Binary Black Holes in Strong Field: Post-Minkowskian, Numerical Relativity and Effective-One-Body

Piero Rettegno

in collaboration with T. Damour, P. Schmidt, G. Pratten, L. Thomas, A. Nagar and A. Placidi

September 18, 2024

IHES
Institut des Hautes Études Scientifiques
UNIVERSITY OF BIRMINGHAM
STUDIUM GENERALE CIVITATIS PERUSII
INFN
Istituto Nazionale di Fisica Nucleare
VIRGO

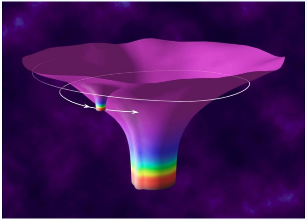
POST-MINKOWSKIAN

Theoretical models

GW signals are generally analyzed using different theoretical predictions, e.g.:

- NR surrogates;
- phenomenological approximants;
- EOB-based models.

I will focus on the latter, which are based on the **effective-one-body (EOB)** approach, first introduced by Buonanno and Damour [1, 2].



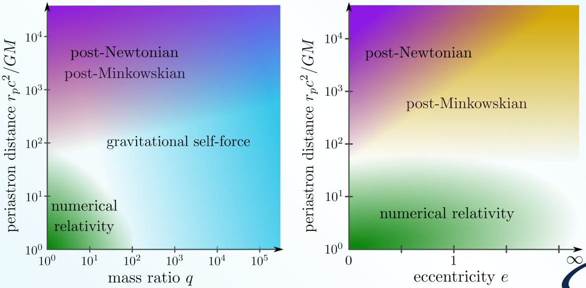
Both phenomenological and EOB models bring together analytical and numerical GR solutions.

Perturbation series

In order to solve Einstein's equations analytically, we need to make use of approximations, such as:

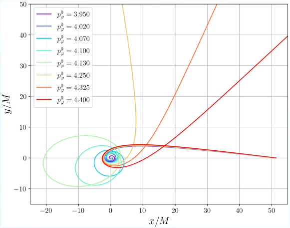
- ✧ **Post-Newtonian (PN)**, assuming small velocities, $\frac{v}{c} \ll 1$;
- ✧ **Post-Minkowskian (PM)**, assuming weak fields, $\frac{GM}{rc^2} \ll 1$;
- ✧ **Gravitational Self-Force (GSF)**, assuming large mass ratios, $\frac{m_2}{m_1} \ll 1$;

and others ...



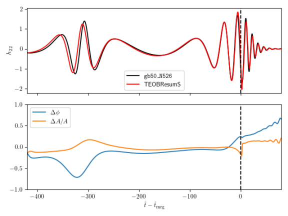
credit: Khalil *et al.* [3]

Post-Minkowskian

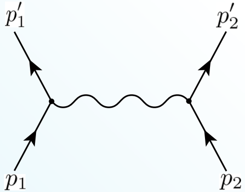


The PM approximation [4, 5, 6], which only assumes weak fields $[GM/(rc^2) \ll 1]$ and **allows for arbitrary large velocities**, is particularly suitable for describing scattering systems.

It hopefully could help improving GW models for eccentric and hyperbolic binaries signals.

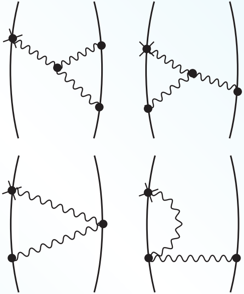


Recent advances



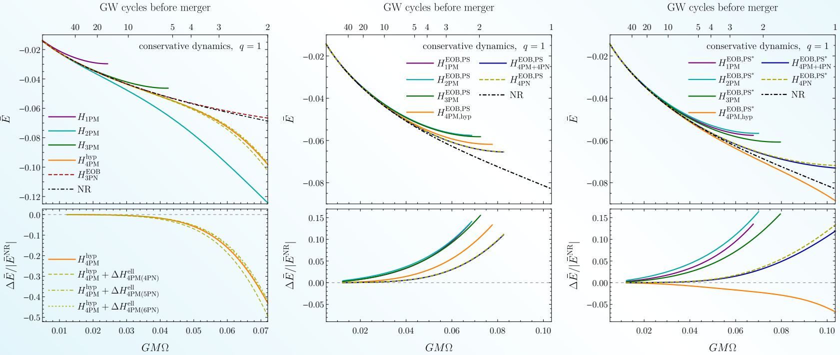
PM results have been computed through various approaches, such as: scattering amplitudes; eikonalization; effective field theory; and worldline (classical or quantum) field theory.

For black hole (BH) binaries, **4PM-accurate results** are available, both for **conservative** [7, 8] and **radiation-reacted dynamics** [6, 9, 10], **including spin-orbit** terms [11, 12].



