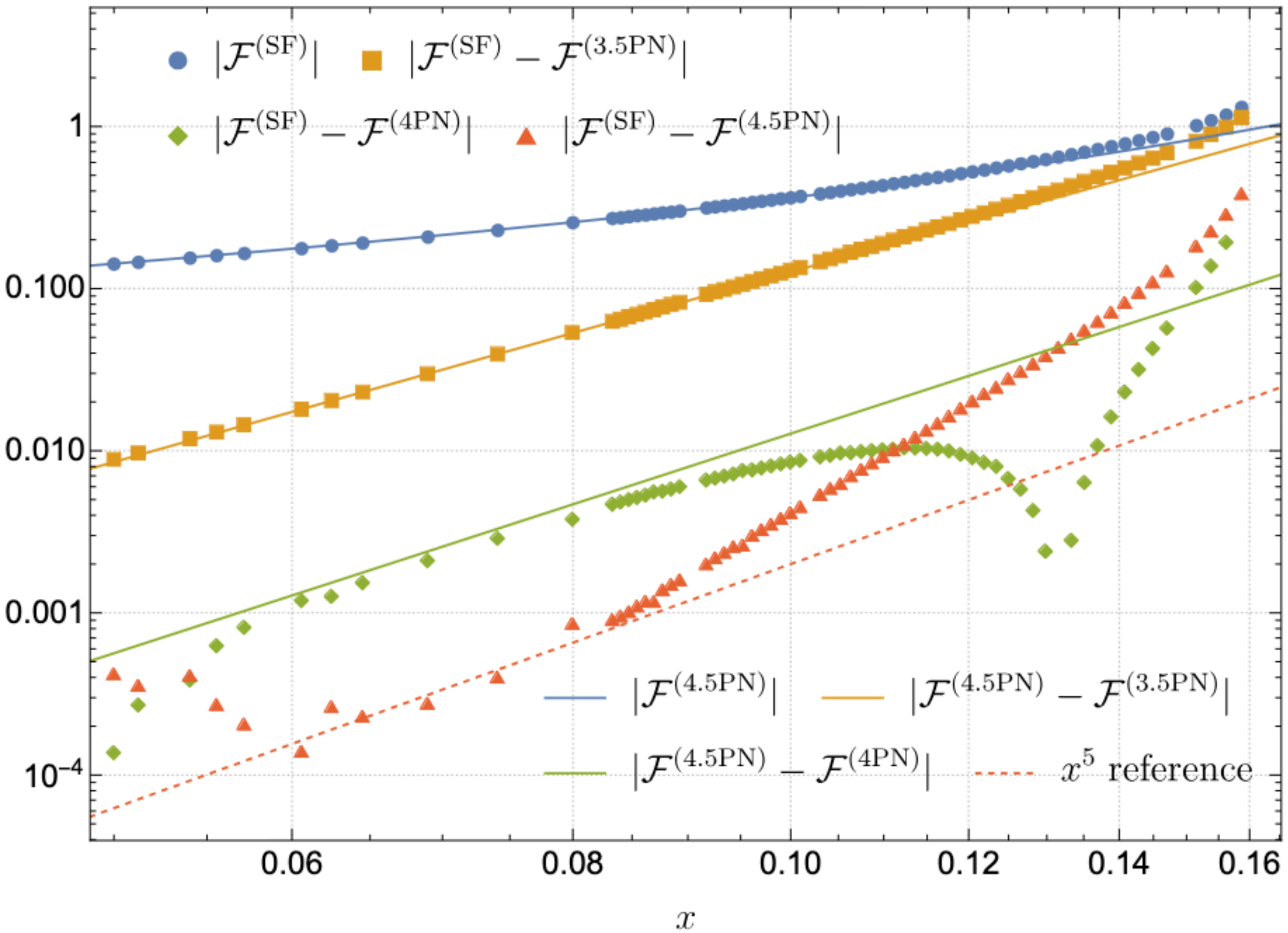
Extra slides