Energy comparison

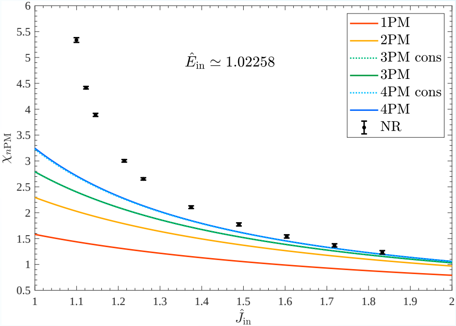
First attempts to use PM results to build EOB-based models for bound orbits were a little disappointing (see, e.g. [13, 3]).



credit: Khalil *et al.* [3]

Scattering angle comparison

The comparison against NR simulations of nonspinning BBH scattering [14] again showed poor agreement [15, 3].



The PM-expansion of scattering angles

$$\chi_{n\mathrm{PM}}(\gamma, j) \equiv \sum_{i=1}^{n} 2\frac{\chi_i(\gamma)}{j^i} = 2\frac{\chi_1(\gamma)}{j} + 2\frac{\chi_2(\gamma)}{j^2} + \dots$$

holds for large angular momenta but loses accuracy in strong-field systems.

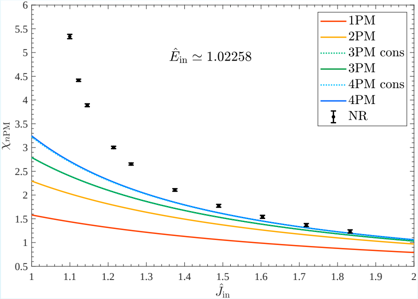
SCATTERING SYSTEMS

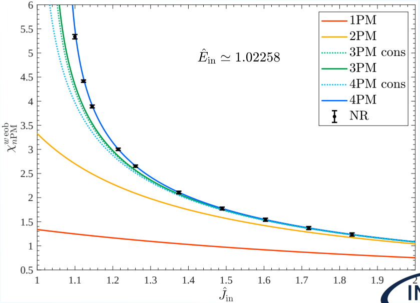
EOB resummation

We first proposed a **resummation** of the PM scattering angles [15], making use of an **EOB gravitational potential** of the form:

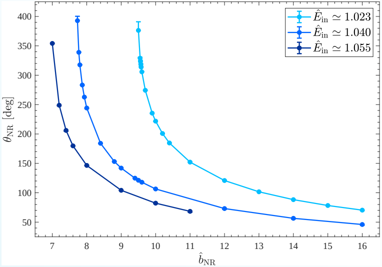
$$w_{n\mathrm{PM}}(\bar{r},\gamma) \equiv \sum_{i=1}^{n} \frac{w_i(\gamma)}{\bar{r}^i} = \frac{w_1(\gamma)}{\bar{r}} + \frac{w_2(\gamma)}{\bar{r}^2} + \dots$$

This reformulation greatly improves the agreement with numerical data (see also [16, 17, 18]):



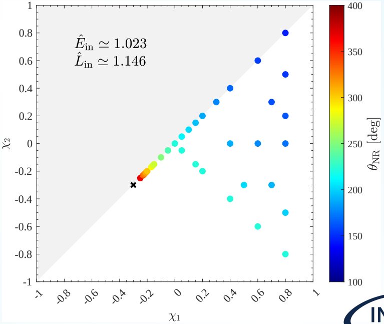


Spinning simulations



We performed **equal-mass, nonspinning simulations** [16] **at higher energies** using the `Einstein Toolkit` [19].

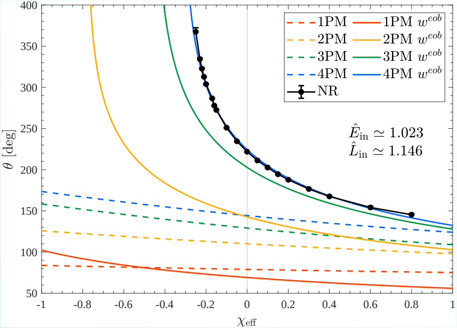
We also computed scattering angles for **equal-mass unequal-spin BBHs** [16].



Extension to spin

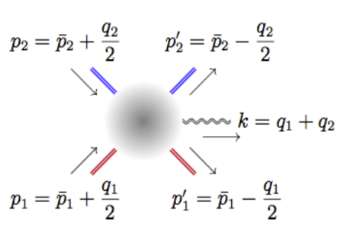
We could extend the EOB potential to take into account (aligned) spin effects:

$$w_{n\mathrm{PM}}(\bar{r},\gamma,\ell,S_i) = w^{\mathrm{orb}}(\bar{r},\gamma) + \frac{\ell\, w^{\mathrm{S}}_{n\mathrm{PM}}(\bar{r},\gamma)}{\bar{r}^2} + \frac{w^{\mathrm{S}^2}_{n\mathrm{PM}}(\bar{r},\gamma)}{\bar{r}^2} + \frac{\ell\, w^{\mathrm{S}^3}_{n\mathrm{PM}}(\bar{r},\gamma)}{\bar{r}^4} + \frac{w^{\mathrm{S}^4}_{n\mathrm{PM}}(\bar{r},\gamma)}{\bar{r}^4}.$$

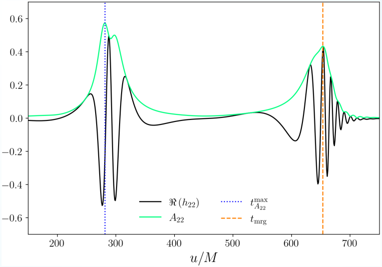


The 4PM EOB-resummed angles are in excellent agreement with numerical data (see also [17]).

Waveforms



Leading-order PM waveform computed long ago by Kovacs and Thorne [20].
One-loop computations $O\left(G^3\right)$ completed recently [21, 22], but not yet in a form useful for GW modelers.



There are also issues in extracting gravitational waveforms and fluxes from numerical simulations [30].

BOUND ORBITS

EOB-PM Hamiltonian: issue #1

It is possible to extract information about closed orbits from PM results [4, 23], generally by informing a (local) Hamiltonian [13, 3, 24].

While PN terms greatly simplify in the EOB frame [25, 26], **PM ones** keep their **convoluted dependence on the effective energy** (γ).

$$g^{\mu\nu}p_\mu p_\nu = -\frac{\gamma^2}{A} + \frac{p_r^2}{B} + \frac{j^2u^2}{C}\,.$$

$$A(u) = 1 - 2u + 2\nu u^3 + \left(\frac{94}{3} - \frac{41\pi^2}{32}\right)\nu u^4 + \left(\left(\frac{2275\pi^2}{512} - \frac{4237}{60} + \frac{128}{5}\gamma_E + \frac{256}{5}\ln 2\right)\nu + \left(\frac{41\pi^2}{32} - \frac{221}{6}\right)\nu^2 + \frac{64}{5}\nu\ln u\right)u^5, \tag{2.7a}$$

$$\bar{D}(u) = 1 + 6\nu u^2 + (52\nu - 6\nu^2)u^3 + \left(\left(-\frac{533}{45} - \frac{23761\pi^2}{1536} + \frac{1184}{15}\gamma_E - \frac{6496}{15}\ln 2 + \frac{2916}{5}\ln 3\right)\nu + \left(\frac{123\pi^2}{16} - 260\right)\nu^2 + \frac{592}{15}\nu\ln u\right)u^4, \tag{2.7b}$$

$$\hat{Q}(\mathbf{r}',\mathbf{p}') = \left(2(4-3\nu)\nu u^2 + \left(\left(-\frac{5308}{15} + \frac{496256}{45}\ln 2 - \frac{33048}{5}\ln 3\right)\nu - 83\nu^2 + 10\nu^3\right)u^3\right)(\mathbf{n}'\cdot\mathbf{p}')^4 + \left(\left(-\frac{827}{3} - \frac{2358912}{25}\ln 2 + \frac{1399437}{50}\ln 3 + \frac{390625}{18}\ln 5\right)\nu - \frac{27}{5}\nu^2 + 6\nu^3\right)u^2(\mathbf{n}'\cdot\mathbf{p}')^6 + O[\nu u(\mathbf{n}'\cdot\mathbf{p}')^8]. \tag{2.7c}$$