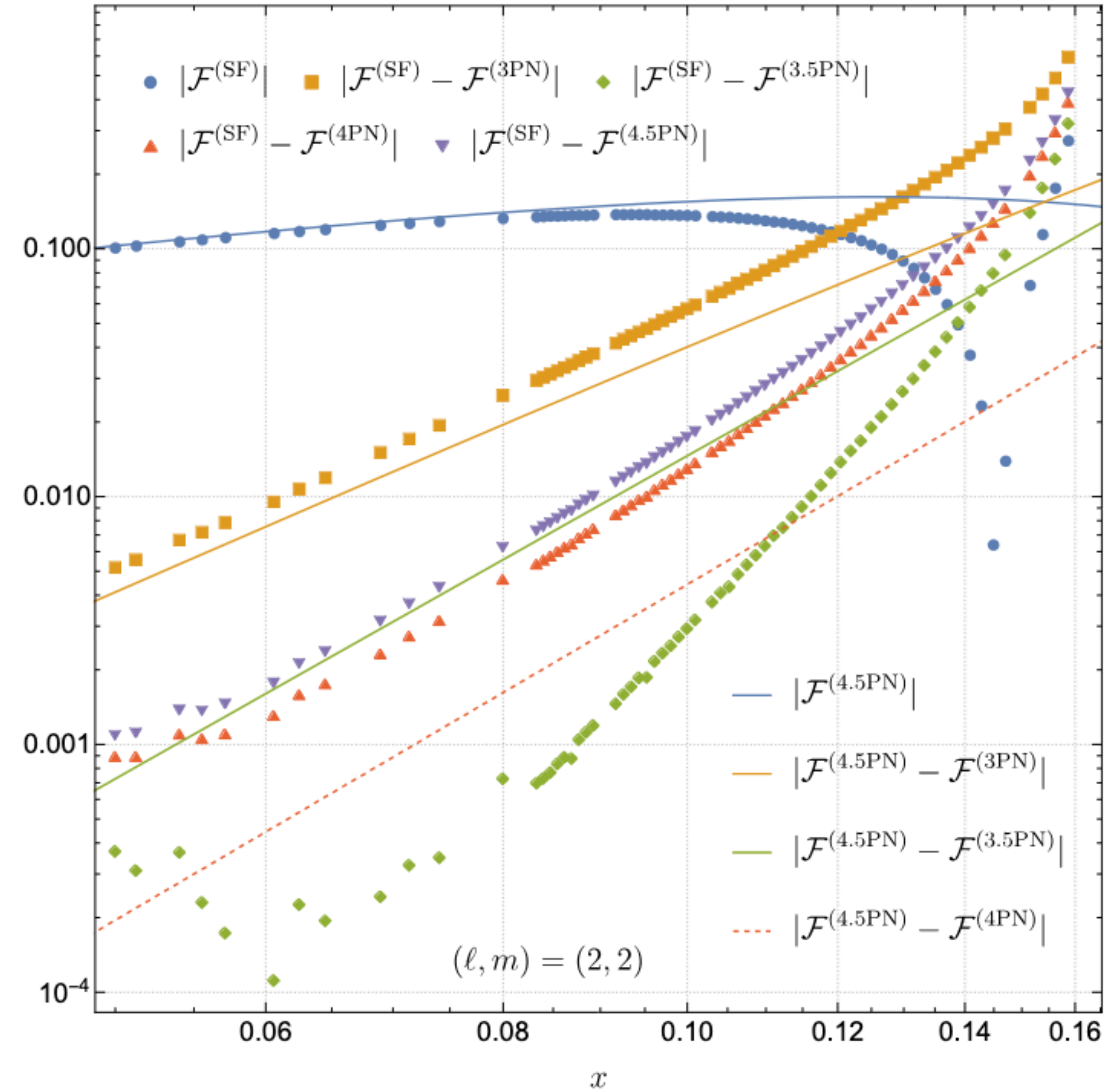
Comparison with NR waveform from SXS collaboration