Hamiltonian potentials up to 4PN

$$\begin{aligned}
\alpha^{(4)}_{(0,0)} &= \frac{7\nu(380\gamma^2+169)}{8(\gamma-1)\gamma^2\Gamma^3}\mathrm{E}^2\left(\frac{\gamma-1}{\gamma+1}\right) + \frac{(1200\gamma^2+2095\gamma+834)\nu}{4\gamma^2(\gamma^2-1)\Gamma^3}\mathrm{K}^2\left(\frac{\gamma-1}{\gamma+1}\right)\\
&+ \frac{(-1200\gamma^3-2660\gamma^2-2929\gamma-1183)\nu}{4\gamma^2(\gamma^2-1)\Gamma^3}\mathrm{E}\left(\frac{\gamma-1}{\gamma+1}\right)\mathrm{K}\left(\frac{\gamma-1}{\gamma+1}\right)\\
&+ \frac{(-25\gamma^6+30\gamma^4+111\gamma^2+20)\nu}{\gamma^2\Gamma^3}\mathrm{Li}_2\left(\frac{1-\gamma}{1+\gamma}\right) + \frac{(\gamma+1)(25\gamma^5-25\gamma^4-5\gamma^3+65\gamma^2+64\gamma+12)\nu}{2\gamma^2\Gamma^3}\mathrm{Li}_2\left(\frac{\gamma-1}{\gamma+1}\right)\\
&+ \frac{(35\gamma^4+120\gamma^3+90\gamma^2+152\gamma+27)\nu}{2\gamma^2\Gamma^3}\log^2\left(\frac{\gamma+1}{2}\right) - \frac{4(2\gamma^2-3)(15\gamma^2-15\gamma+4)\nu}{\gamma(\gamma+1)\Gamma^3}\frac{\mathrm{Ti}_2\left(\sqrt{\frac{1-\gamma}{1+\gamma}}\right)}{\sqrt{1-\gamma^2}}\\
&+ \frac{(2\gamma^2-3)^2(35\gamma^4-30\gamma^2+11)\nu}{8(\gamma^2-1)^3\Gamma^3}\arccos^2\gamma + \frac{2(75\gamma^6-140\gamma^4-283\gamma^2-852)\nu}{3\gamma(\gamma^2-1)\Gamma^3}\log(\gamma)\\
&+ \frac{(210\gamma^6-552\gamma^5+339\gamma^4-912\gamma^3+3148\gamma^2-3336\gamma+1151)\nu}{12\gamma^2(\gamma^2-1)\Gamma^3}\log\left(\frac{u}{4}\right)\\
&+ \left(\frac{(-35\gamma^4-60\gamma^3+150\gamma^2-76\gamma+5)\nu}{2\gamma^2\Gamma^3}\log\left(\frac{u}{4}\right)\right.\\
&\qquad\left.+ \frac{(-75\gamma^7+416\gamma^5+612\gamma^4+739\gamma^3+136\gamma^2+2520\gamma+152)\nu}{3\gamma^2(\gamma^2-1)\Gamma^3}\right)\log\left(\frac{\gamma+1}{2}\right)\\
&+ \left(\frac{(-420\gamma^9+96\gamma^8-48\gamma^7+5328\gamma^6-5279\gamma^5-1584\gamma^4+7142\gamma^3-9360\gamma^2+3453\gamma+720)\nu}{12\gamma^2(\gamma^2-1)^2\Gamma^3}\right.\\
&\qquad - \frac{48(7\gamma^2-5)(4\gamma^4-12\gamma^2-3)(\Gamma-1)\nu}{12\gamma^2(\gamma^2-1)\Gamma^3}\\
&\qquad\left. - \frac{(2\gamma^2-3)(35\gamma^4-30\gamma^2+11)\nu}{4\gamma(1-\gamma^2)\Gamma^3}\log\left(\frac{u}{4}\right) + \frac{4(2\gamma^2-3)(15\gamma^2+2)\nu}{\gamma(1-\gamma^2)\Gamma^3}\log\left(\frac{\gamma+1}{2}\right)\right)\frac{\arccos\gamma}{\sqrt{1-\gamma^2}}\\
&+ (\Gamma-1)\left(\frac{5115\gamma^8-9537\gamma^6+5657\gamma^4-1115\gamma^2+72}{16\gamma^4(\gamma^2-1)^2\Gamma^3}\right.\\
&\qquad\left.+ \frac{(8159\gamma^8-3136\gamma^7-23601\gamma^6-3360\gamma^5+15409\gamma^4+4000\gamma^3-1995\gamma^2+108)\nu}{24(\gamma-1)\gamma^4(\gamma+1)^2\Gamma^3}\right)\\
&+ \frac{\nu}{144\gamma^9(\gamma^2-1)^2\Gamma^3}\Big(-600\pi^2\gamma^{17} + 3600\gamma^{16} + 480(9+4\pi^2)\gamma^{15} + 2(720\pi^2-28843)\gamma^{14} + (36759-5136\pi^2)\gamma^{13}\\
&\qquad + (44698-1056\pi^2)\gamma^{12} + (6624\pi^2-43235)\gamma^{11} + (7702-2208\pi^2)\gamma^{10} - 5(2155+504\pi^2)\gamma^9\\
&\qquad + 2(23947+912\pi^2)\gamma^8 - (45605+288\pi^2)\gamma^7 + 12701\gamma^6 + 648\gamma^5 - 1471\gamma^4 + 207\gamma^2 - 45\Big),
\end{aligned}$$

Hamiltonian contribution at 4PM

EOB-PM Hamiltonian: issue #2

PM computations inherently contain **open-orbit hereditary contributions**.

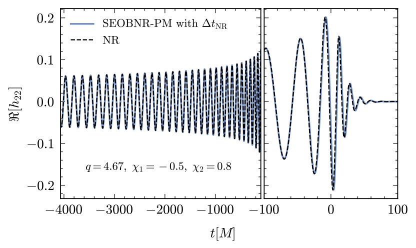
A satisfactory 4PM-accurate splitting between local and nonlocal-in-time terms has not yet been obtained [27, 28].

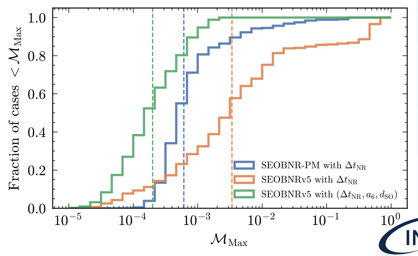
Bini and Damour [28] have obtained the last 4PM term in the local action up to order p_∞^{30}.

A possible EOB-PM Hamiltonian

Buonanno *et al.* [24] proposed an EOB-PM model for bound orbits:

- the spinning Hamiltonian is obtained by iteratively expressing γ as a function of phase-space variables ($\gamma \to \hat{H}_{\mathrm{Schw}} + \hat{H}_{\mathrm{2PM}} + \dots$);
- problematic factors in nonlocal terms are substituted by well-defined quantities [$\log(\gamma^2 - 1) \to \log(u), \dots$];
- the nonspinning Hamiltonian is completed by 4PN terms, both local and nonlocal (for bound orbits);
- the waveform is calibrated to NR simulations.





A new approach to EOB

In an upcoming paper, Damour *et al.* [29] will propose a new way of solving the EOB equations of motions.

Instead of solving the usual Hamilton's equations

$$S[x^\mu, p_\mu] = \int \left[p_i \frac{dx^i}{t_{\text{eff}}} - \hat{H}_{\text{eff}}(x^i, p_i) \right] dt_{\text{eff}} \longrightarrow \begin{cases} \dfrac{dx^i}{dt} = \dfrac{\partial \hat{H}_{\text{eff}}}{\partial p_i} \\ \dfrac{dp_i}{dt} = -\dfrac{\partial \hat{H}_{\text{eff}}}{\partial x^i} + \mathcal{F}_i \end{cases}$$

one can introduce a **Lagrange multiplier** e_{L} and a **constraint** $\mathcal{C}$ such that

$$S[x^\mu, p_\mu, e_{\text{L}}] = \int \left[p_\mu \frac{dx^\mu}{d\tau} - e_{\text{L}}\, \mathcal{C}\,(x^\mu, p_\mu) \right] d\tau \longrightarrow \begin{cases} \dfrac{dx^\mu}{d\tau} = e_{\text{L}} \dfrac{\partial \mathcal{C}}{\partial p_\mu} \\ \dfrac{dp_\mu}{d\tau} = -e_{\text{L}} \dfrac{\partial \mathcal{C}}{\partial x^\mu} + \mathcal{F}_\mu \\ \mathcal{C} = 0 \end{cases}$$

Pro and cons

We can then **avoid to invert the mass-shell condition** (to determine $\hat{H}_{\text{eff}} = \gamma$) at the cost of **solving one additional differential equation** $(d\gamma/dt_{\text{eff}})$.

In general, the EOB mass-shell constraint will look like

$$\mathcal{C} = g^{\mu\nu} p_\mu p_\nu + 1 + Q = -\frac{\gamma^2}{A} + \frac{p_r^2}{B} + \frac{p_\varphi^2 u^2}{C} + 1 + Q = 0.$$

(Schwarzschild is recovered for $A = B^{-1} = 1 - 2u$, $C = 1$, $Q = 0$)

If we can explicitly solve the constraint for $\hat{H}_{\text{eff}}$ (e.g., the EOB potentials do not depend on γ), these new Euler-Lagrange equations are equivalent to Hamilton's equations.

Naturally, PM-informed EOB potential depend in a complicated manner on γ and this approach simplifies computations.

Our choices

We decided to choice a **gauge close to the Schwarzschild case**, i.e.

$$A(\gamma) = B(\gamma)^{-1}, \quad C = 1, \quad Q = 0.$$

Our constraint will be, including spin-orbit interaction:

$$\mathcal{C} = -\frac{[\gamma - p_\varphi G(\gamma) S]^2}{A(\gamma)} + A(\gamma) p_r^2 + p_\varphi^2 u^2 + 1 = 0.$$

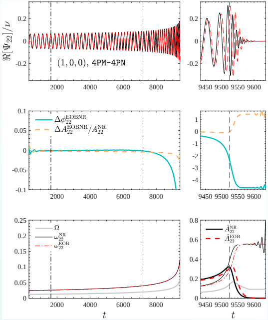
We define A as

$$A_{4\text{PM}}^{4\text{PN}} = A_{\sim 4\text{PM}}^{\text{local}} + \Delta A^{4\text{PN}},$$

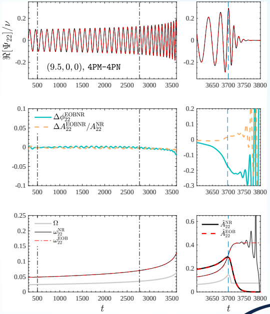
and

- $A_{\sim 4\text{PM}}^{\text{local}}$ is determined by **matching the 4PM local scattering angle** prediction using the almost complete $\chi_{\sim 4\text{PM}}^{\text{local}}$ [28];
- $\Delta A^{4\text{PN}}$ is obtained by **imposing the complete bound-orbit 4PN behavior**.