- Complete quasi-circular 1PA inspiral model with generic (precessing) secondary spin, linear primary spin and evolving m_1 and χ_1
- Precession effects only enter the phase at 2PA (amplitudes effected at 1PA)

Waveform frame





Find agreement with 4.5PN for the total flux but not for the individual modes. Suggestions the calculations are in different frames

Field equations

$$G_{\alpha\beta}[\bar{g}_{\alpha\beta} + \epsilon h^{(1)}_{\alpha\beta} + \epsilon^2 h^{(2)}_{\alpha\beta}] = 8\pi T_{\alpha\beta}$$

Field equations from ϵ^n coefficients:

$$\epsilon^0: \qquad G_{\alpha\beta}[\bar{g}] = 0$$

$$\epsilon^1: \qquad G^1_{\alpha\beta}[h^1] = 8\pi T_{\alpha\beta}$$

$$\epsilon^2: \qquad G^1_{\alpha\beta}[h^2] = -G^2_{\alpha\beta}[h^1, h^1]$$

Field equations

$$G_{\alpha\beta}[\bar{g}_{\alpha\beta} + \epsilon h^{(1)}_{\alpha\beta} + \epsilon^2 h^{(2)}_{\alpha\beta}] = 8\pi T_{\alpha\beta}$$

Field equations from ϵ^n coefficients:

$$\epsilon^0 : \qquad G_{\alpha\beta}[\bar{g}] = 0$$

$$\epsilon^1 : \qquad G^1_{\alpha\beta}[h^{1S} + h^{1R}] = 8\pi T_{\alpha\beta}$$

$$\epsilon^2 : \qquad G^1_{\alpha\beta}[h^{2S} + h^{2R}] = -G^2_{\alpha\beta}[h^1, h^1]$$

Field equations

$$G_{\alpha\beta}[\bar{g}_{\alpha\beta} + \epsilon h^{(1)}_{\alpha\beta} + \epsilon^2 h^{(2)}_{\alpha\beta}] = 8\pi T_{\alpha\beta}$$

Field equations from ϵ^n coefficients:

Mino, Sasaki, Tanaka 1997
Quinn and Wald 1997
MiSaTaQuWa equations

$$\epsilon^0: \quad G_{\alpha\beta}[\bar{g}] = 0$$

$$\epsilon^1: \quad G^1_{\alpha\beta}[h^{1R}] = 8\pi T_{\alpha\beta} - G^1_{\alpha\beta}[h^{1S}]$$

$$\epsilon^2: \quad G^1_{\alpha\beta}[h^{2R}] = -G^2_{\alpha\beta}[h^1, h^1] - G^1_{\alpha\beta}[h^{2S}]$$

Pound 2012
Gralla 2012

Equations of motion

$$u^\beta \nabla_\beta u^\alpha = F^\alpha_{self}[\nabla h^{1R}, \nabla h^{2R}]$$

- Non-compact
- Diverges at the particle

Field equations

- We perform a two-timescale expansion by introducing a "slow time" $\tilde{t} = \epsilon t$. This allows for a frequency domain decomposition:

$$\Box^0 R^{2R}_{lm} = 2\delta^2 G^0_{lm} - \Box^0 R^{2P}_{lm} - \underline{\Box^1 R^1_{lm}}$$

$$h = \sum_{lm} R_{lm}(r) Y_{lm}(\theta, \varphi) e^{-i\omega_m t}$$

$$\partial_{\tilde{t}} h^1 = \dot{r}_0 \partial_{r_0} h^1$$

- As $\dot{r}_0 \propto \dot{\mathscr{E}} \propto \epsilon$ we see that a contribution from the (parametric derivative of the) first-order metric perturbation contributes to the second-order source

- For more complex orbital configurations we will need to compute parametric derivatives with respect to, e.g., (p, e, x_{inc})