Results [preliminary]



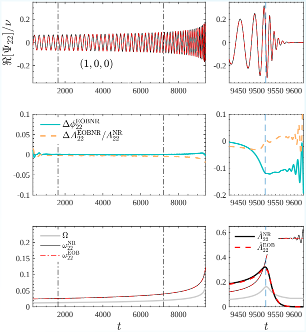
With no NR calibration (up to merger), we are able to reach a reasonable agreement with SXS waveforms (mismatches $\sim 10^{-2}$ → see Andrea's talk).



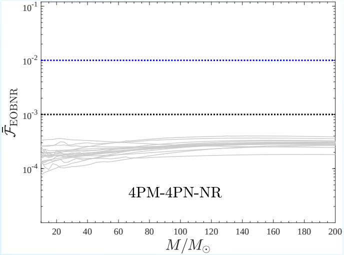
The agreement improves increasing the mass-ratio.

We are working on the inclusion of 8.5PN, 1GSF-accurate terms in the Hamiltonian to assess if we can improve further.

Results [preliminary]



Adding a free 5PM-5PN parameter and fitting it we are able to reach a mismatch of $10^{-4}/10^{-3}$ against all quasi-circular nonspinning SXS waveforms.



[Spinning waveforms almost ready].

Conclusions

✧ **High-order PM results**, once recasted in a particular EOB form, **show excellent agreement with NR scattering** simulations.

✧ The first **EOB-PM models** applied to noncircular (nonprecessing) **bound orbits are under construction**.

However, some things are still missing:

1. **more analytical information**, be it PM, PN or GSF, could help to build a **fully analytical model**;
2. a **PM-based description of the radiative sector** is not yet usable;
3. **additional NR simulations** (and waveforms) are necessary to validate our models throughout the parameter space.

Thank you for your attention

References I

[1] A. Buonanno and T. Damour, “Effective one-body approach to general relativistic two-body dynamics,” *Phys. Rev.*, vol. D59, p. 084006, 1999.

[2] A. Buonanno and T. Damour, “Transition from inspiral to plunge in binary black hole coalescences,” *Phys. Rev.*, vol. D62, p. 064015, 2000.

[3] M. Khalil, A. Buonanno, J. Steinhoff, and J. Vines, “Energetics and scattering of gravitational two-body systems at fourth post-Minkowskian order,” *Phys. Rev. D*, vol. 106, no. 2, p. 024042, 2022.

[4] T. Damour, “Gravitational scattering, post-Minkowskian approximation and Effective One-Body theory,” *Phys. Rev.*, vol. D94, no. 10, p. 104015, 2016.

[5] Z. Bern, C. Cheung, R. Roiban, C.-H. Shen, M. P. Solon, and M. Zeng, “Scattering Amplitudes and the Conservative Hamiltonian for Binary Systems at Third Post-Minkowskian Order,” *Phys. Rev. Lett.*, vol. 122, no. 20, p. 201603, 2019.

[6] D. Bini, T. Damour, and A. Geralico, “Radiative contributions to gravitational scattering,” *Phys. Rev. D*, vol. 104, no. 8, p. 084031, 2021.

[7] Z. Bern, J. Parra-Martinez, R. Roiban, M. S. Ruf, C.-H. Shen, M. P. Solon, and M. Zeng, “Scattering Amplitudes, the Tail Effect, and Conservative Binary Dynamics at O(G4),” *Phys. Rev. Lett.*, vol. 128, no. 16, p. 161103, 2022.

[8] C. Dlapa, G. Kälin, Z. Liu, and R. A. Porto, “Conservative Dynamics of Binary Systems at Fourth Post-Minkowskian Order in the Large-Eccentricity Expansion,” *Phys. Rev. Lett.*, vol. 128, no. 16, p. 161104, 2022.

[9] A. V. Manohar, A. K. Ridgway, and C.-H. Shen, “Radiated Angular Momentum and Dissipative Effects in Classical Scattering,” *Phys. Rev. Lett.*, vol. 129, no. 12, p. 121601, 2022.

[10] C. Dlapa, G. Kälin, Z. Liu, J. Neef, and R. A. Porto, “Radiation Reaction and Gravitational Waves at Fourth Post-Minkowskian Order,” *Phys. Rev. Lett.*, vol. 130, no. 10, p. 101401, 2023.

References II

[11] G. U. Jakobsen, G. Mogull, J. Plefka, B. Sauer, and Y. Xu, “Conservative Scattering of Spinning Black Holes at Fourth Post-Minkowskian Order,” *Phys. Rev. Lett.*, vol. 131, no. 15, p. 151401, 2023.

[12] G. U. Jakobsen, G. Mogull, J. Plefka, and B. Sauer, “Dissipative Scattering of Spinning Black Holes at Fourth Post-Minkowskian Order,” *Phys. Rev. Lett.*, vol. 131, no. 24, p. 241402, 2023.

[13] A. Antonelli, A. Buonanno, J. Steinhoff, M. van de Meent, and J. Vines, “Energetics of two-body Hamiltonians in post-Minkowskian gravity,” *Phys. Rev.*, vol. D99, no. 10, p. 104004, 2019.

[14] T. Damour, F. Guercilena, I. Hinder, S. Hopper, A. Nagar, and L. Rezzolla, “Strong-Field Scattering of Two Black Holes: Numerics Versus Analytics,” *Phys. Rev. D*, vol. 89, no. 8, p. 081503, 2014.

[15] T. Damour and P. Rettegno, “Strong-field scattering of two black holes: Numerical relativity meets post-Minkowskian gravity,” *Phys. Rev. D*, vol. 107, no. 6, p. 064051, 2023.

[16] P. Rettegno, G. Pratten, L. M. Thomas, P. Schmidt, and T. Damour, “Strong-field scattering of two spinning black holes: Numerical relativity versus post-Minkowskian gravity,” *Phys. Rev. D*, vol. 108, no. 12, p. 124016, 2023.

[17] A. Buonanno, G. U. Jakobsen, and G. Mogull, “Post-Minkowskian theory meets the spinning effective-one-body approach for two-body scattering,” *Phys. Rev. D*, vol. 110, no. 4, p. 044038, 2024.

[18] A. Kankani and S. T. McWilliams, “Testing the boundary-to-bound correspondence with numerical relativity,” *Phys. Rev. D*, vol. 110, no. 6, p. 064033, 2024.

[19] R. H. *et al.*, “The einstein toolkit,” Oct. 2022.
To find out more, visit http://einsteintoolkit.org.

[20] S. J. Kovacs and K. S. Thorne, “The Generation of Gravitational Waves. 3. Derivation of Bremsstrahlung Formulas,” *Astrophys. J.*, vol. 217, pp. 252–280, 1977.

References III

[21] A. Brandhuber, G. R. Brown, G. Chen, S. De Angelis, J. Gowdy, and G. Travaglini, "One-loop gravitational bremsstrahlung and waveforms from a heavy-mass effective field theory," *JHEP*, vol. 06, p. 048, 2023.

[22] A. Georgoudis, C. Heissenberg, and I. Vazquez-Holm, "Inelastic exponentiation and classical gravitational scattering at one loop," *JHEP*, vol. 2023, no. 06, p. 126, 2023.

[23] G. Kälin and R. A. Porto, "From Boundary Data to Bound States," *JHEP*, vol. 01, p. 072, 2020.

[24] A. Buonanno, G. Mogull, R. Patil, and L. Pompili, "Post-Minkowskian Theory Meets the Spinning Effective-One-Body Approach for Bound-Orbit Waveforms," 5 2024.

[25] T. Damour, P. Jaranowski, and G. Schäfer, "Fourth post-Newtonian effective one-body dynamics," *Phys. Rev. D*, vol. 91, no. 8, p. 084024, 2015.

[26] T. Damour, P. Jaranowski, and G. Schäfer, "Conservative dynamics of two-body systems at the fourth post-Newtonian approximation of general relativity," *Phys. Rev.*, vol. D93, no. 8, p. 084014, 2016.

[27] C. Dlapa, G. Kälin, Z. Liu, and R. A. Porto, "Local in Time Conservative Binary Dynamics at Fourth Post-Minkowskian Order," *Phys. Rev. Lett.*, vol. 132, no. 22, p. 221401, 2024.

[28] D. Bini and T. Damour, "Fourth post-Minkowskian local-in-time conservative dynamics of binary systems," *Phys. Rev. D*, vol. 110, no. 6, p. 064005, 2024.

[29] T. Damour, A. Nagar, A. Placidi, and P. Rettegno, "A novel Lagrange-multiplier approach to the effective-one-body dynamics
of binary systems in post-Minkowskian gravity," *in prep.*, 2024.

[30] S. Albanesi, A. Rashti, F. Zappa, R. Gamba, W. Cook, B. Daszuta, S. Bernuzzi, A. Nagar, and D. Radice, "Scattering and dynamical capture of two black holes: synergies between numerical and analytical methods," 5 2024.

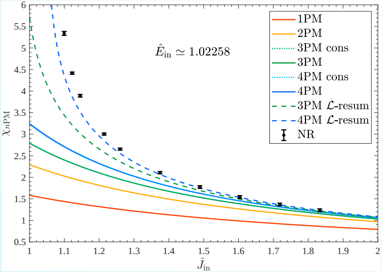
Critical angular momentum

We first introduced [15] a resummation of the PM scattering angles that takes into account the j-singularity due to the boundary between scattering and plunge, such that

$$\chi^{\mathcal{L}}_{n\mathrm{PM}}(\gamma, j) = \mathcal{L}\left(\frac{j_0}{j}\right) \hat{\chi}_{n\mathrm{PM}}(\gamma, j; j_0)\,,$$

with

$$\mathcal{L}(x) \equiv \frac{1}{x} \ln\left[\frac{1}{1-x}\right]\,, \qquad \text{and} \qquad j_0^{n\mathrm{PM}}(\gamma) \equiv \left[n\frac{\chi_n(\gamma)}{\chi_1(\gamma)}\right]^{\frac{1}{n-1}}\,.$$



This procedure already improves the PM-NR agreement.

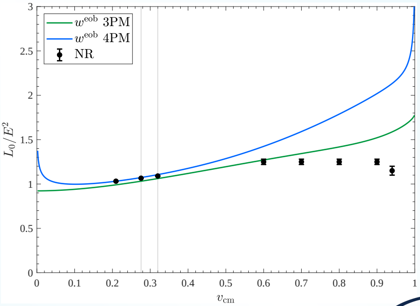
Critical j_0 predictions

We can compare analytical and numerical predictions for the critical angular momentum J_0,

$$\frac{J_0}{E^2} = \frac{\nu\, j_0}{1 + 2\nu\,(\gamma - 1)}\,, \tag{1}$$

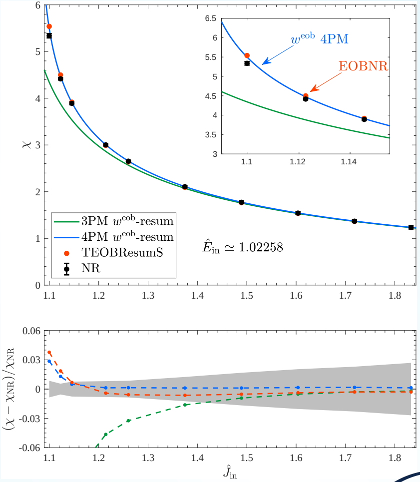
determining the boundary between scattering and plunge.

We were able to extend the parameter-space covered by nonspinning numerical simulations.



Comparison to EOB-NR models

This agreement is outstanding even when compared to PN-based EOB-NR models such as `TEOBResumS`.



Inversion formula

The formula linking the EOB potential and the respective scattering angles is:

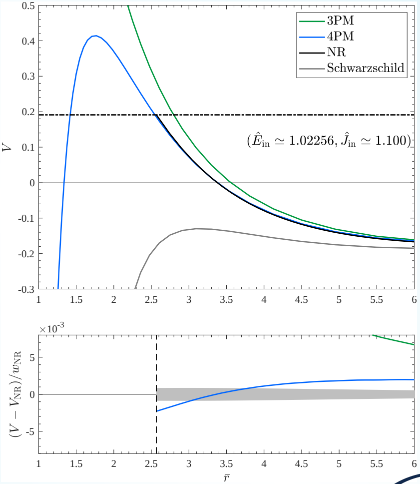
$$\pi + \chi\,(\gamma, j) = 2\,j \int_0^{\bar{u}_{\max}(\gamma,j)} \frac{d\bar{u}}{\sqrt{p_\infty^2 + w(\bar{u},\gamma) - j^2\bar{u}^2}}\,, \quad \text{with} \quad \bar{u} \equiv \frac{1}{\bar{r}}\,.$$

This means we can extract information about the underlying gravitational potential if we know the scattering angles. In particular, we make use of Firsov's inversion formula:

$$\ln\left[1 + \frac{w(\bar{u}, p_\infty)}{p_\infty^2}\right] = \frac{2}{\pi}\int_{\bar{r}|p(\bar{r},\gamma)|}^{\infty} dj\,\frac{\chi(\gamma, j)}{\sqrt{j^2 - \bar{r}^2\,p^2(\bar{r},\gamma)}}\,,$$

NR potential

It is then possible to *invert* the procedure and use the sequence of (constant-energy) scattering angles to obtain an NR gravitational potential.



High E results